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Blowup for semilinear wave equation with space-dependent damping and combined nonlinearities

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Abstract

This paper is concerned with the Cauchy problem for semilinear wave equation with space-dependent scattering damping and combined nonlinearities. The blowup results of solution are established by introducing proper test functions. Moreover, upper bound lifespan estimates of a solution to the Cauchy problem with small initial values are derived. To the best of our knowledge, the results in Theorems 1.1–1.2 are new.

MSC: 35L70; 58J45

Keywords: Space-dependent damping; Semilinear wave equation; Combined nonlinearities; Blowup; Lifespan estimates

1 Introduction and main results

In this work, we consider the following Cauchy problem of wave equation with space-dependent damping and combined nonlinearities:

$$\begin{cases} u_{tt} - \Delta u + \frac{\mu}{(1+|x|)^{\beta}} u_t = |u_t|^p + |u|^q, & (t,x) \in [0,T) \times \mathbb{R}^n, \\ u(0,x) = \varepsilon f(x), \quad u_t(0,x) = \varepsilon g(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $\mu > 0$, $\beta > 1$, $p > 1$, $q > 1$, $n \geq 2$. The compactly supported nonnegative initial values satisfy $(f,g) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ $(n \geq 2)$ and

$$f(x) \geq 0, g(x) \geq 0 \text{ a.e.}, \quad f(x) = g(x) = 0 \quad \text{for } |x| > 1. \tag{1.2}$$

In addition, $f(x), g(x) \not\equiv 0$.

The study of formation of singularity for semilinear wave equation has a long history (see detailed illustrations in [3, 5, 9, 11, 22–25, 27–30, 33, 34, 39–42] and the references therein). In fact, problem (1.1) originates from the following three problems:

$$\begin{cases} u_{tt} - \Delta u = |u|^p, & (t,x) \in [0,T) \times \mathbb{R}^n, n \geq 1, \\ u(0,x) = \varepsilon f(x), \quad u_t(0,x) = \varepsilon g(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.3}$$

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$$\begin{cases} u_{tt} - \Delta u = |u_t|^p, & (t,x) \in [0,T) \times \mathbb{R}^n, n \geq 1, \\ u(0,x) = \varepsilon f(x), & u_t(0,x) = \varepsilon g(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{1.4}$$

and

$$\begin{cases} u_t - \Delta u = |u|^p, & (t,x) \in [0,T) \times \mathbb{R}^n, n \geq 1, \\ u(0,x) = \varepsilon f(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.5}$$

Problem (1.3) is known as the Strauss conjecture (see [35]), which shows that the solution blows up in finite time when $1 < p \leq p_S(n)$ ($n \geq 2$) and $p_S(1) = +\infty$ for $n = 1$, whereas the solution exists globally in time when $p > p_S(n)$. Here $p_S(n)$ is the Strauss critical exponent, which is the positive root of the quadratic equation

$$\gamma(p,n) = -(n-1)p^2 + (n+1)p + 2 = 0.$$

Problem (1.4) is known as the Glassey conjecture (see [6]), where the Glassey critical exponent is $p_G(n) = \frac{n+1}{n-1}$. It is shown in [4] that the Cauchy problem of heat equation (1.5) possesses the Fujita critical exponent $p_F(n) = 1 + \frac{2}{n}$.

Scholars investigated the blowup dynamics of a semilinear wave equation with damping term

$$u_{tt} - \Delta u + h(u_t) = f(u, u_t), \tag{1.6}$$

where $h(u_t) = \frac{\mu}{(1+t)^\beta} u_t$, $\frac{\mu}{(1+|x|)^\beta} u_t$ ($\mu > 0$, $\beta \in \mathbb{R}$) and $f(u,u_t) = |u|^p, |u_t|^p$, $|u_t|^p + |u|^q$ ($p > 1$, $q > 1$). When the critical exponent of the damped wave equation (1.6) is related to the Srauss exponent $p_S(n)$ or the Glassey exponent $p_G(n)$, we say that the equation behaves like that of the wave equation. This means that the damping term in the equation makes no effect. When the critical exponent is related to the Fujita exponent $p_F(n)$, we say that the damping term makes an effect. According to the range of β, we use the following table to show the effect of damping terms (we can see it in [18, 21]).

Blowup and global existence results in connection with the semilinear wave equation with time-dependent damping $\frac{\mu}{(1+t)^\beta} u_t$ are established in [1, 13, 16, 20, 31, 37, 38]. Energy estimates of solution to semilinear wave equation with space-dependent damping are derived in [14, 15, 36]. Nishihara et al. [32] investigated the blowup and global existence for a semilinear wave equation with space- and time-dependent damping. In the present paper, we mainly concentrate on the problem with space-dependent scattering damping case $\frac{\mu}{(1+|x|)^\beta} u_t$ ($\beta > 1$). Namely, the behavior of a solution is similar to that of the wave equation in this case. Lai and Tu [17] considered upper bound lifespan estimates of a solution to the

Table 1 The effect of the damping terms

Range of β	Damping term	
	$\frac{\mu}{(1+t)^\beta} u_t$	$\frac{\mu}{(1+|x|)^\beta} u_t$
$(-\infty,-1)$	Overdamping	Effective
$[-1,1)$	Effective	
1	Scaling invariant	Scaling invariant
$(1,+\infty)$	Scattering	Scattering

wave equation with space-dependent damping $\frac{\mu}{(1+|x|)^\beta}u_t$ ($\beta > 2$, $n \geq 2$) and $f(u,u_t) = |u|^p$, $|u_t|^p$ for both subcritical and critical exponents. Especially, for the power nonlinearity $|u|^p$ ($\frac{n}{n-1} < p \leq p_S(n)$) and derivative-type nonlinearity $|u_t|^p$ ($1 < p \leq p_G(n)$), they obtained the same critical exponents and upper bound lifespan estimates of solutions as in the situation without damping by using the test function method. Lai et al. [17] obtained upper bound lifespan estimate of solution when $f(u,u_t) = |u|^p$ and $\beta > 1$. Meanwhile, the lifespan estimate for the case $1 < p < \frac{n}{n-1}$ was also improved.

We are in the position to present some known results related to the semilinear wave equation (1.6) with combined nonlinearities $f(u,u_t) = |u_t|^p + |u|^q$. Han and Zhou [10] obtained an upper bound lifespan estimate of solution to the Cauchy problem without damping term by constructing a proper test function and solving ordinary differential inequalities. Hidano et al. [12] established the sharp lower bound lifespan estimate of a solution to the problem. Dai et al. [2] derived the sharp lifespan estimate of a solution to the nonlinear wave equation when $p \geq q_S(n)$ and $q = q_S(n)$ ($n = 2,3$), where $q_S(n)$ is the Strauss critical exponent of the semilinear wave equation with power nonlinearity $|u|^q$. Lai and Takamura [19] illustrated blowup results and upper bound lifespan estimates of a solution to the problem with time-dependent damping term $\frac{\mu}{(1+t)^\beta}u_t$ ($\beta > 1$) by using a multiplier and iteration argument. Blowup of a solution to the problem with scale-invariant damping $\frac{\mu}{1+t}u_t$ was investigated by applying test function approach (see [7, 8]). Liu and Wang [26] consider problem (1.1) for the more general nonlinearity $f(u,u_t) = c_1|u_t|^p + c_2|u|^q$ on asymptotically Euclidean manifolds. Upper bound lifespan estimates of solution with different values of c_1 and c_2 are obtained. In addition, the existence of a solution is established.

Inspired by the works [10, 17, 19, 21], we consider blowup and upper bound lifespan estimates of a solution to problem (1.1). To our best knowledge, the blowup for the space-dependent damped wave equation with combined nonlinearities has not been discussed yet. The purpose of this paper is to fill this gap. We establish upper bound lifespan estimates of a solution. It is worth mentioning that in this paper, we employ the test function method different from the technique in [10, 19]. We bear in mind that lifespan estimates of solutions to the problems with space-dependent damping $\frac{\mu}{(1+|x|)^\beta}u_t$ ($\beta > 2$) and $f(u,u_t) = |u|^p, |u_t|^p$ are investigated in [21]. Thanks to the work [17], we obtain upper bound lifespan estimates of a solution to problem (1.1) with $\frac{\mu}{(1+|x|)^\beta}u_t$ ($\beta > 1$) and combined nonlinearities $|u_t|^p + |u|^q$ (see the new results in Theorems 1.1–1.2 in this paper).

The main results in this paper are described as follows.

Theorem 1.1 *Let $n \geq 2$, $\mu > 0$, and $\beta > 1$, and let f and g satisfy* (1.2). *Suppose that problem* (1.1) *has an energy solution u such that*

$$\operatorname{supp}(u,u_t) \subset \bigl\{(t,x) \in [0,T) \times \mathbb{R}^n \big| |x| \leq t+1\bigr\}.$$

Then we have the following lifespan estimates of solution:

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-\frac{2p(q-1)}{-(n-1)pq+(n-1)p+2q+2}}, & \max\{1,\frac{2}{n-1}\} < p < \frac{4n-2}{n-1}, \\ & \frac{n}{n-1} < q < 1 + \frac{4}{(n-1)p-2}, \\ C\varepsilon^{-\frac{2(p-1)}{n+1-(n-1)p}}, & 1 < p < \frac{n+1}{n-1}, 1 < q, \end{cases} \tag{1.7}$$

where C is a positive constant.

Theorem 1.2 *Let $n \geq 2$, $\mu > 0$, and $\beta > 1$, and let f and g satisfy* (1.2). *Suppose that problem* (1.1) *has an energy solution u such that*

$$\operatorname{supp}(u,u_t) \subset \{(t,x) \in [0,T) \times \mathbb{R}^n \mid |x| \leq t+1\}.$$

Then the lifespan estimates of solution satisfy

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-\frac{2q(p-1)}{-(n-1)qp+(n-1)q+2p}}, & 1<p, n=2,3 \text{ or } 1<p<\frac{n-1}{n-3}, n>3, \\ & 1<q<\frac{2p}{(n-1)(p-1)}, \\ C\varepsilon^{-\frac{2(q-1)}{(n+1)-(n-1)q}}, & 1<p, 1<q<\frac{n+1}{n-1}. \end{cases} \tag{1.8}$$

Remark 1.1 In Theorem 1.1, for $\max\{1+\frac{1}{2(n-1)}, \frac{2}{n-1}\} < p < \frac{n+1}{n-1}$ and $\frac{n}{n-1} < q < 2p-1$, we have

$$\frac{2p(q-1)}{-(n-1)pq+(n-1)p+2q+2} < \frac{2(p-1)}{n+1-(n-1)p},$$

where we have used the fact $2p-1 < 1+\frac{4}{(n-1)p-2}$ for $p < \frac{n+1}{n-1}$. When $\max\{1, \frac{2}{n-1}\} < p < \frac{n+1}{n-1}$ and $\max\{2p-1, \frac{n}{n-1}\} < q < 1+\frac{4}{(n-1)p-2}$, we obtain

$$\frac{2p(q-1)}{-(n-1)pq+(n-1)p+2q+2} > \frac{2(p-1)}{n+1-(n-1)p}.$$

We use Fig. 1 to make a simple description for $n=2$.

For $p,q \in B \cup C \cup E$, we have the first lifespan estimate in (1.7). For $p,q \in A \cup B \cup C \cup D$, we obtain the second lifespan estimate in (1.7), whereas for $p,q \in B$, the second lifespan estimate in (1.7) is better than the first one. For $p,q \in C$, the first lifespan estimate in (1.7) is better than the second one.

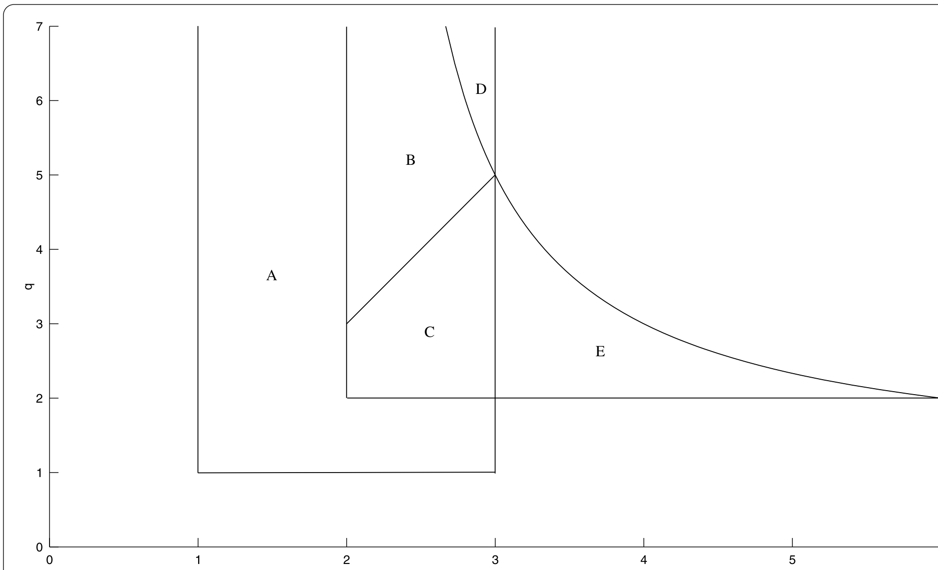

Figure 1 The case $n = 2$ in Theorem 1.1

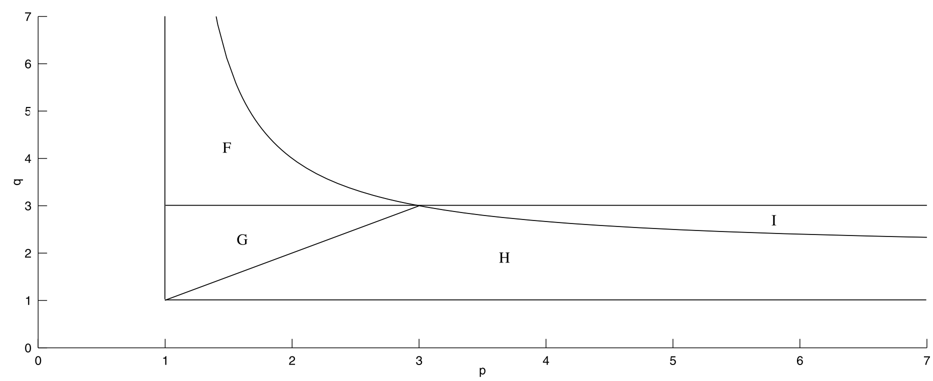

Figure 2 The case $n = 2$ in Theorem 1.2

Remark 1.2 In Theorem 1.2, for $1 < q < p < \frac{n+1}{n-1}$ or $\frac{n+1}{n-1} < p$ $(n = 2, 3)$, $\frac{n+1}{n-1} < p < \frac{n-1}{n-3}$ $(n > 3)$, $1 < q < \frac{2p}{(n-1)(p-1)}$, we have

$$\frac{2(q-1)}{(n+1)-(n-1)q} < \frac{2q(p-1)}{-(n-1)qp+(n-1)q+2p}.$$

When $1 < p < q < \frac{n+1}{n-1}$, we have

$$\frac{2(q-1)}{(n+1)-(n-1)q} > \frac{2q(p-1)}{-(n-1)qp+(n-1)q+2p}.$$

Similarly, we use Fig. 2 to illustrate the specific comparison for $n = 2$.

For $p, q \in F \cup G \cup H$, we obtain the first lifespan estimate in (1.8). For $p, q \in G \cup H \cup I$, we have the second lifespan estimate in (1.8). For $p, q \in G$, the first lifespan estimate in (1.8) is better than the second one, and for $p, q \in H$, the second lifespan estimate in (1.8) is better than the first one.

Remark 1.3 Let $n \geq 2$, $\mu > 0$, and $\beta > 1$. The assumptions in Theorems 1.1 and 1.2 hold. Combining the results in [17, 21] with (1.7) and (1.8), we derive

$$T(\varepsilon) \leq \begin{cases} \exp(C\varepsilon^{-(p-1)}), & p = p_G(n), q > \frac{n+3}{n-1}, \\ C\varepsilon^{-\Gamma_G(n,p)^{-1}}, & q > 2p-1, 1 < p < \frac{n+1}{n-1}, \\ C\varepsilon^{-\Gamma_1(p,q,n)^{-1}}, & \frac{n}{n-1} < q, p \leq q < 2p-1, \Gamma_1(p,q,n) > 0, \\ C\varepsilon^{-\Gamma_S(n,q)^{-1}}, & p > q, \frac{n}{n-1} < q < p_S(n), \\ C\varepsilon^{-\Gamma_2(p,q,n)^{-1}}, & p < q < \frac{n}{n-1}, \\ C\varepsilon^{-\frac{2(q-1)}{(n+1)-(n-1)q}}, & p > q, q < \frac{n}{n-1}, \\ \exp(C\varepsilon^{-q(q-1)}), & p \geq q = p_S(n), \end{cases} \tag{1.9}$$

where

$$\Gamma_1(n,p,q) = \frac{-(n-1)pq+(n-1)p+2q+2}{2p(q-1)},$$

$$\Gamma_2(n,p,q) = \frac{-(n-1)pq+(n-1)q+2p}{2q(p-1)},$$

$$\Gamma_G(n,p) = \frac{n+1-(n-1)p}{2(p-1)},$$
$$\Gamma_S(n,q) = \frac{\gamma(q,n)}{2q(q-1)},$$

$p_S(n)$ denotes the Strauss critical exponent, and $p_G(n)$ represents the Glassey critical exponent.

Throughout this paper, C denotes a positive constant independent of ε, which may vary from line to line.

2 Preliminaries

In this section, we present several basic definitions and lemmas.

Definition 2.1 A function u is called an energy solution of problem (1.1) on $[0,T)$ if

$$u \in \bigcap_{i=0}^{1} C^i\big([0,T);H^{1-i}(\mathbb{R}^n)\big) \cap C^1\big([0,T);L^p(\mathbb{R}^n)\big) \cap L^q_{\mathrm{loc}}\big([0,T)\times\mathbb{R}^n\big)$$

satisfies $u(0,x) = \varepsilon f(x)$ and $u_t(0,x) = \varepsilon g(x)$. Moreover, we have

$$\begin{aligned}
&\varepsilon\int_{\mathbb{R}^n} g(x)\varphi(0,x)\,dx + \int_0^T\int_{\mathbb{R}^n}\big(|u_t|^p + |u|^q\big)\varphi(t,x)\,dx\,dt \\
&\quad + \int_{\mathbb{R}^n}\frac{\mu}{(1+|x|)^\beta}\varepsilon f(x)\varphi(0,x)\,dx \\
&= \int_0^T\int_{\mathbb{R}^n}\big(-\partial_t u(t,x)\partial_t\varphi(t,x) + \nabla u(t,x)\nabla\varphi(t,x)\big)\,dx\,dt \\
&\quad - \int_0^T\int_{\mathbb{R}^n}\frac{\mu}{(1+|x|)^\beta}u(t,x)\partial_t\varphi(t,x)\,dx\,dt,
\end{aligned} \tag{2.1}$$

where $\varphi(t,x) \in C_0^\infty([0,T)\times\mathbb{R}^n)$ and $T \in (1, T(\varepsilon))$. Here $T(\varepsilon)$ represents the upper bound lifespan estimate of a solution to problem (1.1), which satisfies

$$T(\varepsilon) = \sup\{T > 0, \text{there exists an energy solution to problem (1.1)}\}.$$

Definition 2.2 The cutoff function $\eta(t) \in C^\infty([0,\infty))$ is defined by

$$\eta(t) = \begin{cases} 1, & t \le \frac{1}{2}, \\ \text{decreasing}, & t \in (\frac{1}{2}, 1), \\ 0, & t \ge 1, \end{cases}$$

which satisfies $|\eta'(t)|, |\eta''(t)| < C$. Let $\eta_T(t) = \eta(t/T)$ and $\gamma > 1$. We have that

$$\partial_t\eta_T^{2\gamma} = \frac{2\gamma}{T}\eta_T^{2\gamma-1}\eta',$$
$$\partial_t^2\eta_T^{2\gamma} = \frac{2\gamma(2\gamma-1)}{T^2}\eta_T^{2\gamma-2}\big|\eta'\big|^2 + \frac{2\gamma}{T^2}\eta_T^{2\gamma-1}\eta''.$$

Lemma 2.3 (Lemma 3.1 in [21]) *If $\beta > 0$, then for all $\alpha \in \mathbb{R}$ and a fixed constant R, there exists a positive constant C such that*

$$\int_0^{t+R} (1+r)^\alpha e^{-\beta(t-r)}\,dr \le C(t+R)^\alpha. \tag{2.2}$$

Lemma 2.4 (Lemma 2.5 in [17]) *Let $n \ge 2$, $\beta > 1$, and $\mu \ge 0$. Then the equation*

$$\Delta\phi(x) - \frac{\mu}{(1+|x|)^\beta}\phi(x) = \phi(x) \tag{2.3}$$

admits a solution $\phi(x)$. Moreover, there exists a constant $C_1 \in (0,1)$ such that

$$C_1\big(1+|x|\big)^{-\frac{n-1}{2}} e^{|x|} < \phi(x) < C_1^{-1}\big(1+|x|\big)^{-\frac{n-1}{2}} e^{|x|}. \tag{2.4}$$

Let $\psi(t,x) = e^{-t}\phi(x)$. Then we have

$$\partial_t^2\psi(t,x) - \Delta\psi(t,x) - \frac{\mu}{(1+|x|)^\beta}\partial_t\psi(t,x) = 0.$$

3 Proof of Theorem 1.1

In this section, we illustrate the proof of Theorem 1.1.

3.1 Case $p \ge q$

First, we choose $\varphi(t,x) = \eta_T^{2q'}$ as the test function, where q' satisfies $\frac{1}{q} + \frac{1}{q'} = 1$. From (2.1) we obtain

$$\begin{aligned}
&\varepsilon\int_{\mathbb{R}^n} g(x)\,dx + \int_0^T \int_{\mathbb{R}^n} \big(|u_t|^p + |u|^q\big)\eta_T^{2q'}\,dx\,dt + \int_{\mathbb{R}^n} \frac{\mu}{(1+|x|)^\beta}\varepsilon f(x)\,dx \\
&\quad= \int_0^T \int_{\mathbb{R}^n} u\partial_t^2\eta_T^{2q'}\,dx\,dt - \int_0^T \int_{\mathbb{R}^n} \frac{\mu}{(1+|x|)^\beta} u\partial_t\eta_T^{2q'}\,dx\,dt \\
&\quad= I_1 + I_2,
\end{aligned} \tag{3.1}$$

where we have used the fact that $\partial_t\eta_T(0) = 0$ and $\eta_T(T) = 0$.

Using the Hölder and Young inequalities, we have that for $q > \frac{n}{n-1}$,

$$\begin{aligned}
I_1 &= \int_0^T \int_{\mathbb{R}^n} u\partial_t^2\eta_T^{2q'}\,dx\,dt \\
&\le \frac{C}{T^2}\int_0^T \int_{\mathbb{R}^n} \big|u\eta_T^{2q'-2}\big|\,dx\,dt \\
&\le \frac{C}{T^2}\bigg(\int_0^T \int_{\mathbb{R}^n} |u|^q\eta_T^{2q'}\,dx\,dt\bigg)^{\frac{1}{q}}\bigg(\int_0^T \int_{\{r\le t+1\}} 1\,dx\,dt\bigg)^{\frac{1}{q'}} \\
&\le CT^{n+1-2q'} + \frac{1}{3}\int_0^T \int_{\mathbb{R}^n} |u|^q\eta_T^{2q'}\,dx\,dt,
\end{aligned} \tag{3.2}$$

$$
\begin{aligned}
I_2 &= -\int_0^T\int_{\mathbb{R}^n}\frac{\mu}{(1+|x|)^\beta}u\partial_t\eta_T^{2q'}\,dx\,dt \\
&\le \frac{C}{T}\int_0^T\int_{\mathbb{R}^n}\frac{\mu}{(1+|x|)^\beta}\bigl|u\eta_T^{2q'-2}\bigr|\,dx\,dt \\
&\le \frac{C}{T}\biggl(\int_0^T\int_{\mathbb{R}^n}|u|^q\eta_T^{2q'}\,dx\,dt\biggr)^{\frac{1}{q}}\biggl(\int_0^T\int_0^{t+1}\frac{r^{n-1-q'}}{(1+r)^{q'(\beta-1)}}\,dr\,dt\biggr)^{\frac{1}{q'}} \\
&\le CT^{n+1-2q'}+\frac{1}{3}\int_0^T\int_{\mathbb{R}^n}|u|^q\eta_T^{2q'}\,dx\,dt.
\end{aligned}
\tag{3.3}
$$

Combining (3.1)–(3.3), we deduce

$$
C_1(f,g)\varepsilon+\int_0^T\int_{\mathbb{R}^n}\bigl(|u_t|^p+|u|^q\bigr)\eta_T^{2q'}\,dx\,dt\le CT^{n+1-2q'},
\tag{3.4}
$$

where $C_1(f,g)=C(\int_{\mathbb{R}^n}g(x)\,dx+\int_{\mathbb{R}^n}\frac{\mu}{(1+|x|)^\beta}f(x)\,dx)$.

Let $\varphi(t,x)=\partial_t\Phi_1(t,x)$, where $\Phi_1(t,x)=-\eta_T^{2q'}\psi(t,x)=-\eta_T^{2q'}e^{-t}\phi(x)$, and $\psi(t,x)$ is defined in Lemma 2.4. Applying (2.1), we have

$$
\begin{aligned}
&\varepsilon\int_{\mathbb{R}^n}g(x)\phi(x)\,dx+\int_0^T\int_{\mathbb{R}^n}\bigl(|u_t|^p+|u|^q\bigr)\partial_t\Phi_1\,dx\,dt \\
&\qquad+\int_{\mathbb{R}^n}\frac{\mu}{(1+|x|)^\beta}\varepsilon f(x)\phi(x)\,dx \\
&\quad=\int_0^T\int_{\mathbb{R}^n}-\partial_t u\partial_t^2\Phi_1\,dx\,dt+\int_0^T\int_{\mathbb{R}^n}\nabla u\nabla\partial_t\Phi_1\,dx\,dt \\
&\qquad-\int_0^T\int_{\mathbb{R}^n}\frac{\mu}{(1+|x|)^\beta}u\partial_t^2\Phi_1\,dx\,dt,
\end{aligned}
$$

where we have employed the fact $\partial_t\Phi_1(0,x)=\phi(x)$. Since

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{R}^n}-\partial_t u\partial_t^2\Phi_1\,dx\,dt=\int_0^T\int_{\mathbb{R}^n}\partial_t u\bigl(\partial_t^2\eta_T^{2q'}\psi+\eta_T^{2q'}\psi-2\partial_t\eta_T^{2q'}\psi\bigr)\,dx\,dt, \\
&\int_0^T\int_{\mathbb{R}^n}\nabla u\nabla\partial_t\Phi_1\,dx\,dt \\
&\quad=\int_0^T\int_{\mathbb{R}^n}\partial_t(\nabla u\nabla\Phi_1)\,dx\,dt-\int_0^T\int_{\mathbb{R}^n}\nabla u_t\nabla\Phi_1\,dx\,dt \\
&\quad=\int_{\mathbb{R}^n}\varepsilon\nabla f(x)\nabla\phi(x)\,dx-\int_0^T\int_{\mathbb{R}^n}\nabla u_t\nabla\Phi_1\,dx\,dt \\
&\quad=-\varepsilon\int_{\mathbb{R}^n}\Delta\phi f(x)\,dx-\int_0^T\int_{\mathbb{R}^n}\nabla u_t\nabla\Phi_1\,dx\,dt \\
&\quad=-\varepsilon\int_{\mathbb{R}^n}\Delta\phi f(x)\,dx-\int_0^T\int_{\mathbb{R}^n}u_t\eta_T^{2q'}\Delta\psi\,dx\,dt,
\end{aligned}
$$

and

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^n} \frac{\mu}{(1+|x|)^\beta} u \partial_t^2 \Phi_1 \, dx \, dt \\
&= \int_0^T \int_{\mathbb{R}^n} \partial_t \left(\frac{\mu}{(1+|x|)^\beta} u \partial_t \Phi_1 \right) - \frac{\mu}{(1+|x|)^\beta} u_t \partial_t \Phi_1 \, dx \, dt \\
&= -\int_{\mathbb{R}^n} \frac{\mu}{(1+|x|)^\beta} \varepsilon f(x)\phi(x) \, dx - \int_0^T \int_{\mathbb{R}^n} \frac{\mu}{(1+|x|)^\beta} u_t \left(\eta_T^{2q'} \psi - \partial_t \eta_T^{2q'} \psi \right) dx \, dt,
\end{aligned}$$

we have

$$\begin{aligned}
&\varepsilon \int_{\mathbb{R}^n} g(x)\phi(x) \, dx + \int_0^T \int_{\mathbb{R}^n} \left(|u_t|^p + |u|^q \right) \partial_t \Phi_1 \, dx \, dt \\
&\quad + \varepsilon \int_{\mathbb{R}^n} \frac{\mu}{(1+|x|)^\beta} f(x)\phi(x) \, dx \\
&= \int_0^T \int_{\mathbb{R}^n} u_t \left(\partial_t^2 \eta_T^{2q'} \psi + \eta_T^{2q'} \psi - 2\partial_t \eta_T^{2q'} \psi \right) dx \, dt - \int_0^T \int_{\mathbb{R}^n} u_t \eta_T^{2q'} \Delta \psi \, dx \, dt \\
&\quad - \varepsilon \int_{\mathbb{R}^n} \left(1 + \frac{\mu}{(1+|x|)^\beta} \right) f(x)\phi(x) \, dx + \int_{\mathbb{R}^n} \frac{\mu}{(1+|x|)^\beta} \varepsilon f(x)\phi(x) \, dx \\
&\quad + \int_0^T \int_{\mathbb{R}^n} \frac{\mu}{(1+|x|)^\beta} u_t \left[\eta_T^{2q'} \psi - \partial_t \eta_T^{2q'} \psi \right] dx \, dt.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\varepsilon \int_{\mathbb{R}^n} g(x)\phi(x) \, dx + \int_0^T \int_{\mathbb{R}^n} \left(|u_t|^p + |u|^q \right) \partial_t \Phi_1 \, dx \, dt \\
&\quad + \varepsilon \int_{\mathbb{R}^n} \left(1 + \frac{\mu}{(1+|x|)^\beta} \right) f(x)\phi(x) \, dx \\
&= \int_0^T \int_{\mathbb{R}^n} u_t \left(\partial_t^2 \eta_T^{2q'} \psi + 2\partial_t \eta_T^{2q'} \partial_t \psi - \frac{\mu}{(1+|x|)^\beta} \partial_t \eta_T^{2q'} \psi \right) dx \, dt \\
&\quad + \int_0^T \int_{\mathbb{R}^n} u_t \eta_T^{2q'} \left[-\Delta \psi + \psi - \frac{\mu}{(1+|x|)^\beta} \partial_t \psi \right] dx \, dt \\
&= \int_0^T \int_{\mathbb{R}^n} u_t \left(\partial_t^2 \eta_T^{2q'} \psi + 2\partial_t \eta_T^{2q'} \partial_t \psi - \frac{\mu}{(1+|x|)^\beta} \partial_t \eta_T^{2q'} \psi \right) dx \, dt \\
&= I_3 + I_4 + I_5,
\end{aligned} \tag{3.5}$$

where we have applied Lemma 2.4.

We are in the position to derive the estimates for I_3, I_4, and I_5.

Employing Lemma 2.3 leads to

$$\begin{aligned}
I_3 &= \int_0^T \int_{\mathbb{R}^n} u_t \partial_t^2 \eta_T^{2q'} \psi \, dx \, dt \\
&\le \frac{C}{T^2} \int_0^T \int_{\mathbb{R}^n} \left| u_t \eta_T^{2q'-2} \psi \right| dx \, dt \\
&\le \frac{C}{T^2} \left(\int_0^T \int_{\mathbb{R}^n} |u_t|^p \eta_T^{(2q'-2)p} \, dx \, dt \right)^{\frac{1}{p}} \left(\int_0^T \int_{\mathbb{R}^n} |\psi|^{p'} \, dx \, dt \right)^{\frac{1}{p'}}
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
&\le \frac{C}{T^2}\left(\int_0^T\int_{\mathbb{R}^n}|u_t|^p\eta_T^{2q'}\,dx\,dt\right)^{\frac{1}{p}}\left(\int_0^T\int_{\mathbb{R}^n}|\psi|^{p'}\,dx\,dt\right)^{\frac{1}{p'}}\\
&\le CT^{-2+(n-\frac{n-1}{2}p')\frac{1}{p'}}\left(\int_0^T\int_{\mathbb{R}^n}|u_t|^p\eta_T^{2q'}\,dx\,dt\right)^{\frac{1}{p}},
\end{aligned}$$

$$\begin{aligned}
I_4 &= 2\int_0^T\int_{\mathbb{R}^n}u_t\partial_t\eta_T^{2q'}\partial_t\psi\,dx\,dt\\
&\le \frac{C}{T}\int_0^T\int_{\mathbb{R}^n}\left|u_t\eta_T^{2q'-1}\psi\right|\,dx\,dt\\
&\le \frac{C}{T}\left(\int_0^T\int_{\mathbb{R}^n}|u_t|^p\eta_T^{2q'}\,dx\,dt\right)^{\frac{1}{p}}\left(\int_0^T\int_{\mathbb{R}^n}|\psi|^{p'}\,dx\,dt\right)^{\frac{1}{p'}}\\
&\le CT^{-1+(n-\frac{n-1}{2}p')\frac{1}{p'}}\left(\int_0^T\int_{\mathbb{R}^n}|u_t|^p\eta_T^{2q'}\,dx\,dt\right)^{\frac{1}{p}},
\end{aligned} \tag{3.7}$$

$$|I_5| \le CI_4 \le CT^{-1+(n-\frac{n-1}{2}p')\frac{1}{p'}}\left(\int_0^T\int_{\mathbb{R}^n}|u_t|^p\eta_T^{2q'}\,dx\,dt\right)^{\frac{1}{p}}. \tag{3.8}$$

A direct calculation gives rise to

$$\partial_t\Phi_1 = \eta_T^{2q'}\psi - 2q'\eta_T^{2q'-1}\partial_t\eta_T\psi \ge \eta_T^{2q'}\psi > 0.$$

Using (3.5)–(3.8), we have

$$\varepsilon C_2(f,g) \le CT^{-1+(n-\frac{n-1}{2}p')\frac{1}{p'}}\left(\int_0^T\int_{\mathbb{R}^n}|u_t|^p\eta_T^{2q'}\,dx\,dt\right)^{\frac{1}{p}},$$

which implies

$$C\varepsilon^p T^{n-\frac{n-1}{2}p} \le \int_0^T\int_{\mathbb{R}^n}|u_t|^p\eta_T^{2q'}\,dx\,dt, \tag{3.9}$$

where $C_2(f,g) = C(\int_{\mathbb{R}^n}g(x)\phi(x)\,dx + \int_{\mathbb{R}^n}(1+\frac{\mu}{(1+|x|)^\beta})f(x)\phi(x)\,dx)$.

Combining (3.4) and (3.9) and using the assumptions $\max\{1,\frac{2}{n-1}\} < p < \frac{4n-2}{n-1}$, $\frac{n}{n-1} < q < 1+\frac{4}{(n-1)p-2}$, and $q \le p$, we obtain

$$T(\varepsilon) \le C\varepsilon^{-\frac{2p(q-1)}{-(n-1)pq+(n-1)p+2q+2}}.$$

On the other hand, according to (3.5), we derive

$$\begin{aligned}
I_3 &= \int_0^T\int_{\mathbb{R}^n}u_t\partial_t^2\eta_T^{2q'}\psi\,dx\,dt\\
&\le \frac{C}{T^2}\int_0^T\int_{\mathbb{R}^n}\left|u_t\eta_T^{2q'-2}\psi\right|\,dx\,dt\\
&\le \frac{C}{T^2}\left(\int_0^T\int_{\mathbb{R}^n}|u_t|^p\eta_T^{(2q'-2)p}\psi\,dx\,dt\right)^{\frac{1}{p}}\left(\int_0^T\int_{\mathbb{R}^n}\psi\,dx\,dt\right)^{\frac{1}{p'}}\\
&\le CT^{-2+\frac{n+1}{2p'}}\left(\int_0^T\int_{\mathbb{R}^n}|u_t|^p\eta_T^{2q'}\psi\,dx\,dt\right)^{\frac{1}{p}},
\end{aligned} \tag{3.10}$$

$$\begin{aligned} I_4 &= 2\int_0^T\int_{\mathbb{R}^n} u_t\partial_t\eta_T^{2q'}\partial_t\psi\,dx\,dt \\ &\le \frac{C}{T}\int_0^T\int_{\mathbb{R}^n}\big|u_t\eta_T^{2q'-1}\psi\big|\,dx\,dt \\ &\le \frac{C}{T}\left(\int_0^T\int_{\mathbb{R}^n}|u_t|^p\eta_T^{2q'}\psi\,dx\,dt\right)^{\frac{1}{p}}\left(\int_0^T\int_{\mathbb{R}^n}\psi\,dx\,dt\right)^{\frac{1}{p'}} \\ &\le CT^{-1+\frac{n+1}{2p'}}\left(\int_0^T\int_{\mathbb{R}^n}|u_t|^p\eta_T^{2q'}\psi\,dx\,dt\right)^{\frac{1}{p}}, \end{aligned} \tag{3.11}$$

$$|I_5| \le CI_4 \le CT^{-1+\frac{n+1}{2p'}}\left(\int_0^T\int_{\mathbb{R}^n}|u_t|^p\eta_T^{2q'}\psi\,dx\,dt\right)^{\frac{1}{p}}. \tag{3.12}$$

Taking into account (3.5) and (3.10)–(3.12) and using the Young inequality, we have

$$\varepsilon C_2(f,g) \le CT^{-p'+\frac{n+1}{2}}.$$

Therefore, for $1 < p < \frac{n+1}{n-1}$ and $1 < q \le p$, we have that

$$T(\varepsilon) \le C\varepsilon^{-\frac{2(p-1)}{n+1-(n-1)p}}. \tag{3.13}$$

3.2 Case $p < q$

Taking $\varphi(t,x) = \eta_T^{2p'}$ in (2.1) yields

$$\begin{aligned} &\varepsilon\int_{\mathbb{R}^n} g(x)\,dx + \int_0^T\int_{\mathbb{R}^n}\big(|u_t|^p + |u|^q\big)\eta_T^{2p'}\,dx\,dt + \int_{\mathbb{R}^n}\frac{\mu}{(1+|x|)^\beta}\varepsilon f(x)\,dx \\ &\quad= \int_0^T\int_{\mathbb{R}^n} u\partial_t^2\eta_T^{2p'}\,dx\,dt - \int_0^T\int_{\mathbb{R}^n}\frac{\mu}{(1+|x|)^\beta}u\partial_t\eta_T^{2p'}\,dx\,dt \\ &\quad= I_6 + I_7. \end{aligned} \tag{3.14}$$

Applying the fact that $p < q$ and $q > \frac{n}{n-1}$, we deduce

$$\begin{aligned} I_6 &= \int_0^T\int_{\mathbb{R}^n} u\partial_t^2\eta_T^{2p'}\,dx\,dt \\ &\le \frac{C}{T^2}\int_0^T\int_{\mathbb{R}^n}\big|u\eta_T^{2p'-2}\big|\,dx\,dt \\ &\le \frac{C}{T^2}\left(\int_0^T\int_{\mathbb{R}^n}|u|^q\eta_T^{2p'}\,dx\,dt\right)^{\frac{1}{q}}\left(\int_0^T\int_{\{r\le t+1\}}1\,dx\,dt\right)^{\frac{1}{q'}} \\ &\le CT^{n+1-2q'} + \frac{1}{3}\int_0^T\int_{\mathbb{R}^n}|u|^q\eta_T^{2p'}\,dx\,dt, \end{aligned} \tag{3.15}$$

$$\begin{aligned} I_7 &= -\int_0^T\int_{\mathbb{R}^n}\frac{\mu}{(1+|x|)^\beta}u\partial_t\eta_T^{2p'}\,dx\,dt \\ &\le \frac{C}{T}\int_0^T\int_{\mathbb{R}^n}\frac{\mu}{(1+|x|)^\beta}\big|u\eta_T^{2p'-2}\big|\,dx\,dt \\ &\le \frac{C}{T}\left(\int_0^T\int_{\mathbb{R}^n}|u|^q\eta_T^{2p'}\,dx\,dt\right)^{\frac{1}{q}}\left(\int_0^T\int_0^{t+1}\frac{r^{n-1-q'}}{(1+r)^{q'(\beta-1)}}\,dr\,dt\right)^{\frac{1}{q'}} \end{aligned} \tag{3.16}$$

$$\leq CT^{n+1-2q'} + \frac{1}{3}\int_0^T \int_{\mathbb{R}^n} |u|^q \eta_T^{2p'}\, dx\, dt.$$

Combining (3.14)–(3.16), we get

$$C_1(f,g)\varepsilon + \int_0^T \int_{\mathbb{R}^n} \bigl(|u_t|^p + |u|^q\bigr)\eta_T^{2p'}\, dx\, dt \leq CT^{n+1-2q'}. \tag{3.17}$$

We set $\varphi(t,x) = \partial_t \Phi_2(x,t)$ in (2.1), where $\Phi_2(x,t) = -\eta_T^{2p'}\psi(x,t) = -\eta_T^{2p'} e^{-t}\phi(x)$. Therefore we have

$$\begin{aligned}
&\varepsilon \int_{\mathbb{R}^n} g(x)\phi(x)\, dx + \int_0^T \int_{\mathbb{R}^n} \bigl(|u_t|^p + |u|^q\bigr)\partial_t \Phi_2\, dx\, dt \\
&\quad + \varepsilon \int_{\mathbb{R}^n} \biggl(1 + \frac{\mu}{(1+|x|)^\beta}\biggr) f(x)\phi(x)\, dx \\
&\quad = \int_0^T \int_{\mathbb{R}^n} u_t \biggl(\partial_t^2 \eta_T^{2p'}\psi + 2\partial_t \eta_T^{2p'}\partial_t \psi - \frac{\mu}{(1+|x|)^\beta}\partial_t \eta_T^{2p'}\psi\biggr)\, dx\, dt \\
&\quad = I_8 + I_9 + I_{10}.
\end{aligned} \tag{3.18}$$

Similarly to the deduction in (3.9), we obtain

$$C\varepsilon^p T^{n-\frac{n-1}{2}p} \leq \int_0^T \int_{\mathbb{R}^n} |u_t|^p \eta_T^{2p'}\, dx\, dt. \tag{3.19}$$

From (3.17) and (3.19) the conditions $\max\{1, \frac{2}{n-1}\} < p < \frac{4n-2}{n-1}$ and $\max\{p, \frac{n}{n-1}\} < q < 1 + \frac{4}{(n-1)p+2}$ lead to

$$T(\varepsilon) \leq C\varepsilon^{-\frac{2p(q-1)}{-(n-1)pq+(n-1)p+2q+2}}.$$

By (3.18) we have

$$\begin{aligned}
I_8 &= \int_0^T \int_{\mathbb{R}^n} u_t \partial_t^2 \eta_T^{2p'}\psi\, dx\, dt \\
&\leq \frac{C}{T^2}\int_0^T \int_{\mathbb{R}^n} \bigl|u_t \eta_T^{2p'-2}\psi\bigr|\, dx\, dt \\
&\leq \frac{C}{T^2}\biggl(\int_0^T \int_{\mathbb{R}^n} |u_t|^p \eta_T^{(2p'-2)p}\psi\, dx\, dt\biggr)^{\frac{1}{p}} \biggl(\int_0^T \int_{\mathbb{R}^n} \psi\, dx\, dt\biggr)^{\frac{1}{p'}} \\
&\leq CT^{-2+\frac{n+1}{2p'}} \biggl(\int_0^T \int_{\mathbb{R}^n} |u_t|^p \eta_T^{2p'}\psi\, dx\, dt\biggr)^{\frac{1}{p}},
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
I_9 &= 2\int_0^T \int_{\mathbb{R}^n} u_t \partial_t \eta_T^{2p'}\partial_t \psi\, dx\, dt \\
&\leq \frac{C}{T}\int_0^T \int_{\mathbb{R}^n} \bigl|u_t \eta_T^{2p'-1}\psi\bigr|\, dx\, dt \\
&\leq \frac{C}{T}\biggl(\int_0^T \int_{\mathbb{R}^n} |u_t|^p \eta_T^{2p'}\psi\, dx\, dt\biggr)^{\frac{1}{p}} \biggl(\int_0^T \int_{\mathbb{R}^n} \psi\, dx\, dt\biggr)^{\frac{1}{p'}}
\end{aligned} \tag{3.21}$$

$$\le CT^{-1+\frac{n+1}{2p'}}\left(\int_0^T\int_{\mathbb{R}^n}|u_t|^p\eta_T^{2p'}\psi\,dx\,dt\right)^{\frac{1}{p}},$$

$$|I_{10}|\le CI_9\le CT^{-1+\frac{n+1}{2p'}}\left(\int_0^T\int_{\mathbb{R}^n}|u_t|^p\eta_T^{2p'}\psi\,dx\,dt\right)^{\frac{1}{p}}. \tag{3.22}$$

Combining (3.18) and (3.20)–(3.22), for $1<p<\min\{q,\frac{n+1}{n-1}\}$ and $q>1$, we have

$$T(\varepsilon)\le C\varepsilon^{-\frac{2(p-1)}{n+1-(n-1)p}}. \tag{3.23}$$

This completes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

Taking $\varphi(t,x)=\eta_T^{2k}\psi(t,x)$ in (2.1), where $k=\max\{p',q'\}$ and $\psi(t,x)=e^{-t}\phi(x)$, we obtain

$$\begin{aligned}&\varepsilon\int_{\mathbb{R}^n}g(x)\phi(x)\,dx+\int_0^T\int_{\mathbb{R}^n}\left(|u_t|^p+|u|^q\right)\eta_T^{2k}\psi\,dx\,dt\\&\quad+\int_{\mathbb{R}^n}\frac{\mu}{(1+|x|)^\beta}\varepsilon f(x)\phi(x)\,dx\\&=\int_0^T\int_{\mathbb{R}^n}-\partial_t u\partial_t\left[\eta_T^{2k}\psi\right]dx\,dt+\int_0^T\int_{\mathbb{R}^n}\nabla u\nabla\left[\eta_T^{2k}\psi\right]dx\,dt\\&\quad-\int_0^T\int_{\mathbb{R}^n}\frac{\mu}{(1+|x|)^\beta}u\partial_t\left[\eta_T^{2k}\psi\right]dx\,dt.\end{aligned}$$

A direct calculation shows that

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^n}-\partial_t u\partial_t\left[\eta_T^{2k}\psi\right]dx\,dt\\&\quad=-\int_0^T\int_{\mathbb{R}^n}\partial_t\left(u\partial_t\left[\eta_T^{2k}\psi\right]\right)-u\partial_t^2\left[\eta_T^{2k}\psi\right]dx\,dt\\&\quad=-\int_{\mathbb{R}^n}\varepsilon f(x)\phi(x)\,dx+\int_0^T\int_{\mathbb{R}^n}u\left[\partial_t^2\eta_T^{2k}\psi+2\partial_t\eta_T^{2k}\partial_t\psi+\eta_T^{2k}\partial_t^2\psi\right]dx\,dt,\\&\int_0^T\int_{\mathbb{R}^n}\nabla u\nabla\left(\eta_T^{2k}\psi\right)dx\,dt=-\int_0^T\int_{\mathbb{R}^n}u\eta_T^{2k}\Delta\psi\,dx\,dt,\end{aligned}$$

and

$$\begin{aligned}&-\int_0^T\int_{\mathbb{R}^n}\frac{\mu}{(1+|x|)^\beta}u\partial_t\left(\eta_T^{2k}\psi\right)dx\,dt\\&\quad=-\int_0^T\int_{\mathbb{R}^n}\frac{\mu}{(1+|x|)^\beta}u\left[\partial_t\eta_T^{2k}\psi+\eta_T^{2k}\partial_t\psi\right]dx\,dt.\end{aligned}$$

It follows that

$$\begin{aligned}&\varepsilon\int_{\mathbb{R}^n}g(x)\phi(x)\,dx+\int_0^T\int_{\mathbb{R}^n}\left(|u_t|^p+|u|^q\right)\eta_T^{2k}\psi\,dx\,dt\\&\quad+\int_{\mathbb{R}^n}\left(1+\frac{\mu}{(1+|x|)^\beta}\right)\varepsilon f(x)\phi(x)\,dx\end{aligned}$$

$$
\begin{aligned}
&= \int_0^T \int_{\mathbb{R}^n} u\left[\partial_t^2 \eta_T^{2k}\psi + 2\partial_t \eta_T^{2k}\partial_t\psi - \frac{\mu}{(1+|x|)^\beta}\partial_t \eta_T^{2k}\psi\right] dx\,dt \\
&\quad + \int_0^T \int_{\mathbb{R}^n} u\eta_T^{2k}\left[\partial_t^2\psi - \Delta\psi - \frac{\mu}{(1+|x|)^\beta}\partial_t\psi\right] dx\,dt \\
&= \int_0^T \int_{\mathbb{R}^n} u\left[\partial_t^2 \eta_T^{2k}\psi + 2\partial_t \eta_T^{2k}\partial_t\psi - \frac{\mu}{(1+|x|)^\beta}\partial_t \eta_T^{2k}\psi\right] dx\,dt \\
&= I_{11} + I_{12} + I_{13}.
\end{aligned}
\tag{4.1}
$$

Employing (2.2), we have

$$
\begin{aligned}
I_{11} &= \int_0^T \int_{\mathbb{R}^n} u\partial_t^2 \eta_T^{2k}\psi\,dx\,dt \\
&\le \frac{C}{T^2}\int_0^T \int_{\mathbb{R}^n} \left|u\eta_T^{2k-2}\psi\right| dx\,dt \\
&\le \frac{C}{T^2}\left(\int_0^T \int_{\mathbb{R}^n} |u|^q \eta_T^{2k}\psi\,dx\,dt\right)^{\frac{1}{q}} \left(\int_0^T \int_{\mathbb{R}^n} \psi\,dx\,dt\right)^{\frac{1}{q'}} \\
&\le CT^{-2+\frac{n+1}{2q'}}\left(\int_0^T \int_{\mathbb{R}^n} |u|^q \eta_T^{2k}\psi\,dx\,dt\right)^{\frac{1}{q}},
\end{aligned}
\tag{4.2}
$$

$$
\begin{aligned}
I_{12} &= \int_0^T \int_{\mathbb{R}^n} 2\partial_t \eta_T^{2k}\partial_t\psi\,dx\,dt \\
&\le \frac{C}{T}\int_0^T \int_{\mathbb{R}^n} \left|u\eta_T^{2k-1}\psi\right| dx\,dt \\
&\le CT^{-1+\frac{n+1}{2q'}}\left(\int_0^T \int_{\mathbb{R}^n} |u|^q \eta_T^{2k}\psi\,dx\,dt\right)^{\frac{1}{q}},
\end{aligned}
\tag{4.3}
$$

$$
\begin{aligned}
|I_{13}| &\le CI_{12} \\
&\le CT^{-1+\frac{n+1}{2q'}}\left(\int_0^T \int_{\mathbb{R}^n} |u|^q \eta_T^{2k}\psi\,dx\,dt\right)^{\frac{1}{q}}.
\end{aligned}
\tag{4.4}
$$

Combining (4.1)–(4.4), we deduce

$$
C\varepsilon^q T^{q-\frac{n+1}{2}(q-1)} \le \int_0^T \int_{\mathbb{R}^n} |u|^q \eta_T^{2k}\psi\,dx\,dt. \tag{4.5}
$$

On the other hand, we take $\varphi(t,x) = \partial_t\Phi(t,x)$ in (2.1), where $\Phi(t,x) = -\eta_T^{2k}\psi(t,x) = -\eta_T^{2k}e^{-t}\phi(x)$. Similarly to the derivation in (3.5) and (3.18), we acquire

$$
\begin{aligned}
&\varepsilon\int_{\mathbb{R}^n} g(x)\phi(x)\,dx + \int_0^T \int_{\mathbb{R}^n} \left(|u_t|^p + |u|^q\right)\partial_t\Phi\,dx\,dt \\
&\quad + \varepsilon\int_{\mathbb{R}^n} \left(1 + \frac{\mu}{(1+|x|)^\beta}\right) f(x)\phi(x)\,dx \\
&= \int_0^T \int_{\mathbb{R}^n} u_t\left(\partial_t^2 \eta_T^{2k}\psi + 2\partial_t \eta_T^{2k}\partial_t\psi - \frac{\mu}{(1+|x|)^\beta}\partial_t \eta_T^{2k}\psi\right) dx\,dt \\
&= I_{14} + I_{15} + I_{16}.
\end{aligned}
\tag{4.6}
$$

It follows that

$$
\begin{aligned}
I_{14} &= \int_0^T \int_{\mathbb{R}^n} u_t \partial_t^2 \eta_T^{2k} \psi \, dx \, dt \\
&\leq \frac{C}{T^2} \int_0^T \int_{\mathbb{R}^n} \left| u_t \eta_T^{2k-2} \psi \right| dx \, dt \\
&\leq \frac{C}{T^2} \left(\int_0^T \int_{\mathbb{R}^n} |u_t|^p \eta_T^{2k} \psi \, dx \, dt \right)^{\frac{1}{p}} \left(\int_0^T \int_{\mathbb{R}^n} \psi \, dx \, dt \right)^{\frac{1}{p'}} \\
&\leq CT^{-2+\frac{n+1}{2p'}} \left(\int_0^T \int_{\mathbb{R}^n} |u_t|^p \eta_T^{2k} \psi \, dx \, dt \right)^{\frac{1}{p}} \\
&\leq CT^{-2p'+\frac{n+1}{2}} + \frac{1}{3} \int_0^T \int_{\mathbb{R}^n} |u_t|^p \eta_T^{2k} \psi \, dx \, dt.
\end{aligned}
\tag{4.7}
$$

Similarly, we conclude that

$$
\begin{aligned}
I_{15} &= \int_0^T \int_{\mathbb{R}^n} 2u_t \partial_t \eta_T^{2k} \partial_t \psi \, dx \, dt \\
&\leq \frac{C}{T} \int_0^T \int_{\mathbb{R}^n} \left| u_t \eta_T^{2k-2} \psi \right| dx \, dt \\
&\leq \frac{C}{T} \left(\int_0^T \int_{\mathbb{R}^n} |u_t|^p \eta_T^{2k} \psi \, dx \, dt \right)^{\frac{1}{p}} \left(\int_0^T \int_{\mathbb{R}^n} \psi \, dx \, dt \right)^{\frac{1}{p'}} \\
&\leq CT^{-1+\frac{n+1}{2p'}} \left(\int_0^T \int_{\mathbb{R}^n} |u_t|^p \eta_T^{2k} \psi \, dx \, dt \right)^{\frac{1}{p}} \\
&\leq CT^{-p'+\frac{n+1}{2}} + \frac{1}{3} \int_0^T \int_{\mathbb{R}^n} |u_t|^p \eta_T^{2k} \psi \, dx \, dt,
\end{aligned}
\tag{4.8}
$$

$$
\begin{aligned}
|I_{16}| &\leq CI_{15} \\
&\leq CT^{-p'+\frac{n+1}{2}} + \frac{1}{3} \int_0^T \int_{\mathbb{R}^n} |u_t|^p \eta_T^{2k} \psi \, dx \, dt.
\end{aligned}
\tag{4.9}
$$

Employing the fact $\partial_t \Phi = \eta_T^{2k} \psi - 2k \eta_T^{2k-1} \partial_t \eta_T \psi \geq \eta_T^{2k} \psi > 0$ and (4.5)–(4.9), we have

$$
C\varepsilon^q T^{q-\frac{(n+1)(q-1)}{2}} \leq \int_0^T \int_{\mathbb{R}^n} |u|^q \eta_T^{2k} \psi \, dx \, dt \leq CT^{-p'+\frac{n+1}{2}},
$$

which implies

$$
T(\varepsilon) \leq C\varepsilon^{-\frac{2q(p-1)}{-(n-1)qp+(n-1)q+2p}}
\tag{4.10}
$$

for $p > 1$ $(n = 2, 3)$, $1 < p < \frac{n-1}{n-3}$ $(n > 3)$, and $1 < q < \frac{2p}{(n-1)(p-1)}$.

On the other hand, (4.2)–(4.4) yield

$$
\begin{aligned}
I_{11} &\leq CT^{-2+\frac{n+1}{2q'}} \left(\int_0^T \int_{\mathbb{R}^n} |u|^q \eta_T^{2k} \psi \, dx \, dt \right)^{\frac{1}{q}} \\
&\leq CT^{-2q'+\frac{n+1}{2}} + \frac{1}{3} \int_0^T \int_{\mathbb{R}^n} |u|^q \eta_T^{2k} \psi \, dx \, dt,
\end{aligned}
\tag{4.11}
$$

$$
\begin{aligned}
I_{12} &\leq CT^{-1+\frac{n+1}{2q'}}\left(\int_0^T\int_{\mathbb{R}^n}|u|^q\eta_T^{2k}\psi\,dx\,dt\right)^{\frac{1}{q}} \\
&\leq CT^{-q'+\frac{n+1}{2}}+\frac{1}{3}\int_0^T\int_{\mathbb{R}^n}|u|^q\eta_T^{2k}\psi\,dx\,dt,
\end{aligned}
\tag{4.12}
$$

$$
\begin{aligned}
|I_{13}| &\leq CI_{12} \\
&\leq CT^{-q'+\frac{n+1}{2}}+\frac{1}{3}\int_0^T\int_{\mathbb{R}^n}|u|^q\eta_T^{2k}\psi\,dx\,dt.
\end{aligned}
\tag{4.13}
$$

From (4.1) and (4.11)–(4.13) we obtain

$$
\varepsilon C_2(f,g)\leq CT^{-q'+\frac{n+1}{2}},
$$

which implies

$$
T(\varepsilon)\leq C\varepsilon^{-\frac{2(q-1)}{(n+1)-(n-1)q}}
$$

for $p>1$ and $1<q<\frac{n+1}{n-1}$. The proof of Theorem 1.2 is finished.

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Availability of data and materials
Data sharing not applicable to this paper as no data sets were generated or analyzed during the current study.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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References
1. D'Abbicco, M., Lucente, S., Reissig, M.: A shift in the Strauss exponent for semilinear wave equations with a not effective damping. J. Differ. Equ. **259**, 5040–5073 (2015)
2. Dai, W., Fang, D.Y., Wang, C.B.: Global existence and lifespan for semilinear wave equations with mixed nonlinear terms. J. Differ. Equ. **267**, 3328–3354 (2019)
3. Du, Y., Metcalfe, J., Sogge, C.D., Zhou, Y.: Concerning the Strauss conjecture and almost global existence for nonlinear Dirichlet-wave equations in 4-dimensions. Commun. Partial Differ. Equ. **33**(7–9), 1487–1506 (2008)
4. Fujita, H.: On the blowing up of solutions of the Cauchy problem for $u_t=\Delta u+u^{1+\alpha}$. J. Fac. Sci., Univ. Tokyo, Sect. I **13**, 109–124 (1966)

5. Georgiev, V., Lindblad, H., Sogge, C.D.: Weighted Strichartz estimates and global existence for semilinear wave equations. Am. J. Math. **119**, 1291–1319 (1997)
6. Glassey, R.T.: Mathematical reviews to "Global behavior of solutions to nonlinear wave equations in three space dimensions". Sideris, Comm. Part. Diff. Equa., (1983)
7. Hamouda, M., Hamza, M.A.: Blow-up for wave equation with the scale-invariant damping and combined nonlinearities. Math. Methods Appl. Sci. **44**, 1127–1136 (2021)
8. Hamouda, M., Hamza, M.A.: Improvement on the blow-up of the wave equation with the scale invariant damping and combined nonlinearities. Nonlinear Anal., Real World Appl. **59**, 103275 (2021)
9. Han, W.: Concerning the Strauss conjecture for the sub-critical and critical cases on the exterior domain in two space dimensions. Nonlinear Anal. **84**, 136–145 (2013)
10. Han, W., Zhou, Y.: Blow-up for some semilinear wave equations in multi-space dimensions. Commun. Partial Differ. Equ. **39**, 651–665 (2014)
11. Hidano, K., Metcalfe, J., Smith, H.F., Sogge, C.D., Zhou, Y.: On abstract Strichartz estimates and the Strauss conjecture for non-trapping obstacles. Trans. Am. Math. Soc. **362**(5), 2789–2809 (2010)
12. Hidano, K., Wang, C.B., Yokoyama, K.: Combined effects of two nonlinearities in lifespan of small solutions to semilinear wave equations. Math. Ann. **366**, 667–694 (2016)
13. Ikeda, M., Sobajima, M.: Lifespan of solutions to semilinear wave equation with time dependent critical damping for specially localized initial data. Math. Ann. **372**(3–4), 1017–1040 (2018)
14. Ikehata, R.: Some remarks on the wave equation with potential type damping coefficients. Int. J. Pure Appl. Math. **21**, 19–24 (2005)
15. Ikehata, R., Takeda, H.: Uniform energy decay for wave equations with unbounded damping coefficients. Funkc. Ekvacioj **63**, 133–152 (2020)
16. Imai, T., Kato, M., Takamura, H., Wakasa, K.: The lifespan of solutions of semilinear wave equations with the scale-invariant damping in two space dimensions. J. Differ. Equ. **269**, 8387–8424 (2020)
17. Lai, N.A., Liu, M.Y., Tu, Z.H., Wang, C.B.: Lifespan estimates for semilinear wave equations with space dependent damping and potential (2021). arXiv:2102.10257v1
18. Lai, N.A., Takamura, H.: Blow-up for semilinear damped wave equations with sub-Strauss exponent in the scattering case. Nonlinear Anal. **168**, 222–237 (2018)
19. Lai, N.A., Takamura, H.: Non-existence of global solutions of wave equations with weak time dependent damping and combined nonlinearity. Nonlinear Anal., Real World Appl. **45**, 83–96 (2019)
20. Lai, N.A., Takamura, H., Wakasa, K.: Blow-up for semilinear wave equations with the scale invariant damping and super-Fujita exponent. J. Differ. Equ. **263**(9), 5377–5394 (2017)
21. Lai, N.A., Tu, Z.H.: Strauss exponent for semilinear wave equations with scattering space dependent damping. J. Math. Anal. Appl. **489**, 124189 (2020)
22. Lai, N.A., Zhou, Y.: An elementary proof of Strauss conjecture. J. Funct. Anal. **267**(5), 1364–1381 (2014)
23. Lai, N.A., Zhou, Y.: Finite time blow-up to critical semilinear wave equation outside the ball in 3-D. Nonlinear Anal. **125**, 550–560 (2015)
24. Lai, N.A., Zhou, Y.: Non-existence of global solutions to critical semilinear wave equations in exterior domain in high dimensions. Nonlinear Anal., Real World Appl. **143**, 89–104 (2016)
25. Lai, N.A., Zhou, Y.: Blow-up for initial boundary value problem of critical semilinear wave equation in 2-D. Commun. Pure Appl. Anal. **17**(4), 1499–1510 (2018)
26. Liu, M.Y., Wang, C.B.: Blow-up for small-amplitude semilinear wave equations with mixed nonlinearities on asymptotically Euclidean manifolds. J. Differ. Equ. **269**(10), 8573–8596 (2020)
27. Metcalfe, J., Sogge, C.D.: Global existence for high dimensional quasilinear wave equations exterior to star shaped obstacles. Discrete Contin. Dyn. Syst. **28**(4), 1589–1601 (2012)
28. Ming, S., Lai, S.Y., Fan, X.M.: Lifespan estimates of solutions to quasilinear wave equations with scattering damping. J. Math. Anal. Appl. **492**, 124441 (2020)
29. Ming, S., Lai, S.Y., Fan, X.M.: Blow-up for a coupled system of semilinear wave equations with scattering dampings and combined nonlinearities. Appl. Anal. **101**(8), 2996–3016 (2022)
30. Ming, S., Yang, H., Fan, X.M.: Formation of singularities of solutions to the Cauchy problem for semilinear Moore–Gibson–Thompson equations. Commun. Pure Appl. Anal. **21**(5), 1773–1792 (2022)
31. Nishihara, K.: Asymptotic behavior of solutions to the semilinear wave equation with time-dependent damping. Tokyo J. Math. **34**, 327–343 (2011)
32. Nishihara, K., Sobajima, M., Wakasugi, Y.: Critical exponent for the semilinear wave equations with a damping increasing in the far field. Nonlinear Differ. Equ. Appl. **25**(6), 55 (2018)
33. Schaeffer, J.: The equation $\Box u = |u|^p$ for the critical value of p. Proc. R. Soc. Edinb. **101**, 31–44 (1985)
34. Smith, H.F., Soggee, S.D., Wang, C.B.: Strichartz estimates for Dirichlet wave equations in two dimensions with applications. Trans. Am. Math. Soc. **364**, 3329–3347 (2012)
35. Strauss, W.A.: Nonlinear scattering theory at low energy. J. Funct. Anal. **41**(1), 110–133 (1981)
36. Todorova, G., Yordanov, B.: Weighted L^2-estimates for dissipative wave equations with variable coefficients. J. Differ. Equ. **246**, 4497–4518 (2009)
37. Wakasa, K.: The lifespan of solutions to semilinear damped wave equations in one space dimension. Commun. Pure Appl. Anal. **15**, 1265–1283 (2016)
38. Wakasugi, Y.: Critical exponent for the semilinear wave equation with scale invariant damping. Four. Anal. Tren. Math., 375–390 (2014)
39. Wang, C.B.: Long time existence for semilinear wave equations on asymptotically flat space times. Commun. Partial Differ. Equ. **42**(7), 1150–1174 (2017)
40. Yordanov, B., Zhang, Q.S.: Finite time blow-up for critical wave equations in high dimensions. J. Funct. Anal. **231**, 361–374 (2006)
41. Zhou, Y.: Blow-up of solutions to semilinear wave equations with critical exponent in high dimensions. Chin. Ann. Math. **28**, 205–212 (2007)
42. Zhou, Y., Han, W.: Lifespan of solutions to critical semilinear wave equations. Commun. Partial Differ. Equ. **39**, 439–451 (2014)