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Poly-central factorial sequences and poly-central-Bell polynomials

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Abstract

In this paper, we introduce poly-central factorial sequences and poly-central Bell polynomials arising from the polyexponential functions, reducing them to central factorials and central Bell polynomials of the second kind respectively when $k = 1$. We also show some relations: between poly-central factorial sequences and power of x; between poly-central Bell polynomials and power of x; between poly-central Bell polynomials and the poly-Bell polynomials; between poly-central Bell polynomials and higher order type 2 Bernoulli polynomials of second kind; recurrence formula of poly-central Bell polynomials.

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1 Introduction

The central factorial numbers of the first and second kinds consist of the same kind of reciprocity as the corresponding polynomials for the Stirling numbers of the first and second kinds [20]. They have appeared in many different contexts as follows: the approximation theory [2], algebraic geometry [6, 20], and spectral theory of differential operators [7, 18]. The poly-exponential functions were reconsidered by Kim [9] in view of an inverse to the polylogarithm functions which were first studied by Hardy [8]. Kim et al. showed that the degenerate Daehee numbers of order k expressed the degenerate polyexponential functions in [13]. Furthermore, recently, Kim and Kim introduced the poly-Bell polynomials and the poly-Lah–Bell polynomials arising from polyexponential functions respectively in [16, 17]. With this in mind, we introduce poly-central factorial sequences and poly-central Bell polynomials arising from the polyexponential functions, reducing them to central factorials and central Bell polynomials of the second kind respectively when $k = 1$. We also show some relations: between poly-central factorial sequences and power of x; between poly-central Bell polynomials and power of x; between poly-central Bell polynomials and the poly-Bell polynomials; between poly-central Bell polynomials and higher order type 2 Bernoulli polynomials of second kind; recurrence formula of poly-central Bell polynomials.

First, definitions and preliminary properties required in this paper are introduced.

The central factorial $x^{[n]}$ is defined by

$$\begin{aligned} &x^{[0]} = 1, \\ &x^{[n]} = x\left(x+\frac{n}{2}-1\right)\left(x+\frac{n}{2}-2\right)\cdots\left(x-\frac{n}{2}+1\right), \quad (n \geq 1), (\text{see [1, 3, 10, 11, 19]}). \end{aligned} \tag{1}$$

The central factorial $x^{[n]}$ is given by the generating function

$$\sum_{n=0}^{\infty} x^{[n]} \frac{t^n}{n!} = \left(\frac{t}{2}+\sqrt{1+\frac{t^2}{4}}\right)^{2x} \quad (\text{see [10, 11, 19]}). \tag{2}$$

For any nonnegative integer n, the central factorial numbers of the first kind are given by

$$x^{[n]} = \sum_{l=0}^{n} t(n,l)x^l \quad (\text{see [20]}). \tag{3}$$

By (3), we easily get

$$\frac{1}{l!}\left(2\log\left(\frac{t}{2}+\sqrt{1+\frac{t^2}{4}}\right)\right)^l = \sum_{n=l}^{\infty} t(n,l)\frac{t^n}{n!} \quad (\text{see [10, 11, 19]}), \tag{4}$$

where $t \in \mathbb{C}$ with $|t| < 1$.

Let $f(t) = 2\log(\frac{t}{2}+\sqrt{1+\frac{t^2}{4}})$. Then

$$f^{-1}(t) = e^{\frac{t}{2}} - e^{-\frac{t}{2}}, \tag{5}$$

which is the compositional inverse of the function $f(t)$.

From (4) and (5), the central factorial numbers of the second kind are given by

$$\frac{1}{l!}\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^l = \sum_{n=l}^{\infty} T(n,l)\frac{t^n}{n!} \quad (\text{see [10, 11, 19]}). \tag{6}$$

Riordan showed that the central factorial numbers of the second kind $T(n,k)$ are the coefficients in the expansion of x^n in terms of central factorials given by

$$x^n = \sum_{l=0}^{n} T(n,l)x^{[l]} \quad (n \geq l \geq 0), (\text{see [19]}). \tag{7}$$

Kim and Kim introduced the central Bell polynomials $B_n^{(c)}(x)$ defined by

$$B_n^{(c)}(x) = \sum_{k=0}^{n} x^k T(n,k), \quad (n \geq 0), (\text{see [11]}), \tag{8}$$

and the central Bell numbers $B_n^{(c)}$ by $B_n^{(c)}(1)$, so that

$$B_n^{(c)} = \sum_{k=0}^{n} T(n,k), \quad (n \geq 0).$$

From (6) and (8), we note that the generating function for the central Bell polynomials is

$$e^{x(e^{\frac{t}{2}}-e^{-\frac{t}{2}})} = \sum_{n=0}^{\infty} B_n^{(c)}(x)\frac{t^n}{n!} \quad \text{(see [11])}, \tag{9}$$

where $B_n^{(c)}(x) = \sum_{k=0}^{n} T(n,k)x^k$ are the central polynomials Bell polynomials and $B_n^{(c)} = B_n^{(c)}(1)$ are the central Bell numbers.

For $n \geq 0$, the Stirling numbers of the first kind $S_1(n,l)$ are the coefficients of x^l in

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l \quad \text{(see [3, 5, 15])}. \tag{10}$$

From (10), it is easy to see that

$$\frac{1}{k!}\bigl(\log(1+t)\bigr)^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}. \tag{11}$$

In the inverse expression to (10), for $n \geq 0$, the nth power of x can be expressed in terms of the Stirling numbers of the second kind $S_2(n,l)$ as follows:

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l \quad \text{(see [3, 5, 15])}. \tag{12}$$

From (12), it is easy to see that

$$\frac{1}{k!}\bigl(e^t-1\bigr)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}. \tag{13}$$

The nth Bell number B_n $(n \geq 0)$ is the number of ways to partition a set with n elements into nonempty subsets. The Bell polynomials are natural extensions of the Bell numbers as follows:

$$bel_n(x) = \sum_{k=0}^{n} S_2(n,k)x^n. \tag{14}$$

It is well known that the generating function of the Bell polynomials is given by

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} bel_n(x)\frac{t^n}{n!} \quad \text{(see [3, 5, 16])}. \tag{15}$$

Now, as well established within academia, the ordinary Bernoulli polynomials $b_n(x)$ and the Euler polynomials $E_n(x)$, $(n \in \mathbb{N} \cup \{0\})$ are respectively defined by their generating functions as follows (see [3, 5]):

$$\left(\frac{t}{e^t-1}\right)e^{xt}=\sum_{n=0}^{\infty}b_n(x)\frac{t^n}{n!},\qquad \frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}. \tag{16}$$

When $x=0$, $b_n=b_n(0)$ and $E_n=E_n(0)$ are respectively called the Bernoulli numbers and the Euler numbers.

For $r\in\mathbb{R}$, the type 2 Bernoulli polynomials of the second kind with order r are defined by the generating function

$$\left(\frac{(1+t)-(1+t)^{-1}}{2\log(1+t)}\right)^r(1+t)^x=\sum_{n=0}^{\infty}b_n^{*(r)}(x)\frac{t^n}{n!}\quad \text{(see [12])}. \tag{17}$$

When $x=0$, $b_n^{*(r)}(0)$ are called the type 2 Bernoulli numbers of the second kind order r.

Kim and Kim introduced the modified polyexponential function as

$$J_k(x)=\sum_{n=1}^{\infty}\frac{x^n}{(n-1)!n^k}\quad (k\in\mathbb{Z}), \text{(see [9, 14])}. \tag{18}$$

When $k=1$, we see that $\mathrm{Ei}_1(x)=e^x-1$.

Recently, the poly-Bell polynomials were introduced by

$$1+J_k\big(x\big(e^t-1\big)\big)=\sum_{n=0}^{\infty}bel_n^{(k)}(x)\frac{t^n}{n!}\quad \text{(see [16])}, \tag{19}$$

and $bel_0^{(k)}(x)=1$.

When $k=1$, from (15), we note that

$$\begin{aligned}1+J_1\big(x\big(e^t-1\big)\big)&=1+\sum_{n=1}^{\infty}\frac{(x(e^t-1))^n}{(n-1)!n}\\&=\exp\big(x\big(e^t-1\big)\big)=\sum_{n=0}^{\infty}bel_n(x)\frac{t^n}{n!}.\end{aligned} \tag{20}$$

From (20), we have

$$bel_n^{(1)}(x)=bel_n(x).$$

2 Poly-central factorial sequences and poly-central-Bell polynomials

In this section, we define poly-central factorial sequences and poly-central-Bell polynomials respectively by using the degenerate polylogarithm functions and give explicit expressions and recurrence formula of poly-central Bell polynomials.

First, we consider the poly-central factorial sequences $x^{[n](k)}$, which are derived from the polyexponential function to be

$$1+J_k\left(2x\log\left(\frac{t}{2}+\sqrt{1+\frac{t^2}{4}}\right)\right)=\sum_{n=0}^{\infty}x^{[n](k)}\frac{t^n}{n!}\quad\text{and}\quad x^{[0](k)}=1. \tag{21}$$

When $k=1$, since $J_1(x)=e^x-1$, we note that

$$1+J_1\left(2x\log\left(\frac{t}{2}+\sqrt{1+\frac{t^2}{4}}\right)\right)=\left(\frac{t}{2}+\sqrt{1+\frac{t^2}{4}}\right)^{2x}=\sum_{n=0}^{\infty}x^{[n]}\frac{t^n}{n!}. \tag{22}$$

Therefore, by (22), we have $x^{[n](1)}=x^{[n]}$.

Second, we define the poly-central-Bell polynomials $B_n^{(c,k)}(x)$, which arise from the poly-exponential function to be

$$1+J_k\left(x\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)\right)=\sum_{n=0}^{\infty}B_n^{(c,k)}(x)\frac{t^n}{n!} \quad \text{and} \quad B_0^{(c,k)}(x)=1. \tag{23}$$

When $k=1$, since $J_1(x)=e^x-1$, we note that

$$1+J_1\left(x\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)\right)=1+e^{x(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}-1=\sum_{n=0}^{\infty}B_n^{(c)}(x)\frac{t^n}{n!}. \tag{24}$$

By (24), we have $B_n^{(c,1)}(x)=B_n^{(c)}(x)$.

First, we observe relations of poly-falling factorial sequences and powers of x.

Theorem 1 *For $k\in\mathbb{Z}$ and $n\geq 1$, we have*

$$x^{[n](k)}=\sum_{l=1}^{n}\frac{1}{l^{k-1}}t(n,l)x^l,$$

where $t(n,l)$ is the central factorial numbers of the first kind.

Proof By (4) and (21), we observe that

$$\begin{aligned}
1+J_k\left(2x\log\left(\frac{t}{2}+\sqrt{1+\frac{t^2}{4}}\right)\right)&=1+\sum_{l=1}^{\infty}\frac{(2x)^l(\log(\frac{t}{2}+\sqrt{1+\frac{t^2}{4}}))^l}{(l-1)!l^k}\\
&=1+\sum_{l=1}^{\infty}\frac{x^l}{l^{k-1}}\frac{1}{l!}\left(2\log\left(\frac{t}{2}+\sqrt{1+\frac{t^2}{4}}\right)\right)^l\\
&=1+\sum_{l=1}^{\infty}\frac{x^l}{l^{k-1}}\sum_{n=l}^{\infty}t(n,l)\frac{t^n}{n!}\\
&=1+\sum_{n=1}^{\infty}\left(\sum_{l=1}^{n}\frac{1}{l^{k-1}}t(n,l)x^l\right)\frac{t^n}{n!}.
\end{aligned} \tag{25}$$

Combining (21) with (25), we get the desired result. □

In Theorem 1, when $k=1$, we note that

$$x^{[n]}=\sum_{l=0}^{n}t(n,l)x^l.$$

Theorem 2 *For $k \in \mathbb{Z}$ and $n \geq 1$, we have*

$$x^n = n^{k-1} \sum_{l=0}^{n} T(n,l) x^{[l](k)}.$$

Proof By replacing t with $e^{\frac{t}{2}} - e^{-\frac{t}{2}}$ in (21), from (18), the left-hand side of (21) is

$$1 + J_k(xt) = 1 + \sum_{n=1}^{\infty} \frac{x^n t^n}{(n-1)! n^k} = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n^{k-1}} \frac{t^n}{n!}. \tag{26}$$

On the other hand, from (6), the right-hand side of (21) is

$$\begin{aligned} \sum_{m=0}^{\infty} x^{[m](k)} \frac{(e^{\frac{t}{2}} - e^{-\frac{t}{2}})^m}{m!} &= \sum_{m=0}^{\infty} x^{[m](k)} \sum_{n=m}^{\infty} T(n,m) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} x^{[m](k)} T(n,m) \frac{t^n}{n!}. \end{aligned} \tag{27}$$

Comparing with the coefficients of (26) and (27), we have the desired result. □

In Theorem 2, when $k = 1$, we note that

$$x^n = \sum_{l=0}^{n} t(n,l) x^{[l](k)}.$$

Theorem 3 *For $k \in \mathbb{Z}$ and $n \geq 1$, we have*

$$B_n^{(c,k)}(x) = \sum_{l=1}^{n} \frac{1}{l^{k-1}} T(n,l) x^l.$$

Proof From (6) and (23), we observe that

$$\begin{aligned} 1 + J_k\left(x\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)\right) &= 1 + \sum_{l=1}^{\infty} \frac{x^l (e^{\frac{t}{2}} - e^{-\frac{t}{2}})}{(l-1)! l^k} \\ &= 1 + \sum_{l=1}^{\infty} \frac{x^l}{l^{k-1}} \frac{(e^{\frac{t}{2}} - e^{-\frac{t}{2}})^l}{l!} \\ &= 1 + \sum_{l=1}^{\infty} \frac{x^l}{l^{k-1}} \sum_{n=l}^{\infty} T(n,l) \frac{t^n}{n!} \\ &= 1 + \sum_{n=1}^{\infty} \left(\sum_{l=1}^{n} \frac{1}{l^{k-1}} T(n,l) x^l \right) \frac{t^n}{n!}. \end{aligned} \tag{28}$$

Combining (23) with (28), we get

$$B_n^{(c,k)}(x) = \sum_{l=1}^{n} \frac{1}{l^{k-1}} T(n,l) x^l \quad (n \geq 1).$$

□

In Theorem 3, when $k=1$, we note that

$$B_n^{(c)}(x)=\sum_{l=0}^{n}T(n,l)x^l.$$

Theorem 4 *For $k\in\mathbb{Z}$ and $n\geq 1$, we have*

$$B_n^{(c,k)}(x)=\sum_{m=1}^{n}\sum_{l=1}^{m}\binom{n}{m}\left(-\frac{1}{2}\right)^{n-m}l^{n-m-k+1}S_2(m,l)x^l.$$

Proof From (13) and (23), we observe that

$$\begin{aligned}1+J_k\left(x\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)\right)&=1+\sum_{l=1}^{\infty}\frac{x^le^{-\frac{t}{2}l}(e^t-1)^l}{(l-1)!l^k}\\&=1+\sum_{l=1}^{\infty}\frac{x^le^{-\frac{t}{2}l}}{l^{k-1}}\sum_{m=l}^{\infty}S_2(m,l)\frac{t^m}{m!}\\&=1+\sum_{m=1}^{\infty}\sum_{l=1}^{m}\frac{1}{l^{k-1}}S_2(m,l)x^l\frac{t^m}{m!}\sum_{j=0}^{\infty}\left(-\frac{l}{2}\right)^j\frac{t^j}{j!}\\&=1+\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\sum_{l=1}^{m}\binom{n}{m}\frac{1}{l^{k-1}}\left(-\frac{1}{2}l\right)^{n-m}S_2(m,l)x^l\right)\frac{t^n}{n!}.\end{aligned}\tag{29}$$

Combining (23) with (29), we have the desired result. □

In Theorem 4, when $k=1$, we note that

$$B_n^{(c)}(x)=\sum_{m=1}^{n}\sum_{l=1}^{m}\binom{n}{m}\left(-\frac{1}{2}l\right)^{n-m}S_2(m,l)x^l.$$

Theorem 5 *For $k\in\mathbb{Z}$ and $n\geq 1$, we have*

$$B_n^{(c,k)}(x)=\sum_{l=1}^{n}\binom{n}{l}\left(-\frac{1}{2}l\right)^{n-l}bel_{n-l}^{(k)}(x),$$

where $bel_n^{(k)}(x)$ are the poly-Bell polynomials.

Proof From (19) and (23), we get

$$\begin{aligned}1+J_k\left(x\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)\right)&=1+J_k\left(xe^{-\frac{t}{2}}\left(e^t-1\right)\right)=1+\sum_{l=1}^{\infty}\frac{x^le^{-\frac{t}{2}}(e^t-1)^l}{(l-1)!l^k}\\&=1+\sum_{l=1}^{\infty}\frac{(x(e^t-1))^l}{(l-1)!l^k}\sum_{m=0}^{\infty}\left(-\frac{1}{2}l\right)^m\frac{t^m}{m!}\\&=1+\sum_{l=1}^{\infty}bel_l^{(k)}(x)\frac{t^l}{l!}\sum_{m=0}^{\infty}\left(-\frac{1}{2}l\right)^m\frac{t^m}{m!}\\&=1+\sum_{n=1}^{\infty}\sum_{l=1}^{n}\binom{n}{l}\left(-\frac{1}{2}l\right)^{n-l}bel_{n-l}^{(k)}(x)\frac{t^n}{n!}.\end{aligned}\tag{30}$$

Combining (23) with (30), we have the desired result. □

Theorem 6 *For $k \in \mathbb{Z}$ and $n \geq 1$, we have*

$$\sum_{j=1}^{n} 2^j S_1(n,j) B_j^{(c,k)}(x) = \sum_{m=1}^{n} \sum_{l=1}^{m} \binom{n}{m} \frac{2^l x^l}{l^{k-1}} S_1(m,l) b_{n-m}^{*(l)},$$

where $b_n^{*(l)}$ are the order l type 2 Bernoulli polynomials of the second kind.

Proof By replacing t with $2\log(1+t)$ in (23), from (11) and (17), the left-hand side of (23) is

$$\begin{aligned} 1 + J_k\bigl(x\bigl((1+t)-(1+t)^{-1}\bigr)\bigr) &= 1 + \sum_{l=1}^{\infty} \frac{x^l((1+t)-(1+t)^{-1})^l}{(l-1)!l^k} \\ &= 1 + \sum_{l=1}^{\infty} \frac{2^l x^l}{l^{k-1}} \left(\frac{(1+t)-(1+t)^{-1}}{2\log(1+t)} \right)^l \frac{\log(1+t)^l}{l!} \\ &= 1 + \sum_{l=1}^{\infty} \frac{2^l x^l}{l^{k-1}} S_1(m,l) \frac{t^m}{m!} \sum_{j=0}^{\infty} b_j^{*(l)} \frac{t^j}{j!} \\ &= 1 + \sum_{m=1}^{\infty} \sum_{l=1}^{m} \frac{2^l x^l}{l^{k-1}} S_1(m,l) \frac{t^m}{m!} \sum_{j=0}^{\infty} b_j^{*(l)} \frac{t^j}{j!} \\ &= 1 + \sum_{n=1}^{\infty} \sum_{m=1}^{n} \sum_{l=1}^{m} \binom{n}{m} \frac{2^l x^l}{l^{k-1}} S_1(m,l) b_{n-m}^{*(l)} \frac{t^n}{n!}. \end{aligned} \tag{31}$$

On the other hand, by (11), the right-hand side of (23) is

$$\begin{aligned} \sum_{m=0}^{\infty} B_m^{(c,k)}(x) \frac{(2\log(1+t))^m}{m!} &= \sum_{m=0}^{\infty} 2^m B_m^{(c,k)}(x) \sum_{n=m}^{\infty} S_1(n,m) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} 2^m S_1(n,m) B_m^{(c,k)}(x) \frac{t^n}{n!} \\ &= 1 + \sum_{n=1}^{\infty} \sum_{m=0}^{n} 2^m S_1(n,m) B_m^{(c,k)}(x). \end{aligned} \tag{32}$$

Since $S_1(n,0) = 0$ for $n \geq 1$, by comparing with the coefficients of (31) and (32), we have the desired result. □

Theorem 7 *For $k \in \mathbb{Z}$ and $n \geq 1$, we have*

$$\sum_{j=0}^{n} \binom{n}{j} b_j B_{n-j+1}^{(c,k)}(x) = \sum_{m=0}^{n} \binom{n}{m} E_{n-m} B_{m+1}^{(c,k)}(x),$$

where b_n are the ordinary Bernoulli numbers and E_n are the ordinary Euler numbers.

Proof Differentiating with respect to t in (23), the left-hand side of (23) is

$$\begin{aligned}\frac{\partial}{\partial t}\left(1+J_k\left(x\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)\right)\right)&=\sum_{n=1}^{\infty}\frac{x^n(e^{\frac{t}{2}}-e^{-\frac{t}{2}})^{n-1}}{(n-1)!n^{k-1}}\frac{1}{2}\left(e^{\frac{t}{2}}+e^{-\frac{t}{2}}\right)\\&=\frac{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}{2(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\sum_{n=1}^{\infty}\frac{x^n(e^{\frac{t}{2}}-e^{-\frac{t}{2}})^{n}}{(n-1)!n^{k-1}}\\&=\frac{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}{2(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}J_{k-1}\left(x\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)\right)\\&=\frac{e^t+1}{2(e^t-1)}\sum_{n=1}^{\infty}B_n^{(c,k)}(x)\frac{t^n}{n!}.\end{aligned}\tag{33}$$

On the other hand, the right-hand side of (23) is

$$\frac{\partial}{\partial t}\left(\sum_{n=0}^{\infty}B_n^{(c,k)}(x)\frac{t^n}{n!}\right)=\sum_{n=0}^{\infty}B_{n+1}^{(c,k)}(x)\frac{t^n}{n!}.\tag{34}$$

Combining (33) with (34), we get

$$\frac{1}{e^t-1}\sum_{n=1}^{\infty}B_n^{(c,k)}(x)\frac{t^n}{n!}=\frac{2}{e^t+1}\sum_{n=0}^{\infty}B_{n+1}^{(c,k)}(x)\frac{t^n}{n!}.\tag{35}$$

From (16) and (35), we have

$$\sum_{j=0}^{\infty}b_j\frac{t^j}{j!}\sum_{m=0}^{\infty}\frac{1}{m+1}B_{m+1}^{(c,k)}(x)\frac{t^m}{m!}=\sum_{i=0}^{\infty}E_i\frac{t^i}{i!}\sum_{m=0}^{\infty}B_{m+1}^{(c,k)}(x)\frac{t^m}{m!}.\tag{36}$$

By (36), we have

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}b_jB_{n-j+1}^{(c,k)}(x)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}E_{n-m}B_{m+1}^{(c,k)}(x)\frac{t^n}{n!}.\tag{37}$$

By comparing with the coefficients of both sides of (37), we have the desired result. □

3 Further remark

Let $r\in\mathbb{N}\cup\{0\}$, the r-Stirling numbers $S_{2,r}(n,j)$ of the second kind are given by

$$\frac{1}{j!}e^{rt}\left(e^t-1\right)^j=\sum_{n=j}^{\infty}S_{2,r}(n+r,j+r)\frac{t^n}{n!}\quad\text{(see [15])}.\tag{38}$$

In view of (14), the r-Bell polynomials are given by

$$bel_n^{(r)}(x)=\sum_{j=0}^{n}S_{2,r}(n+r,j+r)x^j\quad(n\geq 0),\text{(see [15])}.\tag{39}$$

From (38), it is easy to show that the generating function of degenerate r-Bell polynomials is given by

$$e^{rt}e^{x(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} bel_{n,r}(x|\lambda)\frac{t^n}{n!} \quad \text{(see [14])}. \tag{40}$$

When $x = 1$, $bel_n^{(r)}(\lambda) = bel_n^{(r)}(1|\lambda)$ which are called the degenerate r-Bell numbers.

We can define the extended poly-Bell polynomials $\mathfrak{Bel}_{n,\lambda}^{(k)}(x)$, which are derived from the polyexponential function to be

$$1 + J_k\big(x\big(e^t - 1\big) + rt\big) = \sum_{n=0}^{\infty} \mathfrak{Bel}_n^{(k)}(x)\frac{t^n}{n!} \quad \text{and} \quad \mathfrak{Bel}_0^{(k)}(x) = 1. \tag{41}$$

When $x = 1$, $\mathfrak{Bel}_n^{(k)} = \mathfrak{Bel}_n^{(k)}(1)$ which are called the extended poly-Bell numbers.

When $k = 1$, we note that

$$1 + J_k\big(x\big(e^t - 1\big) + rt\big) = 1 + \sum_{n=1}^{\infty} \frac{(x(e^t-1)+rt)^n}{n!} = e^{x(e^t-1)+rt} = e^{x(e^t-1)}e^{rt} = bel_{n,r}(x).$$

In particular, when $r = -\frac{1}{2}$, $\mathfrak{Bel}_{n,-\frac{1}{2}}(x) = B_n^{(c)}(x)$.

Theorem 8 *For $n \geq 1$, we have*

$$\mathfrak{Bel}_{n,r}^{(k)}(x) = \sum_{l=0}^{m} \binom{n}{m-l}\frac{r^{m-l}}{m^{k-1}} S_2(n+l-m,l).$$

Proof From (13) and (41), we observe that

$$\begin{aligned}
1 + J_k\big(x\big(e^t - 1\big) + rt\big) &= 1 + \sum_{m=1}^{\infty} \frac{(x(e^t-1)+rt)^m}{(m-1)!m^k} \\
&= 1 + \sum_{m=1}^{\infty} \frac{1}{m^{k-1}m!}\sum_{l=0}^{m}\binom{m}{l}x^l\big(e^t-1\big)^l(rt)^{m-l} \\
&= 1 + \sum_{m=1}^{\infty}\sum_{l=0}^{m} \frac{r^{m-l}x^l}{m^{k-1}(m-l)!}t^{m-l}\frac{(e^t-1)^l}{l!} \\
&= 1 + \sum_{m=1}^{\infty}\sum_{l=0}^{m} \frac{r^{m-l}x^l}{m^{k-1}(m-l)!}\sum_{n=l}^{\infty}S_2(n,l)\frac{t^{n+m-l}}{n!} \\
&= 1 + \sum_{m=1}^{\infty}\sum_{l=0}^{m} \frac{r^{m-l}x^l}{m^{k-1}(m-l)!}\sum_{n=0}^{\infty}S_2(n+l,l)\frac{t^{n+m}}{(n+l)!} \\
&= 1 + \sum_{m=1}^{\infty}\sum_{l=0}^{m} \frac{r^{m-l}x^l}{m^{k-1}(m-l)!}\sum_{n=m}^{\infty}S_2(n+l-m,l)\frac{t^{n}}{(n+l-m)!} \\
&= 1 + \sum_{n=1}^{\infty}\sum_{m=1}^{n}\sum_{l=0}^{m}\binom{n}{m-l}\frac{r^{m-l}}{m^{k-1}}S_2(n+l-m,l)\frac{t^n}{n!}.
\end{aligned} \tag{42}$$

By (41) and (42), we get what we want. □

4 Conclusion

To summarize, we introduced poly-central factorial sequences and poly-central Bell polynomials in terms of the polyexponential functions, reducing them to central factorials and central Bell polynomials of the second kind respectively when $k = 1$. We derived relations between poly-central factorial sequences and power of x in Theorems 1, 2. We also obtained relations between poly-central Bell polynomials and power of x in Theorems 3, 4. In addition, we showed several identities between poly-central Bell polynomials and poly-Bell polynomials; between poly-central Bell polynomials and higher order type 2 Bernoulli polynomials of second kind; recurrence formula of poly-central Bell polynomials in Theorems 5, 6, 7.

To conclude, there are various methods for studying special polynomials and numbers, including: generating functions, combinatorial methods, umbral calculus, differential equations, and probability theory [2, 4–7, 18–20]. We are now interested in continuing our research into the application of 'poly' versions of certain special polynomials and numbers in the fields of physics, science, and engineering as well as mathematics.

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Authors' contributions
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