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Existence of solutions for a class of nonlinear fractional difference equations of the Riemann–Liouville type

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Abstract

Nonlinear fractional difference equations are studied deeply and extensively by many scientists by using fixed-point theorems on different types of function spaces. In this study, we combine fixed-point theory with a set of falling fractional functions in a Banach space to prove the existence and uniqueness of solutions of a class of fractional difference equations. The most important part of this article is devoted to correcting a significant mistake made in the literature in using the power rule by providing further conditions for its validity. Also, we provide specific conditions under which difference equations have attractive solutions and the solutions are also asymptotically stable. Furthermore, we construct some fractional difference examples in order to illustrate the validity of the observed results.

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1 Introduction

The idea of discrete fractional calculus is to replace the natural numbers in the order of the difference by fractional orders. However, since the emergence of the theory of discrete fractional calculus, different types of discrete fractional operators have been developed to deal with various situations in the applied and natural sciences due to their great importance as an advanced mathematical tool for the interpretation and modeling of many biological and physical phenomena, such as various biological studies, electrical circuits, mechanical fluids, relaxation processes, and damping-law models (see [1–6]). There are several possible ways to define discrete fractional operators (differences and sums), leading to a diverse and rich field of study (see [7–11]). Here, we shall focus principally on the most commonly used and classical definition, which is known as the Riemann–Liouville (RL) fractional calculus (see, for details, [12, 13]; see also the recent survey-cum-expository review articles [14, 15]).

Fractional difference equations (FDEs) have become a hot research topic in the mathematical and physical sciences. It has been found that the role of FDEs is very important in treating and modeling nonlinear problems with applications in mathematical analysis and various branches of science, including diffusion, plasmas, dynamic systems, nonlinear optics, and many other areas (see [16–21]).

In the last two decades, significant numbers of articles have appeared on this topic, and some of the papers deal with the existence and uniqueness of solutions for difference equation problems (see [22–26]). However, a significant mistake has been made by most of the researchers in using the fractional power rule (see Lemma 2.2).

In some recent articles of Lu et al. [27] and Mohammed [28], some nonlinear RL fractional difference equations were established from the uncertain point of view, and the existence and uniqueness theorems were studied using the scheme of uncertainty theory. In general, the difference equations considered were of the following form:

$$\begin{aligned} &\big({}^{\mathrm{RL}}_{\nu-1}\Delta^{\nu} y\big)(\mathrm{z}) = \psi\big(\mathrm{z}+\nu, y(\mathrm{z}+\nu)\big) \quad \big(\forall \mathrm{z}\in \mathrm{N}_0, \nu\in(0,1)\big), \\ &\big({}^{\mathrm{RL}}_{\nu-1}\Delta^{\nu-1} y\big)(\mathrm{z})\big|_{\mathrm{z}=0} = y_0, \end{aligned} \tag{1.1}$$

where ${}^{\mathrm{RL}}_{a}\Delta^{\nu}$ is the discrete RL fractional difference operator and ψ is supposed to be a real-valued function: $f:[0,\infty)\times\mathbb{R}\to\mathbb{R}$.

Next, in [29] and [30], the problem of the power rule was solved by considering a new difference equation as follows:

$$\begin{aligned} &\big({}^{\mathrm{RL}}_{\nu-1}\Delta^{\nu} y\big)(\mathrm{z}) = \psi\big(\mathrm{z}+\nu-1, y(\mathrm{z}+\nu-1)\big) \quad \big(\forall \mathrm{z}\in \mathrm{N}_0, \nu\in(0,1)\big), \\ &\big({}^{\mathrm{RL}}_{\nu-1}\Delta^{\nu-1} y\big)(\mathrm{z})\big|_{\mathrm{z}=0} = y_0. \end{aligned} \tag{1.2}$$

In the meantime, the existence and uniqueness of the Liouville–Caputo version of the difference equation (1.2) was obtained by Srivastava et al. [31] in the correct way as above.

Moreover, in [32–34] some nonlinear RL and Liouville–Caputo fractional difference equations such as (1.1) and (1.2) were established from the mathematical point of view. Also, the existence and uniqueness theorems were proved there using the scheme of fixed-point theory. Unfortunately, the same mistake as above was made in those articles.

The aim of this work is to present the existence and uniqueness of the solution of the nonlinear fractional difference equation (1.2) using fixed-point theorems in the correct way and correcting the above mistakes. The remainder of the study is structured as follows: In Sect. 2, we give related notations and make some preparations. In Sect. 3, we derive and prove the main theorems of the article: first determining a suitable power-rule condition corresponding to the difference equation (1.2), then proving existence and uniqueness, and finally rewriting the difference equation in such a way that the problem will be more useful in applications. In Sect. 4 we will illustrate our results with several examples of different types, providing specific nonlinear difference equations and conditions under which they have attractive solutions and the solutions are asymptotically stable. In Sect. 5, we conclude the article with some ideas and remarks for future directions of work in this area.

2 Preliminaries

Denote $\rho(\mathrm{z}) := \mathrm{z}-1$, $\sigma(\mathrm{z}) := \mathrm{z}+1$ and $\mathrm{N}_a := \{a, a+1, a+2, \ldots\}$. Let ψ be defined on N_a. Then, the forward and backward difference operators are given by $\Delta\psi(\mathrm{z}) = \psi(\sigma(\mathrm{z})) - \psi(\mathrm{z})$ and $\nabla\psi(\mathrm{z}) = \psi(\mathrm{z}) - \psi(\rho(\mathrm{z}))$ for each $\mathrm{z}\in\mathrm{N}_a$, respectively. There are plenty of possible ways to define discrete fractional differences and sums, leading to a diverse and rich field of study [8–10]. Primarily, we shall focus on the discrete RL fractional operators, which is the most commonly used definition. Here, discrete fractional sums of order $\nu > 0$ are defined by

$$\left({}_a\Delta^{-\nu}\psi\right)(\mathrm{z}) = \frac{1}{\Gamma(\nu)}\sum_{\kappa=a}^{\mathrm{z}-\nu}\left(\mathrm{z}-\sigma(\kappa)\right)^{(\nu-1)}\psi(\kappa) \quad (\forall \mathrm{z} \in \mathrm{N}_{a+\nu}), \tag{2.1}$$

where ψ is defined on N_a and $\mathrm{z}^{(\nu)}$ is the falling factorial function, defined by

$$\mathrm{z}^{(\nu)} = \frac{\Gamma(\mathrm{z}+1)}{\Gamma(\mathrm{z}+1-\nu)} \quad (\forall \mathrm{z} \text{ and } \nu \in \mathbb{R}). \tag{2.2}$$

The discrete RL fractional difference is, as an extension of the discrete fractional sum, defined by

$$\begin{aligned}\left({}_a^{\mathrm{RL}}\Delta^{\nu}\psi\right)(\mathrm{z}) &= \left(\Delta_a\Delta^{-(1-\nu)}\psi\right)(\mathrm{z}) \\ &= \frac{1}{\Gamma(1-\nu)}\Delta\left(\sum_{\kappa=a}^{\mathrm{z}+\nu-1}\left(\mathrm{z}-\sigma(\kappa)\right)^{(-\nu)}\psi(\kappa)\right) \quad (\forall \mathrm{z} \in \mathrm{N}_{a+1-\nu})\end{aligned} \tag{2.3}$$

for $0 \leqq \nu < 1$.

Lemma 2.1 *If* $\nu > 0$, $\mathrm{z}^{(-\nu)}$ *is nonincreasing on* N_0.

Proof From the forward difference operator and (2.2), we have

$$\begin{aligned}\Delta\left(\mathrm{z}^{(-\nu)}\right) = (\mathrm{z}+1)^{(-\nu)} - \mathrm{z}^{(-\nu)} &= \left(\frac{\Gamma(\mathrm{z}+2)}{\Gamma(\mathrm{z}+2+\nu)} - \frac{\Gamma(\mathrm{z}+1)}{\Gamma(\mathrm{z}+1+\nu)}\right) \\ &= \frac{\Gamma(\mathrm{z}+1)}{\Gamma(\mathrm{z}+1+\nu)}\left(\frac{\mathrm{z}+1}{\mathrm{z}+1+\nu} - 1\right) \\ &= -\nu\frac{\Gamma(\mathrm{z}+1)}{\Gamma(\mathrm{z}+2+\nu)} = -\nu\mathrm{z}^{(-\nu-1)}.\end{aligned}$$

Since $\nu > 0$ and $\mathrm{z}^{(-\nu-1)} = \frac{\Gamma(\mathrm{z}+1)}{\Gamma(\mathrm{z}+2+\nu)} \geq 0$, it follows that

$$\Delta\left(\mathrm{z}^{(-\nu)}\right) = (\mathrm{z}+1)^{(-\nu)} - \mathrm{z}^{(-\nu)} = -\nu\mathrm{z}^{(-\nu-1)} \leqq 0,$$

which leads to $(\mathrm{z}+1)^{(-\nu)} \leqq \mathrm{z}^{(-\nu)}$. Thus, the proof is complete. □

Lemma 2.2 (see [9]) *If* $\nu > 0$ *and* $\mu > -1$, *then*

$${}_{a+\mu}\Delta^{-\nu}(\mathrm{z}-a)^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+\nu)}(\mathrm{z}-a)^{(\nu+\mu)} \tag{2.4}$$

for $\mathrm{z} \in \mathrm{N}_{a+\mu+\nu}$.

Lemma 2.3 (see [6, 32, 34]) *The falling factorial function satisfies the following conditions*:

(i) $\mathrm{z}^{(\mu)} \cdot \mathrm{z}^{(-\nu)} \leqq \mathrm{z}^{(\mu-\nu)}$ *for* $\nu, \mu \geq 0$ *and* $\mathrm{z} > \mu - 1$.

(ii) $\mathrm{z}^{(\nu+\mu)} = (\mathrm{z}-\mu)^{(\nu)}\mathrm{z}^{(\mu)}$.

(iii) $(\mathrm{z}+\nu)^{(-\mu)} < \mathrm{z}^{(-\mu)}$ *for each positive value of* ν, μ *and* z.
(iv) $[\mathrm{z}^{(-\gamma)}]^{\beta} \leqq \mathrm{z}^{(-\beta\gamma)}$ *for* $\gamma < 0$ *and* $\beta \in (0,1)$.
Each of the above items can be found in [6, 34] *and* [32], *respectively.*

Definition 2.1 (see [35]) A set Υ of finite or infinite sequences in ℓ_n^∞ is uniformly Cauchy, if, for every $\epsilon > 0$, there exists an integer m such that $|y(i)-y(j)| < \epsilon$ for $i,j > m$ and $y = \{y(n)\}$ in Υ.

The following theorem is known as a discrete Arzela–Ascoli theorem.

Theorem 2.1 (see [36]) *A bounded uniformly Cauchy subset* Υ *of* ℓ_n^∞ *is relatively compact.*

The following theorem is known as the discrete Krasnoselskii fixed-point theorem.

Theorem 2.2 (see [36]) *Let* $\mathtt{S} \neq \emptyset$ *be a bounded, closed and convex subset of the Banach space* Υ *of* ℓ_n^∞. *Let* $\mathtt{A}: \Upsilon \to \Upsilon$ *and* $\mathtt{B}: \mathtt{S} \to \Upsilon$ *be two operators with the following constraints*:
(i) $\mathtt{A}$ *is a contraction mapping with constant* $\mathtt{L} < 1$;
(ii) $\mathtt{B}$ *is continuous and* $\mathtt{BS}$ *resides in a compact subset of* Υ;
(iii) $y = \mathtt{A}y + \mathtt{B}z$, $z \in \mathtt{S}$ *implies that* $y \in \mathtt{S}$.
Then, we can say that the operator equation $\mathtt{A}y + \mathtt{B}z = y$ *has a solution in* $\mathtt{S}$.

Let us now consider the difference equation (1.2) in a more explicit fractional Taylor difference equation form.

Lemma 2.4 (see [29]) *Suppose that* ψ *is a given real-valued function. The difference equation* (1.2) *has one solution if and only if* y *is a solution of the following fractional Taylor difference equation*:

$$y(\mathrm{z}) = \frac{\mathrm{z}^{(\nu-1)}}{\Gamma(\nu)} y_0 + \frac{1}{\Gamma(\nu)} \sum_{\kappa=0}^{\mathrm{z}-\nu} \big(\mathrm{z}-\sigma(\kappa)\big)^{(\nu-1)} \psi\big(\kappa+\nu-1, y(\kappa+\nu-1)\big) \tag{2.5}$$

for $\mathrm{z} \in \mathrm{N}_\nu$.

To proceed, we should define a functional operator $\mathtt{P}$ as follows:

$$(\mathtt{P}y)(\mathrm{z}) := \frac{\mathrm{z}^{(\nu-1)}}{\Gamma(\nu)} y_0 + \frac{1}{\Gamma(\nu)} \sum_{\kappa=0}^{\mathrm{z}-\nu} \big(\mathrm{z}-\sigma(\kappa)\big)^{(\nu-1)} \psi\big(\kappa+\nu-1, y(\kappa+\nu-1)\big) \quad (\forall \mathrm{z} \in \mathrm{N}_\nu). \tag{2.6}$$

Furthermore, we will try to show that the operator $\mathtt{P}$ has a unique fixed point in a possible function space. Let us separate $\mathtt{P}$ into two distinct operators as follows:

$$(\mathtt{A}y)(\mathrm{z}) := \frac{\mathrm{z}^{(\nu-1)}}{\Gamma(\nu)} y_0, \tag{2.7}$$

$$(\mathtt{B}y)(\mathrm{z}) := \frac{1}{\Gamma(\nu)} \sum_{\kappa=0}^{\mathrm{z}-\nu} \big(\mathrm{z}-\sigma(\kappa)\big)^{(\nu-1)} \psi\big(\kappa+\nu-1, y(\kappa+\nu-1)\big) \quad (\forall \mathrm{z} \in \mathrm{N}_\nu). \tag{2.8}$$

It is evident that $\mathtt{P}y = \mathtt{A}y + \mathtt{B}y$ and the operator $\mathtt{A}$ is a contraction mapping with the constant 0, which verifies condition (i) of Theorem 2.2. Moreover, it is clear from (2.5) and (2.6) that y is a fixed point of $\mathtt{P}$ iff y is a solution of (1.2).

3 Establishment of the existence and uniqueness results

Here, we consider the space $\Upsilon := \ell^{\infty}_{\nu+1}$ of functions x that consist of the set of all real sequences $\{x(\mathrm{z})\}_{\mathrm{z}=\nu+1}^{\infty}$. Note that Υ is a Banach space under the norm $\|x\| := \sup_{\mathrm{z}\in\mathrm{N}_{\nu+1}} |x(\mathrm{z})|$. Also, we define

$$\mathtt{S} := \left\{x \in \Upsilon; \left|x(\mathrm{z})\right| \leqq (\mathrm{z}-1)^{(-\gamma)}\ \forall \mathrm{z} \in \mathrm{N}_{\nu+1}, \gamma > 0\right\}. \tag{3.1}$$

It is clear that the set $\mathtt{S}$ is a nonempty bounded and closed subset of Υ.

Theorem 3.1 *Let the following condition on the function ψ hold true*:

(C1) *Suppose that there exist positive constants $\mathtt{C}$ and β, with $\nu + \beta = 1$ and $\beta > \nu$, such that*

$$\left|\psi(\mathrm{z}, y)\right| \leqq \mathtt{C}\mathrm{z}^{(-\beta)} \quad (\forall \mathrm{z} \in \mathrm{N}_{\nu+1}). \tag{3.2}$$

Then, the operator $\mathtt{B}$ is continuous and $\mathtt{B}\mathtt{S}_1$ is a relatively compact subset of $\mathtt{S}_1$ for $\mathrm{z} \in \mathrm{N}_{\nu+n}$, where

$$\mathtt{S}_1 := \left\{x \in \Upsilon; \left|x(\mathrm{z})\right| \leqq (\mathrm{z}-1)^{(-\gamma)}\ \forall \mathrm{z} \in \mathrm{N}_{\nu+n}, \gamma > 0\right\},$$

$\gamma = \frac{\beta-\nu}{2}$ *and n satisfies the following condition*:

$$\frac{(\nu+n+\gamma-1)^{(-0.5)}}{\Gamma(\nu)}|y_0| + \mathtt{C}\frac{\Gamma(1-\beta)}{\Gamma(1+\nu-\beta)}(\nu+n+\gamma-1)^{(-\gamma)} \leqq 1. \tag{3.3}$$

Proof From the definition (2.8) of the operator $\mathtt{B}$, Lemma 2.2 and assumption (C1), we have, for $\mathrm{z} \in \mathrm{N}_{\nu+n}$,

$$\begin{aligned}
\left|(\mathtt{B}y)(\mathrm{z})\right| &\leqq \frac{1}{\Gamma(\nu)}\sum_{\kappa=0}^{\mathrm{z}-\nu}\left(\mathrm{z}-\sigma(\kappa)\right)^{(\nu-1)}\left|\psi\left(\kappa+\nu-1, y(\kappa+\nu-1)\right)\right| \\
&\leqq \frac{\mathtt{C}}{\Gamma(\nu)}\sum_{\kappa=0}^{\mathrm{z}-\nu}\left(\mathrm{z}-\sigma(\kappa)\right)^{(\nu-1)}(\kappa+\nu-1)^{(-\beta)} \\
&= \mathtt{C}\left({}_0\Delta^{-\nu}(\kappa+\nu-1)^{(-\beta)}\right)(\mathrm{z}) \\
&= \mathtt{C}\frac{\Gamma(1-\beta)}{\Gamma(1+\nu-\beta)}(\mathrm{z}+\nu-1)^{(\nu-\beta)} \quad \text{provided that } \nu+\beta = 1.
\end{aligned}$$

Considering ν, $\beta-\nu$ and $\mathrm{z}-1$ are all positive for $\mathrm{z} \in \mathrm{N}_{\nu+n}$, $n = 1, 2, \ldots$, by Lemmas 2.1 and 2.2, and assumption (3.3), we have

$$\begin{aligned}
\left|(\mathtt{B}y)(\mathrm{z})\right| &< \mathtt{C}\frac{\Gamma(1-\beta)}{\Gamma(1+\nu-\beta)}(\mathrm{z}-1)^{(\nu-\beta)} \\
&= \mathtt{C}\frac{\Gamma(1-\beta)}{\Gamma(1+\nu-\beta)}(\mathrm{z}+\gamma-1)^{(-\gamma)}(\mathrm{z}-1)^{(-\gamma)}
\end{aligned}$$

$$\begin{aligned}
&\leqq \mathrm{C}\frac{\Gamma(1-\beta)}{\Gamma(1+\nu-\beta)}(\nu+\gamma+n-1)^{(-\gamma)}(\mathrm{z}-1)^{(-\gamma)} \\
&\leqq (\mathrm{z}-1)^{(-\gamma)}. \qquad (3.4)
\end{aligned}$$

This means that $y \in \mathrm{S}_1$ and thus $\mathrm{B}\mathrm{S}_1 \subseteq \mathrm{S}_1$.

For the continuity of B on S_1, we let $\epsilon > 0$ be given. Then, by using Lemmas 2.1 and 2.2, there exists $m \geq n$ in $\mathbb{N}_1$ such that

$$\mathrm{C}\frac{\Gamma(1-\beta)}{\Gamma(1+\nu-\beta)}(\mathrm{z}-1)^{(\nu-\beta)} < \frac{\epsilon}{2} \quad \text{for } \mathrm{z} \in \mathrm{N}_{\nu+m}. \qquad (3.5)$$

Let $\{y_j\}_{j=\nu+n}^{\infty}$ be a sequence defined on S_1 that converges to y. For $\mathrm{z} \in \mathrm{N}_{\nu+m}$, it follows from assumption (C1) and (3.5) that

$$\begin{aligned}
&\bigl|(\mathrm{B}y_j)(\mathrm{z}) - (\mathrm{B}y)(\mathrm{z})\bigr| \\
&\quad\leqq \frac{1}{\Gamma(\nu)}\sum_{\kappa=0}^{\mathrm{z}-\nu}\bigl(\mathrm{z}-\sigma(\kappa)\bigr)^{(\nu-1)}\Bigl[\bigl|\psi\bigl(\kappa+\nu-1, y_j(\kappa+\nu-1)\bigr)\bigr| \\
&\qquad+ \bigl|\psi\bigl(\kappa+\nu-1, y(\kappa+\nu-1)\bigr)\bigr|\Bigr] \\
&\quad\leqq \frac{2\mathrm{C}}{\Gamma(\nu)}\sum_{\kappa=0}^{\mathrm{z}-\nu}\bigl(\mathrm{z}-\sigma(\kappa)\bigr)^{(\nu-1)}(\kappa+\nu-1)^{(-\beta)} \\
&\quad= 2\mathrm{C}\bigl({}_0\Delta^{-\nu}(\kappa+\nu-1)^{(-\beta)}\bigr)(\mathrm{z}) \\
&\quad= 2\mathrm{C}\frac{\Gamma(1-\beta)}{\Gamma(1+\nu-\beta)}(\mathrm{z}+\nu-1)^{(\nu-\beta)} \\
&\quad< 2\mathrm{C}\frac{\Gamma(1-\beta)}{\Gamma(1+\nu-\beta)}(\mathrm{z}-1)^{(\nu-\beta)} \\
&\quad< \epsilon.
\end{aligned}$$

For the rest of the interval $\mathrm{z} \in \{\nu+n, \nu+n+1, \ldots, \nu+m-1\}$, we use the continuity of ψ and Lemma 2.1 to obtain

$$\begin{aligned}
&\bigl|(\mathrm{B}y_j)(\mathrm{z}) - (\mathrm{B}y)(\mathrm{z})\bigr| \\
&\quad\leqq \frac{1}{\Gamma(\nu)}\sum_{\kappa=0}^{\mathrm{z}-\nu}\bigl(\mathrm{z}-\sigma(\kappa)\bigr)^{(\nu-1)}\bigl|\psi\bigl(\kappa+\nu-1, y_j(\kappa+\nu-1)\bigr) - \psi\bigl(\kappa+\nu-1, y(\kappa+\nu-1)\bigr)\bigr| \\
&\quad\leqq \frac{1}{\Gamma(\nu)}\sum_{\kappa=0}^{\mathrm{z}-\nu}\bigl(\mathrm{z}-\sigma(\kappa)\bigr)^{(\nu-1)} \\
&\qquad\times \max_{\kappa\in\{\nu+n,\nu+n+1,\ldots,\nu+m-1\}}\bigl|\psi\bigl(\kappa+\nu-1, y_j(\kappa+\nu-1)\bigr) - \psi\bigl(\kappa+\nu-1, y(\kappa+\nu-1)\bigr)\bigr| \\
&\quad= \bigl({}_0\Delta^{-\nu}r^{(0)}\bigr)(\mathrm{z}) \\
&\qquad\times \max_{\kappa\in\{\nu+n,\nu+n+1,\ldots,\nu+m-1\}}\bigl|\psi\bigl(\kappa+\nu-1, y_j(\kappa+\nu-1)\bigr) - \psi\bigl(\kappa+\nu-1, y(\kappa+\nu-1)\bigr)\bigr| \\
&\quad= \frac{\mathrm{z}^{(\nu)}}{\Gamma(\nu+1)} \\
&\qquad\times \max_{\kappa\in\{\nu+n,\nu+n+1,\ldots,\nu+m-1\}}\bigl|\psi\bigl(\kappa+\nu-1, y_j(\kappa+\nu-1)\bigr) - \psi\bigl(\kappa+\nu-1, y(\kappa+\nu-1)\bigr)\bigr|
\end{aligned}$$

$$\begin{aligned}
&\leqq \frac{(\nu+m-1)^{(\nu)}}{\Gamma(\nu+1)} \\
&\quad \times \max_{\kappa\in\{\nu+n,\nu+n+1,\dots,\nu+m-1\}} \bigl|\psi\bigl(\kappa+\nu-1, y_j(\kappa+\nu-1)\bigr) - \psi\bigl(\kappa+\nu-1, y(\kappa+\nu-1)\bigr)\bigr| \\
&= \frac{\Gamma(\nu+m)}{\Gamma(\nu+1)\Gamma(m)} \\
&\quad \times \max_{\kappa\in\{\nu+n,\nu+n+1,\dots,\nu+m-1\}} \bigl|\psi\bigl(\kappa+\nu-1, y_j(\kappa+\nu-1)\bigr) - \psi\bigl(\kappa+\nu-1, y(\kappa+\nu-1)\bigr)\bigr|,
\end{aligned}$$

which approaches zero when $j \to \infty$. Therefore, we have proved for each $\mathrm{z} \in \mathbb{N}_{\nu+n}$,

$$\bigl|(\mathtt{B}y_n)(\mathrm{z}) - (\mathtt{B}y)(\mathrm{z})\bigr| \to 0 \quad \text{as } n \to \infty, \tag{3.6}$$

and thus the operator $\mathtt{B}$ is continuous. In the following, we prove that the operator $\mathtt{B}$ is also relatively compact in $\mathtt{S}_1$. Let $\mathrm{z}_1, \mathrm{z}_2 \in \mathbb{N}_{\nu+n}$ with $\mathrm{z}_2 > \mathrm{z}_1$, yielding

$$\begin{aligned}
&\bigl|(\mathtt{B}y)(\mathrm{z}_1) - (\mathtt{B}y)(\mathrm{z}_2)\bigr| \\
&\leqq \frac{1}{\Gamma(\nu)} \sum_{\kappa=0}^{\mathrm{z}_1-\nu} \bigl(\mathrm{z}_1 - \sigma(\kappa)\bigr)^{(\nu-1)} \bigl|\psi\bigl(\kappa+\nu-1, y(\kappa+\nu-1)\bigr)\bigr| \\
&\quad + \frac{1}{\Gamma(\nu)} \sum_{\kappa=0}^{\mathrm{z}_2-\nu} \bigl(\mathrm{z}_2 - \sigma(\kappa)\bigr)^{(\nu-1)} \bigl|\psi\bigl(\kappa+\nu-1, y(\kappa+\nu-1)\bigr)\bigr| \\
&= \mathtt{C}_1\bigl({}_0\Delta^{-\nu}(\kappa+\nu-1)^{(-\beta)}\bigr)(\mathrm{z}_1) + \mathtt{C}_2\bigl({}_0\Delta^{-\nu}(\kappa+\nu-1)^{(-\beta)}\bigr)(\mathrm{z}_2) \\
&= \mathtt{C}_1 \frac{\Gamma(1-\beta)}{\Gamma(1+\nu-\beta)} (\mathrm{z}_1+\nu-1)^{(\nu-\beta)} + \mathtt{C}_2 \frac{\Gamma(1-\beta)}{\Gamma(1+\nu-\beta)} (\mathrm{z}_2+\nu-1)^{(\nu-\beta)} \\
&< \underbrace{\frac{\epsilon}{2} + \frac{\epsilon}{2}}_{\text{according to (3.5)}} = \epsilon.
\end{aligned}$$

Therefore, $\{\mathtt{B}y : y \in \mathtt{S}_1\}$ is a bounded and uniformly Cauchy subset by Definition 2.1. Moreover, $\mathtt{B}\mathtt{S}_1$ is relatively compact in view of Theorem 2.1. Thus, the conclusion follows. □

Theorem 3.2 *Assume that a function ψ of two variables satisfies the assumption* (C1) *stated in Theorem* 3.1. *Then, there exists at least one solution $y(\mathrm{z})$ of the difference equation* (1.2) *for $\mathrm{z} \in \mathbb{N}_{\nu+1}$ in $\mathtt{S}_1$.*

Proof It is enough to show that $y(\mathrm{z})$ is a fixed point of $\mathtt{P}$ in $\mathtt{S}_1$. Let $z \in \mathtt{S}_1$ be fixed. If $y := \mathtt{A}y + \mathtt{B}z$, then we shall show that y is in $\mathtt{S}_1$. By means of (C1), Lemmas 2.1, 2.2 and 2.3(iii) one has for $\mathrm{z} \in \mathbb{N}_{\nu+n}$:

$$\begin{aligned}
\bigl|y(\mathrm{z})\bigr| &\leqq \bigl|(\mathtt{A}y)(\mathrm{z})\bigr| + \bigl|(\mathtt{B}z)(\mathrm{z})\bigr| \\
&\leqq \frac{\mathrm{z}^{(\nu-1)}}{\Gamma(\nu)} |y_0| + \frac{1}{\Gamma(\nu)} \sum_{\kappa=0}^{\mathrm{z}-\nu} \bigl(\mathrm{z} - \sigma(\kappa)\bigr)^{(\nu-1)} \bigl|\psi\bigl(\kappa+\nu-1, z(\kappa+\nu-1)\bigr)\bigr|
\end{aligned}$$

$$\begin{aligned}&\leqq \frac{z^{(\nu-1)}}{\Gamma(\nu)}|y_0| + C\frac{\Gamma(1-\beta)}{\Gamma(1+\nu-\beta)}(z+\nu-1)^{(\nu-\beta)}\\&< \frac{(z-1)^{(\nu-1)}}{\Gamma(\nu)}|y_0| + C\frac{\Gamma(1-\beta)}{\Gamma(1+\nu-\beta)}(z-1)^{(\nu-\beta)}.\end{aligned}$$

By considering the condition (3.3), and Lemmas 2.1 and 2.3(ii), it follows that

$$\begin{aligned}\left|y(z)\right| &\leqq \left[\frac{(z+\gamma-1)^{(-0.5)}}{\Gamma(\nu)}|y_0| + C\frac{\Gamma(1-\beta)}{\Gamma(1+\nu-\beta)}(z+\gamma-1)^{(-\gamma)}\right](z-1)^{(-\gamma)}\\&\leqq \left[\frac{(\nu+n+\gamma-1)^{(-0.5)}}{\Gamma(\nu)}|y_0| + C\frac{\Gamma(1-\beta)}{\Gamma(1+\nu-\beta)}(\nu+n+\gamma-1)^{(-\gamma)}\right](z-1)^{(-\gamma)}\\&\leqq (z-1)^{(-\gamma)},\end{aligned}$$

which means that $y(z) \in S_1$ for $z \in N_{\nu+n}$. By Theorem 3.1 and 2.2, therefore, P has a fixed point in S_1, which means that there exists at least one solution of the difference equation (1.2) on $z \in N_{\nu+n}$. The proof is now completed. □

Theorem 3.3 *Assume that a function ψ of two variables ψ satisfies the assumption* (C1) *stated in Theorem 3.1. Then, the solutions $y(z)$ of the difference equation (1.2) are attractive in S_1.*

Proof By means of Theorem 3.2, the solutions of the difference equation (1.2) exist in S_1. Moreover, each of the functions $y(z)$ tend to 0 as $z \to \infty$. Therefore, the solutions of the difference equation (1.2) tend to 0 as $z \to \infty$. The proof is complete. □

Theorem 3.4 *Let the following condition on the function ψ hold true:*

(C2) *There exist positive constants K and β, with $\nu + \beta = 1$ and $\beta > \nu$, such that*

$$\left|\psi\big(z, y_1(z)\big) - \psi\big(z, y_2(z)\big)\right| \leqq K z^{(-\beta)}\|y - z\| \quad (\forall z \in N_{\nu+1}). \tag{3.7}$$

Then, the solutions of the difference equation (1.2) are stable if

$$\ell := K\frac{\Gamma(1-\beta)}{\Gamma(1+\nu-\beta)}\frac{\Gamma(1+\nu)}{\Gamma(1+\beta)} < 1. \tag{3.8}$$

Proof Let ω, ϖ be two solutions of the difference equation (1.2) and let $\epsilon > 0$. From the assumption (C2) and Lemmas 2.1, 2.2 and 2.3, one has the following for $z \in N_{\nu+1}$:

$$\begin{aligned}\left|\omega(z) - \varpi(z)\right| &\leqq \frac{z^{(\nu-1)}}{\Gamma(\nu)}|\omega_0 - \varpi_0|\\&\quad + \frac{1}{\Gamma(\nu)}\sum_{\kappa=0}^{z-\nu}\big(z-\sigma(\kappa)\big)^{(\nu-1)}\Big|\psi\big(\kappa+\nu-1, \omega(\kappa+\nu-1)\big)\\&\quad - \psi\big(\kappa+\nu-1, \varpi(\kappa+\nu-1)\big)\Big|\\&\leqq \frac{z^{(\nu-1)}}{\Gamma(\nu)}|\omega_0 - \varpi_0| + \frac{\|\omega-\varpi\|}{\Gamma(\nu)}K\sum_{\kappa=0}^{z-\nu}\big(z-\sigma(\kappa)\big)^{(\nu-1)}(\kappa+\nu-1)^{(-\beta)}\\&= \frac{z^{(\nu-1)}}{\Gamma(\nu)}|\omega_0 - \varpi_0| + K\frac{\Gamma(1-\beta)}{\Gamma(1+\nu-\beta)}(z+\nu-1)^{(\nu-\beta)}\|\omega-\varpi\|\end{aligned}$$

$$\leqq \frac{(\nu+1)^{(\nu-1)}}{\Gamma(\nu)}|\omega_0-\varpi_0| + \mathrm{K}\frac{\Gamma(1-\beta)}{\Gamma(1+\nu-\beta)}\nu^{(\nu-\beta)}\|\omega-\varpi\|$$
$$= \frac{\nu(\nu+1)}{2}|\omega_0-\varpi_0| + \mathrm{K}\frac{\Gamma(1-\beta)}{\Gamma(1+\nu-\beta)}\frac{\Gamma(1+\nu)}{\Gamma(1+\beta)}\|\omega-\varpi\|.$$

By using (3.8), it follows that

$$\|\omega-\varpi\| \leqq \frac{\nu(\nu+1)}{2(1-\ell)}|\omega_0-\varpi_0|.$$

Now, chose $\delta = \frac{2(1-\ell)\epsilon}{\nu(\nu+1)}$. Therefore,

$$\begin{aligned}\|\omega-\varpi\| &< \frac{\nu(\nu+1)}{2(1-\ell)}\cdot\delta \quad \text{whenever } |\omega_0-\varpi_0|<\delta\\ &= \epsilon.\end{aligned}$$

Thus, it is proven that the solutions of the difference equation (1.2) are stable. □

Corollary 3.1 *Assume that a function ψ of two variables satisfies the assumptions* (C1) *and* (C2) *stated in Theorems* 3.1 *and* 3.4*, respectively. Then, the solutions of the difference equation* (1.2) *are asymptotically stable.*

Proof Corollary 3.1 follows from Theorems 3.3 and 3.4. □

Remark 3.1 It is important to state explicitly that the power rule (2.4) is used mistakenly in [27, 28, 32–34] as follows:

$$\left({}_0\Delta^{-\nu}(\kappa+\nu)^{(-\beta)}\right)(z) = \frac{\Gamma(1-\beta)}{\Gamma(1+\nu-\beta)}(z+\nu)^{(\nu-\beta)}.$$

In fact, according to Lemma 2.2, it is valid only when $\nu = -\beta$, which contradicts the positivity of ν and β. That is why we have chosen to study such a difference equation of the type (1.2). In this case, we have obtained

$$\left({}_0\Delta^{-\nu}(\kappa+\nu-1)^{(-\beta)}\right)(z) = \frac{\Gamma(1-\beta)}{\Gamma(1+\nu-\beta)}(z+\nu-1)^{(\nu-\beta)},$$

for which we need $\nu+\beta = 1$ according to Lemma 2.2, as we have established in Theorems 3.1 to 3.4.

We now prove a new attractiveness of the solutions of the difference equation (1.2) with a new condition in the following theorem.

Theorem 3.5 *Let the following condition on the function ψ hold true*:

(C3) *There exist positive constants C_2, β and γ, with $\nu+\beta+\gamma = 1$ and $\beta > \nu$, such that*

$$\left|\psi\left(z, y(z)\right)\right| \leqq \mathrm{C}_2(z+\gamma)^{(-\beta)}\left|y(z+1)\right| \quad (\forall z \in \mathrm{N}_{\nu+1}). \tag{3.9}$$

Then, the solutions of the difference equation (1.2) *are attractive.*

Proof To prove this theorem, we will verify the conditions of Theorem 2.2. The first condition is clear because $\mathtt{A}$ is a contraction as we discussed before. Also, the second condition is very similar to the one we proved in Theorem 3.1, so we omit it. Here, we prove the last condition so that $y(\mathrm{z})$ will be a fixed point of $\mathtt{P}$ in $\mathtt{S}_2$, where

$$\mathtt{S}_2 := \{x \in \Upsilon; |x(\mathrm{z})| \leqq (\mathrm{z}-1)^{(-\gamma)}\ \forall \mathrm{z} \in \mathrm{N}_{\nu+n}, \gamma > 0\},$$

where $n \in \mathbb{N}_1$ satisfies the condition that

$$\frac{(\nu+n+\gamma-1)^{(-\beta)}}{\Gamma(\nu)}|y_0| + \mathrm{C}_2\frac{\Gamma(1-\beta-\gamma)}{\Gamma(1+\nu-\beta-\gamma)}(\nu+n+\gamma-1)^{(\nu-\beta)} \leqq 1. \tag{3.10}$$

Let $w \in \mathtt{S}_2$ be fixed. Now, if $y := \mathtt{A}y + \mathtt{B}w$, then we shall show that y is in $\mathtt{S}_2$. By using assumption (C3), Lemmas 2.1, 2.2 and 2.3, we have for $\mathrm{z} \in \mathrm{N}_{\nu+n}$:

$$\begin{aligned}
|y(\mathrm{z})| &\leqq |(\mathtt{A}y)(\mathrm{z})| + |(\mathtt{B}w)(\mathrm{z})| \\
&\leqq \frac{\mathrm{z}^{(\nu-1)}}{\Gamma(\nu)}|y_0| + \frac{1}{\Gamma(\nu)}\sum_{\kappa=0}^{\mathrm{z}-\nu}\bigl(\mathrm{z}-\sigma(\kappa)\bigr)^{(\nu-1)}\bigl|\psi\bigl(\kappa+\nu-1, w(\kappa+\nu-1)\bigr)\bigr| \\
&\leqq \frac{\mathrm{z}^{(\nu-1)}}{\Gamma(\nu)}|y_0| + \frac{\mathrm{C}_2}{\Gamma(\nu)}\sum_{\kappa=0}^{\mathrm{z}-\nu}\bigl(\mathrm{z}-\sigma(\kappa)\bigr)^{(\nu-1)}(\kappa+\nu+\gamma-1)^{(-\beta)}\bigl|w(\kappa+\nu)\bigr| \\
&\leqq \frac{\mathrm{z}^{(\nu-1)}}{\Gamma(\nu)}|y_0| + \frac{\mathrm{C}_2}{\Gamma(\nu)}\sum_{\kappa=0}^{\mathrm{z}-\nu}\bigl(\mathrm{z}-\sigma(\kappa)\bigr)^{(\nu-1)}(\kappa+\nu+\gamma-1)^{(-\beta)}(\kappa+\nu-1)^{(-\gamma)} \\
&\leqq \frac{\mathrm{z}^{(\nu-1)}}{\Gamma(\nu)}|y_0| + \frac{\mathrm{C}_2}{\Gamma(\nu)}\sum_{\kappa=0}^{\mathrm{z}-\nu}\bigl(\mathrm{z}-\sigma(\kappa)\bigr)^{(\nu-1)}(\kappa+\nu-1)^{(-\beta-\gamma)} \\
&\leqq \frac{\mathrm{z}^{(\nu-1)}}{\Gamma(\nu)}|y_0| + \mathrm{C}_2\frac{\Gamma(1-\beta-\gamma)}{\Gamma(1+\nu-\beta-\gamma)}(\mathrm{z}+\nu-1)^{(\nu-\beta-\gamma)} \quad \text{such that } \nu+\beta+\gamma=1 \\
&< \frac{(\mathrm{z}-1)^{(\nu-1)}}{\Gamma(\nu)}|y_0| + \mathrm{C}_2\frac{\Gamma(1-\beta-\gamma)}{\Gamma(1+\nu-\beta-\gamma)}(\mathrm{z}-1)^{(\nu-\beta-\gamma)}.
\end{aligned}$$

By considering condition (3.10), and Lemmas 2.1 and 2.3(ii), it follows that

$$\begin{aligned}
|y(\mathrm{z})| &\leqq \left[\frac{(\mathrm{z}+\gamma-1)^{(-\beta)}}{\Gamma(\nu)}|y_0| + \mathrm{C}_2\frac{\Gamma(1-\beta-\gamma)}{\Gamma(1+\nu-\beta-\gamma)}(\mathrm{z}+\gamma-1)^{(\nu-\beta)}\right](\mathrm{z}-1)^{(-\gamma)} \\
&\leqq \left[\frac{(\nu+n+\gamma-1)^{(-\beta)}}{\Gamma(\nu)}|y_0| + \mathrm{C}_2\frac{\Gamma(1-\beta-\gamma)}{\Gamma(1+\nu-\beta-\gamma)}(\nu+n+\gamma-1)^{(\nu-\beta)}\right](\mathrm{z}-1)^{(-\gamma)} \\
&\leqq (\mathrm{z}-1)^{(-\gamma)}.
\end{aligned}$$

This completes the required result. Therefore, by Theorem 3.1 and 2.2, $\mathtt{P}$ has a fixed point in $\mathtt{S}_2$, which means that there exists at least one solution of the difference equation (1.2) on $\mathrm{z} \in \mathrm{N}_{\nu+n}$. Moreover, by means of Theorem 3.2, each of the functions $y(\mathrm{z})$ in $\mathtt{S}_2$ tend to zero as $\mathrm{z} \to \infty$. Therefore, the solutions of the difference equation (1.2) tend to zero as $\mathrm{z} \to \infty$. This completes the proof. □

Corollary 3.2 *Assume that a function ψ of two variables satisfies the assumptions* (C2) *and* (C3) *stated in Theorems* 3.4 *and* 3.5, *respectively. Then, the solutions of the difference equation* (1.2) *are asymptotically stable such that* (3.8) *holds true.*

Proof This follows from Theorems 3.4 and 3.5. □

Theorem 3.6 *Let the following condition on ψ hold true*:

(C4) *There exist $\eta \in (0,1)$ and the positive constants C_3 and β such that*

$$\left|\psi\big(z, y(z)\big)\right| \leqq C_3 (z+1)^{(-\beta)} \left|y(z+1)\right|^{\eta} \quad (\forall z \in N_{\nu+1}). \tag{3.11}$$

Then, the solutions of the difference equation (1.2) *are attractive.*

Proof We proceed with the same method as that used in Theorem 3.5. We only prove the last condition in 2.2 so that $y(z)$ will be a fixed point of P in S_3, where

$$S_3 := \left\{x \in \Upsilon; \left|x(z)\right| \leqq (z-1)^{(-\gamma)}\ \forall z \in N_{\nu+n}, \gamma > 0\right\},$$

where $\nu + \beta + \gamma\eta = 1$, $\beta > \nu$, $\nu + \gamma \in (0,1)$, $\gamma = \frac{\beta-\nu}{2}$ and $n \in \mathbb{N}_1$ satisfies the condition that

$$\frac{(\nu+n+\gamma-1)^{(\nu+\gamma-1)}}{\Gamma(\nu)}|y_0| + C_3 \frac{\Gamma(1-\beta-\gamma\eta)}{\Gamma(1+\nu-\beta-\gamma\eta)}(\nu+n+\gamma-1)^{(-\gamma)} \leqq 1. \tag{3.12}$$

Let $w \in S_3$ be fixed. Now, if $y := Ay + Bw$, then we shall show that y is in S_3. By using assumption (C4), $\nu < \beta + \gamma\eta < 1$, Lemmas 2.1, 2.2 and 2.3(ii)–(iv), we have for $z \in N_{\nu+n}$:

$$\begin{aligned}
\left|y(z)\right| &\leqq \left|(Ay)(z)\right| + \left|(Bw)(z)\right| \\
&\leqq \frac{z^{(\nu-1)}}{\Gamma(\nu)}|y_0| + \frac{1}{\Gamma(\nu)} \sum_{\kappa=0}^{z-\nu} \big(z-\sigma(\kappa)\big)^{(\nu-1)} \left|\psi\big(\kappa+\nu-1, w(\kappa+\nu-1)\big)\right| \\
&\leqq \frac{z^{(\nu-1)}}{\Gamma(\nu)}|y_0| + \frac{C_3}{\Gamma(\nu)} \sum_{\kappa=0}^{z-\nu} \big(z-\sigma(\kappa)\big)^{(\nu-1)} (\kappa+\nu)^{(-\beta)} \left|w(\kappa+\nu)\right|^{\eta} \\
&\leqq \frac{z^{(\nu-1)}}{\Gamma(\nu)}|y_0| + \frac{C_3}{\Gamma(\nu)} \sum_{\kappa=0}^{z-\nu} \big(z-\sigma(\kappa)\big)^{(\nu-1)} (\kappa+\nu+\gamma\eta-1)^{(-\beta)} \left[(\kappa+\nu-1)^{(-\gamma)}\right]^{\eta} \\
&\leqq \frac{z^{(\nu-1)}}{\Gamma(\nu)}|y_0| + \frac{C_3}{\Gamma(\nu)} \sum_{\kappa=0}^{z-\nu} \big(z-\sigma(\kappa)\big)^{(\nu-1)} (\kappa+\nu+\gamma\eta-1)^{(-\beta)} (\kappa+\nu-1)^{(-\gamma\eta)} \\
&\leqq \frac{z^{(\nu-1)}}{\Gamma(\nu)}|y_0| + \frac{C_3}{\Gamma(\nu)} \sum_{\kappa=0}^{z-\nu} \big(z-\sigma(\kappa)\big)^{(\nu-1)} (\kappa+\nu-1)^{(-\beta-\gamma\eta)} \\
&\leqq \frac{z^{(\nu-1)}}{\Gamma(\nu)}|y_0| + C_3 \frac{\Gamma(1-\beta-\gamma\eta)}{\Gamma(1+\nu-\beta-\gamma\eta)} (z+\nu-1)^{(\nu-\beta-\gamma\eta)} \quad \text{such that } \nu+\beta+\gamma\eta = 1 \\
&\leqq \frac{(z-1)^{(\nu-1)}}{\Gamma(\nu)}|y_0| + C_3 \frac{\Gamma(1-\beta-\gamma\eta)}{\Gamma(1+\nu-\beta-\gamma\eta)} (z-1)^{(\nu-\beta)}.
\end{aligned}$$

Considering condition (3.12), $\nu + \gamma \in (0,1)$, $\beta - \nu = 2\gamma$, and Lemmas 2.1 and 2.3(ii), it follows that

$$\begin{aligned}
|y(z)| &\leqq \left[\frac{(z+\gamma-1)^{(\nu+\gamma-1)}}{\Gamma(\nu)}|y_0| + C_3\frac{\Gamma(1-\beta-\gamma\eta)}{\Gamma(1+\nu-\beta-\gamma\eta)}(z+\gamma-1)^{(-\gamma)}\right](z-1)^{(-\gamma)} \\
&\leqq \left[\frac{(\nu+n+\gamma-1)^{(\nu+\gamma-1)}}{\Gamma(\nu)}|y_0| + C_3\frac{\Gamma(1-\beta-\gamma\eta)}{\Gamma(1+\nu-\beta-\gamma\eta)}(\nu+n+\gamma-1)^{(-\gamma)}\right] \\
&\quad \times (z-1)^{(-\gamma)} \\
&\leqq (z-1)^{(-\gamma)}.
\end{aligned}$$

This proves the required condition (iii) in Theorem 2.2 and thus the proof is completed. □

Remark 3.2 The same mistakes of the power rule, as we discussed in Remark 3.1, are made in Theorems 3.6 and 3.8 in [32]. In those theorems, the used power rule would have been true when $\nu + \beta_3 + \gamma_2 = 0$ and $\nu + \beta_3 + \gamma_2\eta = 0$, respectively. However, these contradict the positivity of ν, β, γ and η. The chosen parameters here are such that $\nu + \beta + \gamma = 1$ in Theorem 3.5 and $\nu + \beta + \gamma\eta = 1$ in Theorem 3.6 have appropriately corrected the above mistakes.

4 Applications

In this section, we present three nonlinear difference examples that illustrate the results established in general above. In each case, Conditions (C1) to (C4) are verified to be true.

Example 4.1 Consider the nonlinear difference equation

$$\begin{aligned}
&\left({}^{\mathrm{RL}}_{-0.75}\Delta^{0.25}y\right)(z) = 0.2(z-0.75)^{(-0.75)}\sin\bigl(y(z-0.75)\bigr) \quad (\forall z \in \mathrm{N}_0), \\
&\left({}_{-0.75}\Delta^{-0.75}y\right)(z)\big|_{z=0} = y_0.
\end{aligned} \tag{4.1}$$

Here, $\nu + \beta = 0.25 + 0.75 = 1$ and $\psi(z, y(z)) = 0.2z^{(-0.75)}\sin(y(z))$. Thus, for $z \in \mathrm{N}_{1.5}$, we have

$$\bigl|\psi\bigl(z, y(z)\bigr)\bigr| = \bigl|0.2z^{(-0.75)}\sin\bigl(y(z)\bigr)\bigr| \leqq 0.2z^{(-0.75)},$$

so (C1) is satisfied. Also, we have

$$\bigl|\psi\bigl(z, y(z)\bigr) - \psi\bigl(z, z(z)\bigr)\bigr| \leqq 0.2z^{(-0.75)}\|y - z\|,$$

so (C2) is satisfied as well. Moreover, in view of Theorem 3.4 with $\mathrm{K} = 0.2$, $\nu = 0.25$ and $\beta = 0.75$, we find that

$$\ell = \mathrm{K}\frac{\Gamma(1-\beta)}{\Gamma(1+\nu-\beta)}\frac{\Gamma(1+\nu)}{\Gamma(1+\beta)} = 0.2\frac{\Gamma(0.25)}{\Gamma(0.5)}\frac{\Gamma(1.25)}{\Gamma(1.75)} = 0.4035 < 1,$$

which verifies (3.8). Therefore, the solutions of the difference equation (4.1) are asymptotically stable according to Corollary 3.1.

Example 4.2 Consider the nonlinear difference equation

$$\begin{aligned} &\left({}^{\mathrm{RL}}_{-0.8}\Delta^{0.2}y\right)(\mathrm{z}) = 0.4(\mathrm{z}+0.2)^{(-0.5)}y(\mathrm{z}-0.8) \quad (\forall \mathrm{z} \in \mathrm{N}_0), \\ &\left({}_{-0.8}\Delta^{-0.8}y\right)(\mathrm{z})\big|_{\mathrm{z}=0} = y_0. \end{aligned} \tag{4.2}$$

From the difference equation, we see that $\nu + \beta + \gamma = 0.2 + 0.5 + 0.3 = 1$ and $\psi(\mathrm{z}, y(\mathrm{z})) = 0.4(\mathrm{z}+1)^{(-0.5)}y(\mathrm{z})$. Since $\mathrm{z}^{(-0.5)}$ is nonincreasing. Then, for $\mathrm{z} \in \mathrm{N}_{1.5}$, we have

$$\left|\psi\left(\mathrm{z}, y(\mathrm{z})\right)\right| = \left|0.4(\mathrm{z}+1)^{(-0.5)}y(\mathrm{z})\right| \leqq 0.4(\mathrm{z}+0.3)^{(-0.5)}\left|y(\mathrm{z})\right|,$$

and so (C3) is satisfied. Also, we see that

$$\left|\psi\left(\mathrm{z}, y(\mathrm{z})\right) - \psi\left(\mathrm{z}, z(\mathrm{z})\right)\right| \leqq 0.4(\mathrm{z}+1)^{(-0.5)}\|y - z\| \leqq 0.4\mathrm{z}^{(-0.5)}\|y - z\|,$$

so (C2) is satisfied. Moreover, in view of Theorem 3.4 with $\mathrm{K} = 0.2$, $\nu = 0.25$ and $\beta = 0.75$, we find that

$$\ell = 0.4\frac{\Gamma(0.5)}{\Gamma(0.7)}\frac{\Gamma(1.2)}{\Gamma(1.5)} = 0.5659 < 1,$$

which verifies the condition (3.8). Therefore, the solutions of the difference equation (4.2) are asymptotically stable according to Corollary 3.2.

Example 4.3 Finally, we consider the following nonlinear difference equation

$$\begin{aligned} &\left({}^{\mathrm{RL}}_{-0.61}\Delta^{0.39}y\right)(\mathrm{z}) = 0.4(\mathrm{z}+0.39)^{(-0.59)}y^{\frac{1}{5}}(\mathrm{z}-0.61) \quad (\forall \mathrm{z} \in \mathrm{N}_0), \\ &\left({}_{-0.61}\Delta^{-0.61}y\right)(\mathrm{z})\big|_{\mathrm{z}=0} = y_0. \end{aligned} \tag{4.3}$$

From the given difference equation, we have $\nu = 0.39$, $\beta = 0.59$, $\gamma = \frac{0.59-0.39}{2} = 0.1$, $\eta = 0.2$ and $\psi(\mathrm{z}, y(\mathrm{z})) = 0.4(\mathrm{z}+1)^{(-0.59)}y^{\frac{1}{5}}(\mathrm{z}+1)$ for $\mathrm{z} \in \mathrm{N}_{1.5}$. Since $\mathrm{z}^{(-0.5)}$ is nonincreasing, for $\mathrm{z} \in \mathrm{N}_{1.5}$, we have

$$\left|\psi\left(\mathrm{z}, y(\mathrm{z})\right)\right| = \left|0.4(\mathrm{z}+1)^{(-0.59)}y^{\frac{1}{5}}(\mathrm{z}+1)\right| \leqq 0.4(\mathrm{z}+1)^{(-0.59)}\left|y(\mathrm{z}+1)\right|^{\frac{1}{5}},$$

so (C4) is satisfied. Therefore, the solutions of the difference equation (4.3) are attractive according to Theorem 3.6.

5 Conclusions and directions for further work

In this work, we dealt with a class of nonlinear fractional difference equations in the sense of Riemann–Liouville. The power-rule mistakes made by some authors in [2, 3] are corrected by providing some further conditions. Having established a set of falling fractional functions that is bounded and closed subsets in a Banach space, and having set some conditions on the nonlinear part and falling fractional functions, we proceeded to prove the existence and uniqueness of the class of nonlinear fractional difference equations.

An important future research direction is to extend the existence and uniqueness of solutions of nonlinear fractional difference equations for other types of discrete fractional calculus. The present work is set within the discrete fractional difference operators of

Riemann–Liouville type, but it may be possible to extend it, applying the same method in other classes of discrete fractional operators such as Liouville–Caputo [8], Caputo–Fabrizio [9] and Atangana–Baleanu [10].

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