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Approximation properties of a new family of Gamma operators and their applications

Reyhan Özçelik¹, Emrah Evren Kara^{1*}, Fuat Usta¹ and Khursheed J. Ansari²

*Correspondence:

eevrenkara@duzce.edu.tr

¹Department of Mathematics,
Faculty of Science and Arts, Düzce
University, 81160, Düzce, Turkey
Full list of author information is
available at the end of the article

Abstract

The present paper introduces a new modification of Gamma operators that protects polynomials in the sense of the Bohman–Korovkin theorem. In order to examine their approximation behaviours, the approximation properties of the newly introduced operators such as Voronovskaya-type theorems, rate of convergence, weighted approximation, and pointwise estimates are presented. Finally, we present some numerical examples to verify that the newly constructed operators are an approximation procedure.

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1 Introduction

Approximation theory is one of the key topics within the framework of mathematical analysis and have been studied by a number of mathematicians. The emergence of this theory dates back to the 19th century. Approximation theory has become increasingly important in the scientific community as it sheds light on scientific problems in many other fields such as engineering. One of the first studies of this topic was by the Russian mathematician P.L. Chebyshev in 1853. While Chebyshev was working on the process of transforming the linear motion of the steam machine into the circular motion of a wheel, and there was no prior study on the existence of the approximation with polynomials, he was looking for a m -degree polynomial that gave the best approximation to an arbitrary continuous function given in a closed interval. With this study of Chebyshev, the best-approximation problem, which has an important place in the theory of approximation, gained meaning. Then, the German mathematician K. Weierstrass made a huge improvement in this field, proving his own name-bearing Weierstrass approximation theory, in 1885. This theorem proved the existence of a sequence of polynomials $\{P_m(x)\}$ that converges uniformly in the interval $[a, b]$ from the compact interval $[a, b]$ to every uniformly continuous function. As the proof of the Weierstrass theorem is so long and complex, a number of mathematicians have dealt with the proof of this theorem in different ways to make it simpler and more understandable [16]. One of the most important proofs of the Weierstrass theorem was given by Bernstein [2] in 1912. Thus, the famous Bernstein operator, whose importance

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is increasing day-by-day, emerged. This was the first step in approximation with positive linear operators. In the following years, positive linear operators were widely used to approximate continuous functions within a closed interval. The Voronovskaya theorem was first introduced by Voronovskaya in 1932 [19] to prove this theorem for Bernstein polynomials, which became the focus of a number of scientific studies. The results obtained in this way later attracted the attention of a number of mathematicians and were used in the construction of different types of positive linear operators. Subsequently, the question arose as to what are the necessary conditions for the sequence $\{L_n\}$ to converge properly to a continuous function. In 1952, Bohman [4] and in 1953 Korovkin [8] found the answer to this question independently. Their theorems show that a positive linear operator sequence can converge to the identity operator under certain conditions. In addition to these, some new operators have been introduced in the literature that led to studies on positive linear operators. One of these, by King [7] in 2003, generalised the Bernstein operator to preserve the function $e_2(x) = x^2$. In this study, King investigated the approximation properties of modified operators and proved that generalised operators have an approximation at least as good as classical Bernstein operators. The interested reader can be referred for instance to [6, 11–13, 15].

In the light of these developments, Lupaş and Müller [10] introduced the Gamma operator, which is one of the operators commonly used in approximation of unknown functions. In more detail, the general Gamma operator is introduced as follows:

$$T_n(g, x) = \frac{x^{n+1}}{\Gamma(n+1)} \int_0^\infty e^{-xv} v^n g\left(\frac{n}{v}\right) dv, \quad \forall x \in \mathbb{R}^+ := (0, \infty), n \in \mathbb{N}. \quad (1.1)$$

Then, other workers introduced various Gamma-type operators in the literature, [1, 3, 17, 18, 20]. The main purpose of this article is to present a new modification of Gamma-type operators, and provide their approximation properties. Finally, we will show that the newly defined operators are successful with some numerical examples.

The rest of this manuscript is constructed as follows. In Sect. 2, the new modification of the Gamma operators is introduced, along with fixing the polynomials. In Sect. 3, the Voronovskaya-type theorem of the new version of Gamma operators is examined, while in Sect. 4 the weighted approximation is reviewed. The rates of convergence are given in Sect. 5, while pointwise estimates are given in Sect. 6. Illustrative examples are discussed in Sect. 7, and in Sect. 8, we conclude the paper.

2 A new family of Gamma operators

In this section, a new modification of the Gamma-type operators and their fundamental approximation properties will be introduced. Throughout the article, $e_k(y) = y^k$ and $\varphi_{x,k}(y) = (y-x)^k$ $x \in (0, \infty)$, $k \in \mathbb{N}$ will be used as polynomial functions. On the other hand, let $C_b(\mathbb{R}^+)$ be the space of all real-valued uniformly continuous and bounded functions on \mathbb{R}^+ endowed with the norm $\|g\| = \sup\{|g| : x \in \mathbb{R}^+\}$.

The modified version of the classic Gamma operator that we are going to use is as follows:

$$T_n^*(g, x) = \frac{x^n}{\Gamma(n+1)} \int_0^\infty e^{-xv^{1/n}} g\left(\frac{n}{v^{1/n}}\right) dv, \quad \forall x \in \mathbb{R}^+, n \in \mathbb{N}. \quad (2.1)$$

It is clear that the newly introduced modified operator is positive and linear. On the other hand, the following equalities are readily obtained:

- (1) $\mathcal{T}_n^*(e_0(y), x) = e_0(x),$
- (2) $\mathcal{T}_n^*(e_1(y), x) = \frac{n}{n-1}e_1(x),$
- (3) $\mathcal{T}_n^*(e_2(y), x) = \frac{n^2}{(n-1)(n-2)}e_2(x),$
- (4) $\mathcal{T}_n^*(e_3(y), x) = \frac{n^3}{(n-1)(n-2)(n-3)}e_3(x),$
- (5) $\mathcal{T}_n^*(e_4(y), x) = \frac{n^4}{(n-1)(n-2)(n-3)(n-4)}e_4(x).$

As a result, we say that the newly introduced operators are an approximation procedure according to the Bohman–Korovkin theorem since the polynomials are preserved both in the limit case and directly. More generally, we present the following lemma without proof.

Lemma 1 *Let $x \in \mathbb{R}^+$ and $k \in \mathbb{N}$. In the circumstances, we have the following equality that is valid for $k \in \mathbb{Z}^+$:*

$$\mathcal{T}_n^*(e_k(y), x) = \frac{n^k \Gamma(n-k)}{\Gamma(n)} e_k(x).$$

Lemma 2 *Let $g \in C_b(\mathbb{R}^+)$. Then, we have*

$$\|\mathcal{T}_n^*(g)\| \leq \|g\|.$$

Proof Using the definition of the newly introduced Gamma operators and the values obtained above, the following inequality is readily obtained.

$$\begin{aligned} \|\mathcal{T}_n^*(g)\| &\leq \frac{x^n}{\Gamma(n+1)} \int_0^\infty e^{-xv^{1/n}} \left| g\left(\frac{n}{v^{1/n}}\right) \right| dv \\ &\leq \|g\| \frac{x^n}{\Gamma(n+1)} \int_0^\infty e^{-xv^{1/n}} dv \\ &= \|g\| \mathcal{T}_n^*(e_0(y), x) \\ &= \|g\|, \end{aligned}$$

which completes the proof. \square

Now, we can present the central moments of the newly constructed operator that will be used in the main theorems of the paper as follows.

Lemma 3 *Let $x \in \mathbb{R}^+$. In the circumstances, we obtained the following equalities for central moments:*

- (1) $\mathcal{T}_n^*(\varphi_{x,0}(y), x) = e_0(x),$
- (2) $\mathcal{T}_n^*(\varphi_{x,1}(y), x) = [\frac{1}{n-1}]e_1(x),$
- (3) $\mathcal{T}_n^*(\varphi_{x,2}(y), x) = [\frac{n+2}{(n-1)(n-2)}]e_2(x),$
- (4) $\mathcal{T}_n^*(\varphi_{x,3}(y), x) = [\frac{7n+6}{(n-1)(n-2)(n-3)}]e_3(x),$
- (5) $\mathcal{T}_n^*(\varphi_{x,4}(y), x) = [\frac{3n^2+46n+24}{(n-1)(n-2)(n-3)(n-4)}]e_4(x).$

Theorem 1 *Let $g \in C_b(\mathbb{R}^+)$. Then, we have*

$$\lim_{n \rightarrow \infty} \mathcal{T}_n^*(g, x) = g(x),$$

for uniformly in each compact subsets of \mathbb{R}^+ .

Proof With the aid of Lemma 1, one can easily obtain the following equality:

$$\lim_{n \rightarrow \infty} \mathcal{T}_n^*(e_k(y), x) = e_k(x),$$

for uniformly in each compact subset of \mathbb{R}^+ for $k = 0, 1, 2$. Then, according to the result of the Bohmans–Korovkin theorem, we deduce the $\lim_{n \rightarrow \infty} \mathcal{T}_n^*(g, x) = g(x)$ for uniformly in each compact subset of \mathbb{R}^+ . This completes the proof of the theorem. \square

3 Transferring the asymptotic formula

One of the fundamental challenges in approximation theory is the calculation of the rate of convergence of positive linear operators to the test functions. For this purpose, we will present and prove the Voronovskaya-type theorem to determine the asymptotic behaviour of newly constructed operators utilising well-recognised Taylor expansion.

Theorem 2 *Let g be bounded and integrable on the interval $x \in \mathbb{R}^+$, g' and g'' exist at a fixed point $x \in \mathbb{R}^+$, in this circumstance the following limit holds:*

$$\lim_{n \rightarrow \infty} n[\mathcal{T}_n^*(g, x) - g(x)] = xg'(x) + \frac{1}{2}x^2g''(x).$$

Proof First, starting with the well-recognised Taylor formula at $y = x$ of function g , we readily deduce that

$$g(y) = g(x) + g'(x)(y - x) + \frac{1}{2}g''(x)(y - x)^2 + \psi(y, x)(y - x)^2, \quad (3.1)$$

where

$$\psi(y, x) = \frac{g''(\xi) - g''(x)}{2}$$

such that ξ lying between, x and y and

$$\lim_{y \rightarrow x} \psi(y, x) = 0.$$

If we apply the new operator $(\mathcal{T}_n^*)_{n \geq 1}$ to the inequality (3.1), we easily obtained that,

$$\mathcal{T}_n^*(g, x) = g(x) + g'(x)\mathcal{T}_n^*((y - x), x) + \frac{1}{2}g''(x)\mathcal{T}_n^*((y - x)^2, x) + \mathcal{T}_n^*(\psi(y, x)(y - x)^2, x).$$

Multiplying each side of the equation here by n will result in the following equality:

$$\begin{aligned} n[\mathcal{T}_n^*(g, x) - g(x)] &= g'(x)n\mathcal{T}_n^*((y - x), x) + \frac{1}{2}g''(x)n\mathcal{T}_n^*((y - x)^2, x) \\ &\quad + n\mathcal{T}_n^*(\psi(y, x)(y - x)^2, x). \end{aligned}$$

If one states this expression in the limit case, we deduce that

$$\begin{aligned} \lim_{n \rightarrow \infty} n[\mathcal{T}_n^*(g, x) - g(x)] &= g'(x) \lim_{n \rightarrow \infty} n\mathcal{T}_n^*((y - x), x) + \frac{1}{2}g''(x) \lim_{n \rightarrow \infty} n\mathcal{T}_n^*((y - x)^2, x) \\ &\quad + \lim_{n \rightarrow \infty} n\mathcal{T}_n^*(\psi(y, x)(y - x)^2, x). \end{aligned}$$

As a consequence of our previous calculations in Lemma 3, the following two expressions can be easily obtained:

$$\lim_{n \rightarrow \infty} nT_n^*((y-x), x) = x, \quad \text{and} \quad \lim_{n \rightarrow \infty} nT_n^*((y-x)^2, x) = x^2.$$

Then, the following is obtained when we replace the information we have obtained above:

$$\lim_{n \rightarrow \infty} n[T_n^*(g, x) - g(x)] = xf'(x) + \frac{1}{2}x^2g''(x) + \lim_{n \rightarrow \infty} nT_n^*(\psi(y, x)\varphi_{x,2}(y), x). \quad (3.2)$$

Finally, if we show,

$$\lim_{n \rightarrow \infty} nT_n^*(\psi(y, x)\varphi_{x,2}(y), x),$$

equals zero, we can conclude the proof. Hence, by applying the well-known Cauchy–Schwarz inequality, we obtain

$$nT_n^*(\psi(y, x)\varphi_{x,2}(y), x) \leq \sqrt{n^2T_n^*(\psi^2(y, x), x)}\sqrt{T_n^*(\varphi_{x,4}(y), x)}. \quad (3.3)$$

Then, with the help of the Korovkin theorem, we can deduce that,

$$\lim_{n \rightarrow \infty} T_n^*(\psi^2(y, x), x) = \psi^2(x, x) = 0, \quad (3.4)$$

since $\psi^2(x, x) = 0$ and $\psi^2(\cdot, x)$ is continuous at $y \in \mathbb{R}^+$ and bounded as $y \rightarrow \infty$ and as $T_n^*(\varphi_{x,4}(y), x) = O(n^{-2})$. As a result, by substituting (3.3) and (3.4) into (3.2), the proof is completed. \square

4 Weighted approximation

After the computation of asymptotic formulae of the introduced operator, we can now provide the Korovkin-type theorem for a weighted approximation. For this purpose, we benefit from the results presented by Gadjevi in [5].

Initially, set $\sigma(x) = 1 + x^2$ as a weight function that is continuous on \mathbb{R} and the $\lim_{|x| \rightarrow \infty} \sigma(x) = \infty$, $\sigma(x) \geq 1$ for all $x \in \mathbb{R}^+$. Then, we shall denote by $C(\mathbb{R}^+)$ the set of all $\mathbb{R}^+ \rightarrow \mathbb{R}$ functions that are continuous. Then let us consider the following weighted spaces. For all $x \in \mathbb{R}^+$, the weighted space of real-valued functions g described on \mathbb{R} with the property $|g(x)| \leq M_g\sigma(x)$, where M_g is a constant depending on the function f defined as

$$B_\sigma(\mathbb{R}^+) = \{g : \mathbb{R}^+ \rightarrow \mathbb{R} : |g(x)| \leq M_g\sigma(x), x \in \mathbb{R}^+\},$$

and

$$C_\sigma(\mathbb{R}^+) = \{g \in B_\sigma(\mathbb{R}^+) : g \text{ is continuous on } \mathbb{R}\} = C(\mathbb{R}^+) \cap B_\sigma(\mathbb{R}^+).$$

These spaces are normed spaces with

$$\|g\|_\sigma = \sup_{x \in \mathbb{R}^+} \frac{|g(x)|}{\sigma(x)}.$$

Since σ is a weight function, $B_\sigma(\mathbb{R}^+)$ and $C_\sigma(\mathbb{R}^+)$ spaces are called weighted spaces. Additionally, if we set that κ_g is a constant dependent on the function g , we can define the following subspace:

$$C_\sigma^\kappa(\mathbb{R}^+) = \left\{ g \in C_\sigma(\mathbb{R}^+) : \lim_{|x| \rightarrow \infty} \frac{g(x)}{\sigma(x)} = \kappa_g \text{ exists and it is finite} \right\}.$$

which is a subspace of space $C_\sigma(\mathbb{R}^+)$. Now, we can provide the following lemma for the new operators.

Lemma 4 *Let $g \in C_\sigma(\mathbb{R}^+)$. Then, the following inequality holds*

$$\|\mathcal{T}_n^*(g)\|_\sigma \leq C\|g\|_\sigma,$$

for the modified operator $\mathcal{T}_n^*(g)$, which means that the sequence of the modified Gamma operators $\mathcal{T}_n^*(g)$ is an approximation process from $C_\sigma(\mathbb{R}^+)$ to $C_\sigma(\mathbb{R}^+)$.

Proof This lemma can be readily proven by using the definition of operators and the results of Lemma 3. Thus, the desired result has been obtained. \square

Now, we can present and prove the main theorem of this section by following Gadjiev's technique for an unbounded interval.

Theorem 3 *Let $g \in C_\sigma^\kappa(\mathbb{R}^+)$. Then, the following equality holds:*

$$\lim_{n \rightarrow \infty} \|\mathcal{T}_n^*(g) - g\|_\sigma = 0,$$

for the modified Gamma operators.

Proof Utilising Gadjiev's [5] theorem, it suffices to demonstrate that $\lim_{n \rightarrow \infty} \|\mathcal{T}_n^*(e_k) - e_k\|_\sigma = 0$ holds for $k = 0, 1, 2$. It is clear that the equation for $k = 0$, which is $\mathcal{T}_n^*(e_0(y)) = e_0(x)$ is initially provided. Secondly, using the result of Lemma 3 for $k = 1$, we readily deduce that,

$$\begin{aligned} \|\mathcal{T}_n^*(e_1) - e_1\|_\sigma &= \sup_{x \in \mathbb{R}^+} \frac{|\mathcal{T}_n^*(e_1) - e_1|}{1 + x^2} \\ &= \sup_{x \in \mathbb{R}^+} \frac{\left| \frac{n}{n-1}x - x \right|}{1 + x^2} \\ &\leq \left| \frac{1}{n-1} \right| \sup_{x \in \mathbb{R}^+} \frac{x}{1 + x^2} \\ &\leq \left| \frac{1}{n-1} \right|. \end{aligned}$$

If we take the limit of the above findings, one can readily express that $\lim_{n \rightarrow \infty} \|\mathcal{T}_n^*(e_1) - e_1\|_\sigma = 0$ as $\lim_{n \rightarrow \infty} \left| \frac{1}{n-1} \right| = 0$. Finally, we need to find an upper bound of $\lim_{n \rightarrow \infty} \|\mathcal{T}_n^*(e_2) - e_2\|_\sigma$.

$e_2\|_{\sigma}$. For that, we have,

$$\begin{aligned}\|\mathcal{T}_n^*(e_2) - e_2\|_{\sigma} &= \sup_{x \in \mathbb{R}^+} \frac{|\mathcal{T}_n^*(e_2) - e_2|}{1+x^2} \\ &= \sup_{x \in \mathbb{R}^+} \frac{\left| \frac{n^2}{(n-1)(n-2)}x^2 - x^2 \right|}{1+x^2} \\ &\leq \left| \frac{3n-2}{n^2-3n+2} \right| \sup_{x \in \mathbb{R}^+} \frac{x^2}{1+x^2} \\ &\leq \left| \frac{3n-2}{n^2-3n+2} \right|,\end{aligned}$$

is obtained in a similar way. In the limit case, we have the desired results, which concludes the proof. \square

5 Rate of convergence

In this section, we provide the convergence rate of the modified Gamma operator in terms of the modulus of continuity. Here, for the closed interval $[0, x_0]$, $x_0 \geq 0$, we denote the standard modulus of continuity of g by $\omega_{x_0}(g, \delta)$ and it can be defined as follows:

$$\omega_{x_0}(g, \delta) = \sup_{|y-x| \leq \delta; x, y \in [0, x_0]} |g(y) - g(x)|.$$

It is obvious that the modulus of continuity $\omega_{x_0}(g, \delta) \rightarrow 0$ as $\delta \rightarrow 0$ for the function $g \in C_b[0, \infty)$. Let us show the corresponding rate of convergence theorem for the newly constructed Gamma operator $(\mathcal{T}_n^*)_{n \geq 1}$. Now, we can provide the main theorem of this section.

Theorem 4 *Let $\omega_{x_0+1}(g, \delta)$ be the modulus of continuity on the finite interval $[0, x_0 + 1] \subset [0, \infty)$ for $x_0 > 0$ and $g \in C_b[0, \infty)$. In the circumstances, the following inequality holds:*

$$|\mathcal{T}_n^*(g, x) - g(x)| \leq 3M_g \left(\frac{n+2}{(n-1)(n-2)} \right) x_0^2 (1+x_0)^2 + 2\omega_{x_0+1} \left(g, \sqrt{\frac{n+2}{(n-1)(n-2)} x_0^2} \right),$$

where M_g is fixed just depending on g .

Proof Now, let $g \in C_b[0, \infty)$, $0 \leq x \leq x_0$ and $y > x_0 + 1$. then, we can deduce that

$$\begin{aligned}|g(y) - g(x)| &\leq |g(y)| + |g(x)| \\ &\leq M_g(\sigma(y) + \sigma(x)) \\ &= M_g(2 + y^2 + x^2) \\ &= M_g((y-x)^2 + 2x(y-x) + 2 + 2x^2) \\ &\leq M_g((y-x)^2 + 2x(y-x)^2 + 2(y-x)^2 + 2x^2(y-x)^2) \\ &= M_g(y-x)^2(2x^2 + 2x + 3) \\ &\leq M_g(y-x)^2(3x_0^2 + 6x_0 + 3) \\ &= 3M_g(y-x)^2(1+x_0)^2\end{aligned}$$

for $y - x > 1$. Then, again let $g \in C_b[0, \infty)$, $0 \leq x \leq x_0$. In the circumstances, the following inequality holds:

$$\begin{aligned} |g(y) - g(x)| &\leq \omega_{x_0+1}(g, |y - x|) \\ &\leq \omega_{x_0+1}(g, \delta) \left(1 + \frac{1}{\delta} |y - x|\right), \end{aligned}$$

for $y \leq x_0 + 1$. As a consequences, from the above inequalities, we deduce that

$$|g(y) - g(x)| \leq 3M_g(y - x)^2(1 + x_0)^2 + \omega_{x_0+1}(g, \delta) \left(1 + \frac{1}{\delta} |y - x|\right), \quad (5.1)$$

for $0 \leq x \leq x_0$ and $0 \leq y < \infty$. Applying \mathcal{T}_n^* and the Cauchy–Schwarz inequality to (5.1), we obtain

$$\begin{aligned} |\mathcal{T}_n^*(g, x) - g(x)| &\leq 3M_g \mathcal{T}_n^*((y - x)^2, x)(1 + x_0)^2 + \omega_{x_0+1}(g, \delta) \left(1 + \frac{1}{\delta} \sqrt{\mathcal{T}_n^*((y - x)^2, x)}\right) \\ &\leq 3M_g \left(\frac{n + 2}{(n - 1)(n - 2)}\right) x_0^2(1 + x_0)^2 + 2\omega_{x_0+1} \left(g, \sqrt{\frac{n + 2}{(n - 1)(n - 2)}} x_0^2\right), \end{aligned}$$

by choosing $\delta = \sqrt{\frac{n+2}{(n-1)(n-2)}} x_0^2$, which completes the proof. \square

6 Pointwise estimates

In this section, let us examine some pointwise estimates of the rates of convergence of the newly defined Gamma operators. First, the local approximation and the relationship between the local smoothness of g are given. For that, let us describe the following. Let $s \in (0, 1]$ and $Q \subset [0, \infty)$. In the circumstances, a function $g \in C_b[0, \infty)$ can be said $\text{Lip}_{M_g}(s)$ on Q if the following condition holds:

$$|g(y) - g(x)| \leq M_{g,s} |y - x|^s, \quad y \in [0, \infty) \text{ and } x \in Q,$$

where $M_{g,s}$ is fixed just depending on g and s .

Theorem 5 *Let $g \in C_b[0, \infty) \cap \text{Lip}_{M_g}(s)$ such that $s \in (0, 1]$ and $Q \subset [0, \infty)$ given as above. In the circumstances, we have the following inequality:*

$$|\mathcal{T}_n^*(g, x) - g(x)| \leq M_{g,s} \left[\left(\frac{n + 2}{(n - 1)(n - 2)} e_2(x) \right)^{s/2} + 2(d(x, Q))^s \right], \quad x \in (0, \infty),$$

where $M_{g,s}$ is defined as above and $d(x, Q)$ is the distance between x and Q described as

$$d(x, Q) = \inf\{|y - x|, y \in Q\}.$$

Proof Let us describe the closure of the set Q as \overline{Q} . Then, one can say that there exists at least one point $y_0 \in \overline{Q}$ such that

$$d(x, Q) = |x - y_0|.$$

Then, utilising the monotonicity properties of $(\mathcal{T}_n^*)_{n \geq 1}$, we deduce that

$$\begin{aligned} |\mathcal{T}_n^*(g, x) - g(x)| &\leq \mathcal{T}_n^*(|g(y) - g(y_0)|, x) + \mathcal{T}_n^*(|g(x) - g(y_0)|, x) \\ &\leq M_{g,s} [\mathcal{T}_n^*(|y - y_0|^s, x) + |x - y_0|^s] \\ &\leq M_{g,s} [\mathcal{T}_n^*(|y - x|^s, x) + 2|x - y_0|^s]. \end{aligned}$$

In the circumstances, with the help of the Hölder inequality, we obtain the following result:

$$\begin{aligned} |\mathcal{T}_n^*(g, x) - g(x)| &\leq M_{g,s} [(\mathcal{T}_n^*(|y - x|^2, x))^{s/2} + 2(d(x, Q))^s] \\ &= M_{g,s} \left[\left(\frac{n+2}{(n-1)(n-2)} e_2(x) \right)^{s/2} + 2(d(x, Q))^s \right], \end{aligned}$$

which finalises the proof. \square

Let us now calculate the local direct estimate of the new modification of Gamma operators. For this purpose, we need to review the Lipschitz-type maximal function of order s given in [9], that is

$$\tilde{\omega}_s(g, x) = \sup_{0 \leq y < \infty, y \neq x} \frac{|g(y) - g(x)|}{|y - x|^s},$$

where $s \in (0, 1]$ and $x \in (0, \infty)$. Now, we can present and prove the theorem.

Theorem 6 *Let $g \in C_b[0, \infty)$ and $s \in (0, 1]$, then the following inequality holds:*

$$|\mathcal{T}_n^*(g, x) - g(x)| \leq \tilde{\omega}_s(g, x) \mathcal{T}_n^* \left(\frac{n+2}{(n-1)(n-2)} e_2(x) \right)^{s/2},$$

for $x \in (0, \infty)$.

Proof Thanks to the definitions of $\tilde{\omega}_s(g, x)$ given above and a well-recognised Hölder inequality, we deduce that,

$$\begin{aligned} |\mathcal{T}_n^*(g, x) - g(x)| &\leq \mathcal{T}_n^*(|g(y) - g(x)|, x) \\ &\leq \tilde{\omega}_s(g, x) \mathcal{T}_n^*(|y - x|^s, x) \\ &\leq \tilde{\omega}_s(g, x) \mathcal{T}_n^*(|y - x|^2, x)^{s/2} \\ &\leq \tilde{\omega}_s(g, x) \mathcal{T}_n^* \left(\frac{n+2}{(n-1)(n-2)} e_2(x) \right)^{s/2}, \end{aligned}$$

thus, the desired result is obtained. \square

Finally, let us consider the following Lipschitz-type space with two parameters, $\alpha, \beta > 0$, such that

$$\text{Lip}_M^{\alpha, \beta}(s) = \left(g \in C[0, \infty) : |g(y) - g(x)| \leq M \frac{|y - x|^s}{(ax^2 + bx + y)^{s/2}}, x, y \in (0, \infty) \right),$$

introduced in [14], where $s \in (0, 1]$ and M is a positive constant.

Theorem 7 Let us consider $g \in \text{Lip}_M^{\alpha,\beta}(s)$ and $x \in (0, \infty)$. Then we have

$$|\mathcal{T}_n^*(g, x) - g(x)| \leq M \left[\frac{\frac{n+2}{(n-1)(n-2)} e_2(x)}{ax^2 + bx} \right]^{s/2},$$

where $\alpha, \beta > 0$.

Proof The proof of this inequality is shown in two steps. First, we take $s = 1$, that is,

$$\begin{aligned} |\mathcal{T}_n^*(g, x) - g(x)| &\leq \mathcal{T}_n^*(|g(y) - g(x)|, x) \\ &\leq M \mathcal{T}_n^*\left(\frac{|y - x|}{\sqrt{ax^2 + bx + y}}, x\right) \\ &\leq \frac{M}{\sqrt{ax^2 + bx}} \mathcal{T}_n^*(|y - x|, x), \end{aligned}$$

$g \in \text{Lip}_M^{\alpha,\beta}(1)$ and $x \in (0, \infty)$. Here, applying the Cauchy–Schwarz inequality, we deduce that,

$$|\mathcal{T}_n^*(g, x) - g(x)| \leq \frac{M}{\sqrt{ax^2 + bx}} [\mathcal{T}_n^*(|y - x|^2, x)]^{1/2} \leq M \left[\frac{\frac{n+2}{(n-1)(n-2)} e_2(x)}{ax^2 + bx} \right]^{1/2},$$

which confirms the proof of the theorem for $s = 1$. Then, let us consider $s \in (0, 1)$. For $g \in \text{Lip}_M^{\alpha,\beta}(s)$ and $x \in (0, \infty)$ we obtain that

$$|\mathcal{T}_n^*(g, x) - g(x)| \leq \frac{M}{(ax^2 + bx)^{s/2}} \mathcal{T}_n^*(|y - x|^s, x).$$

With the help of Hölder inequalities, we obtain the following inequality:

$$|\mathcal{T}_n^*(g, x) - g(x)| \leq \frac{M}{(ax^2 + bx)^{s/2}} \mathcal{T}_n^*(|y - x|^s, x) \leq \frac{M}{(ax^2 + bx)^{s/2}} (\mathcal{T}_n^*(|y - x|, x))^s.$$

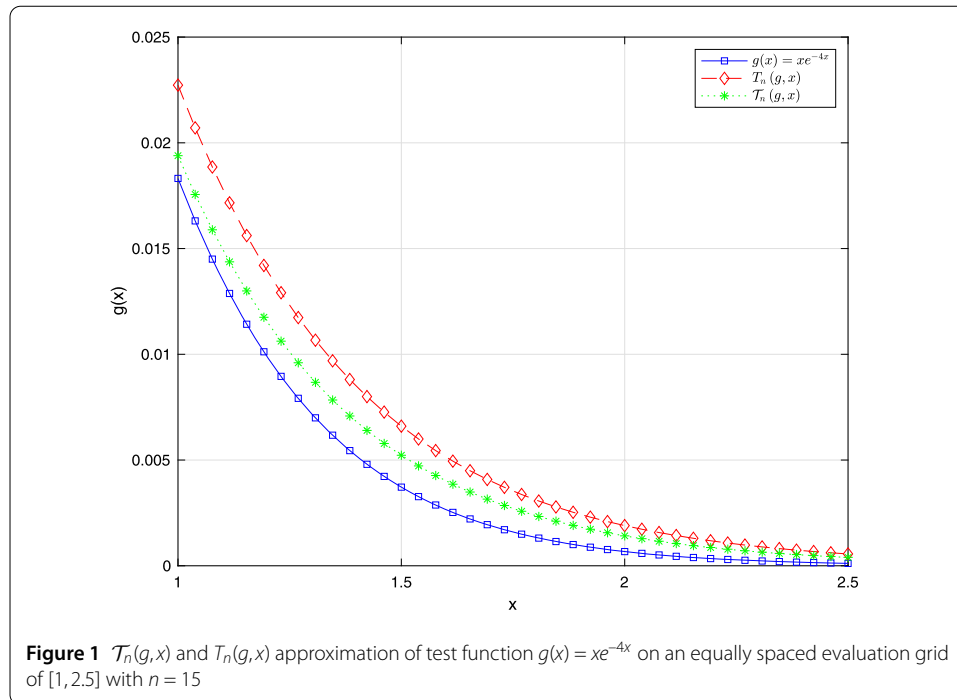
Finally, applying the Cauchy–Schwarz inequality, we have,

$$|\mathcal{T}_n^*(g, x) - g(x)| \leq \frac{M}{(ax^2 + bx)^{s/2}} (\mathcal{T}_n^*(|y - x|^2, x))^{s/2} \leq M \left[\frac{\frac{n+2}{(n-1)(n-2)} e_2(x)}{ax^2 + bx} \right]^{s/2},$$

which completes the proof. \square

7 Numerical examples

As applications, we give some numerical examples to verify the approximation properties of the newly defined Gamma operators in one dimension. In our examples, we compare new modifications of the Gamma operators with its classical correspondence. All of the calculations are performed on an Intel Core i7 personal laptop by running a code implemented by MATLAB 9.7.0.114230202 (R2019b) software. In order to clarify the accuracy and efficiency of the modified Gamma operators, the values of approximations are compared with the values of a test function by plotting them on the same figures.



7.1 Example 1

As a first example, we approximate the function $g : [1, 2.5] \rightarrow \mathbb{R}$ such that

$$g(x) = xe^{-4x},$$

for $n = 15$. Figure 1 shows that the newly defined Gamma operators approximate better in comparison with the classical correspondence for this interval. Of course, it is not possible to say that the newly defined operator is better than the existing operator under all circumstances. However, as can be seen in the examples, the new operator performed better in these selected specific cases.

7.2 Example 2

As a second example, we consider the test function $g : [1, 2.5] \rightarrow \mathbb{R}$ such that

$$g(x) = \cos(x)e^{-3x},$$

for $n = 15$. Similarly, Fig. 2 demonstrates that the introduced modified Gamma operators approximate better in comparison with the standard correspondence for this interval.

8 Concluding remarks

In this manuscript, a new modification of Gamma operators has been introduced and the fundamental properties of them have been analysed. For this purpose, we benefitted from different type function spaces. Finally, we provide a couple of numerical experiments to show the approximation properties of the newly defined operator.

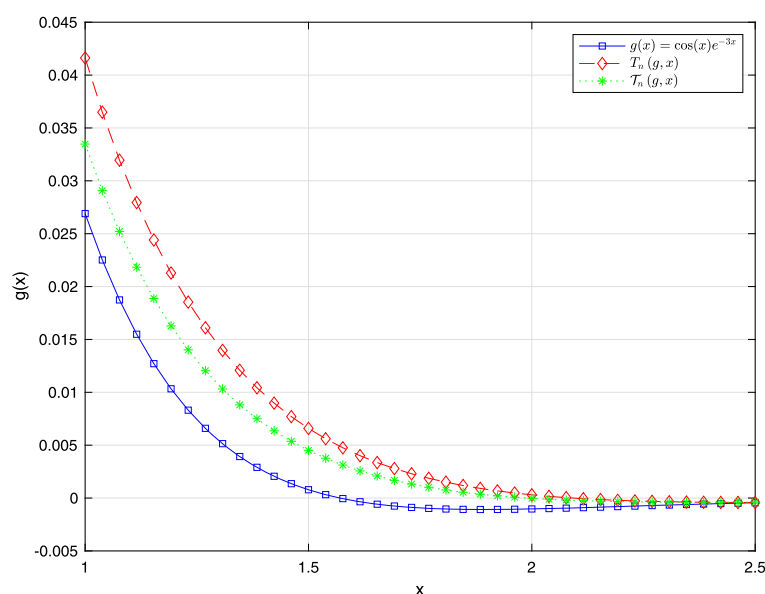


Figure 2 $T_n(g, x)$ and $T_n(g, x)$ approximation of test function $g(x) = \cos(x)e^{-3x}$ on an equally spaced evaluation grid of $[1, 2.5]$ with $n = 15$

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Declarations

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All the authors contributed equally to the writing of this paper. All the authors read and approved the final manuscript.

Author details

¹Department of Mathematics, Faculty of Science and Arts, Düzce University, 81160, Düzce, Turkey. ²Department of Mathematics, College of Science, King Khalid University, 61413 Abha, Saudi Arabia.

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