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The trace of $u \in W^{1,1}_{\mathrm{loc}}(\Omega) \bigcap L^\infty(\Omega)$ and its applications

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Abstract

This paper is concerned with the well-posedness problem of a doubly degenerate parabolic equation with variable exponents. By the parabolically regularized method, the existence of local solution is proved. Moreover, the trace of $u \in W_0^{1,1}(\Omega)$ is generalized to $u \in W^{1,1}_{\mathrm{loc}}(\Omega) \bigcap L^\infty(\Omega)$ in a rational way. Then, a partial boundary value condition matching up with the stability theorem is found.

MSC: 35K55; 35K92; 35K85

Keywords: Doubly degenerate parabolic equation; Partial boundary value condition; Local solution; Stability

1 Introduction

Many problems in physics, mechanics, and biology are described by degenerate parabolic equations. For example, the evolutionary equation

$$u_t - \operatorname{div}\big(a(x)|\nabla u|^{p-2}|\nabla u|\big) + f(x,t,u) = 0 \tag{1}$$

may model the diffusion of a substance in water, soil, or air, heat flow in a material, or diffusion of a population in a habitat. Since the media may not be homogenous, the equation is governed by a diffusion coefficient $a(x)$. When at some points the medium is perfectly insulating, it is natural to assume $a(x) = 0$ at these points. In fact, certain composite material can block the heat at certain points, or a diffusion of a population may degenerate in some locations due to environmental heterogeneity and barriers [5, 7, 12]. In this paper, we consider a more complicate evolutionary equation

$$v_t = \operatorname{div}\big(a(x)|v|^{\alpha(x)}|\nabla v|^{p(x)-2}\nabla v\big) + \sum_{i=1}^N g^i(x,t,v)\frac{\partial v}{\partial x_i} + d(x,t,v), \quad (x,t) \in Q_T, \tag{2}$$

with

$$v|_{t=0} = v_0(x), \quad x \in \Omega, \tag{3}$$

$$v(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{4}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain and $0 < T < \infty$, $Q_T = \Omega \times (0,T)$, $p(x) > 1$ is a $C^1(\overline{\Omega})$ function, both $g^i(x,t,v)$ and $d(x,t,v)$ are continuous on $\overline{Q}_T \times \mathbb{R}$, $a(x) \in C(\overline{\Omega})$ is a nonnegative function. Since both $a(x)$ and $|v|^{\alpha(x)}$ may be degenerate, equation (2) is with double degeneracy. Let us take a quick look at some of the progress that has been made.

If $a(x) = 1$, $\alpha(x) = \alpha$, and $p(x) = p$ are positive constants, equation (2) becomes

$$v_t = \operatorname{div}\left(|v|^{\alpha}|\nabla v|^{p-2}\nabla v\right) + \sum_{i=1}^{N} g^i(x,t,v)\frac{\partial v}{\partial x_i} + d(x,t,v), \tag{5}$$

which is called a polytropic filtration equation with a convection term and a source term. The well-posedness of this equation has been studied widely, one can refer to [6, 11, 14, 24] and the references therein. If $a(x)$ is a nonnegative function satisfying

$$a(x) > 0, \quad x \in \Omega, \qquad a(x) = 0, \quad x \in \partial\Omega, \tag{6}$$

and $\alpha(x) \equiv \alpha$ and $p(x) = p$ are constants, $d(x,t,u) = 0$, then the stability of weak solutions to equation (2) was studied recently in [25, 26].

Also, equation (2) is a simple version of the following equation:

$$v_t = \operatorname{div}\left(a(x,t,v)|\nabla v|^{p(x,t)-2}\nabla v\right) + f(x,t,v,\nabla v), \quad (x,t) \in Q_T, \tag{7}$$

which comes from many applied problems such as the electrorheological fluid theory, the system and method for image depositing [1, 9, 13, 18, 20].

If $0 < a^- \leq a(x,t,v) \leq a^+ < \infty$, $f(x,t,v,\nabla v) = b(x,t)|u|^{\sigma(x,t)-2}$, the existence of local solutions and the blow-up phenomena of equation (7) were studied in [3]. If $a(x,t,v) = |v|^{\alpha} + d_0$, $d_0 > 0$, $\alpha \geq 2$, $p(x)$ is continuous with the logarithmic modulus of continuity, $f(x,t,v,\nabla v) = f(x,t)$, then the existence and uniqueness of weak solutions were showed in [8]. However, when $a(x,t,v) \geq 0$, the uniqueness of weak solution remains open till today. Only when $a(x,t,v) = a(x)|v|^{\alpha(x)}$, some progress has been made by the author recently. Some details are given in what follows.

If $0 \leq \alpha(x) \in C_0^1(\Omega)$ and $p \geq 2$, the well-posedness problem to the following equation:

$$v_t = \operatorname{div}\left(a(x)|v|^{\alpha(x)}|\nabla v|^{p-2}\nabla v\right) + f(x,t,v,\nabla v), \quad (x,t) \in Q_T, \tag{8}$$

was studied in [27, 30]. Very recently, when $\alpha(x) \in C_0^1(\Omega)$ and $p^- = \min_{x\in\overline{\Omega}} p(x) \geq 2$, the existence and uniqueness of weak solutions to the equation

$$v_t = \operatorname{div}\left(a(x)|v|^{\alpha(x)}|\nabla v|^{p(x)-2}\nabla v\right) + f\left(x,t,v,|\nabla v|\right), \quad (x,t) \in Q_T,$$

were proved in [22]. Naturally, there are some other restrictions imposed on $f(x,t,v,\nabla v)$ or $f(x,t,v,|\nabla v|)$ in these papers, in particular, $f(x,t,v,\nabla v) = f(x,t,v) > 0$ when $v < 0$ in [28, 30], $f(x,t,v,\nabla v) = \sum_{i=1}^{N} \frac{\partial b_i(x,t,v)}{\partial x_i}$ with $\frac{\partial b_i(x,t,\cdot)}{\partial x_i} \leq 0$ in [27] and $f(x,t,v,|\nabla v|) \leq 0$ in [22].

However, when $\alpha(x) = 0$, $p(x) = p$, $g^i(x,t,v) = 0$, $i = 1,2,\ldots,N$, and $d(x,t,v) = |v|^{q-1}v$ with $q > p+1$, the solution of equation (2) may blow up in finite time. So, there is only a local weak solution to equation (2). In a word, compared with [22, 27, 28, 30], the main improvements of this paper lie in that the stability of weak solutions is proved when $\alpha(x) \geq 0$

but without the assumption $\alpha(x) \in C_0^1(\Omega)$. Such an improvement is due to the novelty of the classical trace of $u \in in W_0^{1,1}(\Omega)$ being generalized to $u \in W_{\mathrm{loc}}^{1,1}(\Omega) \bigcap L^\infty(\Omega)$.

The contents are arranged as follows. In the first section, we have given some background. In Sect. 2, the definition of weak solution is introduced and the main results are listed. In Sect. 3, Theorem 2 is proved. The stability theorems are proved in Sect. 4.

2 The definitions and the main results

To define the weak solution, we give a basic Banach space which can be found in [4]. For every fixed $t \in [0, T)$, define the Banach space

$$V_t(\Omega) = \left\{u(x,t) : u(x,t) \in L^2(\Omega) \bigcap W_0^{1,1}(\Omega), \left|\nabla u(x,t)\right|^{p(x)} \in L^1(\Omega)\right\},$$
$$\|u\|_{V_t(\Omega)} = \|u\|_{2,\Omega} + \|\nabla u\|_{p(x),\Omega},$$

and denote by $V_t'(\Omega)$ its dual space. At the same time, define the Banach space

$$\begin{cases} \mathbf{W}(Q_T) = \{u : [0,T] \to V_t(\Omega) | u \in L^2(Q_T), |\nabla u|^{p(x)} \in L^1(Q_T), u = 0 \text{ on } \Gamma = \partial\Omega\}, \\ \|u\|_{\mathbf{W}(Q_T)} = \|\nabla u\|_{p(x),Q_T} + \|u\|_{2,Q_T}, \end{cases}$$

and denote by $\mathbf{W}'(Q_T)$ its dual space, and define the norm in $\mathbf{W}'(Q_T)$ by

$$\|v\|_{\mathbf{W}'(Q_T)} = \sup\left\{\langle v, \phi\rangle : \phi \in \mathbf{W}(Q_T), \|\phi\|_{\mathbf{W}(Q_T)} \le 1\right\}.$$

Definition 1 A function $v(x,t)$ is said to be a weak solution of equation (2) with the initial value (3) if

$$v \in L^\infty(Q_T), \qquad \frac{\partial v}{\partial t} \in \mathbf{W}'(Q_T), \qquad a(x)|v|^{\alpha(x)}|\nabla v|^{p(x)} \in L^1(Q_T), \tag{9}$$

$$\nabla v \in L^\infty\left(0, T; L_{\mathrm{loc}}^{p(x)}(\Omega)\right), \tag{10}$$

and for any function $\varphi \in C_0^1(Q_T)$,

$$\begin{aligned} &\iint_{Q_T} \left(\frac{\partial v}{\partial t}\varphi + a(x)|v|^{\alpha(x)}|\nabla v|^{p(x)-2}\nabla v \nabla\varphi\right) dx\,dt \\ &\quad = \sum_{i=1}^N \iint_{Q_T} g^i(x,t,v)\frac{\partial v}{\partial x_i}\varphi\,dx\,dt + \iint_{Q_T} d(x,t,v)\varphi\,dx\,dt. \end{aligned} \tag{11}$$

The initial value (3) is satisfied in the sense

$$\lim_{t\to 0}\int_\Omega v(x,t)\phi(x)\,dx = \int_\Omega v_0(x)\phi(x)\,dx \tag{12}$$

for any $\phi(x) \in C_0^\infty(\Omega)$.

This definition seems to have nothing to do with the boundary value condition (4) we will specify later. We first give the existence theorem here.

Theorem 2 *Suppose that $a(x) \in C^1(\overline{\Omega})$ satisfies (6), $g^i(x,t,s)$ and $d(x,t,s)$ are C^1 functions on $\overline{Q}_{T_0} \times \mathbb{R}$,*

$$\left|g^i(x,t,s)\right| \leq g(x,t)|s|^{\frac{\alpha(x)}{p(x)}}, \quad i = 1,2,\ldots,N, \qquad \left|d(x,t,s)\right| \leq d_0|s|^{\sigma-1} + h(x,t), \tag{13}$$

$$\int_0^{T_0}\int_\Omega \left(\frac{g(x,t)^{p(x)}}{a(x)}\right)^{\frac{1}{p(x)-1}} dx\,dt < \infty, \tag{14}$$

where $\sigma > 2$ and $d_0 > 0$ are constants, $g(x,t)$ is a C^1 function on $\overline{Q}_{T_0}$ and $\|h(x,t)\|_{L^1(0,\theta;L^\infty(\Omega))} \leq c$ for some $\theta > T_0$. If

$$v_0 \in L^\infty(\Omega), \quad a(x)|v_0|^{\alpha(x)}|\nabla v_0|^{p(x)} \in L^1(\Omega), \tag{15}$$

then equation (2) with the initial value (3) has a solution $v(x,t)$ on $\Omega \times [0,T_0]$, where T_0 is a positive constant depending on δ, d_0, $\|v_0\|_{L^\infty(\Omega)}$, $\|h\|_{L^1(0,\theta;L^\infty(\Omega))}$.

Certainly, Theorem 2 only tells us the existence of the local solution. If $\alpha(x) = 0$, $a(x) = 1$, and $p(x) = p$ is a constant, equation (2) becomes the well-known non-Newtonian fluid equation, when $g^i(x,t,v) = 0$ and $|d(x,t,v)| \leq c|v|^{p-1} + \phi(x,t)$, $\phi \in L^r(Q_T)$ with $r > \frac{N+p}{p}$, then the existence of global solution was proved in [31], and the same conclusion was obtained in [13] provided that $p(x) > 1$ is a continuous function. If $g^i(x,t,v) = 0$, $d(x,t,v) = d(x,t) \in L^\infty(Q_T)$, the existence of global solution was proved in [4]. Moreover, if $g^i(x,t,v) = 0$ and $d(x,t,v)$ is a Lipschitz function with $d(x,t,v) > 0$ when $v < 0$, the existence of global solution was proved in [28, 30] recently. If there are not other restrictions on the growth order of $d(x,t,v)$, the weak solutions to equation (2) may blow up, one can refer to [3, 10] for details.

Let $\varphi(x) \in C^1(\overline{\Omega})$ with

$$\varphi(x) = 0, \quad x \in \partial\Omega, \qquad \varphi(x) > 0, \quad x \in \Omega, \tag{16}$$

and define

$$\varphi_\lambda(x) = \begin{cases} 1, & \varphi(x) \geq 2\lambda, \\ \frac{\varphi(x)-\lambda}{\lambda}, & \lambda \leq \varphi(x) \leq 2\lambda, \\ 0, & \varphi(x) \leq \lambda, \end{cases}$$

for small enough positive constant λ. For simplicity, we call the function $\varphi(x)$, which satisfies (16), a weak characteristic function of Ω. Now, we can generalize the classical trace of $v \in W_0^{1,1}(\Omega)$ to that of $v \in W_{\mathrm{loc}}^{1,1}(\Omega) \bigcap L^\infty(\Omega)$ and specify the boundary value condition (4) as follows.

Definition 3 The boundary value condition (4) is true in a general sense of trace if and only if

$$\limsup_{\lambda\to 0} \frac{1}{\lambda}\int_{D_\lambda \setminus D_{2\lambda}} |u|\,dx = 0, \tag{17}$$

where $D_\lambda = \{x \in \Omega : \varphi(x) > \lambda\}$. Moreover, for any $\Sigma_1 \subset \partial\Omega$, we define that

$$u = 0, \quad x \in \Sigma_1 \subset \partial\Omega, \tag{18}$$

$$\limsup_{\lambda\to 0} \frac{1}{\lambda} \int_{D_\lambda^1} |u|\,dx = 0, \tag{19}$$

where $D_\lambda^1 \subset D_\lambda \setminus D_{2\lambda}$ such that

$$\lim_{\lambda\to 0} D_\lambda^1 = \Sigma_1.$$

Let

$$G^i(x,t,u) = \int_0^u g^i(x,t,s)\,ds$$

and $\varphi(x) \in C^1(\overline{\Omega})$ be a weak characteristic function of Ω. Then the partial boundary value condition matching up with equation (2) can be imposed as

$$u(x,t) = 0, \quad (x,t) \in \Sigma_\varphi, \tag{20}$$

where

$$\Sigma_\varphi = \left\{(x,t) \in \partial\Omega \times (0,T) : \sum_{i=1}^N g^i(x,t,0)\varphi_{x_i}(x) > 0\right\}. \tag{21}$$

The main result of this paper is the following theorem.

Theorem 4 *Let $u(x,t)$ and $v(x,t)$ be two solutions of equation (2) with the initial values $u_0(x)$, $v_0(x)$ respectively and with the same partial boundary value condition (20). If $d(x,t,v)$ is a Lipschitz function, $g^i(x,t,\cdot) \in C^1(\overline{Q}_T)$, then*

$$\int_\Omega \bigl|u(x,t) - v(x,t)\bigr|\,dx \le c\int_\Omega \bigl|u_0(x) - v_0(x)\bigr|\,dx, \ a.e.\ t \in [0,T). \tag{22}$$

One can see that, since $a(x)$ satisfies (6), $\varphi(x)$ can be chosen as $a(x)$, $a(x)^k$, or $e^{-\frac{1}{a(x)}}$ in Theorem 4. Naturally, the analytical expression Σ_φ in partial boundary value condition (20) depends on the choice of φ. We conjecture that the best partial boundary value condition matching up with equation (2) should be

$$u(x,t) = 0,\ (x,t) \in \Sigma_1,$$

where $\Sigma_1 = \bigcap \Sigma_\varphi$ and $\varphi(x)$ is a weak characteristic function of Ω.

3 The proof of Theorem 2

Consider the initial-boundary value problem

$$\begin{aligned} &v_{\varepsilon t} - \operatorname{div}\Bigl(\bigl(a(x)+\varepsilon\bigr)\bigl(|v_\varepsilon|^{\alpha(x)}+\varepsilon\bigr)|\nabla v_\varepsilon|^{p(x)-2}\nabla v_\varepsilon\Bigr) - \sum_{i=1}^N g^i(x,t,v_\varepsilon)\frac{\partial v_\varepsilon}{\partial x_i} \\ &\quad = d(x,t,v_\varepsilon), \quad (x,t) \in Q_T, \end{aligned} \tag{23}$$

$$v_\varepsilon(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{24}$$

$$v_\varepsilon(x,0) = v_{\varepsilon 0}(x), \quad x \in \Omega, \tag{25}$$

where $v_{\varepsilon 0} \in C_0^\infty(\Omega)$, $\|v_{\varepsilon 0}\|_{L^\infty(\Omega)} \le \|v_0\|_{L^\infty(\Omega)}$, $a(x)|\nabla v_{\varepsilon 0}|^{p(x)}$ is uniformly convergent to $a(x)|\nabla v_0(x)|^{p(x)}$ in $L^1(\Omega)$.

Definition 5 A function $v(x,t) \in \mathbf{W}(Q_T)\bigcap L^\infty(0,T;L^2(\Omega))$ is said to be a weak solution of problem (23)–(25) if

$$\frac{\partial v}{\partial t} \in \mathbf{W}'(Q_T),$$

and for any function $\varphi \in C_0^1(Q_T)$,

$$\begin{aligned} &\iint_{Q_T} \left(\frac{\partial v}{\partial t}\varphi + \big(a(x)+\varepsilon\big)\big(|v|^{\alpha(x)}+\varepsilon\big)|\nabla v|^{p(x)-2}\nabla v \nabla\varphi \right) dx\,dt \\ &\quad = \iint_{Q_T} \left[\sum_{i=1}^N g^i(x,t,v)\frac{\partial v}{\partial x_i} + d(x,t,v) \right] \varphi\,dx\,dt. \end{aligned} \tag{26}$$

The initial value (24) is satisfied in the sense as (12).

Then, by a similar method as that in [3], we have the following theorem.

Theorem 6 *If $a(x) \in C^1(\overline{\Omega})$ satisfies (6), $g^i(x,t,s)$ and $d(x,t,s)$ satisfy (13)–(14), there is a weak solution v_ε of the initial boundary value problem (23)–(24) on $\Omega \times [0,T^*)$, where*

$$T^* = \sup\big\{\theta : \|u\|_{\infty,Q_\theta} < \infty\big\}. \tag{27}$$

Firstly, we quote the following lemmas.

Lemma 7 *If $u_\varepsilon \in L^\infty(0,T;L^2(\Omega))\bigcap \mathbf{W}(Q_T)$, $\|u_{\varepsilon t}\|_{\mathbf{W}'(Q_T)} \le c$, $\|\nabla(|u_\varepsilon|^{r-1}u_\varepsilon)\|_{p,Q_T} \le c$, then there is a subsequence of $\{u_\varepsilon\}$ which is relatively compact in $L^s(Q_T)$ with $s \in (1,\infty)$. Here, $r \ge 1$, $p > 1$.*

This lemma comes from [19, Sect. 8].

Lemma 8 *Suppose that $p(x) \in C(\overline{\Omega})$ is local Hölder continuous, and denote that*

$$p^+ = \max_{x\in\overline{\Omega}} p(x), \qquad p^- = \min_{x\in\overline{\Omega}} p(x).$$

Then the following facts are true.

(i) *The space $(L^{p(x)}(\Omega), \|\cdot\|_{L^{p(x)}(\Omega)})$, $(W^{1,p(x)}(\Omega), \|\cdot\|_{W^{1,p(x)}(\Omega)})$ and $W_0^{1,p(x)}(\Omega)$ are reflexive Banach spaces.*

(ii) *$p(x)$-Hölder's inequality. Let $q(x) = \frac{p(x)}{p(x)-1}$. Then the conjugate space of $L^{p(x)}(\Omega)$ is $L^{q(x)}(\Omega)$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, there holds*

$$\left|\int_\Omega uv\,dx\right| \le 2\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{q(x)}(\Omega)}.$$

(iii)
If $\|u\|_{L^{p(x)}(\Omega)} = 1$, *then* $\int_\Omega |u|^{p(x)}\,dx = 1$.
If $\|u\|_{L^{p(x)}(\Omega)} > 1$, *then* $\|u\|^{p^-}_{L^{p(x)}(\Omega)} \le \int_\Omega |u|^{p(x)}\,dx \le \|u\|^{p^+}_{L^{p(x)}(\Omega)}$.
If $\|u\|_{L^{p(x)}(\Omega)} < 1$, *then* $\|u\|^{p^+}_{L^{p(x)}(\Omega)} \le \int_\Omega |u|^{p(x)}\,dx \le \|u\|^{p^-}_{L^{p(x)}(\Omega)}$.

This lemma can be found in [9, 20] etc.

Secondly, we give the details of the proof of Theorem 2.

Proof of Theorem 2 According to Theorem 6, there is a weak solution v_ε of the initial boundary value problem (23)–(24), and

$$\|v_\varepsilon\|_{\infty,Q_{T_0}} \le c(T_0),$$

where $T_0 < T^*$ is a given positive constant, $c(T_0)$ is a constant that may depend on T_0.

By multiplying (23) by v_ε, one has

$$\begin{aligned} &\frac{1}{2}\int_\Omega v_\varepsilon^2\,dx + \int_0^{T_0}\int_\Omega \bigl(a(x)+\varepsilon\bigr)\bigl(|v_\varepsilon|^{\alpha(x)}+\varepsilon\bigr)|\nabla v_\varepsilon|^{p(x)}\,dx\,dt \\ &\quad = \frac{1}{2}\int_\Omega v_{\varepsilon 0}^2\,dx + \iint_{Q_{T_0}} d(x,t,v_\varepsilon)v_\varepsilon\,dx\,dt + \sum_{i=i}^N \int_0^{T_0}\int_\Omega g^i(x,t,v_\varepsilon)v_\varepsilon \frac{\partial v_\varepsilon}{\partial x_i}\,dx\,dt. \end{aligned} \tag{28}$$

Since

$$\bigl|g^i(x,t,v_\varepsilon)\bigr| \le g(x,t)|v_\varepsilon|^{\frac{\alpha(x)}{p(x)}}, \quad i=1,2,\dots,N,$$

and $g(x,t) \in C(\overline{Q_{T_0}})$ satisfies (13), then one has

$$\left|g^i(x,t,v_\varepsilon)v_\varepsilon\frac{\partial v_\varepsilon}{\partial x_i}\right| \le \left|g(x,t)|v_\varepsilon|^{\frac{\alpha(x)}{p(x)}}\frac{\partial v_\varepsilon}{\partial x_i}\right| \le c(\varepsilon) + \varepsilon|v|^{\alpha(x)}|\nabla v_\varepsilon|^{p(x)}. \tag{29}$$

By that $|d(x,t,s)| \le d_0|s|^{\sigma-1} + h(x,t)$, $\|h\|_{L^1(0,\theta;L^\infty(\Omega))} \le c$, one has

$$\begin{aligned} \left|\iint_{Q_{T_0}} d(x,t,v_\varepsilon)v_\varepsilon\,dx\,dt\right| &\le \iint_{Q_{T_0}} \bigl[d_0|s|^{\sigma-1}+h(x,t)\bigr]|v_\varepsilon|\,dx\,dt \\ &\le c(T_0)\iint_{Q_{T_0}} \bigl[d_0|s|^{\sigma-1}+h(x,t)\bigr]\,dx\,dt \\ &\le c(T_0). \end{aligned} \tag{30}$$

Then formulas (28),(29), and (30) imply

$$\begin{aligned} \iint_{Q_{T_0}} a(x)|v_\varepsilon|^{\alpha(x)}|\nabla v_\varepsilon|^{p(x)}\,dx\,dt &\le \iint_{Q_{T_0}} \bigl(a(x)+\varepsilon\bigr)\bigl(|v_\varepsilon|^{\alpha(x)}+\varepsilon\bigr)|\nabla v_\varepsilon|^{p(x)}\,dx\,dt \\ &\le c(T_0), \end{aligned} \tag{31}$$

accordingly, one has

$$\begin{aligned}\iint_{Q_{T_0}} a(x)\left|\nabla v_\varepsilon^{\frac{\alpha(x)}{p(x)}+1}\right|^{p(x)} dx\,dt &\le c\iint_{Q_{T_0}} a(x)\left[\left|\nabla\frac{\alpha(x)}{p(x)}\right|\left|v_\varepsilon^{\frac{\alpha(x)}{p(x)}+1}\ln v_\varepsilon(x)\right|\right.\\ &\quad\left.+\left(\frac{\alpha(x)}{p(x)}+1\right)v_\varepsilon^{\frac{\alpha(x)}{p(x)}}|\nabla v_\varepsilon|\right]^{p(x)} dx\,dt\\ &\le c+c\iint_{Q_{T_0}} a(x)|v_\varepsilon|^{\alpha(x)}|\nabla v_\varepsilon|^{p(x)}\,dx\,dt\\ &\le c(T_0). \end{aligned} \tag{32}$$

Now, for any $u\in C_0^1(Q_{T_0})$, $\|u\|_{W(Q_{T_0})}=1$, one has

$$\begin{aligned}\langle v_{\varepsilon t},u\rangle &= -\iint_{Q_{T_0}}\big(a(x)+\varepsilon\big)\big(|v_\varepsilon|^{\alpha(x)}+\varepsilon\big)|\nabla v_\varepsilon|^{p(x)-2}\nabla v_\varepsilon\nabla u\,dx\,dt\\ &\quad+\sum_{i=1}^N\iint_{Q_{T_0}} g^i(x,t,v_\varepsilon)\frac{\partial v_\varepsilon}{\partial x_i}u\,dx\,dt+\iint_{Q_{T_0}} d(x,t,v_\varepsilon)u\,dx\,dt. \end{aligned} \tag{33}$$

Since $g^i(x,t,v_\varepsilon)$ and $d(x,t,v_\varepsilon)$ satisfy (13)(14), by $\|v_\varepsilon\|_{\infty,Q_{T_0}}\le c(T_0)$, using the Hölder inequality, one obtains

$$\begin{aligned}\left|\iint_{Q_{T_0}} g^i(x,t,v_\varepsilon)\frac{\partial v_\varepsilon}{\partial x_i}u\,dx\,dt\right| &\le \iint_{Q_{T_0}} g(x,t)|v_\varepsilon|^{\frac{\alpha(x)}{p(x)}-1}\left|\frac{\partial v_\varepsilon}{\partial x_i}u\right|dx\,dt\\ &\le c(T_0)\left(\iint_{Q_{T_0}}\left(\frac{g(x,t)^{p(x)}}{a(x)}\right)^{\frac{1}{p(x)-1}}dx\,dt\right)^{\frac{1}{q_1}}\\ &\le c(T_0), \end{aligned}$$

where $q_1=\max_{x\in\overline{\Omega}}\frac{p(x)}{p(x)-1}$ or $\min_{x\in\overline{\Omega}}\frac{p(x)}{p(x)-1}$ according to (iii) of Lemma 8, and

$$\left|\iint_{Q_{T_0}} d(x,t,v_\varepsilon)u\,dx\,dt\right|\le\iint_{Q_{T_0}}\big(d_0|v_\varepsilon|^{\sigma-1}+h(x,t)\big)|u|\,dx\,dt\le c(T_0).$$

By the above discussion, one has

$$\begin{aligned}\big|\langle v_{\varepsilon t},u\rangle\big| &\le c(T_0)\left[\iint_{Q_{T_0}}\big(a(x)+\varepsilon\big)\big(|v_\varepsilon|^{\alpha(x)}+\varepsilon\big)|\nabla v_\varepsilon|^{p(x)}\,dx\,dt\right.\\ &\quad\left.+\iint_{Q_{T_0}}\big(|u|^{p(x)}+|\nabla u|^{p(x)}\big)\,dx\,dt+1\right]\\ &\le c(T_0). \end{aligned}$$

Since $C_0^1(Q_{T_0})$ is dense on $W(Q_{T_0})$, one has

$$\|v_{\varepsilon t}\|_{\mathbf{W}'(Q_{T_0})}\le c(T_0)$$

and

$$\left\| v_{\varepsilon t}^{\frac{\alpha(x)}{p(x)}+1} \right\|_{\mathbf{W}'(Q_{T_0})} \le c(T_0). \tag{34}$$

If one denotes $d(x) = \operatorname{dist}(x, \partial\Omega)$ as the distance function from the boundary $\partial\Omega$, sets

$$\Omega_\lambda = \{x \in \Omega : d(x) > \lambda\}$$

for small $\lambda > 0$, and defines $\varphi \in C_0^1(\Omega)$, $0 \le \varphi \le 1$ such that

$$\varphi \mid_{\Omega_{2\lambda}} = 1, \qquad \varphi \mid_{\Omega\setminus\Omega_\lambda} = 0, \tag{35}$$

then, for any $\varphi \in C_0^1(\Omega)$ satisfying (35), $0 \le \varphi \le 1$, one has

$$\left\| \left(\varphi v_{\varepsilon}^{\frac{\alpha(x)}{p(x)}+1}\right)_t \right\|_{\mathbf{W}'(Q_{T_0})} \le c(\lambda, T_0). \tag{36}$$

Once again, since $a(x) > 0$ when $x \in \Omega$, by (32), one has

$$\iint_{Q_{T_0}} \left|\nabla\left(\varphi v_{\varepsilon}^{\frac{\alpha(x)}{p(x)}+1}\right)\right|^{p(x)} dx\,dt \le c(\lambda, T_0)\left(1 + \int_0^T \int_{D_\lambda} |\nabla v_\varepsilon|^{p(x)}\, dx\,dt\right) \le c(\lambda, T_0), \tag{37}$$

and so

$$\left\|\nabla\left(\varphi v_{\varepsilon}^{\frac{\alpha(x)}{p(x)}+1}\right)\right\|_{p^-,Q_{T_0}} \le \left\|\nabla\left(\varphi v_{\varepsilon}^{\frac{\alpha(x)}{p(x)}+1}\right)\right\|_{p(x),Q_{T_0}} \le c(T_0). \tag{38}$$

If one denotes $v_{1\varepsilon} = v_{\varepsilon}^{\frac{\alpha(x)}{p(x)}+1}$, then, from (36) and (38), Lemma 7 yields that $\varphi v_{1\varepsilon} \to \varphi v_1$ a.e. in Q_T. By the arbitrariness of φ, one has $v_{1\varepsilon} = v_{\varepsilon}^{\frac{\alpha(x)}{p(x)}+1} \to v_1$ a.e. in Q_{T_0}. By (12), $v \in L^\infty(Q_{T_0})$ and

$$v_\varepsilon \rightharpoonup v, \quad \text{weakly star in } L^\infty(Q_{T_0}). \tag{39}$$

By the weak convergence theory, one has

$$v_1 = v^{\frac{\alpha(x)}{p(x)}+1}.$$

Thus, $v_\varepsilon \to v$ a.e. in Q_{T_0}, and then

$$g^i(x,t,v_\varepsilon) \to g^i(x,t,v), \qquad d(x,t,v_\varepsilon) \to d(x,t,v), \quad \text{a.e. in } Q_{T_0}. \tag{40}$$

Moreover, since $a(x) \in C^1(\overline{\Omega})$ and $a(x)|_{x\in\Omega} > 0$, one has

$$\nabla v_{\varepsilon}^{\frac{\alpha(x)}{p(x)}+1} \rightharpoonup \nabla v^{\frac{\alpha(x)}{p(x)}+1} \quad \text{in } L^1\left(0,T;L_{\mathrm{loc}}^{p(x)}(\Omega)\right). \tag{41}$$

Now, similar as the techniques used in [22, 27, 28, 30], if one chooses $(v_\varepsilon^{\frac{\alpha(x)}{p(x)}+1} - v^{\frac{\alpha(x)}{p(x)}+1})\phi$ as the test function where $\phi(x) \in C_0^1(\Omega)$, then there holds

$$\begin{aligned}
&\int_0^{T_0}\int_\Omega \frac{\partial v_\varepsilon}{\partial t}\big(v_\varepsilon^{\frac{\alpha(x)}{p(x)}+1} - v^{\frac{\alpha(x)}{p(x)}+1}\big)\phi\,dx\,dt \\
&\quad + \int_0^{T_0}\int_\Omega \phi(x)\big(a(x)+\varepsilon\big)\big(|v_\varepsilon|^{\alpha(x)}+\varepsilon\big)|\nabla v_\varepsilon|^{p(x)-2}\nabla v_\varepsilon \nabla\big(v_\varepsilon^{\frac{\alpha(x)}{p(x)}+1} - v^{\frac{\alpha(x)}{p(x)}+1}\big)\,dx\,dt \\
&\quad + \int_0^{T_0}\int_\Omega \big(a(x)+\varepsilon\big)\big(|v_\varepsilon|^{\alpha(x)}+\varepsilon\big)|\nabla v_\varepsilon|^{p(x)-2}\nabla v_\varepsilon\big(v_\varepsilon^{\frac{\alpha(x)}{p(x)}+1} - v^{\frac{\alpha(x)}{p(x)}+1}\big)\nabla\phi\,dx\,dt \\
&\quad - \sum_{i=1}^N\int_0^{T_0}\int_\Omega g^i(x,t,v_\varepsilon)\frac{\partial v_\varepsilon}{\partial x_i}\big(v_\varepsilon^{\frac{\alpha(x)}{p(x)}+1} - v^{\frac{\alpha(x)}{p(x)}+1}\big)\phi\,dx\,dt \\
&= \int_0^{T_0}\int_\Omega d(x,t,v_\varepsilon)\big(v_\varepsilon^{\frac{\alpha(x)}{\alpha(x)}+1} - v^{\frac{\alpha(x)}{\alpha(x)}+1}\big)\phi\,dx\,dt.
\end{aligned}\tag{42}$$

Since

$$\big|g^i(x,t,v_\varepsilon)\big| \le g(x,t)|v_\varepsilon|^{\frac{\alpha(x)}{p(x)}},\quad i=1,2,\dots,N,$$

and using (14), it can be deduced that

$$\int_0^{T_0}\int_\Omega \phi(x)a(x)|v_\varepsilon|^{\alpha(x)}|\nabla v_\varepsilon|^{p(x)}\nabla v_\varepsilon\nabla v^{\frac{\alpha(x)}{p(x)}+1}\,dx\,dt \le c.\tag{43}$$

By the arbitrariness of ϕ and $|v_\varepsilon|^{\alpha(x)}|\nabla v_\varepsilon|^{p(x)-2}\nabla v_\varepsilon \in L^1(0,T_0;L_{\mathrm{loc}}^{\frac{p(x)}{p(x)-1}}(\Omega))$, one has

$$\nabla v \in L^\infty\big(0,T;L_{\mathrm{loc}}^{p(x)}(\Omega)\big).\tag{44}$$

By this property, one can show that

$$g^i(x,t,v_\varepsilon)\frac{\partial v_\varepsilon}{\partial x_i} \rightharpoonup g^i(x,t,v)\frac{\partial v}{\partial x_i}\quad \text{in } L^1(Q_{T_0}).\tag{45}$$

The details are omitted here.

Thus, there are functions $v(x,t)$ and ζ_i satisfying

$$v(x,t)\in L^\infty(Q_{T_0}),\qquad \big|\zeta_i(x,t)\big| \in L^1\big(0,T_0;L^{\frac{p(x)}{p(x)-1}}(\Omega)\big)$$

such that

$$\begin{aligned}
&v_\varepsilon \rightharpoonup v,\quad \text{weakly star in } L^\infty(Q_{T_0}),\\
&g^i(x,t,v_\varepsilon)\to g^i(x,t,v),\qquad d(x,t,v_\varepsilon)\to d(x,t,v),\quad \text{a.e. in } Q_{T_0},\\
&\big(a(x)+\varepsilon\big)\big(|v_\varepsilon|^{\alpha(x)}+\varepsilon\big)|\nabla v_\varepsilon|^{p(x)-2}\nabla v_\varepsilon \rightharpoonup \vec{\zeta},\quad \text{in } L^1\big(0,T_0;L^{\frac{p(x)}{p(x)-1}}(\Omega)\big).
\end{aligned}$$

Moreover, by the important property (44), it is not difficult to show that

$$\iint_{Q_{T_0}} a(x)|v|^{\alpha(x)}|\nabla v|^{p(x)-2}\nabla v\cdot\nabla\varphi\,dx\,dt = \iint_{Q_{T_0}} \vec{\zeta}\cdot\nabla\varphi\,dx\,dt$$

for any given function $\varphi \in C_0^1(Q_{T_0})$. Then v is a weak solution of equation (2) with the initial value (3). □

4 The stability

For small $\eta > 0$, let

$$S_\eta(s) = \int_0^s h_\eta(\tau)\,d\tau, \qquad h_\eta(s) = \frac{2}{\eta}\left(1 - \frac{|s|}{\eta}\right)_+.$$

Certainly, $h_\eta(s) \in C(\mathbb{R})$ and

$$\lim_{\eta\to 0} S_\eta(s) = \operatorname{sgn} s, \qquad \lim_{\eta\to 0} s h_\eta(s) = 0, \qquad \lim_{\eta\to 0} H_\eta(s) = |s|, \tag{46}$$

where $H_\eta(s) = \int_0^s S_\eta(\tau)\,d\tau$. In this section, $\lim_{\lambda\to 0}$ represents $\limsup_{\lambda\to 0}$.

Proof of Theorem 4 Let $\varphi(x,t) = S_\eta(u-v)\varphi_\lambda(x)$ in (26) and denote $D_\lambda = \{x \in \Omega : \varphi(x) \geq \lambda\}$ for small enough λ. Then

$$\begin{aligned}
&\int_0^t \int_\Omega \frac{\partial(u-v)}{\partial t}\varphi_\lambda(x) S_\eta(u-v)\,dx\,dt \\
&\quad + \int_0^t \int_{D_\lambda} a(x)\varphi_\lambda(x)|u|^{\alpha(x)}\left(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\right)\nabla(u-v)h_\eta(u-v)\,dx\,dt \\
&\quad + \int_0^t \int_{D_\lambda} a(x)\varphi_\lambda(x)\left(|u|^{\alpha(x)} - |v|^{\alpha(x)}\right)|\nabla v|^{p(x)-2}\nabla v\nabla(u-v)h_\eta(u-v)\,dx\,dt \\
&\quad + \int_0^t \int_{D_\lambda\setminus D_{2\lambda}} a(x)|u|^{\alpha(x)}\left(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\right)\nabla\varphi_\lambda S_\eta(u-v)\,dx\,dt \\
&\quad + \int_0^t \int_{D_\lambda\setminus D_{2\lambda}} a(x)\left(|u|^{\alpha(x)} - |v|^{\alpha(x)}\right)|\nabla v|^{p(x)-2}\nabla v\nabla\varphi_\lambda S_\eta(u-v)\,dx\,dt \\
&\quad - \sum_{i=1}^N \int_0^t \int_{D_\lambda} \left[g^i(x,t,u)u_{x_i} - g^i(x,t,v)v_{x_i}\right] S_\eta(u-v)\varphi_\lambda(x)\,dx\,dt \\
&= \int_0^t \int_{D_\lambda} \left[d(x,t,u) - d(x,t,v)\right]\varphi_\lambda(x) S_\eta\left((u-v)\right)dx\,dt.
\end{aligned} \tag{47}$$

The monotonicity of the $p(x)$-Laplacian operator yields

$$\int_{D_\lambda} \varphi_\lambda(x)a(x)|u|^{\alpha(x)}\left(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\right)\nabla(u-v)h_\eta(u-v)\,dx \geq 0. \tag{48}$$

By the definition of $\varphi_\lambda(x)$, there exists

$$\begin{aligned}
&\lim_{\lambda\to 0}\lim_{\eta\to 0} \int_0^t \int_\Omega \varphi_\lambda \frac{\partial H_\eta(u-v)}{\partial t}\,dx\,dt \\
&\quad = \int_\Omega \left|u(x,t) - v(x,t)\right|dx - \int_\Omega \left|u_0(x) - v_0(x)\right|dx.
\end{aligned} \tag{49}$$

By that

$$\int_{D_\lambda} a(x)|u|^{\alpha(x)}|\nabla u|^{p(x)}\,dx \le c(\lambda), \qquad \int_{D_\lambda} a(x)|v|^{\alpha(x)}|\nabla v|^{p(x)}\,dx \le c(\lambda),$$

using the Lebesgue dominated convergence theorem, one has

$$\begin{aligned}
&\lim_{\eta\to 0}\left|\int_{D_\lambda} \varphi_\lambda(x)a(x)\big(|u|^{\alpha(x)}-|v|^{\alpha(x)}\big)|\nabla v|^{p(x)-2}\nabla v\nabla(u-v)h_\eta(u-v)\,dx\right| \\
&\quad\le \lim_{\eta\to 0}\int_{D_\lambda} a(x)\big||u|^{\alpha(x)}-|v|^{\alpha(x)}\big||\nabla v|^{p(x)-1}\big(|\nabla u|+|\nabla v|\big)h_\eta(u-v)\,dx \\
&\quad\le c\lim_{\eta\to 0}\int_{D_\lambda} a(x)||u|^{\alpha(x)}-|v|^{\alpha(x)}||\nabla v|^{p(x)-1}|\nabla u|h_\eta(u-v)\,dx \\
&\qquad+\int_{D_\lambda} a(x)\big||u|^{\alpha(x)}-|v|^{\alpha(x)}\big||\nabla v|^{p(x)}h_\eta(u-v)\,dx \\
&\quad\le c\lim_{\eta\to 0}\left(\int_{D_\lambda} a(x)\big||u|^{\alpha(x)}-|v|^{\alpha(x)}\big||\nabla v|^{p(x)}h_\eta(u-v)\,dx\right)^{\frac{1}{q^+}} \\
&\qquad\cdot\left(\int_{D_\lambda} a(x)||u|^{\alpha(x)}-|v|^{\alpha(x)}||\nabla u|^{p(x)}h_\eta(u-v)\,dx\right)^{\frac{1}{p^+}} \\
&\qquad+\lim_{\eta\to 0}\int_{D_\lambda} a(x)\big||\nabla v|^{p(x)}||u|^{\alpha(x)}-|v|^{\alpha(x)}\big||h_\eta(u-v)\,dx \\
&\quad\le c\lim_{\eta\to 0}\left(\int_{D_{1\lambda}} a(x)\alpha(x)\xi^{\alpha(x)-1}|\nabla v|^{p(x)}\big|(u-v)h_\eta(u-v)\big|\,dx\right)^{\frac{1}{q^+}} \\
&\qquad\cdot\left(\int_{D_{1\lambda}} a(x)\alpha(x)\xi^{\alpha(x)-1}|\nabla u|^{p(x)}\big|(u-v)h_\eta(u-v)\big|\,dx\right)^{\frac{1}{p^+}} \\
&\qquad+\lim_{\eta\to 0}\int_{D_{1\lambda}} a(x)\alpha(x)\xi^{\alpha(x)-1}|\nabla v|^{p(x)}\big|(u-v)h_\eta(u-v)\big|\,dx \\
&\quad=0,
\end{aligned}\tag{50}$$

where $D_{1\lambda}=\{x\in D_\lambda:\alpha(x)\neq 0, u(x)\neq v(x)\}$, $0<\xi\in(v,u)$, $x\in D_{1\lambda}$ is the mean value.
Since

$$\nabla\varphi_\lambda(x)=0, \quad x\in D_{2\lambda},$$

one has

$$\begin{aligned}
&\int_{D_\lambda} a(x)|u|^{\alpha(x)}\big(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v\big)\nabla\varphi_\lambda(u-v)h_\eta(u-v)\,dx \\
&\quad=\frac{1}{\lambda}\int_{D_\lambda\setminus D_{2\lambda}} a(x)|u|^{\alpha(x)}\big(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v\big)\nabla\varphi(u-v)h_\eta(u-v)\,dx \\
&\quad\to 0
\end{aligned}\tag{51}$$

as $\eta\to 0$.

Similarly, one has

$$\lim_{\eta\to 0}\int_{D_\lambda} a(x)\big(|u|^{\alpha(x)} - |u|^{\alpha(x)}\big)|\nabla v|^{p(x)-2}\nabla v\nabla\varphi_\lambda(u-v)h_\eta(u-v)\,dx = 0. \tag{52}$$

Moreover, for the sixth term on the left hand side of (47), by that

$$\frac{\partial G^i(x,t,u)}{\partial x_i} = g^i(x,t,u)u_{x_i} + \int_0^u \frac{\partial g^i(x,t,s)}{\partial x_i}\,ds,$$

one has

$$\begin{aligned}
&\int_{D_\lambda} g^i(x,t,u)u_{x_i}S_\eta(u-v)\varphi_\lambda(x)\,dx\\
&\quad= \int_{D_\lambda}\left[\frac{\partial G^i(x,t,u)}{\partial x_i} - \int_0^u \frac{\partial g^i(x,t,s)}{\partial x_i}\,ds\right]S_\eta(u-v)\varphi_\lambda(x)\,dx\\
&\quad= -\int_{D_\lambda} G^i(x,t,u)\big[h_\eta(u-v)(u-v)_{x_i}\varphi_\lambda(x) + S_\eta(u-v)\varphi_{\lambda x_i}(x)\big]\,dx\\
&\qquad- \int_{D_\lambda}\int_0^u \frac{\partial g^i(x,t,s)}{\partial x_i}ds S_\eta(u-v)\varphi_\lambda(x)\,dx
\end{aligned}\tag{53}$$

and

$$\begin{aligned}
&\int_{D_\lambda} g^i(x,t,v)v_{x_i}S_\eta(u-v)\varphi_\lambda(x)\,dx\\
&\quad= \int_{D_\lambda}\left[\frac{\partial G^i(x,t,v)}{\partial x_i} - \int_0^v \frac{\partial g^i(x,t,s)}{\partial x_i}\,ds\right]S_\eta(u-v)\varphi_\lambda(x)\,dx\\
&\quad= -\int_{D_\lambda} G^i(x,t,v)\big[h_\eta(u-v)(u-v)_{x_i}\varphi_\lambda(x) + S_\eta(u-v)\varphi_{\lambda x_i}(x)\big]\,dx\\
&\qquad- \int_{D_\lambda}\int_0^v \frac{\partial g^i(x,t,s)}{\partial x_i}ds S_\eta(u-v)\varphi_\lambda(x)\,dx.
\end{aligned}\tag{54}$$

Let (53) minus (54). Firstly, by the definition of φ_λ,

$$|u_{x_i}|\in L^1(D_\lambda),\qquad |v_{x_i}|\in L^1(D_\lambda),\quad i=1,2,\ldots,N,$$

using the Lebesgue dominated convergence theorem, one has

$$\begin{aligned}
&\lim_{\eta\to 0}\int_0^t\int_{D_\lambda}\big[G^i(x,t,u) - G^i(x,t,v)\big]h_\eta(u-v)(u-v)_{x_i}\varphi_\lambda(x)\,dx\,dt\\
&\quad= \lim_{\eta\to 0}\int_0^t\int_{D_\lambda}\big[G^i(x,t,u) - G^i(x,t,v)\big]h_\eta(u-v)(u-v)_{x_i}\varphi_\lambda(x)\,dx\,dt = 0.
\end{aligned}\tag{55}$$

Secondly, by the partial boundary value condition (20)–(21), one has

$$\begin{aligned}
&\lim_{\lambda\to 0}\lim_{\eta\to 0}\int_0^t\int_{D_\lambda\setminus D_{2\lambda}}\sum_{i=1}^N\big[G^i(x,t,u) - G^i(x,t,v)\big]S_\eta(u-v)\varphi_{\lambda x_i}(x)\,dx\,dt\\
&\quad= \lim_{\lambda\to 0}\lim_{\eta\to 0}\int_0^t\frac{1}{\lambda}\int_{D_\lambda\setminus D_{2\lambda}}\sum_{i=1}^N\big[G^i(x,t,u) - G^i(x,t,v)\big]S_\eta(u-v)\varphi_{x_i}(x)\,dx\,dt
\end{aligned}$$

$$\begin{aligned}
&= \lim_{\lambda\to 0}\int_0^t \frac{1}{\lambda}\int_{D_\lambda\setminus D_{2\lambda}} \sum_{i=1}^N g^i(x,t,\xi)|u-v|\varphi_{x_i}(x)\,dx\,dt \\
&\le \lim_{\lambda\to 0}\int_0^t \frac{1}{\lambda}\int_{D_\lambda\setminus D_{2\lambda}\bigcap\{x:\sum_{i=1}^N g^i(x,t,\xi)\varphi_{x_i}(x)>0\}} |u-v|\sum_{i=1}^N g^i(x,t,\xi)\varphi_{x_i}(x)\,dx\,dt \\
&= \int_0^t \int_{\partial\Omega\bigcap\{x\in\partial\Omega:\sum_{i=1}^N g^i(x,t,0)\varphi_{x_i}(x)>0\}} |u-v|\sum_{i=1}^N g^i(x,t,0)\varphi_{x_i}(x)\,dx\,dt \\
&= 0.
\end{aligned} \tag{56}$$

Thus, it can be deduced that

$$\begin{aligned}
&\lim_{\lambda\to 0}\lim_{\eta\to 0}\int_0^t \int_\Omega \big[g^i(x,t,u)u_{x_i} - g^i(x,t,v)v_{x_i}\big]S_\eta(u-v)\varphi_\lambda(x)\,dx\,dt \\
&\quad = \lim_{\lambda\to 0}\lim_{\eta\to 0}\int_0^t \int_\Omega \big[G^i(x,t,u) - G^i(x,t,v)\big] \\
&\qquad \times \big[h_\eta(u-v)(u-v)_{x_i}\varphi_\lambda(x) + S_\eta(u-v)\varphi_{\lambda x_i}(x)\big]\,dx\,dt \\
&\qquad - \lim_{\lambda\to 0}\lim_{\eta\to 0}\int_0^t \int_\Omega \int_v^u \frac{\partial g^i(x,t,s)}{\partial x_i}\,ds S_\eta(u-v)\varphi_\lambda(x)\,dx\,dt \\
&\quad \le c\int_\Omega |u-v|\,dx.
\end{aligned} \tag{57}$$

Thirdly,

$$\begin{aligned}
&\lim_{\lambda\to 0}\lim_{\eta\to 0}\left|\int_0^t \int_\Omega \left[\int_0^u \frac{\partial g^i(x,t,s)}{\partial x_i}\,ds - \int_0^v \frac{\partial g^i(x,t,s)}{\partial x_i}\,ds\right]\varphi_\lambda(x)S_\eta(u-v)\,dx\,dt\right| \\
&\quad \le c\int_0^t \int_\Omega \big|u(x,t)-v(x,t)\big|\,dx\,dt.
\end{aligned} \tag{58}$$

At last, by that $d(x,t,s)$ is a Lipschitz function, we have

$$\begin{aligned}
&\lim_{\lambda\to 0}\lim_{\eta\to 0}\left|\int_0^t \int_\Omega \big[d(x,t,u)-d(x,t,v)\big]\varphi_\lambda(x)S_\eta(u-v)\,dx\,dt\right| \\
&\quad \le c\int_0^t \int_\Omega \big|u(x,t)-v(x,t)\big|\,dx\,dt.
\end{aligned} \tag{59}$$

After letting $\eta \to 0$ in (47), let $\lambda \to 0$. By (49)–(59), we have

$$\int_\Omega \big|u(x,t)-v(x,t)\big|\,dx \le \int_\Omega \big|u_0(x)-v_0(x)\big|\,dx + c\int_0^t \int_\Omega \big|u(x,t)-v(x,t)\big|\,dx\,dt,$$

the well-known Gronwall inequality yields

$$\int_\Omega \big|u(x,t)-v(x,t)\big|\,dx \le \int_\Omega \big|u_0(x)-v_0(x)\big|\,dx. \qquad \square$$

5 Conclusion and a simple comment

It is well known that, in order to study the well-posedness problem of a polytropic filtration equation

$$v_t = \operatorname{div}\big(|v|^{\alpha}|\nabla v|^{p-2}\nabla v\big) + f(x,t,v,\nabla v), \quad (x,t)\in Q_T, \tag{60}$$

one generally transfers it to the following type:

$$\big(|u|^{\beta-1}u\big)_t = \delta\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big), \quad (x,t)\in Q_T, \tag{61}$$

as [6, 21], where $\beta = (p-1)(\alpha+p-1)^{-1}$, $\delta = \beta^{p-1}$. Then the methods and techniques used in the study of the well-posedness of non-Newtonian fluid equations may be valid. But, since equation (2) contains the nonlinear term $|v|^{\alpha(x)}$ and the variable exponent $p(x)$, to transfer equation (2) to another equation similar to equation (61) is impossible. At the same time, compared with our previous works [27, 28, 30], the key assumption $\alpha(x)\in C_0(\Omega)$ in [27, 28, 30] has been weakened to $\alpha(x)\in C(\Omega)$. Moreover, the classical trace of $u\in W_0^{1,1}(\Omega)$ is generalized to $u\in W_{\mathrm{loc}}^{1,1}(\Omega)\bigcap L^{\infty}(Q_T)$, and basing on such a generalization, a reasonable partial boundary value condition is found to match up with equation (2). The methods used to prove the stability of the weak solutions also are valid to prove the corresponding stability theorems related to the degenerate parabolic equation appearing in [2, 22, 27, 28, 30].

At the end of the paper, we give a simple comment on the definition of the trace.

For a linear degenerate elliptic equation [15–17]

$$\sum_{r,s=1}^{N+1} a^{rs}(x)\frac{\partial^2 u}{\partial x_r\partial x_s} + \sum_{r=1}^{N+1} b_r(x)\frac{\partial u}{\partial x_r} + c(x)u = f(x), \quad x\in\widetilde{\Omega}\subset\mathbb{R}^{N+1},$$

it is well known that an appropriate partial boundary condition is

$$u|_{\sum_2\bigcup\sum_3} = g(x). \tag{62}$$

Here, $\{n_s\}$ is the unit inner normal vector of $\partial\widetilde{\Omega}$ and

$$\Sigma_2 = \big\{x\in\partial\widetilde{\Omega} : a^{rs}n_rn_s = 0, \big(b_r - a^{rs}_{x_s}\big)n_r < 0\big\},$$
$$\Sigma_3 = \big\{x\in\partial\widetilde{\Omega} : a^{rs}n_sn_r > 0\big\}.$$

It means that if the matrix $((a^{rs}))$ is positive definite, then condition (62) is just the usual Dirichlet boundary condition. Thus, for a classical parabolic equation

$$u_t = \sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2 u}{\partial x_i\partial x_j} + \sum_{i=1}^{N} b_i(x,t)\frac{\partial u}{\partial x_i} - c(x,t)u + f(x,t), \tag{63}$$

when the matrix $((a^{ij}))$ is positive definite, then we should impose the following initial-boundary condition:

$$u(x,0) = u_0(x), \quad x\in\Omega, \tag{64}$$

$$u(x,t) = g(x,t), \quad (x,t)\in\partial\Omega\times[0,T). \tag{65}$$

Naturally, the solutions of equations (60) and (63) are the classical solutions, and conditions (61)(65) are true in the sense of continuity. However, for nonlinear degenerate parabolic equations, the solutions generally are in a weak sense, the boundary value condition cannot be true in the sense of continuity. Moreover, since $C_0^\infty(\Omega)$ is dense in a Sobolev space $W_0^{1,p}(\Omega)$, the trace of $f(x) \in W_0^{1,p}(\Omega)$ on the boundary $\partial\Omega$ is defined as the limit of a sequence $f_\varepsilon(x)$ as

$$f(x)|_{x\in\partial\Omega} = \lim_{\varepsilon\to 0} f_\varepsilon(x)|_{x\in\partial\Omega} = 0. \tag{66}$$

If the weak solution of a nonlinear equation belongs to a Sobolev space $W_0^{1,p}(\Omega)$, then the Dirichlet boundary value condition is true in the sense of (66). Actually, let $\mathrm{BV}(\Omega)$ be the BV function space, i.e., $|\frac{\partial f}{\partial x_i}|$ is a regular measure, and

$$\mathrm{BV}(\Omega) = \left\{ f(x) : \int_\Omega \left| \frac{\partial f}{\partial x_i} \right| < c, i = 1, 2, \dots, N \right\}.$$

Then the BV function space is the weakest space such that the trace of $u \in \mathrm{BV}(\Omega)$ can be defined as (66) (when $u = 0$ on the boundary $\partial\Omega$).

If a weak solution of a nonlinear equation does not belong to a Sobolev space $W_0^{1,p}(\Omega)$, how to impose a suitable boundary condition has been an important and difficult problem for a long time. A typical example is evolutionary p-Laplacian equations of the form

$$\frac{\partial u}{\partial t} - \operatorname{div}\bigl(\alpha(x)|\nabla u|^{p-2}\nabla u\bigr) - b_i(x)D_i u + c(x,t)u = f(x,t), \quad (x,t) \in Q_T, \tag{67}$$

where $\alpha(x) \in C(\Omega)$, $\alpha(x) > 0$ in Ω but may be equal to 0 on the boundary $\partial\Omega$. The authors of [23] classified the boundary $\partial\Omega$ into three parts: the nondegenerate boundary Σ_3,

$$\Sigma_3 = \bigl\{ x \in \partial\Omega : \alpha(x) > 0 \bigr\},$$

the weakly degenerate boundary

$$\Sigma_4 = \left\{ x \in \partial\Omega : \alpha(x) = 0, \text{there exists } r > 0, \text{such that } \int_{\Omega\bigcap B_r(x)} a(y)^{-\frac{1}{p-1}}\, dy < +\infty \right\},$$

and the strongly degenerate boundary

$$\Sigma^0 = \partial\Omega \setminus (\Sigma_3 \cup \Sigma_4) = \left\{ x \in \partial\Omega : \text{for any small } r > 0, \int_{\Omega\bigcap B_r(x)} a(y)^{-\frac{1}{p-1}}\, dy = +\infty \right\},$$

where $B_r(x) = \{y : d(x,y) < r\}$. Denote by $\mathbf{B}$ the closure of the set $C_0^\infty(Q_T)$ with respect to the norm

$$\|u\|_{\mathbf{B}} = \iint_{Q_T} a(x)\bigl(\bigl|u(x,t)\bigr|^p + \bigl|\nabla u(x,t)\bigr|^p\bigr)\, dx\, dt, \quad u \in \mathbf{B}.$$

In [23], the trace of $u \in \mathbf{B}$, $u(x,t) = 0$ on the boundary is defined as

$$\operatorname{ess\,sup} \lim_{\lambda\to 0} \int_{\{x\in\partial\Omega_\lambda : \sum_{i=1}^N b_i(x)n_i(x)<0\}} u^2 \sum_{i=1}^N b_i(x)n_i(x)\, d\sigma = 0, \tag{68}$$

where $\operatorname{ess\,sup}\lim_{\lambda\to 0} f(\lambda) = \inf_{\delta>0}\{\operatorname{ess\,sup}\{f(\lambda) : |\lambda| < \delta\}\}$ is the super limit.

Meanwhile, by defining

$$\Sigma_0 = \left\{x \in \Sigma^0 : \sum_{i=1}^{N} b_i(x)n_i(x) = 0\right\},$$
$$\Sigma_1 = \left\{x \in \Sigma^0 : \sum_{i=1}^{N} b_i(x)n_i(x) > 0\right\},$$

and

$$\Sigma_2 = \left\{x \in \Sigma^0 : \sum_{i=1}^{N} b_i(x)n_i(x) < 0\right\},$$

they imposed a partial boundary value condition

$$u(x,t) = 0, \quad (x,t) \in \left(\Sigma_2 \bigcup \Sigma_3 \bigcup \Sigma_4\right) \times (0,T), \tag{69}$$

where $\vec{n} = \{n_i(x)\}$ is the inner normal vector of $\partial\Omega$.

Along this way, the author of this paper has given another generalization of the trace to the functional space $L^\infty(0,T;W^{1,p}_{\mathrm{loc}}(\Omega))$ in [29] recently. However, such a generalization of the trace is based on the convection term $b_i(x)D_iu$. Once a nonlinear evolutionary equation is without a convection term, for example, if considering the equation

$$v_t = \operatorname{div}\left(a(x)|v|^{\alpha(x)}|\nabla v|^{p(x)-2}\nabla v\right) + f(x,t,v), \quad (x,t) \in Q_T, \tag{70}$$

then the definition of (68) cannot be used. On the other hand, the general trace defined as Definition 3 is valid for equation (70) and any other equations appearing in this paper. So, we think the trace defined as Definition 3 is more natural and novel.

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Declarations

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I am the only author of the paper. The author read and approved the final manuscript.

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