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Applications of some new Krasnoselskii-type fixed-point results for generalized expansive and equiexpansive mappings

Niaz Ahmad[1], Nayyar Mehmood[1] and Ali Akgül[2*]

*Correspondence:
aliakgul00727@gmail.com
[2]Department of Mathematics, Art and Science Faculty, Siirt University, 56100 Siirt, Turkey
Full list of author information is available at the end of the article

Abstract

We consider Ω as a subset of a Banach space W and Λ as a function of Ω into W. Let $\mathcal{F}$ be a function whose image values lie in W and domain is $\Lambda(\Omega) \times \Omega$ or $\Omega \times \Omega$. In this paper, we establish some fixed-point results for a generalized expansive and equiexpansive operator $\mathcal{F}$ such that $\Omega \subseteq \mathcal{F}(\Lambda\omega, \Omega)$ or $\Omega \subseteq \mathcal{F}(\omega, \Omega)$. We apply our results to acquire the solutions of fractional evolution equations and certain types of integral equations. We demonstrate our results with examples, and plot approximate and exact solutions with errors.

Keywords: Generalized equiexpansive; Krasnoselskii; Expansive; Contraction; Fixed points

1 Introduction

Fixed-point theory has been playing a vital role to analyze the stability and existence of many linear and nonlinear problems over the last few decades. Two of the most celebrated results of fixed-point theory are the Banach contraction principle and Schauder's fixed-point result. In addition to the beauty of the Banach contraction principle to determine the stability and uniqueness of the solution, Schauder's result is still more applicable. Both the results have their own importance and domain. During the analysis of solutions of neutral and delayed differential equations, Krasnoselskii observed that, in most of the cases, the solutions of these equations might be expressed as a sum of contractive and compact operators. Therefore, in [10] Krasnoselskii generalized both results of Banach and Schauder, for finding the fixed points of the sum of contractive and compact operators. Huge implementations in the existence theory of differential and integral equations can be seen in the following monographs [9, 17, 29].

Fractional evolution equations can provide a unique way to analyze the well-posedness of many complex systems. Most differential systems with fractional derivatives provide useful tools for the representation of memory and inherited properties. The real physical systems of fractional order are always more applicable than the classical integer-order systems. Recently, the existence of many fractional-order controlled problems and fractional evolution differential equations have been studied, see [5, 13, 14, 22, 24]. Due to the involvement of contractive operators in Krasnoselskii's result, the existence results cover a small class of physical problems. To overcome this issue, some new results are required involving other less restrictive conditions like expansive operators. In this regard, we describe a new class of generalized equiexpansive mappings, assume F is an implicit function of two variables with some conditions, we present many variants of Krasnoselskii's fixed-point theorem.

Krasnoselskii, in 1958, combined the Schauder theorem and the Banach contraction principle to consider the fixed-point problem:

$$\Lambda\omega + B\omega = \omega, \quad \omega \in \Omega,$$

where Ω is a subset of Banach space W. Keeping the importance of this type of problems, many researchers have solved the operator equation $\Lambda\omega + B\omega = \omega$ under different assumptions on Λ, B and Ω. These types of results can be seen in [2, 6, 7, 16, 18, 23, 25–28]. The Krasnoselskii theorem can be stated in the following way.

Theorem 1 ([21]) *Suppose that Ω is a nonempty closed convex subset of a Banach space W. Λ and B are mappings of Ω into W such that*
(1) $\Lambda\omega + B\nu \in \Omega$ *for all* $\omega, \nu \in \Omega$;
(2) Λ *is compact and continuous*;
(3) B *is a contraction*.
Then, there is $\omega \in \Omega$ such that $\Lambda\omega + B\omega = \omega$.

Xiang and Yuan ([26], see also in [25]) proved the following theorem that is a variant of the Banach contraction principle.

Theorem 2 *Let $\Omega \subseteq W$ be a nonempty closed set, where W is a complete metric space and $\Upsilon : \Omega \longrightarrow W$ is an expansive operator such that $\Upsilon(\Omega) \supseteq \Omega$. Thus, there is a unique $\omega \in \Omega$ such that $\Upsilon\omega = \omega$.*

In most fixed-point results the condition $\Upsilon(\Omega) \subseteq \Omega$ is used, for example, in the Banach contraction principle, the Schauder fixed-point theorem and the Sadovskii fixed-point theorem. Xiang and Yuan [26] discussed and proved the fixed-point result for an expansive operator satisfying the reverse condition $\Upsilon(\Omega) \supseteq \Omega$. They (in [26]) also proved the following fixed-point result that is a variant of the Krasnoselskii fixed-point theorem.

Theorem 3 *Let $\Omega \subseteq W$ be a nonempty closed and convex set, where W is a Banach space. Assume that $\Lambda : \Omega \longrightarrow W$ and $B : \Omega \longrightarrow W$ are mappings such that*
(1) Λ *is continuous and* $\Lambda(\Omega) \subseteq C$, *where C is a compact subset of W*;
(2) B *is an expansive mapping*;
(3) $\Omega \subseteq \upsilon + B(\Omega)$ *for all* $\upsilon \in \Lambda(\Omega)$.
Thus, there is $\omega \in \Omega$ such that $\Lambda\omega + B\omega = \omega$.

In Theorem 1, $\Lambda\omega + B\nu \in \Omega$ and B is a contraction, while in Theorem 3, $\Omega \subseteq \upsilon + B(\Omega)$ and B is an expansive mapping. The single operator $\Lambda\omega + B\nu$ combined from two operators Λ and B can be considered in the general form $\digamma(\omega, \nu) = \Lambda\omega + B\nu$. The fixed-point study for this type of operator $\digamma$ can be seen in [4, 8, 12, 15].

Let W be a Banach space and Ω be a subset of W. Suppose $\digamma : \Omega \times \Omega \longrightarrow W$ or $\digamma : \Lambda(\Omega) \times \Omega \longrightarrow W$. We, in this paper, study the fixed-point results of the equations of types $\omega = \digamma(\Lambda\omega, \omega)$ or $\omega = \digamma(\omega, \omega)$ by using generalized expansive and equiexpansive conditions on the operator $\digamma$ and such that $\Omega \subseteq \digamma(\Lambda\omega, \Omega)$ or $\Omega \subseteq \digamma(\omega, \Omega)$.

2 Main results

Definition 2.1 ([26]) Let (W, d_W) be a metric space and Ω be a subset of W. A mapping Υ of Ω into W is said to be expansive if there exists a constant $h > 1$ such that $d_W(\Upsilon\omega, \Upsilon\nu) \geq hd_W(\omega, \nu)$.

Definition 2.2 Let (W, d_W) be a metric space and Ω be a subset of W. A mapping Υ of Ω into W is said to be generalized expansive if

$$\begin{aligned} &cd_W(\Upsilon\omega, \Upsilon\nu) + ad_W(\Upsilon\omega, \omega) + bd_W(\Upsilon\nu, \nu) \\ &\quad \geq hd_W(\omega, \nu) \quad a, b \geq 0, c, h > 0, a + b + c < h. \end{aligned}$$

Below in Example 30, it is proved that a generalized expansive mapping may not be an expansive mapping. Also, note that every expansive mapping is one to one but every generalized expansive mapping may not be one to one. In the case of a generalized expansive mapping or an expansive mapping we need to have one to one mapping. We pose here a question, whether in the case of a generalized expansive, does a mapping has a fixed point if it is not one to one?

Theorem 4 *Assume that Ω is a nonempty closed subset of a complete metric space W and Υ is a one to one mapping of Ω into W such that*

(1) $\Omega \subseteq \Upsilon(\Omega)$;

(2) $cd_W(\Upsilon\omega, \Upsilon\nu) + ad_W(\Upsilon\omega, \omega) + bd_W(\Upsilon\nu, \nu) \geq hd_W(\omega, \nu)$, $a, b \geq 0$, $c, h > 0$, $a + b + c < h$.

Thus, there is a unique $\omega \in \Omega$ such that $\Upsilon\omega = \omega$.

Proof Since Υ is one to one, the inverse of $\Upsilon : \Omega \longrightarrow \Upsilon(\Omega)$ exists. For $\upsilon, \vartheta \in \Upsilon(\Omega)$ there are $\omega, \nu \in \Omega$ such that $\Upsilon\omega = \upsilon$ and $\Upsilon\nu = \vartheta$, then $\omega = \Upsilon^{-1}\upsilon$ and $\nu = \Upsilon^{-1}\vartheta$. From (2)

$$hd_W\left(\Upsilon^{-1}\upsilon, \Upsilon^{-1}\vartheta\right) \leq ad_W\left(\upsilon, \Upsilon^{-1}\upsilon\right) + bd_W\left(\vartheta, \Upsilon^{-1}\vartheta\right) + cd_W(\upsilon, \vartheta)$$

for all $\upsilon, \vartheta \in \Upsilon(\Omega)$. Thus,

$$d_W\left(\Upsilon^{-1}\upsilon, \Upsilon^{-1}\vartheta\right) \leq a'd_W\left(\upsilon, \Upsilon^{-1}\upsilon\right) + b'd_W\left(\vartheta, \Upsilon^{-1}\vartheta\right) + c'd_W(\upsilon, \vartheta),$$

where $a' + b' + c' = \frac{a}{h} + \frac{b}{h} + \frac{c}{h} < 1$. Since $\Omega \subseteq \Upsilon(\Omega)$, therefore $\Upsilon^{-1}|_\Omega : \Omega \longrightarrow \Omega$ is the Reich contraction. Hence, by [19] there is $\omega \in \Omega$ such that $\Upsilon^{-1}\omega = \omega$ and $\omega = \Upsilon\omega$. For uniqueness, let $\omega, \nu \in \Omega$ such that $\Upsilon\omega = \omega$ and $\Upsilon\nu = \nu$. Thus, from (2), $cd_W(\omega, \nu) \geq hd_W(\omega, \nu)$, which implies that $\omega = \nu$. □

Corollary 5 *Let Ω be a nonempty closed subset of a complete metric space W and Υ is a mapping of Ω into W such that*

(1) $\Omega \subseteq \Upsilon(\Omega)$;
(2) $d_W(\Upsilon\omega, \Upsilon\nu) \geq h' d_W(\omega, \nu)$, $h' > 1$.
Thus, there is a unique $\omega \in \Omega$ such that $\Upsilon\omega = \omega$.

Proof From (2) we deduce that Υ is one to one. Using the above theorem, we can take $a = b = 0$ and $h > c > 0$ such that $h' = \frac{h}{c}$. Thus, there is a unique $\omega \in \Omega$ such that $\Upsilon\omega = \omega$. □

Remark 6 The above corollary is Theorem 2.1 in [26] (see also Theorem 2.1 in [25]).

Definition 7 Suppose Ω is a nonempty subset of a Banach space W. Let F be a mapping of $\Omega \times \Omega$ into W. F is called equiexpansive if there exists a constant $h > 1$ such that

$$\left\| F(\omega, \nu) - F\left(\omega, \nu'\right) \right\| \geq h \left\| \nu - \nu' \right\|$$

for all (ω, ν), (ω, ν') in the domain of F.

If we define F by $F(\omega, \nu) = \Lambda\omega + B\nu$ where $B : \Omega \longrightarrow W$ is expansive, then F is an equiexpansive mapping.

Definition 8 Suppose Ω is a nonempty subset of a Banach space W. Let F be a mapping of $\Omega \times \Omega$ into W. F is called generalized equiexpansive if

$$c\left\| F(\omega, \nu) - F\left(\omega, \nu'\right) \right\| + a\left\| F(\omega, \nu) - \nu \right\| + b\left\| F\left(\omega, \nu'\right) - \nu' \right\| \geq h\left\| \nu - \nu' \right\|$$

for all (ω, ν), (ω, ν') in the domain of F, $a, b \geq 0$, $c, h > 0$, $a + b + c < h$.

Theorem 9 *Suppose Ω is a nonempty closed subset of a Banach space W. Let F be a generalized equiexpansive mapping of $\Omega \times \Omega$ into W such that $\Omega \subseteq F(\omega, \Omega)$ for all $\omega \in \Omega$ and*
(1) *For each $\omega \in \Omega$, $F(\omega, \nu) = F(\omega, \nu')$ yields $\nu = \nu'$;*
(2) $\|F(\omega, \nu) - F(\omega', \nu)\| \leq \|\omega - \omega'\|$ *for all $(\omega, \nu), (\omega, \nu') \in \Omega \times \Omega$.*
Then, there is a unique $\omega \in \Omega$ such that $F(\omega, \omega) = \omega$.

Proof For $\omega \in \Omega$, define a mapping H of Ω into W such that $H(\nu) = F(\omega, \nu)$. From (1)

$$c\left\| H(\nu) - H\left(\nu'\right) \right\| + a\left\| H(\nu) - \nu \right\| + b\left\| H\left(\nu'\right) - \nu' \right\| \geq h\left\| \nu - \nu' \right\|.$$

Also

$$\Omega \subseteq F(\omega, \Omega) = H(\Omega)$$

and, using (2), H is a one to one function. Thus, (by Theorem 4) there is a unique $G\omega \in \Omega$ such that $G\omega = F(\omega, G\omega)$. Now,

$$\begin{aligned} &\left\| G(\omega) - G\left(\omega'\right) \right\| \\ &\quad = \left\| F(\omega, G\omega) - F\left(\omega', G\omega'\right) \right\| \end{aligned}$$

$$
\begin{aligned}
&= \left\| \left(F\left(\omega', G\omega\right) - F\left(\omega', G\omega'\right)\right) - \left(F\left(\omega', G\omega\right) - F(\omega, G\omega)\right) \right\| \\
&\geq \left\| F\left(\omega', G\omega\right) - F\left(\omega', G\omega'\right) \right\| - \left\| F\left(\omega', G\omega\right) - F(\omega, G\omega) \right\| \\
&\geq \frac{h}{c}\left\| G\omega - G\omega' \right\| - \frac{a}{c}\left\| F\left(\omega', G\omega\right) - G\omega \right\| - \frac{b}{c}\left\| F\left(\omega', G\omega'\right) - G\omega' \right\| \\
&\quad - \left\| F\left(\omega', G\omega\right) - F(\omega, G\omega) \right\| \\
&= \frac{h}{c}\left\| G\omega - G\omega' \right\| - \frac{a}{c}\left\| F\left(\omega', G\omega\right) - F(\omega, G\omega) \right\| - \frac{b}{c}\left\| G\omega' - G\omega' \right\| \\
&\quad - \left\| F\left(\omega', G\omega\right) - F(\omega, G\omega) \right\|.
\end{aligned}
$$

Simplifying

$$
\left\| G(\omega) - G\left(\omega'\right) \right\| \leq \frac{a+c}{h-c}\left\| F\left(\omega', G\omega\right) - F(\omega, G\omega) \right\| \leq \frac{a+c}{h-c}\left\| \omega - \omega' \right\|,
$$

which yields that G is a contraction mapping of Ω into Ω. Thus, there is a unique $\omega \in \Omega$ such that $G\omega = \omega$. Since for this $\omega \in \Omega$ there is a unique $G\omega \in \Omega$ such that $G\omega = F(\omega, G\omega)$, there is a unique ω satisfying $F(\omega, \omega) = \omega$. □

Corollary 10 *Suppose Ω is a nonempty closed subset of a Banach space W. Let F be a mapping of $\Omega \times \Omega$ into W such that $\Omega \subseteq F(\omega, \Omega)$ for all $\omega \in \Omega$ and*

(1) *$c\|F(\omega, \nu) - F(\omega, \nu')\| \geq h\|\nu - \nu'\|$ for all (ω, ν), (ω, ν') in the domain of F, $c, h > 0$, $2c < h$;*
(2) *$\|F(\omega, \nu) - F(\omega', \nu)\| \leq \|\omega - \omega'\|$ for all (ω, ν), $(\omega, \nu') \in \Omega \times \Omega$.*

Then, there is a unique $\omega \in \Omega$ such that $F(\omega, \omega) = \omega$.

Proof From (1), $F(\omega, \nu) = F(\omega, \nu')$ yields $\nu = \nu'$. Putting $a = b = 0$ in the generalized equiexpansive condition, used in Theorem 9, we acquire the required result. □

Corollary 11 *Suppose Ω is a nonempty closed subset of a Banach space W. Λ and B are mappings of Ω into W such that $\Omega \subseteq \Lambda\omega + B(\Omega)$ for all $\omega \in \Omega$ and*

(1) *$c\|B\nu - B\nu'\| + a\|\Lambda\omega - (I - B)\nu\| + b\|\Lambda\omega - (I - B)\nu'\| \geq h\|\nu - \nu'\|$ for all (ω, ν), (ω, ν') in the domain of F, $a, b \geq 0$, $c, h > 0$, $a + b + c < h$, $a + c < h - c$;*
(2) *B is a one to one mapping;*
(3) *$\|\Lambda\omega - \Lambda\omega'\| \leq \|\omega - \omega'\|$ for all $\omega, \omega' \in \Omega$.*

Thus, there is a unique $\omega \in \Omega$ such that $\Lambda\omega + B\omega = \omega$.

Proof Define $F(\omega, \nu) = \Lambda\omega + B\nu$. Clearly F is a mapping of $\Omega \times \Omega$ into W. Also, $\Omega \subseteq \Lambda\omega + B(\Omega) = F(\omega, \Omega)$. From (1)

$$
\begin{aligned}
h\left\| \nu - \nu' \right\| &\leq c\left\| B\nu - B\nu' \right\| + a\left\| \Lambda\omega - (I - B)\nu \right\| + b\left\| \Lambda\omega - (I - B)\nu' \right\| \\
&= c\left\| (\Lambda\omega + B\nu) - \left(\Lambda\omega + B\nu'\right) \right\| + a\left\| \nu - (\Lambda\omega + B\nu) \right\| \\
&\quad + b\left\| \nu' - \left(\Lambda\omega + B\nu'\right) \right\| \\
&= c\left\| F(\omega, \nu) - F\left(\omega, \nu'\right) \right\| + a\left\| F(\omega, \nu) - \nu \right\| + b\left\| F\left(\omega, \nu'\right) - \nu' \right\|.
\end{aligned}
$$

Since B is a one to one mapping, $B\nu = B\nu'$ implies $\nu = \nu'$ and hence $F(\omega,\nu) = F(\omega',\nu)$ implies $\nu = \nu'$. Also,

$$\|F(\omega,\nu) - F(\omega',\nu)\| = \|\Lambda\omega - \Lambda\omega'\| \leq \|\omega - \omega'\|.$$

Hence, (by Theorem 9) there is a unique $\omega \in \Omega$ such that $\Lambda\omega + B\omega = \omega$. □

Remarkk 11.1 If $\Lambda = O$ in the above corollary, then we acquire the result like Theorem 4.

Corollary 12 *Suppose Ω is a nonempty closed subset of a Banach space W. Λ and B are mappings of Ω into W such that $\Omega \subseteq \Lambda\omega + B(\Omega)$ for all $\omega \in \Omega$ and*

(1) *$c\|B\nu - B\nu'\| \geq h\|\nu - \nu'\|$ for all (ω,ν), (ω,ν') in the domain of F, $c, h > 0$, $2c < h$;*
(2) *$\|\Lambda\omega - \Lambda\omega'\| \leq \|\omega - \omega'\|$ for all $\omega, \omega' \in \Omega$.*

Then, there is a unique $\omega \in \Omega$ such that $\Lambda\omega + B\omega = \omega$.

Proof From (1) we deduce that B is one to one. Hence, the result can be obtained by taking $a = b = 0$ in Corollary 11. □

Theorem 13 *Suppose Ω is a nonempty closed subset of a Banach space W. Let F be a mapping of $\Lambda(\Omega) \times \Omega$ into W such that $\Omega \subseteq F(\Lambda\omega, \Omega)$ for all $\Lambda\omega \in \Lambda(\Omega)$ such that*

(1) *$c\|F(\Lambda\omega,\nu) - F(\Lambda\omega,\nu')\| + a\|F(\Lambda\omega,\nu) - \nu\| + b\|F(\Lambda\omega,\nu') - \nu'\| \geq h\|\nu - \nu'\|$ for all (ω,ν), (ω,ν') in the domain of F, $a, b \geq 0$, $c, h > 0$, $a + b + c < h$, $a + c < h - c$;*
(2) *For each $\Lambda\omega \in \Omega$, $F(\Lambda\omega,\nu) = F(\Lambda\omega,\nu')$ implies $\nu = \nu'$ for all $\nu, \nu' \in \Omega$;*
(3) *$\|F(\Lambda\omega,\nu) - F(\Lambda\omega',\nu)\| \leq \|\omega - \omega'\|$ for all $(\Lambda\omega,\nu)$, $(\Lambda\omega',\nu) \in \Lambda(\Omega) \times \Omega$ and Λ is a mapping of Ω into W.*

Then, there is a unique $\omega \in \Omega$ such that $F(\Lambda\omega,\omega) = \omega$.

Proof For $\Lambda\omega \in \Lambda(\Omega)$, define a mapping H of Ω into W such that $H(\nu) = F(\Lambda\omega,\nu)$. From (1)

$$c\|H(\nu) - H(\nu')\| + a\|H(\nu) - \nu\| + b\|H(\nu') - \nu'\| \geq h\|\nu - \nu'\|.$$

Also,

$$\Omega \subseteq F(\Lambda\omega, \Omega) = H(\Omega)$$

and, using (2), H is a one to one function. Thus, (by Theorem 4) there is a unique point $G(\Lambda\omega) \in \Omega$ such that $G(\Lambda\omega) = F(\Lambda\omega, G(\Lambda\omega))$. Now,

$$\begin{aligned}
&\|G(\Lambda\omega) - G(\Lambda\omega')\| \\
&\quad = \|F(\Lambda\omega, G(\Lambda\omega)) - F(\Lambda\omega', G(\Lambda\omega'))\| \\
&\quad = \|(F(\Lambda\omega', G(\Lambda\omega)) - F(\Lambda\omega', G(\Lambda\omega'))) - (F(\Lambda\omega', G(\Lambda\omega)) - F(\Lambda\omega, G(\Lambda\omega)))\| \\
&\quad \geq \|F(\Lambda\omega', G(\Lambda\omega)) - F(\Lambda\omega', G(\Lambda\omega'))\| - \|F(\Lambda\omega', G(\Lambda\omega)) - F(\Lambda\omega, G(\Lambda\omega))\| \\
&\quad \geq \left(\frac{h}{c}\|G(\Lambda\omega) - G(\Lambda\omega')\| - \frac{a}{c}\|F(\Lambda\omega', G(\Lambda\omega)) - G(\Lambda\omega)\|\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{b}{c}\big\|\digamma\big(\Lambda\omega', G(\Lambda\omega')\big) - G\big(\Lambda\omega'\big)\big\| - \big\|\digamma\big(\Lambda\omega', G(\Lambda\omega)\big) - \digamma\big(\Lambda\omega, G(\Lambda\omega)\big)\big\|\bigg) \\
&= \bigg(\frac{h}{c}\big\|G(\Lambda\omega) - G\big(\Lambda\omega'\big)\big\| - \frac{a}{c}\big\|\digamma\big(\Lambda\omega', G(\Lambda\omega)\big) - \digamma\big(\Lambda\omega, G(\Lambda\omega)\big)\big\| \\
&\quad - \frac{b}{c}\big\|G\big(\Lambda\omega'\big) - G\big(\Lambda\omega'\big)\big\| - \big\|\digamma\big(\Lambda\omega', G(\Lambda\omega)\big) - \digamma\big(\Lambda\omega, G(\Lambda\omega)\big)\big\|\bigg).
\end{aligned}
$$

Simplifying

$$
\begin{aligned}
\big\|(G\circ\Lambda)\omega) - (G\circ\Lambda)\omega')\big\| &\le \frac{a+c}{h-c}\big\|\digamma\big(\Lambda\omega', G(\Lambda\omega)\big) - \digamma\big(\Lambda\omega, G(\Lambda\omega)\big)\big\| \\
&\le \frac{a+c}{h-c}\big\|\omega - \omega'\big\|,
\end{aligned}
$$

which implies that $G\circ\Lambda$ is a contraction mapping of Ω into Ω and there is a unique $\omega\in\Omega$ such that $(G\circ\Lambda)\omega = \omega$ or $G(\Lambda\omega) = \omega$. Also, for $\Lambda\omega\in\Lambda(\Omega)$ there is a unique $G(\Lambda\omega)\in\Omega$ such that $G(\Lambda\omega) = \digamma(\Lambda\omega, G(\Lambda\omega))$. Hence, there is a unique ω such that $\omega = \digamma(\Lambda\omega,\omega)$. □

Definition 14 ([3]) Assume that $\widetilde{W}$ is the family of bounded subsets of a Banach space W. A mapping μ of $\widetilde{W}$ into $[0,+\infty)$ is called a measure of noncompactness (MNC) if the following properties hold for $A, B\in\widetilde{W}$.

(1) $\mu(A) = 0$, $\Longleftrightarrow$ A is precompact;
(2) $\mu(A) = \mu(\overline{A})$;
(3) $\mu(A\cup B) = \max\{\mu(A), \mu(B)\}$.

We can also deduce the following properties;

(4) $A\subseteq B$ implies $\mu(A)\le\mu(B)$;
(5) $\mu(A+B)\le\mu(A)+\mu(B)$.

Definition 15 ([26]) Assume that Ω is a subset of a Banach space W and Υ is a mapping of Ω into W. Υ is said to be k-set contractive if $\mu(\Upsilon(A))\le k\mu(A)$ for any bounded subset of A of Ω and Υ is bounded and continuous.

Υ is said to be strictly k-set contractive if Υ is k-set contractive and $\mu(\Upsilon(A)) < k\mu(A)$ for all bounded subsets A of Ω with $\mu(A)\neq 0$. Υ is called a condensing map if Υ is strictly 1-set contractive.

Theorem 16 *Suppose that Ω is a nonempty bounded closed convex subset of a Banach space W. $\digamma$ is a mapping of $\Lambda(\Omega)\times\Omega$ into W such that $\Omega\subseteq\digamma(\Lambda\omega,\Omega)$ for all $\Lambda\omega\in\Lambda(\Omega)$ such that*

(1) *$c\|\digamma(\Lambda\omega,\nu) - \digamma(\Lambda\omega,\nu')\| + a\|\digamma(\Lambda\omega,\nu) - \nu\| + b\|\digamma(\Lambda\omega,\nu') - \nu'\|\ge h\|\nu-\nu'\|$ for all (ω,ν), (ω,ν') in the domain of $\digamma$, $a, b\ge 0$, $c, h > 0$, $a+b+c < h$;*
(2) *For each $\omega\in\Omega$, $\digamma(\Lambda\omega,\nu) = \digamma(\Lambda\omega,\nu')$ implies $\nu=\nu'$ for all $\nu,\nu'\in\Omega$.*
(3) *$\|\digamma(\Lambda\omega,\nu) - \digamma(\Lambda\omega',\nu)\|\le\|\Lambda\omega - \Lambda\omega'\|$ for all $(\Lambda\omega,\nu)$, $(\Lambda\omega,\nu')\in\Lambda(\Omega)\times\Omega$ and Λ is a k-set contractive mapping of Ω into W for $k < \frac{h-c}{a+c}$.*

Then, there is $\omega\in\Omega$ such that $\digamma(\Lambda\omega,\omega) = \omega$.

Proof For $\Lambda\omega\in\Omega$, define a mapping H of Ω into W such that $H(\nu) = \digamma(\Lambda\omega,\nu)$. From (1)

$$
c\big\|H(\nu) - H\big(\nu'\big)\big\| + a\big\|H(\nu) - \nu\big\| + b\big\|H\big(\nu'\big) - \nu'\big\| \ge h\big\|\nu - \nu'\big\|.
$$

Also,

$$\Omega \subseteq F(\Lambda\omega, \Omega) = H(\Omega)$$

and, using (2), H is a one to one function. Thus, (by Theorem 4) there is a unique $G(\Lambda\omega) \in \Omega$ such that $G(\Lambda\omega) = F(\Lambda\omega, G(\Lambda\omega))$. Now,

$$\begin{aligned}
&\|G(\Lambda\omega) - G(\Lambda\omega')\| \\
&\quad = \|F(\Lambda\omega, G(\Lambda\omega)) - F(\Lambda\omega', G(\Lambda\omega'))\| \\
&\quad = \|(F(\Lambda\omega', G(\Lambda\omega)) - F(\Lambda\omega', G(\Lambda\omega'))) \\
&\qquad - (F(\Lambda\omega', G(\Lambda\omega)) - F(\Lambda\omega, G(\Lambda\omega)))\| \\
&\quad \geq \|F(\Lambda\omega', G(\Lambda\omega)) - F(\Lambda\omega', G(\Lambda\omega'))\| \\
&\qquad - \|F(\Lambda\omega', G(\Lambda\omega)) - F(\Lambda\omega, G(\Lambda\omega))\| \\
&\quad \geq \biggl(\frac{h}{c}\|G(\Lambda\omega) - G(\Lambda\omega')\| - \frac{a}{c}\|F(\Lambda\omega', G(\Lambda\omega)) - G\Lambda\omega\| \\
&\qquad - \frac{b}{c}\|F(\Lambda\omega', G(\Lambda\omega')) - G(\Lambda\omega')\| - \|F(\Lambda\omega', G(\Lambda\omega)) - F(\Lambda\omega, G(\Lambda\omega))\|\biggr) \\
&\quad = \biggl(\frac{h}{c}\|G(\Lambda\omega) - G(\Lambda\omega')\| - \frac{a}{c}\|F(\Lambda\omega', G(\Lambda\omega)) - F(\Lambda\omega, G(\Lambda\omega))\| \\
&\qquad - \frac{b}{c}\|G(\Lambda\omega') - G(\Lambda\omega')\| - \|F(\Lambda\omega', G(\Lambda\omega)) - F(\Lambda\omega, G(\Lambda\omega))\|\biggr).
\end{aligned}$$

Simplifying

$$\begin{aligned}
\|(G \circ \Lambda)\omega) - (G \circ \Lambda)\omega')\| &\leq \frac{a+c}{h-c}\|F(\Lambda\omega', G(\Lambda\omega)) - F(\Lambda\omega, G(\Lambda\omega))\| \\
&\leq \frac{a+c}{h-c}\|\Lambda\omega - \Lambda\omega'\|,
\end{aligned}$$

which implies that $G \circ \Lambda$ is a continuous mapping of Ω into Ω. Now,

$$\begin{aligned}
\mu(G \circ \Lambda(N)) &= \mu(G(\Lambda(N))) \\
&\leq \frac{a+c}{h-c}\mu(\Lambda(N)) \\
&\leq k\frac{a+c}{h-c}\mu(N) < \mu(N)
\end{aligned}$$

for all $N \subseteq \Omega$. By the Sadovskii fixed-point theorem there is $\omega \in \Omega$ such that $(G \circ \Lambda)\omega = \omega$ or $G(\Lambda\omega) = \omega$. Now, for $\Lambda\omega \in \Lambda(\Omega)$ there is a unique $G(\Lambda\omega) \in \Omega$ such that $G(\Lambda\omega) = F(\Lambda\omega, G(\Lambda\omega))$. Hence, $\omega = G(\Lambda\omega) = F(\Lambda\omega, G(\Lambda\omega)) = F(\Lambda\omega, \omega)$. □

Corollary 17 *Suppose that Ω is a nonempty bounded closed convex subset of a Banach space W. F is a mapping of $\Lambda(\Omega) \times \Omega$ into W such that $\Omega \subseteq F(\Lambda\omega, \Omega)$ for all $\Lambda\omega \in \Lambda(\Omega)$ such that*

(1) $c\|F(\Lambda\omega, \nu) - F(\Lambda\omega, \nu')\| \geq h\|\nu - \nu'\|$ *for all* (ω, ν), (ω, ν') *in the domain of* F, $c, h > 0$, $c < h$;

(2) $\|F(\Lambda\omega,\nu) - F(\Lambda\omega',\nu)\| \leq \|\Lambda\omega - \Lambda\omega'\|$ *for all* $(\Lambda\omega,\nu), (\Lambda\omega',\nu) \in \Lambda(\Omega)\times\Omega$ *and* Λ *is a k-set contractive mapping of* Ω *into* W *for* $k < \frac{h-c}{c}$.

Then, there is $\omega \in \Omega$ *such that* $F(\Lambda\omega,\omega) = \omega$.

Proof From (1), $F(\Lambda\omega,\nu) = F(\Lambda\omega,\nu')$ implies $\nu = \nu'$. Putting $a = b = 0$ in Theorem 16, we acquire the required result. □

Corollary 18 *Suppose that* Ω *is a nonempty bounded closed convex subset of a Banach space* W. Λ *and* B *are mappings of* Ω *into* W *such that* $\Omega \subseteq \Lambda\omega + B(\Omega)$ *for all* $\omega \in \Omega$ *and*

(1) *B is a one to one mapping*;

(2) $c\|B\nu - B\nu'\| + a\|\Lambda\omega - (I-B)\nu\| + b\|\Lambda\omega - (I-B)\nu'\| \geq h\|\nu - \nu'\|$ *for all* $(\omega,\nu), (\omega,\nu')$ *in the domain of* F, $a, b \geq 0$, $c, h > 0$, $a+b+c < h$;

(3) Λ *is a k-set contractive mapping of* Ω *into* W *for* $k < \frac{h-c}{a+c}$. *Then, there is* $\omega \in \Omega$ *such that* $F(\Lambda\omega,\omega) = \omega$.

Proof Define $F(\Lambda\omega,\nu) = \Lambda\omega + B\nu$. Clearly F is a mapping of $\Lambda(\Omega)\times\Omega$ into W. Also, $\Omega \subseteq \Lambda\omega + B(\Omega) = F(\Lambda\omega,\Omega)$. From (2)

$$\begin{aligned} h\|\nu - \nu'\| &\leq c\|B\nu - B\nu'\| + a\|\Lambda\omega - (I-B)\nu\| + b\|\Lambda\omega - (I-B)\nu'\| \\ &= c\|(\Lambda\omega + B\nu) - (\Lambda\omega + B\nu')\| + a\|\nu - (\Lambda\omega + B\nu)\| + b\|\nu' - (\Lambda\omega + B\nu')\| \\ &= c\|F(\Lambda\omega,\nu) - F(\Lambda\omega,\nu')\| + a\|F(\Lambda\omega,\nu) - \nu\| + b\|F(\Lambda\omega,\nu') - \nu'\|. \end{aligned}$$

Since B is a one to one mapping, $B\nu = B\nu'$ implies $\nu = \nu'$, and hence $F(\omega,\nu) = F(\omega',\nu)$ implies $\nu = \nu'$. Also,

$$\|F(\omega,\nu) - F(\omega',\nu)\| = \|\Lambda\omega - \Lambda\omega'\| \leq \|\Lambda\omega - \Lambda\omega'\|.$$

Hence, (by Theorem 16) there is $\omega \in \Omega$ such that $\Lambda\omega + B\omega = \omega$. □

Corollary 19 *Suppose* Ω *is a nonempty bounded closed convex subset of a Banach space* W. Λ *and* B *are mappings of* Ω *into* W *such that*

(1) $\Omega \subseteq \Lambda\omega + B(\Omega)$ *for all* $\omega \in \Omega$;

(2) $c\|B\nu - B\nu'\| \geq h\|\nu - \nu'\|$ *for all* $(\omega,\nu), (\omega,\nu')$ *in the domain of* F, $c, h > 0$, $c < h$;

(3) Λ *is a k-set contractive mapping of* Ω *into* W *for* $k < \frac{h-c}{c}$.

Then, there is $\omega \in \Omega$ *such that* $\Lambda\omega + B\omega = \omega$.

Proof (2) implies that B is a one to one mapping. Putting $a = b = 0$ in Corollary 18, we acquire the required result. □

Remarkk 19.1 Corollary 19 is Theorem 2.6 in [26].

Corollary 20 *Suppose* Ω *is a nonempty bounded closed convex subset of a Banach space* W. *Let* Λ *and* B *be the mappings of* Ω *into* W *such that*

(1) $\Omega \subseteq \Lambda\omega + B(\Omega)$ *for all* $\omega \in \Omega$;

(2) Λ *is continuous and* $\Lambda(\Omega)$ *lies in a compact subset of* W;

(3) *$c\|B\nu - B\nu'\| \geq h\|\nu - \nu'\|$ for all (ω, ν), (ω, ν') in the domain of $\digamma$, $c, h > 0$, $c < h$;*
Then, there is $\omega \in \Omega$ such that $\Lambda\omega + B\omega = \omega$.

Proof We show that all the conditions of Corollary 17 are satisfied. Define $\digamma : \Lambda(\Omega) \times \Omega \longrightarrow W$ by $\digamma(\Lambda\omega, \nu) = \Lambda\omega + B\nu$. From (1), since $\Omega \subseteq \Lambda\omega + B(\Omega)$, therefore $\Omega \subseteq \Lambda\omega + B(\Omega) = \digamma(\Lambda\omega, \Omega)$. Now, using (3)

$$c\left\|\digamma(\Lambda\omega, \nu) - \digamma\left(\Lambda\omega, \nu'\right)\right\| = c\left\|B\nu - B\nu'\right\| \geq h\left\|\nu - \nu'\right\|$$

and

$$\left\|\digamma(\Lambda\omega, \nu) - \digamma\left(\Lambda\omega', \nu\right)\right\| = \left\|\Lambda\omega - \Lambda\omega'\right\| \leq \left\|\Lambda\omega - \Lambda\omega'\right\|.$$

Since $\Lambda(\Omega)$ lies in a compact subset C of W, therefore for every $N \subseteq \Omega$

$$\mu\left(\Lambda(N)\right) \leq \mu\left(\Lambda(\Omega)\right) \leq \mu(C) = 0.$$

Hence, Λ is a k-set contractive mapping for $k = 0$. Hence, there is $\omega \in \Omega$ such that $\omega = \digamma(\Lambda\omega, \omega) = \Lambda\omega + B\omega$. □

Remarkk 20.1 Corollary 20 is Theorem 2.2 in [26] with the assumption that Ω is a bounded set.

Theorem 20* ([11]. p. 392) *If (W, d_W) is a complete metric space, $0 \leq q < 1/2$ and $\Upsilon : W \longrightarrow W$ is a map such that*

$$d_W(\Upsilon\omega, \Upsilon\nu) \leq q\left[d_W(\omega, \Upsilon\omega) + d_W(\nu, \Upsilon\nu)\right]$$

for all $\omega, \nu \in W$. Then, Υ has a unique fixed point.

Theorem 21 *Let Ω be nonempty closed subset of a complete metric space W and Υ is a one to one mapping of Ω into W such that*
(1) $\Omega \subseteq \Upsilon(\Omega)$;
(2) $d_W(\Upsilon\omega, \omega) + d_W(\Upsilon\nu, \nu) \geq h d_W(\omega, \nu)$, $h > 2$.
Then, there is a unique $\omega \in \Omega$ such that $\Upsilon\omega = \omega$.

Proof By Theorem 4 and utilizing the result of Theorem 20*, we deduce the required result. □

Theorem 22 *Suppose that Ω is a nonempty closed subset of a Banach space W. $\digamma$ is a mapping of $\Lambda(\Omega) \times \Omega$ into W such that $\Omega \subseteq \digamma(\Lambda\omega, \Omega)$ for all $\Lambda\omega \in \Lambda(\Omega)$ and*
(1) *$\|\digamma(\Lambda\omega, \nu) - \nu\| + \|\digamma(\Lambda\omega, \nu') - \nu'\| \geq h\|\nu - \nu'\|$ for all (ω, ν), (ω, ν') in the domain of $\digamma$, $h > 2$;*
(2) *For each $\omega \in \Omega$, $\digamma(\Lambda\omega, \nu) = \digamma(\Lambda\omega, \nu')$ implies $\nu = \nu'$ for all $\nu, \nu' \in \Omega$;*
(3) *$\|\digamma(\Lambda\omega, \nu) - \digamma(\Lambda\omega', \nu)\| \leq \|\omega - \omega'\|$ for all $(\Lambda\omega, \nu)$, $(\Lambda\omega', \nu) \in \Lambda(\Omega) \times \Omega$ and Λ is a mapping of Ω into W.*
Then, there is a unique $\omega \in \Omega$ such that $\digamma(\Lambda\omega, \omega) = \omega$.

Proof For $\Lambda\omega \in \Omega$, define a mapping H of Ω into W such that $H(\nu) = \digamma(\Lambda\omega, \nu)$. From (1)

$$\|H(\nu) - \nu\| + \|H(\nu') - \nu'\| \geq h\|\nu - \nu'\|.$$

Also,

$$\Omega \subseteq \digamma(\Lambda\omega, \Omega) = H(\Omega)$$

and, using (2), H is a one to one function. Therefore, there is a unique $G(\Lambda\omega) \in \Omega$ such that $G(\Lambda\omega) = \digamma(\Lambda\omega, G(\Lambda\omega))$. Now, putting $\nu = G(\Lambda\omega)$ and $\nu' = G(\Lambda\omega')$ in (1) we deduce

$$\begin{aligned}
&h\|G(\Lambda\omega) - G(\Lambda\omega')\| \\
&\quad\leq \|\digamma(\Lambda\omega, G(\Lambda\omega)) - G(\Lambda\omega)\| + \|\digamma(\Lambda\omega, G(\Lambda\omega')) - G(\Lambda\omega')\| \\
&\quad\leq \|G(\Lambda\omega) - G(\Lambda\omega)\| + \|\digamma(\Lambda\omega, G(\Lambda\omega')) - \digamma(\Lambda\omega', G(\Lambda\omega'))\| \\
&\quad= \|\digamma(\Lambda\omega, G(\Lambda\omega')) - \digamma(\Lambda\omega', G(\Lambda\omega'))\| \\
&\quad\leq \|\omega - \omega'\|.
\end{aligned}$$

This gives

$$\|(G \circ \Lambda)\omega - (G \circ \Lambda)\omega'\| \leq \frac{1}{h}\|\omega - \omega'\|,$$

which shows that $G \circ \Lambda$ is a contraction mapping and there is a unique $\omega \in \Omega$ such that $(G \circ \Lambda)\omega = \omega$ or $G(\Lambda\omega) = \omega$. Also, for $\Lambda\omega \in \Lambda(\Omega)$ there is a unique $G(\Lambda\omega) \in \Omega$ such that $G(\Lambda\omega) = \digamma(\Lambda\omega, G(\Lambda\omega))$. Hence, there is a unique ω such that $\omega = G(\Lambda\omega) = \digamma(\Lambda\omega, G(\Lambda\omega)) = \digamma(\Lambda\omega, \omega)$. □

Theorem 23 *Suppose that Ω is a nonempty closed subset of a Banach space W. $\digamma$ is a mapping of $\Omega \times \Omega$ into W such that $\Omega \subseteq \digamma(\omega, \Omega)$ for all $\omega \in \Omega$ and*

(1) $\|\digamma(\omega, \nu) - \nu\| + \|\digamma(\omega, \nu') - \nu'\| \geq h\|\nu - \nu'\|$ *for all* (ω, ν), (ω, ν') *in the domain of* $\digamma$, $h > 2$;
(2) *For each* $\omega \in \Omega$, $\digamma(\omega, \nu) = \digamma(\omega, \nu')$ *implies* $\nu = \nu'$;
(3) $\|\digamma(\omega, \nu) - \digamma(\omega', \nu)\| \leq \|\omega - \omega'\|$ *for all* (ω, ν), $(\omega', \nu) \in \Omega \times \Omega$.

Thus, there is a unique $\omega \in \Omega$ such that $\digamma(\omega, \omega) = \omega$.

Proof For $\omega \in \Omega$, define a mapping H of Ω into W such that $H(\nu) = \digamma(\omega, \nu)$. From (1)

$$\|H(\nu) - \nu\| + \|H(\nu') - \nu'\| \geq h\|\nu - \nu'\|.$$

Also,

$$\Omega \subseteq \digamma(\omega, \Omega) = H(\Omega)$$

and, using (2), H is a one to one function. Thus, there is a unique $G\omega \in \Omega$ such that $G\omega = \digamma(\omega, G\omega)$. Now, putting $\nu = G\omega$ and $\nu' = G\omega'$ in (1) we deduce

$$h\|G\omega - G\omega'\| \leq \|\digamma(\omega, G\omega) - G\omega\| + \|\digamma(\omega, G\omega') - G\omega'\|$$

$$
\begin{aligned}
&\leq \|G\omega - G\omega\| + \left\| F\left(\omega, G\omega'\right) - F\left(\omega', G\omega'\right)\right\| \\
&= \left\| F\left(\omega, G\omega'\right) - F\left(\omega', G\omega'\right)\right\| \\
&\leq \left\|\omega - \omega'\right\|,
\end{aligned}
$$

which shows that G is a contraction. Hence, (by similar arguments as in Theorem 22) there is a unique $\omega \in \Omega$ such that $F(\omega, \omega) = \omega$. □

Remark 24 Theorems 4 and 9 are extensions of Theorem 2.1 in [26] (see also Theorem 2.1 in [25]).

Remark 25 Theorem 16 is an extension of Theorem 2.6 and Theorem 2.2 in [26].

3 Applications

In this section, first we prove the existence result for the solutions of Cauchy problem (A) given below, with suitable conditions on given functions. Then, we propose a general class of integral equations and using our main Theorem 9, we will discuss the existence of solutions. This integral equations presents many kinds of evolution equations. We summarize the discussion in the form of Theorems at the end.

Problem 26 Consider the following Cauchy problem:

$$
\begin{aligned}
&{}^{ABC}D^{\rho}\omega(\sigma) = A\omega(\sigma) + g\big(\sigma, \omega(\sigma)\big) + \Phi(\sigma)\upsilon(\sigma), \qquad \text{(A)}\\
&\omega(0) = \omega_0.
\end{aligned}
$$

The mild solution of the above problem is given as;

$$
\begin{aligned}
\omega(\sigma) = {} & GT_{\rho}(\sigma)\omega_0 + \frac{KG(1-\rho)}{B(\rho)\Gamma(\rho)}\int_0^{\sigma}(\sigma-\varrho)^{\rho-1}g\big(\varrho, \omega(\varrho)\big)\,d\varrho \\
&+ \frac{\rho G^2}{B(\rho)}\int_0^{\sigma} S_{\rho}(\sigma-\varrho)g\big(\varrho, \omega(\varrho)\big)\,d\varrho \\
&+ \frac{KG(1-\rho)}{B(\rho)\Gamma(\rho)}\int_0^{\sigma}(\sigma-\varrho)^{\rho-1}\Phi(\varrho)\upsilon(\varrho) \\
&+ \frac{\rho G^2}{B(\rho)}\int_0^{\sigma} S_{\rho}(\sigma-\varrho)\Phi(\varrho)\upsilon(\varrho)\,d\varrho,
\end{aligned}
$$

where

$$G = \mu(\mu I - A)^{-1}$$

and

$$K = -\mu A(\mu I - A)^{-1}$$

with $\mu = \frac{B(\rho)}{1-\rho}$ and

$$T_{\rho}(\sigma) = E_{\rho}\left(-K\sigma^{\rho}\right) = \frac{1}{2\pi i}\int_{\Delta} e^{\delta\sigma}\delta^{\rho-1}\left(\delta^{\rho}I - K\right)^{-1}d\delta,$$

$$S_\rho(\sigma) = \sigma^{\rho-1}E_\rho\big(-K\sigma^\rho\big) = \frac{1}{2\pi i}\int_\Delta e^{\delta\sigma}\big(\delta^\rho I - K\big)^{-1}\,d\delta,$$

here, Δ is a specific path lying on $E_{(\beta,\varpi)}$ and $g \in C(J \times E, E)$, $\Phi \in C(J,E)$ where $J = [0,T]$ for some $T > 0$, see [1]. If $A \in A^\rho(\beta_0, \varpi_0)$, thus $\|T_\rho(\sigma)\| \le \Omega e^{\varpi\sigma}$ and $\|S_\rho(\sigma)\| \le C_1 e^{\varpi\sigma}(1 + \sigma^{\rho-1})$ for every $\sigma > 0$ and $\varpi > \varpi_0$. Set $\Omega = \sup_{\sigma>0}\|T_\rho(\sigma)\|$ and $\Omega_1 = \sup_{\sigma>0} C_1 e^{\varpi\sigma}(1 + \sigma^{\rho-1})$ then we obtain

$$\big\|T_\rho(\sigma)\big\| \le \Omega \quad \text{and} \quad \big\|S_\rho(\sigma)\big\| \le \Omega_1.$$

Define $\digamma : B_\delta \times B_\delta \to W$ by

$$\begin{aligned}
&\digamma(\omega,\upsilon)(\sigma) \\
&\quad= GT_\rho(\sigma)\omega_0 + \frac{KG(1-\rho)}{B(\rho)\Gamma(\rho)}\int_0^\sigma(\sigma-\varrho)^{\rho-1}g\big(\varrho,\omega(\varrho)\big)\,d\varrho \\
&\qquad+ \frac{\rho G^2}{B(\rho)}\int_0^\sigma S_\rho(\sigma-\varrho)g\big(\varrho,\omega(\varrho)\big)\,d\varrho \\
&\qquad+ \frac{KG(1-\rho)}{B(\rho)\Gamma(\rho)}\int_0^\sigma(\sigma-\varrho)^{\rho-1}\Phi(\varrho)\upsilon(\varrho)\,d\varrho \\
&\qquad+ \frac{\rho G^2}{B(\rho)}\int_0^\sigma S_\rho(\sigma-\varrho)\Phi(\varrho)\upsilon(\varrho)\,d\varrho. \qquad\qquad \text{(B)}
\end{aligned}$$

Set $\min\{\frac{KG(1-\rho)}{B(\rho)\Gamma(\rho)}, \frac{\rho G^2}{B(\rho)}\} = K_G$.

Assume that the following conditions hold:

(H1) $\|\int_0^\sigma\{(\sigma-\varrho)^{\rho-1} + S_\rho(\sigma-\varrho)\}\Phi(\varrho)(\upsilon(\varrho) - \vartheta(\varrho))\,d\varrho\| \ge K_S\|\upsilon - \upsilon\|$ for some $K_S > 0$.

(H2) $\|g(\varrho,\omega_1(\varrho)) - g(\varrho,\omega_2(\varrho))\| \le L_g\|\omega_1 - \omega_2\|$ for some $L_g > 0$.

(H3) $(\frac{\|G\|}{B(\rho)}\|\frac{K(1-\rho)}{\Gamma(\rho)}\|\frac{1}{\rho} + \rho\|G\|\Omega_1)L_g \le 1$ and $2 < K_G K_S$,

where B_δ is a closed ball with radius δ and center at 0 in W.

First, we show that $\digamma$ satisfies (1) of Corollary 10,

$$\begin{aligned}
&c\big\|\digamma(\omega,\upsilon) - \digamma(\omega,\vartheta)\big\| \\
&\quad= c\left\|\begin{matrix}\frac{KG(1-\rho)}{B(\rho)\Gamma(\rho)}\int_0^\sigma(\sigma-\varrho)^{\rho-1}\Phi(\varrho)\upsilon(\varrho)\,d\varrho + \frac{\rho G^2}{B(\rho)}\int_0^\sigma S_\rho(\sigma-\varrho)\Phi(\varrho)\upsilon(\varrho)\,d\varrho \\ -\frac{KG(1-\rho)}{B(\rho)\Gamma(\rho)}\int_0^\sigma(\sigma-\varrho)^{\rho-1}\Phi(\varrho)\vartheta(\varrho)\,d\varrho + \frac{\rho G^2}{B(\rho)}\int_0^\sigma S_\rho(\sigma-\varrho)\Phi(\varrho)\vartheta(\varrho)\,d\varrho\end{matrix}\right\| \\
&\quad= c\left\|\frac{KG(1-\rho)}{B(\rho)\Gamma(\rho)}\int_0^\sigma(\sigma-\varrho)^{\rho-1}\Phi(\varrho)\big(\upsilon(\varrho)-\vartheta(\varrho)\big)\,d\varrho\right. \\
&\qquad\left.+ \frac{\rho G^2}{B(\rho)}\int_0^\sigma S_\rho(\sigma-\varrho)\Phi(\varrho)\big(\upsilon(\varrho)-\vartheta(\varrho)\big)\,d\varrho\right\| \\
&\quad\ge cK_G\left\|\int_0^\sigma(\sigma-\varrho)^{\rho-1}\Phi(\varrho)\big(\upsilon(\varrho)-\vartheta(\varrho)\big)\,d\varrho\right. \\
&\qquad\left.+ \int_0^\sigma S_\rho(\sigma-\varrho)\Phi(\varrho)\big(\upsilon(\varrho)-\vartheta(\varrho)\big)\,d\varrho\right\| \\
&\quad= cK_G\left\|\int_0^\sigma\big\{(\sigma-\varrho)^{\rho-1} + S_\rho(\sigma-\varrho)\big\}\Phi(\varrho)\big(\upsilon(\varrho)-\vartheta(\varrho)\big)\,d\varrho\right\| \\
&\quad\ge cK_GK_S\|\upsilon - \vartheta\|.
\end{aligned}$$

Now, we show that $\mathcal{F}$ satisfies (1) of Corollary 10, consider

$$\begin{aligned}
&\left\|\mathcal{F}(\upsilon,\omega_1)-\mathcal{F}(\upsilon,\omega_2)\right\| \\
&=\left\|\begin{array}{c} GT_\rho(\sigma)\omega_0+\frac{KG(1-\rho)}{B(\rho)\Gamma(\rho)}\int_0^\sigma(\sigma-\varrho)^{\rho-1}g(\varrho,\omega_1(\varrho))\,d\varrho \\ +\frac{\rho G^2}{B(\rho)}\int_0^\sigma S_\rho(\sigma-\varrho)g(\varrho,\omega_1(\varrho))\,d\varrho+\upsilon(\sigma) \\ -GT_\rho(\sigma)\omega_0-\frac{KG(1-\rho)}{B(\rho)\Gamma(\rho)}\int_0^\sigma(\sigma-\varrho)^{\rho-1}g(\varrho,\omega_2(\varrho))\,d\varrho \\ -\frac{\rho G^2}{B(\rho)}\int_0^\sigma S_\rho(\sigma-\varrho)g(\varrho,\omega_2(\varrho))\,d\varrho-\upsilon(\sigma)\end{array}\right\| \\
&\le\left\|\frac{KG(1-\rho)}{B(\rho)\Gamma(\rho)}\int_0^\sigma(\sigma-\varrho)^{\rho-1}\bigl[g\bigl(\varrho,\omega_1(\varrho)\bigr)-g\bigl(\varrho,\omega_2(\varrho)\bigr)\bigr]\,d\varrho\right. \\
&\quad\left.+\frac{\rho G^2}{B(\rho)}\int_0^\sigma S_\rho(\sigma-\varrho)\bigl[g\bigl(\varrho,\omega_1(\varrho)\bigr)-g\bigl(\varrho,\omega_2(\varrho)\bigr)\bigr]\,d\varrho\right\| \\
&\le\left\|\frac{KG(1-\rho)}{B(\rho)\Gamma(\rho)}\right\|\frac{1}{\rho}\left\|g\bigl(\varrho,\omega_1(\varrho)\bigr)-g\bigl(\varrho,\omega_2(\varrho)\bigr)\right\| \\
&\quad+\left\|\frac{\rho G^2}{B(\rho)}\right\|\Omega_1\left\|g\bigl(\varrho,\omega_1(\varrho)\bigr)-g\bigl(\varrho,\omega_2(\varrho)\bigr)\right\| \\
&\le\left(\frac{\|G\|}{B(\rho)}\left\|\frac{K(1-\rho)}{\Gamma(\rho)}\right\|\frac{1}{\rho}+\rho\|G\|\Omega_1\right)\left\|g\bigl(\varrho,\omega_1(\varrho)\bigr)-g\bigl(\varrho,\omega_2(\varrho)\bigr)\right\| \\
&\le\left(\frac{\|G\|}{B(\rho)}\left\|\frac{K(1-\rho)}{\Gamma(\rho)}\right\|\frac{1}{\rho}+\rho\|G\|\Omega_1\right)L_g\|\omega_1-\omega_2\|.
\end{aligned}$$

Since $(\frac{\|G\|}{B(\rho)}\|\frac{K(1-\rho)}{\Gamma(\rho)}\|\frac{1}{\rho}+\rho\|G\|\Omega_1)L_g\le 1$, therefore $\mathcal{F}$ satisfies (2). All the conditions of Corollary 10 are satisfied to obtain a point $\gamma\in B_\delta$ such that $\gamma=\mathcal{F}(\gamma,\gamma)$, which is the solution of our main problem (A).

Example 27 We take into consideration

$$\begin{cases}\frac{\partial^\alpha\upsilon(\sigma,\omega)}{\partial\sigma^\alpha}=\frac{\partial^2\upsilon(\sigma,\omega)}{\partial\omega^2}+f(\sigma,\upsilon(\sigma,\omega))+\Phi(\sigma,\upsilon(\sigma)),\\ \upsilon(\sigma,0)=\upsilon(\sigma,\pi),\qquad \upsilon'(\sigma,0)=\upsilon'(\sigma,\pi),\\ \upsilon(0,\omega)=\upsilon_0(\omega),\end{cases}\tag{P}$$

where $\sigma\in J=[0,1]$, $\omega\in(0,\pi)$, $0<\alpha<1$, let $W=L^2([0,\pi])$ and consider the operator $A:D(A)\subseteq W\to W$ defined by

$$A(\upsilon)=\frac{\partial^2\upsilon}{\partial\omega^2}$$

with domain

$$D(A)=\left\{\upsilon,\frac{\partial\upsilon}{\partial\omega},\frac{\partial^2\upsilon}{\partial\omega^2}\in W\right\}.$$

Clearly, A is densely defined in W and is the infinitesimal generator of a resolvent family $\{T_\alpha(\sigma)\}_{\sigma\ge0}$ on W [20] and let $\upsilon,\vartheta\in C(J,W)$. Define the operators f by

$$f(\sigma,\upsilon)=\frac{1}{(\sigma+2)^2}\tan^{-1}\upsilon.$$

Clearly,

$$\left|f(\sigma,\upsilon)-f(\sigma,\vartheta)\right|\leq\frac{1}{4}|\vartheta-\upsilon|.$$

All the conditions of the above theorem are satisfied to acquire the solution of given controlled problem (P).

Problem 28 More generally, the above solution (B) can be represented by the following integral equation:

$$\omega(\sigma)=\zeta(\sigma)\omega_0+\lambda\int_0^\sigma h(\sigma)g\bigl(\varrho,\omega(\varrho)\bigr)\,d\varrho+\mu\int_0^\sigma k(\sigma)g\bigl(\varrho,\omega(\varrho)\bigr)\,d\varrho+h\bigl(\upsilon(\sigma)\bigr)$$

or

$$\omega(\sigma)=\zeta(\sigma)\omega_0+\int_0^\sigma\bigl[\lambda h(\sigma)+\mu k(\sigma)\bigr]g\bigl(\varrho,\omega(\varrho)\bigr)\,d\varrho+h\bigl(\upsilon(\sigma)\bigr),\tag{C}$$

where $h:C(J,\mathbb{R})\to\mathbb{R}$, $\zeta,k,\upsilon\in C(J,\mathbb{R})$ and $g\in C(J\times\mathbb{R},\mathbb{R})$. The integral equation (C) and the fixed-point problem $\omega=\digamma(\omega,\omega)$ are equivalent, where

$$\digamma(\omega,\upsilon)=\zeta(\sigma)\omega_0+\int_0^\sigma\bigl[\lambda h(\sigma)+\mu k(\sigma)\bigr]g\bigl(\varrho,\omega(\varrho)\bigr)\,d\varrho+h\bigl(\upsilon(\sigma)\bigr).\tag{D}$$

We assume the following:

(C1) h is injective and there exists $\eta>0$ such that

$$\bigl\|h(\upsilon)-h\bigl(\upsilon'\bigr)\bigr\|\geq\eta\bigl\|\upsilon-\upsilon'\bigr\|,\quad\text{for all }\upsilon,\upsilon'\in B_\delta.$$

(C2)

$$\bigl|g\bigl(\sigma,\omega(\sigma)\bigr)-g\bigl(\sigma,\omega'(\sigma)\bigr)\bigr|\leq\bigl|\omega(\sigma)-\omega'(\sigma)\bigr|\quad\text{for all }\omega,\omega'\in B_\delta\text{ and }\sigma\in J.$$

(C3) $a+b+c<2+\eta c$ and $T(\mu+\lambda)\Omega\leq 1$, where $\Omega=\max\{\|k\|,\|h\|\}$.

We apply Theorem 9, to show that there exists ω such that $\omega=\digamma(\omega,\omega)$.

For this purpose consider

$$\begin{aligned}
&\bigl|\digamma(\omega,\upsilon)-\digamma\bigl(\omega',\upsilon\bigr)\bigr|\\
&\quad=\left|\begin{matrix}\zeta(\sigma)\omega_0+\int_0^\sigma[\lambda h(\sigma)+\mu k(\sigma)]g(\varrho,\omega(\varrho))\,d\varrho+h(\upsilon(\sigma))\\-\zeta(\sigma)\omega_0-\int_0^\sigma[\lambda h(\sigma)+\mu k(\sigma)]g(\varrho,\omega'(\varrho))\,d\varrho-h(\upsilon(\sigma))\end{matrix}\right|\\
&\quad\leq\left|\int_0^\sigma\bigl[\lambda h(\sigma)+\mu k(\sigma)\bigr]g\bigl(\varrho,\omega(\varrho)\bigr)\,d\varrho-\int_0^\sigma\bigl[\lambda h(\sigma)+\mu k(\sigma)\bigr]g\bigl(\varrho,\omega'(\varrho)\bigr)\,d\varrho\right|\\
&\quad\leq(\mu+\lambda)\Omega\int_0^\sigma\bigl|g\bigl(\varrho,\omega(\varrho)\bigr)-g\bigl(\varrho,\omega'(\varrho)\bigr)\bigr|\,d\varrho,
\end{aligned}$$

which implies using (C2)–(C3) that

$$\bigl\|\digamma(\omega,\upsilon)-\digamma\bigl(\omega',\upsilon\bigr)\bigr\|\leq T(\mu+\lambda)\Omega\bigl\|\omega-\omega'\bigr\|\leq\bigl\|\omega-\omega'\bigr\|,\quad\text{since }T(\mu+\lambda)\Omega\leq 1.$$

This shows that condition (2) of Theorem 9 is satisfied. Since h is injective, $\digamma(\omega,\upsilon) = \digamma(\omega,\upsilon')$ implies $\upsilon = \upsilon'$, which is condition (1) of Theorem 9. Set

$$\begin{aligned}&\min\begin{bmatrix} a\|\zeta(\sigma)\omega_0 + \int_0^\sigma[\lambda h(\sigma)+\mu k(\sigma)]g(\varrho,\omega(\varrho))\,d\varrho + h(\upsilon(\sigma)) - \upsilon(\sigma)\|, \\ b\|\zeta(\sigma)\omega_0 + \int_0^\sigma[\lambda h(\sigma)+\mu k(\sigma)]g(\varrho,\omega(\varrho))\,d\varrho + h(\upsilon'(\sigma)) - \upsilon'(\sigma)\|\end{bmatrix}\\ &\quad\geq \|\upsilon-\upsilon'\|.\end{aligned}$$

Also, as

$$\begin{aligned}&c\|\digamma(\omega,\upsilon) - \digamma(\omega,\upsilon')\|\\ &\quad= c\left\|\begin{matrix}\zeta(\sigma)\omega_0 + \int_0^\sigma[\lambda h(\sigma)+\mu k(\sigma)]g(\varrho,\omega(\varrho))\,d\varrho + h(\upsilon(\sigma))\\ -\zeta(\sigma)\omega_0 - \int_0^\sigma[\lambda h(\sigma)+\mu k(\sigma)]g(\varrho,\omega(\varrho))\,d\varrho - h(\upsilon'(\sigma))\end{matrix}\right\|\\ &\quad= c\|h(\upsilon(\sigma)) - h(\upsilon'(\sigma))\|,\\ &a\|\digamma(\omega,\upsilon)-\upsilon\|\\ &\quad= a\left\|\zeta(\sigma)\omega_0 + \int_0^\sigma[\lambda h(\sigma)+\mu k(\sigma)]g(\varrho,\omega(\varrho))\,d\varrho + h(\upsilon(\sigma)) - \upsilon(\sigma)\right\|,\end{aligned}$$

and

$$\begin{aligned}&b\|\digamma(\omega,\upsilon')-\upsilon'\|\\ &\quad= b\left\|\zeta(\sigma)\omega_0 + \int_0^\sigma[\lambda h(\sigma)+\mu k(\sigma)]g(\varrho,\omega(\varrho))\,d\varrho + h(\upsilon'(\sigma)) - \upsilon'(\sigma)\right\|.\end{aligned}$$

Then, clearly

$$\begin{aligned}&c\|\digamma(\omega,\upsilon) - \digamma(\omega,\upsilon')\| + a\|\digamma(\omega,\upsilon)-\upsilon\| + b\|\digamma(\omega,\upsilon')-\upsilon'\|\\ &\quad\geq 2\|\upsilon-\upsilon'\| + c\|h(\upsilon(\sigma)) - h(\upsilon'(\sigma))\|\\ &\quad\geq (2+\eta c)\|\upsilon-\upsilon'\|.\end{aligned}$$

All the conditions of Theorem 9 are satisfied to obtain $\omega \in B_\delta$ such that $\omega = \digamma(\omega,\omega)$, which is a solution of (C). We can obtain the following consequences.

In the following example we will use the above theorem and iterative scheme to find the approximate solution of the given integral equation.

Example 29 Consider the following integral equation;

$$\nu(t) = (t+e^{-t}) - t\sin(\nu(t)) + \int_0^t \frac{\sin(\nu(t))\nu(s)}{(s+e^{-s})}\,ds$$

a special kind of the above integral equation (D). Clearly, it satisfies all the conditions of the above theorem for $t \in [0,1]$. We have the following iterative sequence

$$\nu_{n+1}(t) = (t+e^{-t}) - t\sin(\nu_n(t)) + \int_0^t \frac{\sin(\nu_n(t))\nu_n(s)}{(s+e^{-s})}\,ds.$$

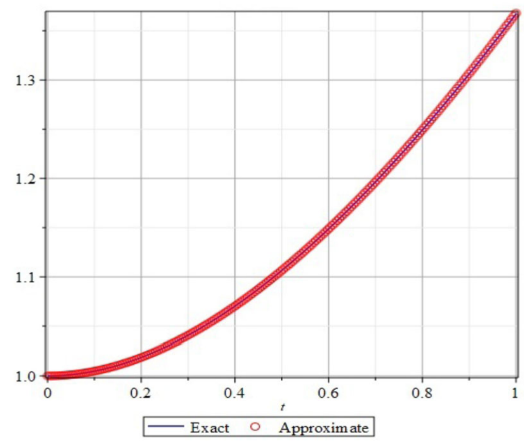

Figure 1 Exact and approximate solution

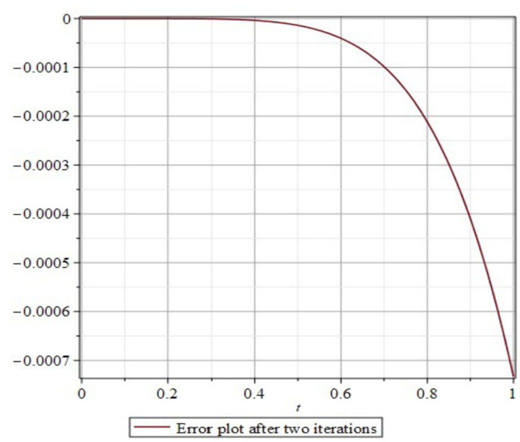

Figure 2 Error plot

We take an initial guess $\nu_0(t) = 1 + \frac{t^2}{2} - \frac{t^3}{6}$, after two iterations we have the approximate solution $\nu_2(t)$ and exact solution $\nu(t) = t + e^{-t}$ given in Fig. 1. The error is plotted in Fig. 2.

Example 30 Consider the mapping

$$\Upsilon x = 1 - kx, \quad x \in \left[0, \frac{1}{2}\right], \text{for some } k \leq 1.$$

Then, Υ is a generalized expansive mapping but not expansive. Being a closed subset of $\mathbb{R}$ with the usual metric, the set $[0, \frac{1}{2}]$ is complete. Also, Υ satisfies the following condition:

$$\begin{aligned} &c d_W(\Upsilon\omega, \Upsilon\nu) + a d_W(\Upsilon\omega, \omega) + b d_W(\Upsilon\nu, \nu) \\ &\quad \geq h d_W(\omega, \nu), \quad a, b \geq 0, c, h > 0, a + b + c < h, \end{aligned} \tag{E}$$

for $a = 1.1$, $b = 1.1$, $c = 1$ and $h = 3.21 > a + b + c$. Figure 3 shows the above inequality (E) holds for different values of k, but note that we can not find, $h > 1$ for $a = b = 0$, to show

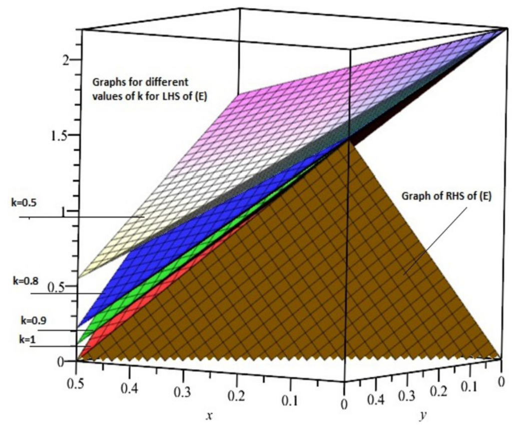

Figure 3 Inequality (E) for different values of k

that Υ is an expansive mapping. Therefore, Υ is not an expansive mapping. For $k = 1$ the mapping is neither expansive nor contractive.

Theorem 31 *Let ζ, h, k, g be as defined above, and $\digamma$ is defined in* (D). *If* (C1), (C2) *and* (C3) *are satisfied, there is a solution $\omega \in C(J, \mathbb{R})$ of integral equation* (C).

Remark 32 Comparing equations (B) with (D), we can choose specific values of ζ, h, k, g, in this case, the above theorem also provides the existence of solutions of the evolution equation (A).

4 Conclusion

In this article, we proved some new variants of Krasnoselskii's fixed-point theorem for equiexpansive and generalized equiexpansive mappings. We consider $\digamma$ a mapping with domain either $\Lambda(\Omega) \times \Omega$ or $\Omega \times \Omega$ and range in a Banach space W, where Λ is an operator from Ω into W. The operator equation $\omega = \digamma(\Lambda\omega, \omega)$ is solved with the assumption that $\digamma$ is either equiexpansive of generalized equiexpansive. We also apply our main results to obtain the solutions of a general class of integral equations, which represents solutions of many evolution equations of fractional order. Some examples are established to validate the results.

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Author details
[1]Department of Mathematics and Statistics, International Islamic University, H-10, Islamabad, Pakistan. [2]Department of Mathematics, Art and Science Faculty, Siirt University, 56100 Siirt, Turkey.

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