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Applying fixed point methodologies to solve a class of matrix difference equations for a new class of operators

Hasanen A. Hammad[1,2], Mohamed Elmursi[2], Rashwan A. Rashwan[3] and Hüseyin Işık[4*]

*Correspondence:
isikhuseyin76@gmail.com
[4]Department of Engineering Science, Bandırma Onyedi Eylül University, Bandırma 10200, Balıkesir, Turkey
Full list of author information is available at the end of the article

Abstract

The goal of this paper is to present a new class of operators satisfying the Prešić-type rational η-contraction condition in the setting of usual metric spaces. New fixed point results are also obtained for these operators. Our results generalize, extend, and unify many papers in this direction. Moreover, two examples are derived to support and document our theoretical results. Finally, to strengthen our paper and its contribution to applications, some convergence results for a class of matrix difference equations are investigated.

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1 Introduction and preliminaries

In 1922, Banach [1] presented his principle which states: A self-mapping $\beth$ defined on a complete metric space $(\mho,\varpi)$ has a unique fixed point (FP), i.e., $\zeta^*\in\mho$, $\zeta^*=\beth\zeta^*$, provided $\beth$ is a contraction, that is, for a constant $\alpha\in(0,1)$, we have

$$\varpi(\beth\zeta_1,\beth\zeta_2)\le\alpha\varpi(\zeta_1,\zeta_2),\quad\forall\zeta_1,\zeta_2\in\mho.$$

Due to the ease of this principle and its essence, which is related to many applications in various branches of mathematics, many researchers have created various supplements and additions. For example, see [2–8].

From now until the end of our manuscript, we will consider z as a positive integer and $(\mho,\varpi)$ as a complete metric space.

In 1965, Banach FP theorem was extended by Prešić [9]. He used his results to ensure the convergence of a certain type of sequences as follows:

Theorem 1.1 ([9]) *Let* $\beth:\mho^z\to\mho$ *be a mapping satisfying the condition below*:

$$\varpi\big(\beth(\zeta_1,\zeta_2,\dots,\zeta_z),\beth(\zeta_2,\dots\zeta_z,\zeta_{z+1})\big)$$

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$$\le \gamma_1 \varpi(\zeta_1,\zeta_2) + \gamma_2 \varpi(\zeta_2,\zeta_3) + \cdots + \gamma_z \varpi(\zeta_z,\zeta_{z+1}) \tag{1.1}$$

for all $\zeta_1,\ldots,\zeta_{z+1} \in \mho$, *where* $\gamma_1,\gamma_2,\ldots,\gamma_z$ *are nonnegative constants so that* $\sum_{j=1}^{z}\gamma_j < 1$. *Then there is a unique point* $\zeta^* \in \mho$ *so that* $\beth(\zeta^*,\ldots,\zeta^*) = \zeta^*$. *Also if, for any chosen points* $\zeta_1,\ldots,\zeta_z$ *in* $\mho$ *and for* $l \in \mathbb{N}$,

$$\zeta_{l+z} = \beth(\zeta_l,\zeta_{l+1},\ldots,\zeta_{l+z-1}), \tag{1.2}$$

then the sequence $\{\zeta_l\}$ *is convergent and* $\zeta^* = \lim_{l\to\infty}\zeta_l = \beth(\lim_{l\to\infty}\zeta_l, \lim_{l\to\infty}\zeta_l,\ldots, \lim_{l\to\infty}\zeta_l)$.

Remark 1.2 From Theorem 1.1, we note the following:

- An operator $\beth : \mho^z \to \mho$ fulfilling (1.1) is said to be a Prešić operator;
- A point $\zeta^* \in \mho$ is called an FP of $\beth$ if $\beth(\zeta^*,\ldots,\zeta^*) = \zeta^*$;
- If we put $z = 1$, we directly obtain the Banach contraction principle (BCP).

The results of Prešić [9] have been generalized by Ćirić and Prešić [10] as follows:

Theorem 1.3 ([10]) *Assume that* $\beth : \mho^z \to \mho$ *satisfies*

$$\begin{aligned}&\varpi\bigl(\beth(\zeta_1,\zeta_2,\ldots,\zeta_z), \beth(\zeta_2,\ldots,\zeta_z,\zeta_{z+1})\bigr)\\ &\quad\le \gamma \max\bigl\{\varpi(\zeta_1,\zeta_2), \varpi(\zeta_2,\zeta_3),\ldots,\varpi(\zeta_z,\zeta_{z+1})\bigr\},\end{aligned} \tag{1.3}$$

for any $\zeta_1,\ldots,\zeta_{z+1} \in \mho$, *where* $\gamma \in (0,1)$. *Then there is* $\zeta^* \in \mho$ *so that* $\beth(\zeta^*,\ldots,\zeta^*) = \zeta^*$. *Further, for any chosen points* $\zeta_1,\ldots,\zeta_z \in \mho$, *the sequence* $\{\zeta_l\}$ *described in* (1.2) *is convergent and*

$$\lim_{l\to\infty}\zeta_l = \beth\Bigl(\lim_{l\to\infty}\zeta_l, \lim_{l\to\infty}\zeta_l,\ldots,\lim_{l\to\infty}\zeta_l\Bigr).$$

Also, if

$$\varpi\bigl(\beth(\zeta^*,\ldots,\zeta^*), \beth(\zeta',\ldots,\zeta')\bigr) < \varpi(\zeta^*,\zeta')$$

holds for all $\zeta^*,\zeta' \in \mho$ *with* $\zeta^* \ne \zeta'$, *then the FP is unique in* $\mho$.

Pâcurar [11] was able to present a convergence theorem for Prešić–Kannan contraction. For more details along this line of research, we refer the readers to [12–14].

Theorem 1.4 ([11]) *Suppose that* $\beth : \mho^z \to \mho$. *If there is* $\gamma \in \mathbb{R}$ *with* $\gamma z(1+z) \in (0,1)$ *so that the inequality*

$$\varpi\bigl(\beth(\zeta_1,\ldots,\zeta_z), \beth(\zeta_2,\ldots,\zeta_{z+1})\bigr) \le \gamma \sum_{j=1}^{z+1} \varpi\bigl(\zeta_j, \beth(\zeta_j,\ldots,\zeta_j)\bigr), \tag{1.4}$$

is satisfied for each $(\zeta_1,\ldots,\zeta_{z+1}) \in \mho^{z+1}$, *then*

- $\sqsupset$ *possesses a unique FP* $\zeta^* \in \mho$,
- *for any chosen points* $\zeta_1, \dots, \zeta_z \in \mho$, *the sequence* $\{\zeta_l\}$ *described in* (1.2) *converges to* ζ^*.

In 2014, a new contribution to the extension of the BCP was highlighted by Jleli and Samet [15]. They presented a new type of contraction, called η-contraction, and obtained generalized results under appropriate conditions.

Definition 1.5 ([15]) Let $\eta : (0,\infty) \to (1,\infty)$ be a mapping such that:

(η_1) η is nondecreasing;

(η_2) for each $\{r_l\} \subseteq \mathbb{R}^+$, $\lim_{l\to\infty} \eta(r_l) = 1$ iff $\lim_{l\to\infty} r_l = 0$;

(η_3) there are $\ell \in (0,1)$ and $u \in (0,\infty)$ so that $\lim_{r\to 0^+}(\frac{\eta(r)-1}{r^\ell}) = u$.

A mapping $\sqsupset : \mho \to \mho$ is called an η-contraction if $\exists \gamma \in (0,1)$ and a function η satisfying (η_1)–(η_3) so that

$$\sqsupset\zeta \neq \sqsupset\vartheta \quad \Longrightarrow \quad \eta\big(\varpi(\sqsupset\zeta, \sqsupset\vartheta)\big) \leq \big[\eta\big(\varpi(\zeta,\vartheta)\big)\big]^\gamma, \quad \text{for all } \zeta,\vartheta \in \mho.$$

The set of all mappings $\eta : (0,\infty) \to (1,\infty)$ will be denoted by ∇.

Theorem 1.6 ([15]) *An operator* $\sqsupset : \mho \to \mho$ *has a unique FP provided that* $\sqsupset$ *is an* η-*contraction.*

Numerous academics have discussed the concept of η-contraction, and important theoretical and practical findings have been documented that justify the use of FPs in nonlinear analysis under different spatial constraints. We advise the reader to view [16, 17] for further details.

In [18], two classes of matrix equations have been investigated by Ran and Reurings as follows:

$$\xi = \eth \pm \sum_{j=1}^{m} \wp_j^* \xi \wp_j, \tag{1.5}$$

where $\eth$ is an $n \times n$ positive definite matrix and $\wp_j$ are arbitrary $n \times n$ matrices. Under some hypotheses, they established the existence and uniqueness of positive definite solutions to (1.5). Duan et al. [19] generalized system (1.5) by making a small change as follows:

$$\xi = \eth \pm \sum_{j=1}^{m} \wp_j^* \xi^{\rho_j} \wp_j,$$

where $0 < |\rho_j| < 1$. They investigated the existence and uniqueness of a positive definite solution to such an equation on the basis of a fixed point theorem for mixed monotone mappings. This form of matrix equation frequently occurs in a variety of fields, including ladder networks [20, 21], dynamic programming [22, 23], control theory [24, 25], etc.

In the setting of η-contraction and matrix equations, our paper is organized as follows. Section 1 is devoted to providing previous contributions to our studied problem in terms of definitions and theory useful in understanding our manuscript. In Sect. 2, the convergence of iterative sequences of the Presšíc-type rational η-contraction mappings in complete metric spaces is discussed. Also, nontrivial examples are obtained to support the theoretical results. Ultimately, in Sect. 3, the obtained results are applied to obtain convergence results for a class of matrix difference equations as a kind of application.

2 Main results

We start this part with the following definition:

Definition 2.1 We say that a mapping $\sqsupset : \mho^z \to \mho$ is a Prešić-type rational η-contraction (*PTR η-C*, for short) if there is some $\gamma \in (0,1)$ so that

$$\eta\big(\varpi(\sqsupset(\zeta_1,\ldots,\zeta_z),\sqsupset(\zeta_2,\ldots,\zeta_{z+1}))\big) \le \left\{\eta\left(\max\left\{\frac{\varpi(\zeta_j,\zeta_{j+1})}{1+\varpi(\zeta_j,\zeta_{j+1})} : 1 \le j \le z\right\}\right)\right\}^{\gamma} \tag{2.1}$$

for each $(\zeta_1,\ldots,\zeta_{z+1}) \in \mho^{z+1}$ with $\sqsupset(\zeta_1,\ldots,\zeta_z) \neq \sqsupset(\zeta_2,\ldots,\zeta_{z+1})$.

It should be noted that if $\eta(r) = e^{\sqrt{r}}$, then *PTR η-C* reduces to

$$\varpi(\sqsupset(\zeta_1,\ldots,\zeta_z),\sqsupset(\zeta_2,\ldots,\zeta_{z+1})) \le \gamma^2\left(\max\left\{\frac{\varpi(\zeta_j,\zeta_{j+1})}{1+\varpi(\zeta_j,\zeta_{j+1})} : 1 \le j \le z\right\}\right), \tag{2.2}$$

for each $(\zeta_1,\ldots,\zeta_{z+1}) \in \mho^{z+1}$, $\sqsupset(\zeta_1,\ldots,\zeta_z) \neq \sqsupset(\zeta_2,\ldots,\zeta_{z+1})$.

In addition, if $(\zeta_1,\ldots,\zeta_{z+1}) \in \mho^{z+1}$ is such that $\sqsupset(\zeta_1,\ldots,\zeta_z) = \sqsupset(\zeta_2,\ldots,\zeta_{z+1})$, then condition (2.2) is more general than (1.3), so the mapping $\sqsupset$ in (2.2) extends and unifies Cirić–Prešić contraction.

Remark 2.2 Every *PTR η-C* $\sqsupset$ is a Prešić mapping by (η_1) and (1.4), that is,

$$\begin{aligned}\varpi(\sqsupset(\zeta_1,\ldots,\zeta_z),\sqsupset(\zeta_2,\ldots,\zeta_{z+1})) &\le \gamma \max\left\{\frac{\varpi(\zeta_j,\zeta_{j+1})}{1+\varpi(\zeta_j,\zeta_{j+1})} : 1 \le j \le z\right\} \\ &< \max\left\{\varpi(\zeta_j,\zeta_{j+1}) : 1 \le j \le z\right\}.\end{aligned}$$

for each $(\zeta_1,\ldots,\zeta_{z+1}) \in \mho^{z+1}$ with $\sqsupset(\zeta_1,\ldots,\zeta_z) \neq \sqsupset(\zeta_2,\ldots,\zeta_{z+1})$. Thus, each *PTR η-C* $\sqsupset$ is a continuous function.

Now, our first result is as follows:

Theorem 2.3 *Suppose that $\sqsupset : \mho^z \to \mho$ is a PTR η-C. Then for any chosen points $\zeta_1,\ldots,\zeta_z \in \mho$, the sequence $\{\zeta_l\}$ described in (1.2) is convergent to $\zeta^* \in \mho$ and ζ^* is an FP of $\sqsupset$. In addition, if $\sqsupset(\zeta^*,\ldots,\zeta^*) \neq \sqsupset(\zeta',\ldots,\zeta')$ with*

$$\eta\big(\varpi(\sqsupset(\zeta^*,\ldots,\zeta^*),\sqsupset(\zeta',\ldots,\zeta'))\big) \le \big[\eta\big(\varpi(\zeta^*,\zeta')\big)\big]^{\gamma}$$

for $\zeta^*,\zeta' \in \mho$ such that $\zeta^* \neq \zeta'$, then the point ζ^* is unique.

Proof Let $\zeta_1,\ldots,\zeta_z$ be arbitrary z elements in $\mho$ and for $l \in \mathbb{N}$ the sequence $\{\zeta_l\}$ is defined in (1.2). If for some $l_0 = \{1,2,\ldots,z\}$ one has $\zeta_{l_0} = \zeta_{l_0+1}$, then

$$\zeta_{l_0+z} = \sqsupset(\zeta_{l_0},\zeta_{l_0+1},\ldots,\zeta_{l_0+z-1}) = \sqsupset(\zeta_{l_0+z},\zeta_{l_0+z},\ldots,\zeta_{l_0+z}),$$

which means that ζ_{l_0+z} is an FP of $\sqsupset$ and there is no further proof needed. So, we consider $\zeta_{l+z} \neq \zeta_{l+z+1}$ for all $l \in \mathbb{N}$. Put $\beth_{l+z} = \varpi(\zeta_{l+z}, \zeta_{l+z+1})$ and

$$\phi = \max\left\{\frac{\varpi(\zeta_1,\zeta_2)}{1+\varpi(\zeta_1,\zeta_2)}, \frac{\varpi(\zeta_2,\zeta_3)}{1+\varpi(\zeta_2,\zeta_3)}, \dots, \frac{\varpi(\zeta_z,\zeta_{z+1})}{1+\varpi(\zeta_z,\zeta_{z+1})}\right\}.$$

Then for all $l \in \mathbb{N}$ and $\phi > 0$, we have $\beth_{l+z} > 0$. Thus, for $l \leq z$, we obtain

$$\begin{aligned}
1 &< \eta(\beth_{z+1}) \\
&= \eta\big(\varpi(\zeta_{z+1}, \zeta_{z+2})\big) \\
&= \eta\big(\varpi\big(\sqsupset(\zeta_1,\zeta_2,\dots,\zeta_z), \sqsupset(\zeta_2,\zeta_3,\dots,\zeta_{z+1})\big)\big) \\
&\leq \left[\eta\left(\max\left\{\frac{\varpi(\zeta_j,\zeta_{j+1})}{1+\varpi(\zeta_j,\zeta_{j+1})} : 1 \leq j \leq z\right\}\right)\right]^{\gamma} \\
&= \big[\eta(\phi)\big]^{\gamma}.
\end{aligned}$$

Also,

$$\begin{aligned}
1 &< \eta(\beth_{z+2}) \\
&= \eta\big(\varpi(\zeta_{z+2}, \zeta_{z+3})\big) \\
&= \eta\big(\varpi\big(\sqsupset(\zeta_2,\zeta_3,\dots,\zeta_{z+1}), \sqsupset(\zeta_3,\zeta_4,\dots,\zeta_{z+2})\big)\big) \\
&\leq \left[\eta\left(\max\left\{\frac{\varpi(\zeta_j,\zeta_{j+1})}{1+\varpi(\zeta_j,\zeta_{j+1})} : 2 \leq j \leq z+1\right\}\right)\right]^{\gamma} \\
&= \big[\eta(\phi)\big]^{\gamma^2}.
\end{aligned}$$

Continuing in the same pattern, for $l \geq 1$, we get

$$\begin{aligned}
1 &< \eta(\beth_{z+l}) \\
&= \eta\big(\varpi(\zeta_{l+z}, \zeta_{l+z+1})\big) \\
&= \eta\big(\varpi\big(\sqsupset(\zeta_l,\zeta_{l+1},\dots,\zeta_{l+z-1}), \sqsupset(\zeta_{l+1},\zeta_{l+2},\dots,\zeta_{l+z})\big)\big) \\
&\leq \big[\eta(\phi)\big]^{\gamma^l}.
\end{aligned} \tag{2.3}$$

Taking $l \to \infty$ in (2.3) and using (η_2), we have

$$\lim_{l\to\infty} \eta(\beth_{z+l}) = 1 \quad \Longleftrightarrow \quad \lim_{l\to\infty} \beth_{z+l} = 0.$$

Based on (η_3), there are $\ell \in (0,1)$ and $u \in (0,\infty)$ so that

$$\lim_{l\to\infty}\left(\frac{\eta(\beth_{z+l}) - 1}{\beth_{z+l}^{\ell}}\right) = u.$$

Assume that $u < \infty$ and $\nu = \frac{u}{2} > 0$. By the definition of the limit, there is $l_1 \in \mathbb{N}$ such that

$$\left|\frac{\eta(\beth_{z+l}) - 1}{\beth_{z+l}^{\ell}} - u\right| \leq \nu, \quad \forall l > l_1.$$

It follows that

$$\frac{\eta(\daleth_{z+l})-1}{\daleth_{z+l}^{\ell}} \geq u-\nu = \frac{u}{2} = \nu, \quad \forall l > l_1.$$

Set $\frac{1}{\nu} = q$, then

$$l\daleth_{z+l}^{\ell} \leq lq\big(\eta(\daleth_{z+l})-1\big), \quad \forall l > l_1.$$

Suppose that $u = \infty$ and $\nu > 0$. By the definition of the limit, there is $l_1 \in \mathbb{N}$ such that

$$\nu \leq \frac{\eta(\daleth_{z+l})-1}{\daleth_{z+l}^{\ell}}, \quad \forall l > l_1.$$

This implies after taking $\frac{1}{\nu} = q$ that

$$l\daleth_{z+l}^{\ell} \leq lq\big(\eta(\daleth_{z+l})-1\big), \quad \forall l > l_1.$$

Thus, in both cases, there are $l_1 \in \mathbb{N}$ and $q > 0$ so that

$$l\daleth_{z+l}^{\ell} \leq lq\big(\eta(\daleth_{z+l})-1\big), \quad \forall l > l_1.$$

Applying (2.3), we get

$$l\daleth_{z+l}^{\ell} \leq lq\big(\big[\eta(\phi)\big]^{\gamma^l}-1\big), \quad \forall l > l_1,$$

and, when $l \to \infty$, have

$$\lim_{l\to\infty} l\daleth_{z+l}^{\ell} = 0.$$

Thus, there is $l_2 \in \mathbb{N}$ and $q > 0$ such that

$$l\daleth_{z+l}^{\ell} \leq 1, \quad \forall l > l_2.$$

Hence we can write

$$\daleth_{z+l} \leq \frac{1}{l^{\frac{1}{\ell}}}, \quad \forall l > l_2.$$

Now, we clarify that $\{\zeta_l\}$ is a Cauchy sequence. For $b > l > l_2$, one can write

$$\begin{aligned}
\varpi(\zeta_{z+l},\zeta_{z+b}) &= \varpi\big(\sqsupset(\zeta_l,\ldots,\zeta_{z+l-1}),\sqsupset(\zeta_b,\ldots,\zeta_{z+b-1})\big)\\
&\leq \varpi\big(\sqsupset(\zeta_l,\ldots,\zeta_{z+l-1}),\sqsupset(\zeta_{l+1},\ldots,\zeta_{z+l})\big)\\
&\quad + \varpi\big(\sqsupset(\zeta_{l+1},\ldots,\zeta_{z+l}),\sqsupset(\zeta_{l+2},\ldots,\zeta_{z+l+1})\big)\\
&\quad + \cdots + \varpi\big(\sqsupset(\zeta_{b-1},\ldots,\zeta_{z+b-2}),\sqsupset(\zeta_b,\ldots,\zeta_{z+b-1})\big)\\
&= \varpi(\zeta_{z+l},\zeta_{z+l+1}) + \varpi(\zeta_{z+l+1},\zeta_{z+l+2}) + \cdots + \varpi(\zeta_{z+b-1},\zeta_{z+b})
\end{aligned}$$

$$\begin{aligned}&= \daleth_{l+z} + \daleth_{l+z+1} + \cdots + \daleth_{z+b-1}\\&= \sum_{s=l}^{b-1} \daleth_{s+z} < \sum_{s=l}^{\infty} \daleth_{s+z} \le \sum_{s=l}^{\infty} \frac{1}{s^{\frac{1}{\ell}}} < \infty,\end{aligned}$$

hence it follows that $\{\zeta_l\}$ is a Cauchy sequence in $(\mho, \varpi)$. The completeness of $\mho$ yields that there is $\zeta^* \in \mho$ such that

$$\lim_{l,b\to\infty} \varpi(\zeta_l, \zeta_b) = \lim_{l\to\infty} \varpi\left(\zeta_l, \zeta^*\right) = 0.$$

Because $\sqsupset$ is continuous, we have

$$\begin{aligned}\hbar &= \lim_{l\to\infty} \zeta_{l+z}\\&= \lim_{l\to\infty} \sqsupset(\zeta_l, \zeta_{l+1}, \ldots, \zeta_{z+l-1})\\&= \sqsupset\Big(\lim_{l\to\infty} \zeta_l, \lim_{l\to\infty} \zeta_{l+1}, \ldots, \lim_{l\to\infty} \zeta_{z+l-1}\Big)\\&= \sqsupset\big(\zeta^*, \zeta^*, \ldots, \zeta^*\big).\end{aligned}$$

For uniqueness, assume that ζ^* and ζ' are two distinct FP of the mapping $\sqsupset$, i.e., $\zeta^* = \sqsupset(\zeta^*, \zeta^*, \ldots, \zeta^*)$ and $\zeta' = \sqsupset(\zeta', \zeta', \ldots, \zeta')$ with $\zeta^* \ne \zeta'$. Hence, by hypothesis (2.1), we can write

$$\begin{aligned}\eta\big(\varpi(\zeta^*, \zeta')\big) &= \eta\big(\varpi\big(\sqsupset(\zeta^*, \zeta^*, \ldots, \zeta^*), \sqsupset(\zeta', \zeta', \ldots, \zeta')\big)\big)\\&\le \left[\eta\left(\frac{\varpi(\zeta^*, \zeta')}{1 + \varpi(\zeta^*, \zeta')}\right)\right]^{\gamma}\\&\le \big[\eta\big(\varpi(\zeta^*, \zeta')\big)\big]^{\gamma},\end{aligned}$$

a contradiction, as $\gamma \in (0, 1)$. Therefore, $\zeta^* = \zeta'$. This ends the proof. □

The following examples support Theorem 2.3.

Example 2.4 Let $\{\zeta_l\}$ be a sequence defined as follows:

$$\begin{cases}\zeta_1 = 3,\\ \zeta_2 = 3 + 7,\\ \vdots\\ \zeta_l = 3 + 7 + 11 + \cdots + (4l - 1) = l(2l + 1).\end{cases}$$

Assume that $\mho = \{\zeta_l : l \in \mathbb{N}\}$ and $\varpi(\widetilde{\zeta}, \widehat{\zeta}) = |\widetilde{\zeta} - \widehat{\zeta}|$. Clearly, $(\mho, \varpi)$ is a complete metric space. Define a mapping $\sqsupset : \mho^3 \to \mho$ by

$$\sqsupset(\zeta_l, \widetilde{\zeta}_l, \widehat{\zeta}_l) = \begin{cases}\frac{\zeta_{l-1} + \widetilde{\zeta}_{l-1} + \widehat{\zeta}_{l-1}}{3}, & \text{when } l > 1,\\ \frac{\zeta_1 + \widetilde{\zeta}_1 + \widehat{\zeta}_1}{3}, & \text{otherwise.}\end{cases}$$

For $l > 5$, we have

$$\begin{aligned}
&\varpi\left(\sqsupset(\zeta_{l-4},\zeta_{l-3},\zeta_{l-2}),\sqsupset(\zeta_{l-2},\zeta_{l-1},\zeta_{l})\right)\\
&\quad=\varpi\left(\frac{\zeta_{l-5}+\zeta_{l-4}+\zeta_{l-3}}{3},\frac{\zeta_{l-3}+\zeta_{l-2}+\zeta_{l-1}}{3}\right)\\
&\quad=\frac{1}{3}\Big|\big((l-5)(2l-9)+(l-4)(2l-7)+(l-3)(2l-5)\big)\\
&\qquad-\big((l-3)(2l-5)+(l-2)(2l-3)+(l-1)(2l-1)\big)\Big|\\
&\quad=\frac{1}{3}\Big|\big(6l^2-45l+88\big)-\big(6l^2-21l+22\big)\Big|\\
&\quad=\frac{1}{3}|24l-66|=8l-22,
\end{aligned}$$

and

$$\begin{aligned}
&\max\left\{\varpi\left((\zeta_{l-4},\zeta_{l-3},\zeta_{l-2}),(\zeta_{l-2},\zeta_{l-1},\zeta_{l})\right)\right\}\\
&\quad=\max\left\{\begin{array}{c}\big|(l-4)(2l-7)-(l-2)(2l-3)\big|,\\ \big|(l-3)(2l-5)-(l-1)(2l-1)\big|,\\ \big|(l-2)(2l-3)-l(2l+1)\big|\end{array}\right\}\\
&\quad=\max\big\{(8l-22),(8l-14),(6l-6)\big\}=(8l-14).
\end{aligned}$$

Now,

$$\lim_{l\to\infty}\frac{\varpi(\sqsupset(\zeta_{l-4},\zeta_{l-3},\zeta_{l-2}),\sqsupset(\zeta_{l-2},\zeta_{l-1},\zeta_{l}))}{\max\{\varpi((\zeta_{l-4},\zeta_{l-3},\zeta_{l-2}),(\zeta_{l-2},\zeta_{l-1},\zeta_{l}))\}}=\lim_{l\to\infty}\frac{8l-22}{8l-14}=1.$$

Thus,

$$\varpi\left(\sqsupset(\zeta_{l-4},\zeta_{l-3},\zeta_{l-2}),\sqsupset(\zeta_{l-2},\zeta_{l-1},\zeta_{l})\right)\le\gamma\max\left\{\varpi\left((\zeta_{l-4},\zeta_{l-3},\zeta_{l-2}),(\zeta_{l-2},\zeta_{l-1},\zeta_{l})\right)\right\}$$

does not hold for $\gamma\in(0,1)$, which implies that assumption (1.1) of Theorem 1.1 is not fulfilled. Now, define the mapping $\eta:(0,\infty)\to(1,\infty)$ by $\eta(s)=e^{\frac{se^s}{1+s}}$. We can easily verify that $\eta\in\nabla$ and $\sqsupset$ is *PTR* η-*C*. Indeed, the inequality

$$\begin{aligned}
&e^{\sqrt{\varpi(\sqsupset(\zeta_i,\zeta_{i+1},\zeta_{i+2}),\sqsupset(\zeta_{i+2},\zeta_{i+3},\zeta_{i+4}))\frac{e^{\varpi(\sqsupset(\zeta_i,\zeta_{i+1},\zeta_{i+2}),\sqsupset(\zeta_{i+2},\zeta_{i+3},\zeta_{i+4}))}}{1+\varpi(\sqsupset(\zeta_i,\zeta_{i+1},\zeta_{i+2}),\sqsupset(\zeta_{i+2},\zeta_{i+3},\zeta_{i+4}))}}}\\
&\quad\le e^{\gamma\sqrt{\varpi((\zeta_i,\zeta_{i+1},\zeta_{i+2}),(\zeta_{i+2},\zeta_{i+3},\zeta_{i+4}))\frac{e^{\varpi((\zeta_i,\zeta_{i+1},\zeta_{i+2}),(\zeta_{i+2},\zeta_{i+3},\zeta_{i+4}))}}{1+\varpi((\zeta_i,\zeta_{i+1},\zeta_{i+2}),(\zeta_{i+2},\zeta_{i+3},\zeta_{i+4}))}}},
\end{aligned}\tag{2.4}$$

holds for $\sqsupset(\zeta_i,\zeta_{i+1},\zeta_{i+2})\neq\sqsupset(\zeta_{i+2},\zeta_{i+3},\zeta_{i+4})$, $i=1,2,\ldots$, and for some $\gamma\in(0,1)$. Inequality (1.1) is equivalent to

$$\begin{aligned}
&\varpi\left(\sqsupset(\zeta_i,\zeta_{i+1},\zeta_{i+2}),\sqsupset(\zeta_{i+2},\zeta_{i+3},\zeta_{i+4})\right)e^{\frac{\varpi(\sqsupset(\zeta_i,\zeta_{i+1},\zeta_{i+2}),\sqsupset(\zeta_{i+2},\zeta_{i+3},\zeta_{i+4}))}{1+\varpi(\sqsupset(\zeta_i,\zeta_{i+1},\zeta_{i+2}),\sqsupset(\zeta_{i+2},\zeta_{i+3},\zeta_{i+4}))}}\\
&\quad\le\gamma^2\max\left\{\varpi\left((\zeta_i,\zeta_{i+1},\zeta_{i+2}),(\zeta_{i+2},\zeta_{i+3},\zeta_{i+4})\right)\right\}e^{\frac{\max\{\varpi((\zeta_i,\zeta_{i+1},\zeta_{i+2}),(\zeta_{i+2},\zeta_{i+3},\zeta_{i+4}))\}}{1+\max\{\varpi((\zeta_i,\zeta_{i+1},\zeta_{i+2}),(\zeta_{i+2},\zeta_{i+3},\zeta_{i+4}))\}}}.
\end{aligned}$$

So, for some $\gamma \in (0,1)$, we can write

$$\frac{\varpi(\sqsupset(\zeta_i,\zeta_{i+1},\zeta_{i+2}),\sqsupset(\zeta_{i+2},\zeta_{i+3},\zeta_{i+4}))e^{\frac{\varpi(\sqsupset(\zeta_i,\zeta_{i+1},\zeta_{i+2}),\sqsupset(\zeta_{i+2},\zeta_{i+3},\zeta_{i+4}))}{1+\varpi(\sqsupset(\zeta_i,\zeta_{i+1},\zeta_{i+2}),\sqsupset(\zeta_{i+2},\zeta_{i+3},\zeta_{i+4}))}}}{\max\{\varpi((\zeta_i,\zeta_{i+1},\zeta_{i+2}),(\zeta_{i+2},\zeta_{i+3},\zeta_{i+4}))\}e^{\frac{\max\{\varpi((\zeta_i,\zeta_{i+1},\zeta_{i+2}),(\zeta_{i+2},\zeta_{i+3},\zeta_{i+4}))\}}{1+\max\{\varpi((\zeta_i,\zeta_{i+1},\zeta_{i+2}),(\zeta_{i+2},\zeta_{i+3},\zeta_{i+4}))\}}}} \le \gamma^2.$$

Now, we will discuss the following cases:

(i) If $i = l = 1$, we get

$$\begin{aligned}
&\frac{\varpi(\sqsupset(\zeta_1,\zeta_2,\zeta_3),\sqsupset(\zeta_3,\zeta_4,\zeta_5))e^{\frac{\varpi(\sqsupset(\zeta_1,\zeta_2,\zeta_3),\sqsupset(\zeta_3,\zeta_4,\zeta_5))}{1+\varpi(\sqsupset(\zeta_1,\zeta_2,\zeta_3),\sqsupset(\zeta_3,\zeta_4,\zeta_5))}}}{\max\{\varpi((\zeta_1,\zeta_2,\zeta_3),(\zeta_3,\zeta_4,\zeta_5))\}e^{\frac{\max\{\varpi((\zeta_1,\zeta_2,\zeta_3),(\zeta_3,\zeta_4,\zeta_5))\}}{1+\max\{\varpi((\zeta_1,\zeta_2,\zeta_3),(\zeta_3,\zeta_4,\zeta_5))\}}}} \\
&\quad = \frac{\varpi(\frac{\zeta_1+\zeta_2+\zeta_3}{3},\frac{\zeta_3+\zeta_4+\zeta_5}{3})e^{\frac{\varpi(\frac{\zeta_1+\zeta_2+\zeta_3}{3},\frac{\zeta_3+\zeta_4+\zeta_5}{3})}{1+\varpi(\frac{\zeta_1+\zeta_2+\zeta_3}{3},\frac{\zeta_3+\zeta_4+\zeta_5}{3})}}}{\max\{\varpi((\zeta_1,\zeta_2,\zeta_3),(\zeta_3,\zeta_4,\zeta_5))\}e^{\frac{\max\{\varpi((\zeta_1,\zeta_2,\zeta_3),(\zeta_3,\zeta_4,\zeta_5))\}}{1+\max\{\varpi((\zeta_1,\zeta_2,\zeta_3),(\zeta_3,\zeta_4,\zeta_5))\}}}} \\
&\quad = \frac{\varpi(\frac{34}{3},\frac{112}{3})e^{\frac{\varpi(\frac{34}{3},\frac{112}{3})}{1+\varpi(\frac{34}{3},\frac{112}{3})}}}{\max\{\varpi((3,10,21),(21,36,55))\}e^{\frac{\max\{\varpi((3,10,21),(21,36,55))\}}{1+\max\{\varpi((3,10,21),(21,36,55))\}}}} \\
&\quad \le \frac{26e^{26}}{34e^{34}} = \frac{13}{17}e^{-8} < e^{-2}.
\end{aligned}$$

(ii) If $i = l > 1$, we obtain

$$\begin{aligned}
&\frac{\varpi(\sqsupset(\zeta_l,\zeta_{l+1},\zeta_{l+2}),\sqsupset(\zeta_{l+2},\zeta_{l+3},\zeta_{l+4}))e^{\frac{\varpi(\sqsupset(\zeta_l,\zeta_{l+1},\zeta_{l+2}),\sqsupset(\zeta_{l+2},\zeta_{l+3},\zeta_{l+4}))}{1+\varpi(\sqsupset(\zeta_l,\zeta_{l+1},\zeta_{l+2}),\sqsupset(\zeta_{l+2},\zeta_{l+3},\zeta_{l+4}))}}}{\max\{\varpi((\zeta_l,\zeta_{l+1},\zeta_{l+2}),(\zeta_{l+2},\zeta_{l+3},\zeta_{l+4}))\}e^{\frac{\max\{(\zeta_l,\zeta_{l+1},\zeta_{l+2}),(\zeta_{l+2},\zeta_{l+3},\zeta_{l+4})\}}{1+\max\{(\zeta_l,\zeta_{l+1},\zeta_{l+2}),(\zeta_{l+2},\zeta_{l+3},\zeta_{l+4})\}}}} \\
&\quad = \frac{\varpi(\frac{\zeta_{l-1}+\zeta_l+\zeta_{l+1}}{3},\frac{\zeta_{l+1}+\zeta_{l+2}+\zeta_{l+3}}{3})e^{\frac{\varpi(\frac{\zeta_{l-1}+\zeta_l+\zeta_{l+1}}{3},\frac{\zeta_{l+1}+\zeta_{l+2}+\zeta_{l+3}}{3})}{1+\varpi(\frac{\zeta_{l-1}+\zeta_l+\zeta_{l+1}}{3},\frac{\zeta_{l+1}+\zeta_{l+2}+\zeta_{l+3}}{3})}}}{\max\{\varpi((\zeta_l,\zeta_{l+1},\zeta_{l+2}),(\zeta_{l+2},\zeta_{l+3},\zeta_{l+4}))\}e^{\frac{\max\{\varpi((\zeta_l,\zeta_{l+1},\zeta_{l+2}),(\zeta_{l+2},\zeta_{l+3},\zeta_{l+4}))\}}{1+\max\{\varpi((\zeta_l,\zeta_{l+1},\zeta_{l+2}),(\zeta_{l+2},\zeta_{l+3},\zeta_{l+4}))\}}}} \\
&\quad = \frac{|\frac{6l^2+3l+4}{3}-\frac{6l^2+27l+34}{3}|e^{\frac{|\frac{6l^2+3l+4}{3}-\frac{6l^2+27l+34}{3}|}{1+|\frac{6l^2+3l+4}{3}-\frac{6l^2+27l+34}{3}|}}}{\max\{|8l+10|,|8l+18|,|8l+26|\}e^{\frac{\max\{|8l+10|,|8l+18|,|8l+26|\}}{1+\max\{|8l+10|,|8l+18|,|8l+26|\}}}} \\
&\quad = \frac{(8l+10)e^{\frac{(8l+10)}{1+(8l+10)}}}{(8l+26)e^{\frac{(8l+26)}{1+(8l+26)}}} \le \frac{(8l+10)e^{(8l+10)}}{(8l+26)e^{(8l+26)}}e^{-16} < e^{-2},
\end{aligned}$$

with $\gamma = \frac{1}{e}$. Hence all requirements of Theorem 2.3 are fulfilled and the point $(1,1,1)$ is the unique FP of $\sqsupset$.

Example 2.5 Assume that $\mho = [0,1]$, $\varpi(\widetilde{\zeta},\widehat{\zeta}) = |\widetilde{\zeta}-\widehat{\zeta}|$, and $\sqsupset : \mho^z \to \mho$ is described by

$$\sqsupset(\zeta_1,\dots,\zeta_l) = \frac{\zeta_1+\zeta_l}{8l}, \quad \forall \zeta_1,\dots,\zeta_l \in \mho.$$

Let $\eta:(0,\infty)\to(1,\infty)$ be a mapping defined by $\eta(s)=e^{\sqrt{\frac{s}{1+s}}}$. Since $e^{\sqrt{\frac{s}{1+s}}}\leq e^{\sqrt{s}}$, we can see from [15] that $\eta\in\nabla$. Now, for $\zeta_1,\zeta_2,\dots,\zeta_{l+1}\in\mho$, one can write

$$\varpi\big(\beth(\zeta_1,\dots,\zeta_l),\beth(\zeta_2,\dots,\zeta_{l+1})\big)>0,$$

and

$$\begin{aligned}
&\eta\big(\varpi\big(\beth(\zeta_1,\dots,\zeta_l),\beth(\zeta_2,\dots,\zeta_{l+1})\big)\big)\\
&\quad=e^{\sqrt{\frac{\varpi(\beth(\zeta_1,\dots,\zeta_l),\beth(\zeta_2,\dots,\zeta_{l+1}))}{1+\varpi(\beth(\zeta_1,\dots,\zeta_l),\beth(\zeta_2,\dots,\zeta_{l+1}))}}}\\
&\quad=e^{\sqrt{\frac{(\frac{1}{8l})|(\zeta_1-\zeta_2)+(\zeta_l-\zeta_{l+1})|}{1+|(\zeta_1-\zeta_2)+(\zeta_l-\zeta_{l+1})|}}}\\
&\quad=e^{(\frac{1}{2\sqrt{2l}})\sqrt{\frac{|(\zeta_1-\zeta_2)+(\zeta_l-\zeta_{l+1})|}{1+|(\zeta_1-\zeta_2)+(\zeta_l-\zeta_{l+1})|}}}\\
&\quad\leq e^{(\frac{1}{\sqrt{2}})\sqrt{\frac{\max\{\varpi(\zeta_1,\zeta_2),\varpi(\zeta_l,\zeta_{l+1})\}}{1+\max\{\varpi(\zeta_1,\zeta_2),\varpi(\zeta_l,\zeta_{l+1})\}}}}\\
&\quad\leq e^{(\frac{1}{\sqrt{2}})\sqrt{\max\{\frac{\varpi(\zeta_j,\zeta_{j+1})}{1+\varpi(\zeta_j,\zeta_{j+1})}:1\leq j\leq z\}}}\\
&\quad=\left[\eta\left(\max\left\{\frac{\varpi(\zeta_j,\zeta_{j+1})}{1+\varpi(\zeta_j,\zeta_{j+1})}:1\leq j\leq z\right\}\right)\right]^{\gamma},
\end{aligned}$$

with $\gamma=\frac{1}{\sqrt{2}}$. In addition, for all $\zeta^*,\zeta'\in\mho$ with $\zeta^*\neq\zeta'$, we obtain

$$\varpi\big(\beth(\zeta^*,\zeta^*,\dots,\zeta^*),\beth(\zeta',\zeta',\dots,\zeta')\big)=\frac{|\zeta^*-\zeta'|}{8l}>0,$$

and

$$\begin{aligned}
\eta\big(\varpi\big(\beth(\zeta^*,\zeta^*,\dots,\zeta^*),\beth(\zeta',\zeta',\dots,\zeta')\big)\big)&=\eta\left(\frac{|\zeta^*-\zeta'|}{8l}\right)\\
&=e^{\sqrt{(\frac{\frac{|\zeta^*-\zeta'|}{8l}}{1+\frac{|\zeta^*-\zeta'|}{8l}})}}\\
&\leq e^{(\frac{1}{2\sqrt{2l}})\sqrt{(\frac{|\zeta^*-\zeta'|}{1+|\zeta^*-\zeta'|})}}\\
&\leq e^{\frac{1}{\sqrt{2}}\sqrt{(\frac{|\zeta^*-\zeta'|}{1+|\zeta^*-\zeta'|})}}\\
&=\big[\eta\big(\varpi(\zeta^*,\zeta')\big)\big]^{\gamma},
\end{aligned}$$

with $\gamma=\frac{1}{\sqrt{2}}$. Hence, all assumptions of Theorem 2.3 are fulfilled. In addition, for some chosen $\zeta_1,\dots,\zeta_l\in\mho$, the sequence $\{\zeta_l\}$ defined in (2.3) converges to $\zeta^*=0$, which is the unique FP of $\beth$.

If we put $\eta(s)=e^{\sqrt{s}}$ in Theorem 2.3, we get the result below.

Corollary 2.6 *Consider $\daleth : \mho^z \to \mho$ is a given mapping and suppose there is $\gamma \in (0,1)$ such that*

$$\varpi\big(\daleth(\zeta_1,\dots,\zeta_z),\daleth(\zeta_2,\dots,\zeta_{z+1})\big) \le \gamma^2\left(\max\left\{\frac{\varpi(\zeta_j,\zeta_{j+1})}{1+\varpi(\zeta_j,\zeta_{j+1})} : 1 \le j \le z\right\}\right). \tag{2.5}$$

Then for any chosen points $\zeta_1,\dots,\zeta_z \in \mho$, the sequence $\{\zeta_l\}$ described in (1.2) converges to $\zeta^* \in \mho$ and $\zeta^* = \daleth(\zeta^*,\dots,\zeta^*)$. Moreover, if

$$\varpi\big(\daleth(\zeta^*,\dots,\zeta^*),\daleth(\zeta',\dots,\zeta')\big) \le \gamma^2\varpi(\zeta^*,\zeta')$$

holds for all $\zeta^*,\zeta' \in \mho$ with $\zeta^* \neq \zeta'$, Then the point ζ^* is a unique FP of the mapping $\daleth$.

Corollary 2.7 *Assume that $\daleth : \mho^z \to \mho$ is a given mapping and there are nonnegative constants $\gamma_1,\gamma_2,\dots,\gamma_z$ with $\gamma_1+\gamma_2+\dots+\gamma_z < 1$ such that*

$$\begin{aligned}\varpi\big(\daleth(\zeta_1,\dots,\zeta_z),\daleth(\zeta_2,\dots,\zeta_{z+1})\big) \le{}& \gamma_1\frac{\varpi(\zeta_1,\zeta_2)}{1+\varpi(\zeta_1,\zeta_2)}+\gamma_2\frac{\varpi(\zeta_2,\zeta_3)}{1+\varpi(\zeta_2,\zeta_3)}\\&+\dots+\gamma_z\frac{\varpi(\zeta_z,\zeta_{z+1})}{1+\varpi(\zeta_z,\zeta_{z+1})},\end{aligned} \tag{2.6}$$

for each $(\zeta_1,\dots,\zeta_{z+1}) \in \mho^{z+1}$ with $\daleth(\zeta_1,\dots,\zeta_z) \neq \daleth(\zeta_2,\dots,\zeta_{z+1})$. Then for any chosen points $\zeta_1,\dots,\zeta_z \in \mho$, the sequence $\{\zeta_l\}$, given by (1.2) converges to $\zeta^* \in \mho$, where ζ^* is a unique FP of $\daleth$.

Proof It is clear that (2.6) implies (2.5) with $\gamma^2 = \gamma_1+\gamma_2+\dots+\gamma_z$.

Now, suppose that $\zeta^*,\zeta' \in \mho$ with $\zeta^* \neq \zeta$, Based on (2.6), one can obtain

$$\begin{aligned}&\varpi\big(\daleth(\zeta^*,\zeta^*,\dots,\zeta^*),\daleth(\zeta',\zeta',\dots,\zeta')\big)\\&\quad= \varpi\big(\daleth(\zeta^*,\dots,\zeta^*),\daleth(\zeta^*,\dots,\zeta^*,\zeta')\big)\\&\qquad+\varpi\big(\daleth(\zeta^*,\dots,\zeta^*,\zeta'),\daleth(\zeta^*,\dots,\zeta^*,\zeta',\zeta')\big)\\&\qquad+\dots+\varpi\big(\daleth(\zeta^*,\dots,\zeta',\zeta'),\daleth(\zeta',\dots,\zeta',\zeta')\big)\\&\quad\le(\gamma_z+\gamma_{z-1}+\dots+\gamma_z)\frac{\varpi(\zeta^*,\zeta')}{1+\varpi(\zeta^*,\zeta')}\\&\quad\le\gamma^2\varpi(\zeta^*,\zeta').\end{aligned}$$

Thus, the conditions of Corollary 2.6 hold. □

If we take a large class of functions ∇, for example,

$$\eta(s) = 2-\frac{2}{\pi}\arctan\left(\frac{1}{s^\theta}\right),$$

where $\theta \in (0,1)$ and $s > 0$, we obtain the following theorem from Theorem 2.3.

Theorem 2.8 *Suppose that $\daleth : \mho^z \to \mho$ is a given mapping. If there are a mapping $\eta \in \nabla$ and constants $\gamma, \theta \in (0,1)$ such that*

$$2 - \frac{2}{\pi}\arctan\left(\frac{1}{[\varpi(\daleth(\zeta_1,\ldots,\zeta_z),\daleth(\zeta_2,\ldots,\zeta_{z+1}))]^\theta}\right) \\ \leq \left[2 - \frac{2}{\pi}\arctan\left(\frac{1}{[\max\{\frac{\varpi(\zeta_j,\zeta_{j+1})}{1+\varpi(\zeta_j,\zeta_{j+1})} : 1 \leq j \leq z\}]^\theta}\right)\right]^\gamma,$$

for each $(\zeta_1,\ldots,\zeta_{z+1}) \in \mho^{z+1}$ with $\daleth(\zeta_1,\ldots,\zeta_z) \neq \daleth(\zeta_2,\ldots,\zeta_{z+1})$, then for any chosen points $\zeta_1,\ldots,\zeta_z \in \mho$, the sequence $\{\zeta_l\}$, given by (1.2) converges to $\zeta^* \in \mho$. Then ζ^* is a unique FP of $\daleth$. Moreover, if

$$2 - \frac{2}{\pi}\arctan\left(\frac{1}{[\varpi(\daleth(\zeta^*,\ldots,\zeta^*),\daleth(\zeta',\ldots,\zeta'))]^\theta}\right) \\ \leq \left[2 - \frac{2}{\pi}\arctan\left(\frac{1}{(\varpi(\zeta^*,\zeta'))^\theta}\right)\right]^\gamma,$$

holds for $\zeta^*, \zeta' \in \mho$ with $\zeta^* \neq \zeta'$, then the point ζ^* is a unique FP of the mapping $\daleth$.

Remark 2.9 It should be noted that:

- Our Theorem 2.3 unifies and extends Theorem 1.3 in [10] and Theorem 1.2 in [9].
- Corollary 1 in [15] can be obtained directly from Theorem 2.3 putting $\gamma = 1$ and neglecting the denominator of the contractive condition (2.1).
- If we take $\gamma = 1$ and neglect the denominators of the contractivity conditions of Corollaries 2.6 and 2.7, we obtain the BCP [1].

3 Application to matrix difference equations

In this part, the symbols $\aleph(N)$ for $(N \geq 2)$, $\eth$, $\wp$, $\wp^*$, and φ refer to the family of $N \times N$ Hermitian positive definite matrices, an $N \times N$ Hermitian positive semidefinite matrix, an $N \times N$ nonsingular matrix, the conjugate transpose of $\wp$, and the function from $\aleph(N)$ to $\aleph(N)$, respectively.

Now, the definition of the equilibrium point is stated as follows:

Definition 3.1 Consider $\daleth : \mho^z \to \mho$ as a given mapping. For any $\zeta_1,\ldots,\zeta_z \in \mho$, define a recursive sequence $\{\zeta_l\}$ by

$$\zeta_{l+z} = \daleth(\zeta_l, \zeta_{l+1}, \ldots, \zeta_{l+z-1}), \tag{3.1}$$

for each $l \in \mathbb{N}$. We say that a point $\overline{\zeta} \in \mho$ is an equilibrium point of (3.1) if the hypothesis below holds:

$$\overline{\zeta} = \daleth(\overline{\zeta}, \overline{\zeta}, \ldots, \overline{\zeta}). \tag{3.2}$$

Definition 3.2 If for all $\zeta_1, \zeta_2, \ldots, \zeta_z \in \mho$ one has $\varpi(\zeta_l, \overline{\zeta}) \to 0$ as $l \to \infty$, then an equilibrium point is called a global attractor.

Here, we explore the global attractivity for the following recursive sequence:

$$\zeta_{l+z} = \eth + \frac{1}{z}\sum_{j=0}^{z-1} \wp^{*}\varphi\left(\frac{\zeta_{l+j}}{1+\zeta_{l+j}}\right)\wp, \quad \forall l \geq 1. \tag{3.3}$$

Before applying the theoretical results, we analyze the Thompson metric ϖ on $\aleph(N)$, which is described as

$$\varpi(\wp_1,\wp_2) = \max\left\{\log W\left(\frac{\wp_1}{\wp_2}\right), \log W\left(\frac{\wp_2}{\wp_1}\right)\right\},$$

for $\wp_1,\wp_2 \in \aleph(N)$, where $W(\frac{\wp_1}{\wp_2}) = \inf\{\Lambda > 0 : \wp_1 \leq \Lambda\wp_2\} = \Lambda^{+}(\wp_2^{-\frac{1}{2}}\wp_1^{\frac{-1}{2}}\wp_2)$, that is, W is the maximal eigenvalue of $\wp_2^{-\frac{1}{2}}\wp_1^{\frac{-1}{2}}\wp_2$. Here $\wp_1 \leq \wp_2$ means that $\wp_2 - \wp_1$ is positive semidefinite and $\wp_1 < \wp_2$ means that $\wp_2 - \wp_1$ is positive definite.

Based on ϖ, which defined as

$$\varpi(\wp_1,\wp_2) = \left\|\ln\left(\wp_1^{-\frac{1}{2}}\wp_2\wp_1^{\frac{-1}{2}}\right)\right\|,$$

$\aleph(N)$ is a complete metric space [26], where $\|\cdot\|$ is a spectral norm [27].

Now, let us start with the exciting characteristics of ϖ, i.e., for any nonsingular matrix W,

$$\varpi(\wp_1,\wp_2) = \varpi\left(\wp_1^{-1},\wp_2^{-1}\right) = \varpi\left(W^{*}\wp_1 W, W^{*}\wp_2 W\right). \tag{3.4}$$

The second important result is the nonpositive curvature property of ϖ in the form of

$$\varpi\left(\wp_1^{r},\wp_2^{r}\right) \leq h\varpi(\wp_1,\wp_2), \quad h \in [0,1]. \tag{3.5}$$

Based on (3.4) and (3.5), we can write

$$\varpi\left(W^{*}\wp_1^{r}W, W^{*}\wp_2^{r}W\right) \leq |h|\varpi\left(\wp_1^{r},\wp_2^{r}\right), \quad h \in [-1,1],$$

for all $\wp_1,\wp_2 \in \aleph(N)$.

Lemma 3.3 ([28]) *For each $\wp_1,\wp_2,\wp_3,\wp_4 \in \aleph(N)$, we get*

$$\varpi(\wp_1+\wp_2,\wp_3+\wp_4) \leq \max\left\{\varpi(\wp_1,\wp_3),\varpi(\wp_2,\wp_4)\right\}.$$

Moreover, for all positive semidefinite $\wp_1,\wp_2,\wp_3 \in \aleph(N)$, we have

$$\varpi(\wp_1+\wp_2,\wp_1+\wp_3) \leq \varpi(\wp_2,\wp_3).$$

Consider that $\varphi : \aleph(N) \to \aleph(N)$ is an η-contraction related with ϖ. For $W_1, W_2, \ldots, W_z \in \aleph(N)$, let $\{W_l\} \subset \aleph(N)$ be a sequence defined by (3.3).

Now, we can state and prove our main theorem for this part.

Theorem 3.4 *Equation* (3.3) *has a global attractor* $\overline{\zeta} \in \aleph(N)$, *which is a unique equilibrium point.*

Proof Define an operator $\beth : \aleph(N)^z \to \aleph(N)$ by

$$\begin{aligned}
&\beth(W_1, W_2, \dots, W_z)\\
&\quad = \eth + \frac{1}{z}\left[\wp^*\varphi\left(\frac{W_1}{1+W_1}\right)\wp + \wp^*\varphi\left(\frac{W_2}{1+W_2}\right)\wp + \cdots + \wp^*\varphi\left(\frac{W_z}{1+W_z}\right)\wp\right],
\end{aligned}$$

for all $W_1, W_2, \dots, W_z \in \aleph(N)$.

Let $W_1, W_2, \dots, W_{z+1} \in \aleph(N)$. According to Lemma 3.3, we obtain

$$\begin{aligned}
&\varpi\left(\beth(W_1, W_2, \dots, W_z), \beth(W_2, W_3, \dots, W_{z+1})\right)\\
&\quad = \varpi\left(\eth + \frac{1}{z}\sum_{j=1}^{z}\wp^*\varphi\left(\frac{W_i}{1+W_i}\right)\wp, \eth + \frac{1}{z}\sum_{k=2}^{z+1}\wp^*\varphi\left(\frac{W_k}{1+W_k}\right)\wp\right)\\
&\quad \le \varpi\left(\frac{1}{z}\sum_{j=1}^{z}\wp^*\varphi\left(\frac{W_i}{1+W_i}\right)\wp, \frac{1}{z}\sum_{k=2}^{z+1}\wp^*\varphi\left(\frac{W_k}{1+W_k}\right)\wp\right)\\
&\quad = \varpi\left(\sum_{j=1}^{z}\left(\frac{1}{\sqrt{z}}\wp\right)^*\varphi\left(\frac{W_i}{1+W_i}\right)\left(\frac{1}{\sqrt{z}}\wp\right), \sum_{k=2}^{z+1}\left(\frac{1}{\sqrt{z}}\wp\right)^*\varphi\left(\frac{W_k}{1+W_k}\right)\left(\frac{1}{\sqrt{z}}\wp\right)\right).
\end{aligned}$$

Set $\Delta = \frac{1}{\sqrt{z}}\wp$. Then by Lemma 3.3, we have

$$\begin{aligned}
&\varpi\left(\beth(W_1, W_2, \dots, W_z), \beth(W_2, W_3, \dots, W_{z+1})\right)\\
&\quad \le \varpi\left(\sum_{j=1}^{z}\Delta^*\varphi\left(\frac{W_i}{1+W_i}\right)\Delta, \sum_{k=2}^{z+1}\Delta^*\varphi\left(\frac{W_k}{1+W_k}\right)\Delta\right)\\
&\quad = \varpi\begin{pmatrix}\Delta^*\varphi(\frac{W_1}{1+W_1})\Delta + \Delta^*\varphi(\frac{W_2}{1+W_2})\Delta + \cdots + \Delta^*\varphi(\frac{W_z}{1+W_z})\Delta,\\ \Delta^*\varphi(\frac{W_2}{1+W_2}) + \Delta^*\varphi(\frac{W_3}{1+W_{32}})\Delta + \cdots + \Delta^*\varphi(\frac{W_{k+1}}{1+W_{k+1}})\Delta\end{pmatrix}\\
&\quad \le \max\left\{\begin{array}{l}\varpi\left(\Delta^*\varphi\left(\frac{W_1}{1+W_1}\right)\Delta, \Delta^*\varphi\left(\frac{W_2}{1+W_2}\right)\Delta\right),\\ \varpi\left(\Delta^*\varphi\left(\frac{W_3}{1+W_3}\right)\Delta, \Delta^*\varphi\left(\frac{W_4}{1+W_4}\right)\Delta\right),\\ \varpi\left(\Delta^*\varphi\left(\frac{W_z}{1+W_z}\right)\Delta, \Delta^*\varphi\left(\frac{W_{k+1}}{1+W_{k+1}}\right)\Delta\right)\end{array}\right\}\\
&\quad = \max\left\{\varpi\left(\Delta^*\varphi\left(\frac{W_k}{1+W_k}\right)\Delta, \Delta^*\varphi\left(\frac{W_{k+1}}{1+W_{k+1}}\right)\Delta\right)\right\},
\end{aligned} \tag{3.6}$$

for $k = 1, 2, \dots, z$. Because $\wp$ is nonsingular, Δ is also nonsingular. Using (3.6) for all $k = 1, 2, \dots, z$, we can write

$$\varpi\left(\Delta^*\varphi\left(\frac{W_k}{1+W_k}\right)\Delta, \Delta^*\varphi\left(\frac{W_{k+1}}{1+W_{k+1}}\right)\Delta\right) = \varpi\left(\varphi\left(\frac{W_k}{1+W_k}\right), \varphi\left(\frac{W_{k+1}}{1+W_{k+1}}\right)\right).$$

Since φ is an η-contraction, for all $k = 1, 2, \dots, z$, we have

$$\eta\left(\varpi\left(\Delta^*\varphi\left(\frac{W_k}{1+W_k}\right)\Delta, \Delta^*\varphi\left(\frac{W_{k+1}}{1+W_{k+1}}\right)\Delta\right)\right) \leq \left[\eta\left(\frac{\varpi(W_k, W_{k+1})}{1+\varpi(W_k, W_{k+1})}\right)\right]^{\lambda},$$

for some $\gamma \in (0,1)$. Thus, we get

$$\begin{aligned}
&\eta\big(\varpi\big(\beth(W_1, W_2, \dots, W_z), \beth(W_2, W_3, \dots, W_{z+1})\big)\big) \\
&\quad \leq \left[\eta\left(\max\left\{\frac{\varpi(W_k, W_{k+1})}{1+\varpi(W_k, W_{k+1})}\right\} : 1 \leq j \leq z\right)\right]^{\gamma},
\end{aligned}$$

for $W_1, W_2, \dots, W_{z+1} \in \aleph(N)$. Hence, by Theorem 2.3, there exists an FP of the mapping $\beth$, which is a global attractor equilibrium point $\overline{\zeta} \in \aleph(N)$. Moreover, for $W_1, W_2 \in \aleph(N)$, such that $\beth(W_1, W_1, \dots, W_1) \neq \beth(W_2, W_2, \dots, W_2)$, one can write

$$\begin{aligned}
&\eta\big(\varpi\big(\beth(W_1, W_1, \dots, W_1), \beth(W_2, W_2, \dots, W_2)\big)\big) \\
&\quad = \eta\left(\varpi\left(\eth + \wp^*\varphi\left(\frac{W_1}{1+W_1}\right)\wp, \eth + \wp^*\varphi\left(\frac{W_2}{1+W_2}\right)\wp\right)\right) \\
&\quad \leq \eta\left(\varpi\left(\wp^*\varphi\left(\frac{W_1}{1+W_1}\right)\wp, \wp^*\varphi\left(\frac{W_2}{1+W_2}\right)\wp\right)\right) \\
&\quad = \eta\left(\varpi\left(\varphi\left(\frac{W_1}{1+W_1}\right), \varphi\left(\frac{W_2}{1+W_2}\right)\right)\right) \\
&\quad \leq \left[\eta\left(\varpi\left(\frac{W_1}{1+W_1}, \frac{W_2}{1+W_2}\right)\right)\right]^{\gamma} \\
&\quad \leq \big[\eta\big(\varpi(W_1, W_2)\big)\big].
\end{aligned}$$

Again, based on Theorem 2.3, the equilibrium point is unique. □

4 Conclusion and future work

In this study, a new concept of Prešić-type rational η-contraction mappings has been introduced and the convergence of iterative sequences of such contractions has been discussed in the setting of complete metric spaces. The new theory improves and extends many results existing in the literature. Some nontrivial examples have been provided to support the results obtained herein. Moreover, some convergence results for a class of matrix difference equations have been derived. As future works, the authors are looking for generalizations of these results to multivalued Prešić-type rational η-contraction mappings and studying the convergence of the matrix difference equations numerically.

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Author details
[1]Department of Mathematics, Unaizah College of Sciences and Arts, Qassim University, Buraydah 52571, Saudi Arabia. [2]Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt. [3]Department of Mathematics, Faculty of Science, Assiut University, Assiut 71516, Egypt. [4]Department of Engineering Science, Bandırma Onyedi Eylül University, Bandırma 10200, Balıkesir, Turkey.

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