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On a degenerate parabolic equation from double phase convection

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Abstract

The initial-boundary value problem of a degenerate parabolic equation arising from double phase convection is considered. Let $a(x)$ and $b(x)$ be the diffusion coefficients corresponding to the double phase respectively. In general, it is assumed that $a(x) + b(x) > 0$, $x \in \overline{\Omega}$ and the boundary value condition should be imposed. In this paper, the condition $a(x) + b(x) > 0$, $x \in \overline{\Omega}$ is weakened, and sometimes the boundary value condition is not necessary. The existence of a weak solution u is proved by parabolically regularized method, and $u_t \in L^2(Q_T)$ is shown. The stability of weak solutions is studied according to the different integrable conditions of $a(x)$ and $b(x)$. To ensure the well-posedness of weak solutions, the classical trace is generalized, and that the homogeneous boundary value condition can be replaced by $a(x)b(x)|_{x\in\partial\Omega} = 0$ is found for the first time.

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1 Introduction

Consider the parabolic equation with a nonlinear convective term

$$u_t = \operatorname{div}\big(a(x)|\nabla u|^{p-2}\nabla u + b(x)|\nabla u|^{q-2}\nabla u\big) + \sum_{i=1}^{N} \frac{\partial f_i(u,x,t)}{\partial x_i}, \quad (x,t) \in Q_T, \tag{1.1}$$

which arises from the double phase problems, as well as from the flows of incompressible turbulent fluids etc. [3]. In this paper, $Q_T = \Omega \times (0,T)$, Ω is a smooth bounded domain in $\mathbb{R}^N$, $p, q > 1$, $a(x), b(x) \in C(\overline{\Omega})$, $f_i(s,x,t)$ is a Lipschitz function when $|s|$ is bounded.

Though the initial-boundary value problem of the non-Newtonian fluid equation

$$u_t = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big), \quad (x,t) \in Q_T, \tag{1.2}$$

has been studied far and widely [12, 13, 24], as a generalized case, equation (1.1) has not provoked researchers' attention until recent years. Since the authors of [20] pointed out that the methods used in studying the well-posedness problem of equation (1.2) are invalid for equation (1.1), more and more works related to equation (1.1) have appeared, one can refer to [4–6, 8–10, 35]. First of all, let us give a simple review of [20]. If $q \geq p > 1$ and

$$a(x) \geq 0, \qquad b(x) \geq 0, \qquad a(x) + b(x) > 0, \quad x \in \overline{\Omega}, \tag{1.3}$$

then

$$a(x)|\nabla u|^p + b(x)|\nabla u|^q \geq c|\nabla u|^p,$$

provided that $|\nabla u| \geq 1$. By its coercivity, we can minimize, with fixed boundary values, the integral

$$F(u) = \int_\Omega \bigl(a(x)|\nabla u|^p + b(x)|\nabla u|^q\bigr)\,dx$$

and the local minimizers in the Sobolev class $W^{1,p}_{\mathrm{loc}}(\Omega)$. It is expected (however it is not always true!) that any local minimizer u is also a weak solution to the corresponding Euler's first variation, i.e., the PDE in a divergence form

$$\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, \nabla u) = 0, \quad x \in \Omega,$$

where $a(x, \nabla u) = (a_i(x, \nabla u))$, $i = 1, 2, \ldots, N$, is given by

$$a(x, \xi) = \bigl\{pa(x)|\nabla u|^{p-2} + qb(x)|\nabla u|^{q-2}\bigr\}\nabla u,$$

satisfying that

$$\bigl|a(x, \xi)\bigr| \leq c\bigl(1 + |\xi|^{q-1}\bigr), \quad \xi \in \mathbb{R}^N.$$

If $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$, we can obtain

$$\bigl|a(x, \nabla u)\bigr| \leq M\bigl(1 + |\nabla u|^{q-1}\bigr) \in L^{\frac{q}{q-1}}_{\mathrm{loc}}(\Omega) = L^{q'}_{\mathrm{loc}}(\Omega),$$

and $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ would satisfy the (correct) weak form of the equation

$$\int_\Omega \sum_{i=1}^N a_i(x, \nabla u)\frac{\partial \varphi}{\partial x_i}\,dx = 0, \quad \forall \varphi \in W^{1,q}_0(\Omega), \operatorname{supp}\varphi \subset \Omega.$$

But the fact is that a minimizer of the functional (1.3) is only a function of class $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$! This is a difference (and a difficulty) with respect to equation (1.2) in which $p = q$.

Of course, there is a similar difficulty in the evolution problems. Emphasizing the fact that an evolution problem is usually formulated by a differential equation and not as a minimization, the authors of [20] adopted a different point of view and posed this philosophical question: does a counterpart of the minimization property exist in evolution problems? i.e., may a solution of an evolution problem be a variational minimizer? By introducing a new kind of weak solution (called as variational solution in [20]), such a problem has been perfectly solved in [20].

In this paper, we study the well-posedness problem of equation (1.1) only assuming that

$$\big[a(x)+b(x)\big]|_{x\in\partial\Omega}=0, \qquad \big[a(x)+b(x)\big]|_{x\in\Omega}>0, \tag{1.4}$$

or

$$\big[a(x)b(x)\big]|_{x\in\partial\Omega}=0, \qquad \big[a(x)b(x)\big]|_{x\in\partial\Omega}>0. \tag{1.5}$$

Since $a(x)\geq 0$, $b(x)\geq 0$, on the boundary $\partial\Omega$, if $[a(x)+b(x)]|_{x\in\partial\Omega}=0$ implies $[a(x)b(x)]|_{x\in\partial\Omega}=0$. But generally, $[a(x)b(x)]|_{x\in\partial\Omega}=0$ does not imply $[a(x)+b(x)]|_{x\in\partial\Omega}=0$. Let us give some special cases when $N=2$, $\Omega\subset\mathbb{R}^2$ is bounded. For example, we can choose a small constant $\delta<\frac{1}{4}$ and

$$\Omega_1=\big\{x=(x_1,x_2):x_1^2+x_2^2<1\big\},$$

$0\leq a_1(x)\in C^1(\overline{\Omega_1})$, and on the boundary it is defined as

$$a_1(x)=\begin{cases}0, & \text{if } x\in\{1\geq x_2\geq -\delta, x_1^2+x_2^2=1\},\\ \text{smooth connected}, & \text{if } x\in\{-\delta\geq x_2\geq -2\delta, x_1^2+x_2^2=1\},\\ >0, & \text{if } x\in\{-2\delta>x_2\geq -1, x_1^2+x_2^2=1\}.\end{cases}$$

While $0\leq b_1(x)\in C^1(\overline{\Omega_1})$, and on the boundary it is defined as

$$b_1(x)=\begin{cases}0, & \text{if } x\in\{-1\leq x_2\leq \delta, x_1^2+x_2^2=1\},\\ \text{smooth connected}, & \text{if } x\in\{2\delta\geq x_2\geq \delta, x_1^2+x_2^2=1\},\\ >0, & \text{if } x\in\{2\delta<x_2\leq 1, x_1^2+x_2^2=1\}.\end{cases}$$

Then $a_1(x)b_1(x)=0$, $x\in\partial\Omega_1$, but

$$a_1(x)+b_1(x)>0, \quad x\in\big\{-2\delta>x_2\geq -1, x_1^2+x_2^2=1\big\}\cup\big\{2\delta<x_2\leq 1, x_1^2+x_2^2=1\big\}.$$

So, (1.5) is true, whether or not (1.4) is right.

For a simpler example, let

$$\Omega_2=\big\{x=(x_1,x_2):1<x_1^2+x_2^2<4\big\}$$

and

$$a_2(x)=x_1^2+x_2^2-1, \qquad b_2(x)=4-\big(x_1^2+x_2^2\big).$$

Clearly, $a_2(x)+b_2(x)=3>0$ and $a_2(x)b_2(x)=0$ when $x\in\partial\Omega_2$. Also (1.5) is true, whether or not (1.4) is right.

Certainly, if $a(x)=0$, $b(x)=0$ on the boundary $\partial\Omega$, then conditions (1.4) and (1.5) can be true at the same time.

We would like to enlarge a bit upon this point. If $a(x)$ and $b(x)$ satisfy

$$a(x) > 0, \qquad b(x) > 0, \quad x \in \overline{\Omega}, \tag{1.6}$$

to study the well-posedness of the solutions of equation (1.1), besides the initial value condition

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.7}$$

similar to the usual non-Newtonian fluid equation (1.2), in the sense of the classical trace, the boundary value condition

$$u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T) \tag{1.8}$$

should be imposed.

On the other hand, if $a(x)$ and $b(x)$ only satisfy (1.4) or (1.5), equation (1.1) may be degenerate on the boundary $\partial\Omega$, how to define a suitable boundary value condition instead of (1.8) becomes an important problem. In fact, for a degenerate parabolic equation, the weak solution $u(x,t)$ may be too weak to define the trace on the boundary. For example, the authors of [25] pointed out that if the weak solution $u(x,t)$ only has the property

$$\int_\Omega \alpha(x)|\nabla u|^p\,dx < \infty, \int_\Omega \alpha(x)^{-\frac{1}{p-1}}\,dx = +\infty \tag{1.9}$$

with that

$$\alpha(x) \geq 0, \quad x \in \overline{\Omega},$$

then $C_0^\infty(Q_T)$ is not dense in the space $\mathbf{B} = \{u : u \text{ satisfies } (1.9)\}$, and so one cannot define the trace on $\partial\Omega$ in the classical way. The author of [25] gave a new way to deal with the boundary value condition, and we will introduce the related content in the last section of this paper.

In recent years, the author of this paper has been interested in the stability of weak solutions to the following equation:

$$u_t = \operatorname{div}\bigl(\alpha(x)|\nabla u|^{p(x)-2}\nabla u\bigr) + f(x,t,u,\nabla u), \qquad (x,t) \in Q_T, \tag{1.10}$$

including the special cases of that $p(x) = p$ is a constant, provided that

$$\alpha(x) = 0, \quad x \in \partial\Omega, \qquad \alpha(x) > 0, \quad x \in \Omega. \tag{1.11}$$

If the weak solutions of (1.10) only satisfy (1.9), we also cannot define the trace on the boundary in the classical way. To solve this problem, we have avoided to use the boundary value condition (1.8). Instead, we have found that, to study the uniqueness of weak solution of equation (1.10), condition (1.11) can take place of the boundary value condition (1.8) [26, 29–32].

Actually, for a degenerate parabolic equation, how to deal with the boundary value condition (1.8) has been an important problem for a long time, and there are many papers devoted to this question, one can refer to [14, 15, 18, 21, 27, 30] etc. for the details.

2 The definition of weak solution and the main results

In the first place, we give some basic concepts.

Assume that $\nu(x)$ is a positive measurable function defined in Ω. Define the weighted Lebesgue space $L^p(\nu,\Omega)$, $1 < p < \infty$, as the space of all real-valued functions u for which

$$\|u\|_{p,\nu} = \left(\int_\Omega \nu(x)\left|u(x)\right|^p\,dx\right)^{1/p} < \infty.$$

Further we suppose that

$$\nu(x) \in L^1_{\mathrm{loc}}(\Omega), \qquad \left[\nu(x)\right]^{-1} \in L^{1/(p-1)}_{\mathrm{loc}}(\Omega). \tag{2.1}$$

Now, we denote by $W^{1,p}(\nu,\Omega)$ the space of all real-valued functions u such that the derivative in the sense of distributions fulfills

$$u \in L^p(\Omega) \quad \text{and} \quad \nu^{1/p}|\nabla u| \in L^p(\Omega)$$

with the norm

$$\|u\|_{1,p,\nu} = \left(\int_\Omega \left|u(x)\right|^p\,dx + \int_\Omega \nu(x)\left|\nabla u(x)\right|^p\,dx\right)^{1/p}. \tag{2.2}$$

By (2.1) we can introduce the subspace $W^{1,p}_0(\nu,\Omega)$ of $W^{1,p}(\nu,\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.2). Moreover, conditions (2.1) imply that $W^{1,p}(\nu,\Omega)$ as well as $W^{1,p}_0(\nu,\Omega)$ are reflexive Banach spaces [19].

Lemma 2.1 *Let us suppose that* (2.1) *holds and*

$$\left[\nu(x)\right]^{-1} \in L^{g^*}(\Omega) \tag{2.3}$$

with some $g^* \geq \frac{1}{p-1}$. *Then* $W^{1,p}(\nu,\Omega)$ *is continuously imbedded into* $W^{1,p_1}(\Omega)$, *where* $p_1 = \frac{pg^*}{g^*+1}$.

Remark 2.2 By virtue of compact imbedding theorems (see [6]) and Lemma 2.1, we obtain that the imbedding

$$W^{1,p}(\nu,\Omega) \to L^q(\Omega) \tag{2.4}$$

is compact for $1 \leq q < \frac{Np_1}{N-p_1}$ if $N > p_1$, for $1 \leq q < \infty$ if $N = p_1$, for $1 \leq q < \infty$ if $N < p_1$. Therefore, if we also suppose that the number g^* from Lemma 2.1 satisfies $g^* > \frac{N}{p}$. then $W^{1,p}(\mu,\Omega)$ is compactly imbedded into $L^p(\Omega)$.

In the second place, we introduce the definition of weak solution.

Definition 2.3 If $u(x,t)$ is a $L^\infty(Q_T)$ function, it satisfies

$$u \in L^p\bigl(0,T;W^{1,p}\bigl(a(x),\Omega\bigr)\bigr) \cap L^q\bigl(0,T;W^{1,q}\bigl(b(x),\Omega\bigr)\bigr), \quad u_t \in L^2(Q_T) \tag{2.5}$$

and

$$\begin{aligned}
&\iint_{Q_T} u_t\varphi \,dx\,dt \\
&\quad + \iint_{Q_T}\Biggl[\bigl(a(x)|\nabla u|^{p-2}\nabla u + b(x)|\nabla u|^{q-2}\nabla u\bigr)\nabla\varphi + \sum_{i=1}^{N} f_i(u,x,t)\varphi_{x_i}\Biggr]\,dx\,dt \\
&= 0, \quad \forall\varphi\in C_0^1(Q_T),
\end{aligned} \tag{2.6}$$

then $u(x,t)$ is said to be the weak solution of equation (1.1) with the initial boundary values (1.7)–(1.8) provided that

$$\lim_{t\to 0}\int_\Omega \bigl(u(x,t)-u_0(x)\bigr)\phi(x)\,dx = 0, \quad \forall\phi(x)\in C_0^\infty(\Omega), \tag{2.7}$$

and the boundary value condition (1.8) is satisfied in the sense of trace.

In the third place, we give the main results.

Theorem 2.4 *Suppose that $q \ge p > 2$, $a(x), b(x) \in C(\overline{\Omega})$ satisfies (1.5), $f_i(s,x,t)$ is a C^1 function on $\mathbb{R}\times\overline{Q}_T$ satisfies*

$$\bigl|f_i(u_\varepsilon,x,t)\bigr| \le c\min\bigl\{a(x),b(x)\bigr\}, \quad i=1,2,\ldots,N, \tag{2.8}$$

$u_0(x)$ satisfies

$$u_0\in L^\infty(\Omega), \quad |\nabla u_0|\in L^q(\Omega). \tag{2.9}$$

If

$$\int_\Omega a(x)^{-g_1*}\,dx < \infty, \qquad g_1* \ge \frac{2}{p-2}, \tag{2.10}$$

or

$$\int_\Omega b(x)^{-g_2*}\,dx < \infty, \qquad g_2* \ge \frac{2}{q-2}, \tag{2.11}$$

then the initial-boundary value problem (1.1)–(1.7)–(1.8) has a solution.

If we do not require $u_t\in L^2(Q_T)$, instead, $u_t\in L^1(0,T;W^{-1,q}(\Omega))$ (or a more general Banach space), condition (2.10) or condition (2.11) may not be necessary. Also, condition (2.8) is only used in the proof of the L^∞-norm estimate of u, and we conjecture it can be replaced by the condition

$$\sum_{i=1}^{N}\frac{\partial}{\partial x_i} f_i(\cdot,x,t) < 0. \tag{2.12}$$

In this case, if $u_0(x) \geq 0$, then by the maximal value theorem, one may prove that there is a nonnegative weak solution $u(x,t)$ satisfying

$$0 \leq u(x,t) \leq \|u_0(x)\|_{L^\infty(\Omega)}.$$

Moreover, by considering the minimality for the variational solution, the existence and the regularity of weak solutions was studied in [20] when

$$2 \leq p \leq q < p + \min\left\{1, \frac{4}{N}\right\}.$$

However, the main aim of this paper is not to study the existence of weak solution to equation (1.1), we do not pay attention to whether conditions (2.10)(2.11) are optimal or not. Also, we do not try to compare Theorem 2.4 with the results of weak solutions given in [20]. We only give a result on the existence of weak solution for the completeness of the paper. We mainly focus on the stability of weak solutions to equation (1.1) when the coefficients $a(x)$ and $b(x)$ may be degenerate on the boundary $\partial\Omega$.

Theorem 2.5 *Let $q \geq p > 1$, $a(x), b(x) \in C(\overline{\Omega})$ satisfy*

$$\int_\Omega \left[a(x)^{-\frac{1}{p-1}} + b(x)^{-\frac{1}{q-1}}\right] dx < \infty, \tag{2.13}$$

$f_i(s,x,t)$ be a Lipschitz function when $|s| \leq c$, $i = 1,2,\ldots,N$. If $u(x,t)$ and $v(x,t)$ are two solutions of equation (1.1) with the same homogeneous boundary value condition

$$u(x,t) = v(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \tag{2.14}$$

and with different initial values $u_0(x)$ and $v_0(x)$ respectively, then

$$\int_\Omega |u(x,t) - v(x,t)|\, dx \leq \int_\Omega |u_0(x) - v_0(x)|\, dx, \quad t \in [0,T). \tag{2.15}$$

If condition (2.13) is invalid, there are three cases

(a)

$$\int_\Omega a(x)^{-\frac{1}{p-1}}\, dx < \infty, \qquad \int_\Omega b(x)^{-\frac{1}{q-1}}\, dx = \infty;$$

(b)

$$\int_\Omega a(x)^{-\frac{1}{p-1}}\, dx = \infty, \qquad \int_\Omega b(x)^{-\frac{1}{q-1}}\, dx < \infty;$$

(c)

$$\int_\Omega a(x)^{-\frac{1}{p-1}}\, dx = \infty, \qquad \int_\Omega b(x)^{-\frac{1}{q-1}}\, dx = \infty.$$

By Proposition 3.3, in cases (a) and (b), we still can impose the boundary value condition (2.14) and obtain stability (2.15). If $a(x)$, $b(x)$ satisfy (c), we cannot impose the boundary value condition (2.14) generally. Fortunately, if there are some restrictions between $a(x)$, $b(x)$ and $f_i(s,x,t)$, we are still able to prove the following stability of weak solutions without (2.14).

Theorem 2.6 *Let $q \geq p > 1$, $a(x), b(x) \in C^1(\overline{\Omega})$ satisfy (1.5), $f_i(s,x,t)$ be a Lipschitz function when $|s| \leq c$. Suppose that $u(x,t)$ and $v(x)$ are two solutions of (1.1) with the initial values $u_0(x)$ and $v_0(x)$ respectively. If $a(x)$, $b(x)$ and $f_i(s,x,t)$ satisfy*

$$\bigl|f_i(u,x,t) - f_i(v,x,t)\bigr| \leq c\bigl[a(x)^{\frac{1}{p}} + b(x)^{\frac{1}{q}}\bigr]|u-v|, \quad i = 1,2,\dots,N, \tag{2.16}$$

and for η small enough,

$$\int_{\Omega\setminus\Omega_\eta} a(x)^{1-p}|\nabla a|^p\,dx \leq c, \qquad \int_{\Omega\setminus\Omega_\eta} a(x)b(x)^{-p}|\nabla b|^p\,dx \leq c, \tag{2.17}$$

$$\int_{\Omega\setminus\Omega_\eta} b(x)^{1-q}|\nabla b|^p\,dx \leq c, \qquad \int_{\Omega\setminus\Omega_\eta} b(x)a(x)^{-q}|\nabla a|^q\,dx \leq c, \tag{2.18}$$

then stability (2.15) is true.

Here, η is a small constant and $\Omega_\eta = \{x \in \Omega : a(x)b(x) > \eta\}$.

Theorem 2.7 *Let $q \geq p > 1$, $a(x), b(x) \in C^1(\overline{\Omega})$ satisfy (1.4), $f_i(s,x,t)$ be a Lipschitz function when $|s| \leq c$. Suppose that $u(x,t)$ and $v(x)$ are two solutions of (1.1) with the initial values $u_0(x)$ and $v_0(x)$, respectively. If $a(x)$, $b(x)$ and $f_i(s,x,t)$ satisfy*

$$\begin{aligned}&\bigl|f_i(u,x,t) - f_i(v,x,t)\bigr| \\ &\quad\leq c\bigl[\bigl(a(x)+b(x)\bigr)^{\frac{1}{p}} + \bigl(a(x)+b(x)\bigr)^{\frac{1}{q}}\bigr]|u-v|, \quad i = 1,2,\dots,N,\end{aligned} \tag{2.19}$$

and for η small enough,

$$\frac{1}{\eta}\biggl(\int_{\Omega\setminus D_\eta} \bigl|\nabla\bigl(a(x)+b(x)\bigr)^{1+\frac{1}{p}}\bigr|^p\,dx\biggr)^{\frac{1}{p}} \leq c, \tag{2.20}$$

$$\frac{1}{\eta}\biggl(\int_{\Omega\setminus D_\eta} \bigl|\nabla\bigl(a(x)+b(x)^{1+\frac{1}{q}}\bigr)\bigr|^q\,dx\biggr)^{\frac{1}{q}} \leq c, \tag{2.21}$$

then stability (2.15) is true.

Here, η is a small constant and $D_\eta = \{x \in \Omega : a(x)+b(x) > \eta\}$. There is an essential difference between Theorem 2.6 and Theorem 2.7. In Theorem 2.6, $a(x), b(x) \in C^1(\overline{\Omega})$ satisfy (1.5), and so

$$a(x) > 0, \qquad b(x) > 0, \quad x \in \Omega;$$

while in Theorem 2.7 $a(x), b(x) \in C^1(\overline{\Omega})$ satisfy (1.4), and so

$$a(x) = 0, \qquad b(x) = 0, \quad x \in \partial\Omega.$$

But it is possible that there is $x_0 \in \Omega$,

$$a(x_0) + b(x_0) > 0, \qquad a(x_0)b(x_0) = 0. \tag{2.22}$$

Naturally, condition (2.16) (or (2.19)) may not be necessary. In the last section of this paper, by giving a generalization of the classical trace of $u \in BV(Q_T)$, we will use a reasonable partial boundary value condition instead of condition (2.16) (or (2.19)) to study the stability of weak solutions.

3 The existence of weak solutions

In this section, we want to prove Theorem 2.4. Let us first consider the following Cauchy–Dirichlet problem:

$$\begin{aligned} &u_{\varepsilon t} - \operatorname{div}\Big(\big(a(x)+\varepsilon\big)\big(|\nabla u_\varepsilon|^2+\varepsilon\big)^{\frac{p-2}{2}}\nabla u_\varepsilon + \big(b(x)+\varepsilon\big)\big(|\nabla u_\varepsilon|^2+\varepsilon\big)^{\frac{q-2}{2}}\nabla u_\varepsilon\Big) \\ &\qquad - \sum_{i=1}^{N}\frac{\partial f_i(u_\varepsilon,x,t)}{\partial x_i} \\ &\quad = 0, (x,t) \in Q_T, \end{aligned} \tag{3.1}$$

$$u_\varepsilon(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{3.2}$$

$$u_\varepsilon(x,0) = u_{\varepsilon 0}(x), \quad x \in \Omega, \tag{3.3}$$

where $u_{\varepsilon 0} \in C_0^\infty(\Omega)$, $\|u_{\varepsilon 0}\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)}$, $|\nabla u_{\varepsilon 0}|$ converges to $|\nabla u_0(x)|$ in $L^q(\Omega)$.

Since the convection function $f_i(s,x,t)$ is a C^1 function on $\mathbb{R} \times \overline{Q}_T$, $i = 1,2,\dots,N$, by the classical existence theory for parabolic equations [17], similar to [8], we know there is a unique weak solution $u_\varepsilon \in C^0([0,T];L^2(\Omega)) \cap L^q(0,T;W_0^{1,q}(\Omega))$ with $\partial_t u_\varepsilon \in L^{q'}(0,T;W^{-1,q'}(\Omega))$. Now, let us show that

$$\|u_\varepsilon\|_{L^\infty(Q_T)} \leq c.$$

Lemma 3.1 *Assume that* a_1, b_1, λ *are positive constants, where* $\lambda > \frac{1}{2} + \frac{b_1}{a_1}$. *Define*

$$\varphi(s) = \begin{cases} e^{\lambda s - 1} & s \geq 0, \\ -e^{-\lambda s} + 1 & s \leq 0. \end{cases} \tag{3.4}$$

Then the following properties hold:

1. *For any* $s \in \mathbb{R}$, *we have*

$$\big|\varphi(s)\big| \geq \lambda|s|, \qquad a_1\varphi'(s) - b_1\big|\varphi(s)\big| \geq \frac{a_1}{2}e^{\lambda|s|}. \tag{3.5}$$

2. *For any* $s \geq d$, *there hold constants* $d \geq 0$, $M > 1$, *we have*

$$\varphi'(s) \leq \lambda M\left[\varphi\left(\frac{s}{l}\right)\right]^l, \qquad \varphi(s) \leq M\left[\varphi\left(\frac{s}{l}\right)\right]^l, \tag{3.6}$$

where $l > 1$.

3. *Let* $\Phi(s) = \int_0^s \varphi(\sigma)\,d\sigma$. *For any* $s \geq 0$, *if* $l > 2$, *here holds constant* $c^* > 0$, *we have*

$$\Phi(s) \geq c^*\left[\varphi\left(\frac{s}{l}\right)\right]^l. \tag{3.7}$$

If $1 < l < 2$, *then there exist* $d \geq 0$ *and* $c^* = c^*(q,d)$ *such that*

$$\begin{cases} \Phi(s) \geq c^*[\varphi(\frac{s}{l})]^l, & \forall s \geq d, \\ \Phi(s) \geq c^*[\varphi(\frac{s}{l})]^2, & \forall 0 \leq s \leq d. \end{cases} \tag{3.8}$$

Lemma 3.1 can be found in [16].

Lemma 3.2 *Assume that* u_ε *is a weak solution of* (3.1), *then there is a constant* c (*which is independent of* ε) *such that*

$$\|u_\varepsilon\|_{L^\infty(Q_T)} \leq \|u_0\|_{L^\infty(\Omega)} + c. \tag{3.9}$$

Proof We only give the proof provided that condition (2.10) is true. When condition (2.11) is true, this lemma can be verified in a similar way. Let k be a real number and $\|u_0\|_{L^\infty(\Omega)} \leq k$, the function φ be defined as (3.4). Define

$$G_k(u_\varepsilon) = \begin{cases} u_\varepsilon - k, & u_\varepsilon > k, \\ u_\varepsilon + k, & u_\varepsilon < -k, \\ 0, & |u_\varepsilon| \leq k. \end{cases}$$

We can see $\varphi(G_k(u_\varepsilon)) \in V \cap L^\infty(Q_T)$. So, for any $\tau \in [0,T]$, we can choose $v = \varphi(G_k(u_\varepsilon))\chi_{[0,\tau]}$ as a test function (where χ_A is an eigenfunction on the set A). At the same time, we know that $v_{x_i} = \chi_{[0,\tau]}\chi\{|u_\varepsilon| > k\}\varphi'(G_k(u_\varepsilon))u_{\varepsilon x_i}$ and $\nabla v = \chi_{[0,\tau]}\chi\{|u_\varepsilon| > k\}\varphi'(G_k(u_\varepsilon))\nabla u_\varepsilon$. Since $f_i(u_\varepsilon,x,t)$ satisfies (2.8), we have

$$\begin{aligned} &\int_0^\tau \langle u_{\varepsilon t}, \varphi(G_k(u_\varepsilon))\rangle\,dt \\ &\quad + \int_0^\tau \int_\Omega \Big[\big(b(x)+\varepsilon\big)\big(|\nabla u_\varepsilon|^2+\varepsilon\big)^{\frac{q-2}{2}}|\nabla u_\varepsilon|^2 \\ &\quad + \big(a(x)+\varepsilon\big)\big(|\nabla u_\varepsilon|^2+\varepsilon\big)^{\frac{p-2}{2}}|\nabla u_\varepsilon|^2\Big] \\ &\quad \cdot \varphi'\big(G_k(u_\varepsilon)\big)\chi\big\{|u_\varepsilon| > k\big\}\,dx\,dt \\ &= -\sum_{i=1}^N \int_0^\tau \int_\Omega f_i(u_\varepsilon,x,t)\chi\big\{|u_\varepsilon| > k\big\}\varphi'\big(G_k(u_\varepsilon)\big)u_{\varepsilon x_i}\,dx\,dt \\ &\leq \int_0^\tau \int_\Omega \left[\frac{1}{p}a(x)|\nabla u_\varepsilon|^p + \frac{1}{q}a(x)\right]\chi\big\{|u_\varepsilon| > k\big\}\varphi'\big(G_k(u_\varepsilon)\big)\,dx\,dt, \end{aligned} \tag{3.10}$$

where $\langle u_{\varepsilon t}, \varphi(G_k(u_\varepsilon))\rangle$ is the dyadic interaction between $L^p(0,T;W_0^{1,p}(\Omega))$ and $L^{p'}(0,T;W^{-1,p'}(\Omega))$.

Let $A_k(t) = \{x \in \Omega : |u_\varepsilon(x,t)| > k\}$ depend on k. We have

$$\begin{aligned}\int_0^\tau \langle u_{\varepsilon t}, \varphi(G_k(u_\varepsilon))\rangle\,dt &= \int_\Omega \Phi(G_k(u_\varepsilon))(\tau)\,dx - \int_\Omega \Phi(G_k(u_{\varepsilon 0}))\,dx \\ &= \int_{A_k(\tau)} \Phi(G_k(u_\varepsilon))(\tau)\,dx - \int_{A_k(0)} \Phi(G_k(u_{\varepsilon 0}))\,dx \\ &= \int_{A_k(\tau)} \Phi(G_k(u_\varepsilon)(\tau)\,dx.\end{aligned} \tag{3.11}$$

Substituting (3.11) into (3.10), using Lemma 2.1, we can deduce that

$$\begin{aligned}&\int_{A_k(\tau)} \Phi(G_k(u_\varepsilon))(\tau)\,dx + \int_0^\tau \int_{A_k(t)} |\nabla u_\varepsilon|^p \varphi'\,dx\,dt \\ &\quad\leq \int_{A_k(\tau)} \Phi(G_k(u_\varepsilon))(\tau)\,dx + \int_0^\tau \int_{A_k(t)} a(x)|\nabla u_\varepsilon|^p \varphi'\,dx\,dt \\ &\qquad + \int_0^\tau \int_{A_k(t)} b(x)|\nabla u_\varepsilon|^q \varphi'\,dx\,dt \\ &\quad\leq c\int_{A_k(\tau)} \Phi(G_k(u_\varepsilon))(\tau)\,dx \\ &\qquad + c\int_0^\tau \int_{A_k(t)} \Big[(b(x)+\varepsilon)(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{q-2}{2}}|\nabla u_\varepsilon|^2 \\ &\qquad + (a(x)+\varepsilon)(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla u_\varepsilon|^2\Big]\varphi'\,dx\,dt \\ &\quad\leq c\int_0^\tau \int_{A_k(t)} a(x)\varphi'(G_k(u_\varepsilon))\,dx\,dt.\end{aligned} \tag{3.12}$$

Let $\omega_k = \varphi(\frac{|G_k(u_\varepsilon)|}{p})$. Then

$$\begin{aligned}\int_0^\tau \int_{A_k(t)} |\nabla u_\varepsilon|^p \varphi'\,dx\,dt &\geq \frac{1}{2}\int_0^\tau \int_{A_k(t)} \Big|e^{\lambda \frac{|G_k(u_\varepsilon)|}{p}} \nabla u\Big|^p\,dx\,dt \\ &= \frac{1}{2}\int_0^\tau \int_{A_k(t)} \Big|\frac{p}{\lambda}\Big|^p |\nabla \omega_k|^p\,dx\,dt \\ &\geq \frac{1}{2}\Big(\frac{1}{\lambda}\Big)^p \int_0^\tau \int_{A_k(t)} |\nabla \omega_k|^p\,dx\,dt.\end{aligned} \tag{3.13}$$

By definition we know that $A_k(t)\backslash A_{k+d}(t) = \{x \in \Omega : k < |u_\varepsilon(x,t)| \leq k+d\}$. So, in the set of $A_k(t)\backslash A_{k+d}(t)$, we get $0 < |G_k(u_\varepsilon)| \leq d$, $\varphi'(G_k(u_\varepsilon)) = \lambda e^{\lambda|G_k(u_\varepsilon)|} \leq \lambda e^{\lambda d}$. Combining (3.6) with (3.12) and (3.13), we have

$$\begin{aligned}&\int_{A_k(\tau)} \Phi(G_k(u_\varepsilon))(\tau)\,dx + \frac{1}{2}\Big(\frac{1}{\lambda}\Big)^p \int_0^\tau \int_{A_k(t)} |\nabla \omega_k|^p\,dx\,dt \\ &\quad\leq c\lambda \int_0^\tau \int_{A_{k+d}(t)} |w_k|^p\,dx\,dt + c\lambda e^{\lambda d}\int_0^\tau \int_{A_k(t) A_{k+d}(t)} dx\,dt.\end{aligned} \tag{3.14}$$

Since $p \geq 2$, by (3.7), then

$$\int_{A_k(\tau)} \Phi\big(G_k(u_\varepsilon)\big)(\tau)\,dx \geq c^* \int_{A_k(\tau)} |\omega_k|^{p^-}\,dx. \tag{3.15}$$

Plugging (3.15) into (3.14) and taking the supremum for $\tau \in [0, t_1]$, with $t_1 \leq T$ to be determined later, we have

$$\begin{aligned} &\sup_{\tau\in[0,t_1]} \int_{A_k(\tau)} |\omega_k|^{p^-}\,dx + \frac{1}{2}\left(\frac{1}{\lambda}\right)^p \int_0^{t_1}\int_{A_k(t)} |\nabla\omega_k|^p\,dx\,dt \\ &\quad\leq c\lambda \int_0^\tau \int_{A_{k+d}(t)} |w_k|^p\,dx\,dt + c_1\lambda e^{\lambda d}\int_0^\tau \int_{A_k(t)A_{k+d}(t)}\,dx\,dt. \end{aligned} \tag{3.16}$$

Let $\psi_k = \int_0^{t_1} \mu(A_k(t))\,dt$. By choosing

$$c_1\big(t_1\mu(\Omega)\big)^{\frac{p}{N+p}} \leq \frac{1}{2}, \tag{3.17}$$

where $\mu(\Omega)$ is the Lebesgue measure of Ω. Now, using the embedding inequality [16, 24], we can deduce that

$$\left(\int_0^{t_1}\int_{A_k(t)} |\omega_k|^{p\frac{N+p}{N}}\,dx\,dt\right)^{\frac{N}{N+p}} \leq \gamma\left(\sup_{\tau\in[0,t_1]}\int_{A_k(\tau)} |\omega_k|^p\,dx + \int_0^{t_1}\int_{A_k(t)} |\nabla\omega|^p\,dx\,dt\right),$$

where γ is a constant independent of t_1, similar to the proof of Theorem 2.2 in [16], it follows from (3.16) that

$$\psi_l \leq \frac{c}{(l-k)^{\frac{p(N+p)}{N}}}\psi_k^{(1-\frac{1}{r})\frac{N+p}{N}}, \tag{3.18}$$

where $r > \frac{N+p}{N}$ is a constant, and so

$$\left(1-\frac{1}{r}\right)\frac{N+p}{N} > 1.$$

Therefore, thanks to the iteration lemma in [21], from (3.18), we eventually obtain that

$$\psi_{(\|u_0\|_{L^\infty(\Omega)}+D)} = 0,$$

where $D > 0$ is a constant depending only on p, N, t_1, r, Ω. This proves that, for fixed λ validating Lemma 2.1,

$$\big\|u(x,t)\big\|_{L^\infty(Q_{t_1})} \leq \|u_{0\varepsilon}\|_{L^\infty(\Omega)} + D. \tag{3.19}$$

Finally, we partition the time interval $[0, T]$ into finite subintervals $[0, t_1], [t_1, t_2], \ldots, [t_{n-1}, T]$ such that the conditions similar to (3.17) are available for each subinterval $[t_i, t_{i+1}]$, then we deduce an inequality of the form (3.19). Eventually, we have conclusion (3.9). □

Proof of Theorem 2.4 Multiplying (3.1) by u_ε and integrating it over Q_T yield

$$\begin{aligned}
&\frac{1}{2}\int_\Omega u_\varepsilon^2\,dx + \iint_{Q_T}\Big[\big(a(x)+\varepsilon\big)\big(|\nabla u_\varepsilon|^2+\varepsilon\big)^{\frac{p-2}{2}}|\nabla u_\varepsilon|^2 \\
&\quad + \big(b(x)+\varepsilon\big)\big(|\nabla u_\varepsilon|^2+\varepsilon\big)^{\frac{q-2}{2}}|\nabla u_\varepsilon|^2\Big]\,dx\,dt \\
&\quad + \sum_{i=1}^N \iint_{Q_T}\frac{\partial f_i(u_\varepsilon,x,t)}{\partial x_i}u_\varepsilon\,dx\,dt \\
&= \frac{1}{2}\int_\Omega u_{\varepsilon 0}^2\,dx.
\end{aligned} \tag{3.20}$$

Since $f_i(s,x,t)$ is a Lipschitz function when $|s|\le c$, $\frac{\partial f_i(s,x,t)}{\partial s}$ exists almost everywhere and is bounded. If $\int_\Omega a(x)^{-\frac{2}{p-2}}\,dx<\infty$, then $\int_\Omega a(x)^{-\frac{1}{p-1}}\,dx<\infty$, and

$$\begin{aligned}
\left|\int_\Omega \frac{\partial f_i(u_\varepsilon,x,t)}{\partial x_i}u_\varepsilon\,dx\right| &\le \int_\Omega \left|\frac{\partial f_i(s,x,t)}{\partial s}\Big|_{s=u_\varepsilon} u_{\varepsilon x_i}\right||u_\varepsilon|\,dx + c \\
&\le c\int_\Omega \left|\frac{\partial f_i(s,x,t)}{\partial s}\Big|_{s=u_\varepsilon} u_{\varepsilon x_i}\right|\,dx + c \\
&\le \frac{1}{4}\int_\Omega a(x)|\nabla u_\varepsilon|^p\,dx + c\int_\Omega a(x)^{-\frac{1}{p-1}}\,dx + c \\
&\le \frac{1}{4}\int_\Omega a(x)|\nabla u_\varepsilon|^p\,dx + c.
\end{aligned}$$

Or similarly, if $\int_\Omega b(x)^{-\frac{2}{q-2}}\,dx<\infty$, we know that $\int_\Omega b(x)^{-\frac{1}{q-1}}\,dx<\infty$ and

$$\begin{aligned}
\left|\int_\Omega \frac{\partial f_i(u_\varepsilon,x,t)}{\partial x_i}u_\varepsilon\,dx\right| &\le \int_\Omega \left|\frac{\partial f_i(s,x,t)}{\partial s}\Big|_{s=u_\varepsilon} u_{\varepsilon x_i}\right||u_\varepsilon|\,dx + c \\
&\le c\int_\Omega \left|\frac{\partial f_i(s,x,t)}{\partial s}\Big|_{s=u_\varepsilon} u_{\varepsilon x_i}\right|\,dx + c \\
&\le \frac{1}{4}\int_\Omega b(x)|\nabla u_\varepsilon|^q\,dx + c\int_\Omega b(x)^{-\frac{1}{q-1}}\,dx + c \\
&\le \frac{1}{4}\int_\Omega b(x)|\nabla u_\varepsilon|^q\,dx + c.
\end{aligned}$$

Accordingly, based on condition (2.10) or condition (2.10), by (3.20), we obtain

$$\begin{aligned}
&\int_\Omega u_\varepsilon^2\,dx + \iint_{Q_T} a(x)|\nabla u_\varepsilon|^p\,dx\,dt + \iint_{Q_T} b(x)|\nabla u_\varepsilon|^q\,dx\,dt \\
&\quad\le \int_\Omega u_\varepsilon^2\,dx + \iint_{Q_T}\big(a(x)+\varepsilon\big)\big(|\nabla u_\varepsilon|^2+\varepsilon\big)^{\frac{p-2}{2}}|\nabla u_\varepsilon|^2\,dx\,dt \\
&\qquad + \iint_{Q_T}\big(b(x)+\varepsilon\big)\big(|\nabla u_\varepsilon|^2+\varepsilon\big)^{\frac{q-2}{2}}|\nabla u_\varepsilon|^2\,dx\,dt \\
&\quad\le c.
\end{aligned} \tag{3.21}$$

Let $\Omega_1 \subset\subset \Omega$. Then there exists a constant $c(\Omega_1)$ such that

$$a(x) \geq c(\Omega_1) > 0, \qquad b(x) \geq c(\Omega_1) > 0.$$

By $q \geq p > 2$, (3.21) yields

$$\begin{aligned}
&\int_0^T \int_{\Omega_1} |\nabla u_\varepsilon|^2 \,dx\,dt \\
&\quad\leq c\left(\int_0^T \int_{\Omega_1} |\nabla u_\varepsilon|^p \,dx\,dt\right)^{\frac{2}{p}} \\
&\quad\leq c(\Omega_1)\left[\left(\int_0^T \int_{\Omega_1} a(x)|\nabla u_\varepsilon|^p \,dx\,dt\right)^{\frac{2}{p}} + \left(\int_0^T \int_{\Omega_1} b(x)|\nabla u_\varepsilon|^q \,dx\,dt\right)^{\frac{2}{q}}\right] \\
&\quad\leq c.
\end{aligned} \tag{3.22}$$

Multiplying (2.9) by $u_{\varepsilon t}$, we have

$$\begin{aligned}
&\iint_{Q_T} |u_{\varepsilon t}|^2 \,dx\,dt \\
&\quad= \sum_{i=1}^N \iint_{Q_T} \frac{\partial}{\partial x_i}\Big[\big(a(x)+\varepsilon\big)\big(|\nabla u_\varepsilon|^2+\varepsilon\big)^{\frac{p-2}{2}} u_{\varepsilon x_i} \\
&\qquad+ \big(b(x)+\varepsilon\big)\big(|\nabla u_\varepsilon|^2+\varepsilon\big)^{\frac{q-2}{2}} u_{\varepsilon x_i}\Big] u_{\varepsilon t} \,dx\,dt \\
&\qquad+ \sum_{i=1}^N \iint_{Q_T} u_{\varepsilon t} \frac{\partial f_i(u_\varepsilon, x, t)}{\partial x_i} \,dx\,dt.
\end{aligned} \tag{3.23}$$

For every term in (1.7), firstly, we have

$$\begin{aligned}
&\sum_{i=1}^N \iint_{Q_T} \frac{\partial}{\partial x_i}\Big(\big(a(x)+\varepsilon\big)\big(|\nabla u_\varepsilon|^2+\varepsilon\big)^{\frac{p-2}{2}} u_{\varepsilon x_i}\Big) u_{\varepsilon t} \,dx\,dt \\
&\quad= -\sum_{i=1}^N \iint_{Q_T} \big(a(x)+\varepsilon\big)\big(|\nabla u_\varepsilon|^2+\varepsilon\big)^{\frac{p-2}{2}} u_{\varepsilon x_i} u_{\varepsilon x_i t} \,dx\,dt \\
&\quad= -\frac{1}{2}\sum_{i=1}^N \iint_{Q_T} \big(a(x)+\varepsilon\big) \frac{d}{dt} \int_0^{|\nabla u_\varepsilon|^2+\varepsilon} s^{\frac{p-2}{2}} \,ds\,dx\,dt
\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}
&\sum_{i=1}^N \iint_{Q_T} \frac{\partial}{\partial x_i}\Big(\big(b(x)+\varepsilon\big)\big(|\nabla u_\varepsilon|^2+\varepsilon\big)^{\frac{q-2}{2}} u_{\varepsilon x_i}\Big) u_{\varepsilon t} \,dx\,dt \\
&\quad= -\sum_{i=1}^N \iint_{Q_T} \big(b(x)+\varepsilon\big)\big(|\nabla u_\varepsilon|^2+\varepsilon\big)^{\frac{q-2}{2}} u_{\varepsilon x_i} u_{\varepsilon x_i t} \,dx\,dt \\
&\quad= -\frac{1}{2}\sum_{i=1}^N \iint_{Q_T} \big(b(x)+\varepsilon\big) \frac{d}{dt} \int_0^{|\nabla u_\varepsilon|^2+\varepsilon} s^{\frac{q-2}{2}} \,ds\,dx\,dt.
\end{aligned} \tag{3.25}$$

Secondly, if $\int_\Omega a(x)^{-\frac{2}{p-2}}(x)\,dx < \infty$, by the Hölder inequality, we have

$$\begin{aligned}
&\iint_{Q_T} |\nabla u_\varepsilon|^2\,dx\,dt \\
&\quad = c\iint_{Q_T} a(x)^{-\frac{2}{p}} a(x)^{\frac{2}{p}} |\nabla u_\varepsilon|^2\,dx\,dt \\
&\quad \le c\biggl(\iint_{Q_T} a(x)^{-\frac{2}{p-2}}\,dx\,dt\biggr)^{\frac{p-2}{p}} \biggl(\iint_{Q_T} a(x)|\nabla u_\varepsilon|^p\,dx\,dt\biggr)^{\frac{2}{p}} \\
&\quad \le c.
\end{aligned} \tag{3.26}$$

Similarly, if $\int_\Omega b(x)^{-\frac{2}{q-2}}(x)\,dx < \infty$, we have

$$\begin{aligned}
&\iint_{Q_T} |\nabla u_\varepsilon|^2\,dx\,dt \\
&\quad = c\iint_{Q_T} b(x)^{-\frac{2}{q}} b(x)^{\frac{2}{q}} |\nabla u_\varepsilon|^2\,dx\,dt \\
&\quad \le c\biggl(\iint_{Q_T} b(x)^{-\frac{2}{q-2}}\,dx\,dt\biggr)^{\frac{q-2}{q}} \biggl(\iint_{Q_T} b(x)|\nabla u_\varepsilon|^q\,dx\,dt\biggr)^{\frac{2}{q}} \\
&\quad \le c.
\end{aligned} \tag{3.27}$$

Thirdly, we have

$$\begin{aligned}
&\sum_{i=1}^N \iint_{Q_T} u_{\varepsilon t}\frac{\partial f_i(u_\varepsilon,x,t)}{\partial x_i}\,dx\,dt \\
&\quad \le \sum_{i=1}^N \iint_{Q_T} \bigl|f_{iu_\varepsilon}(u_\varepsilon,x,t)\bigr||u_{\varepsilon x_i}||u_{\varepsilon t}|\,dx\,dt \\
&\qquad + \sum_{i=1}^N + \iint_{Q_T} \bigl|f_{ix_i}(u_\varepsilon,x,t)\bigr||u_{\varepsilon t}|\,dx\,dt \\
&\quad \le \frac{1}{2}\iint_{Q_T} |u_{\varepsilon t}|^2\,dx\,dt + c\iint_{Q_T} |\nabla u_\varepsilon|^2\,dx\,dt + c.
\end{aligned} \tag{3.28}$$

Combining inequalities (3.24)–(3.28) with (3.23), we can extrapolate that

$$\begin{aligned}
&\iint_{Q_T} |u_{\varepsilon t}|^2\,dx\,dt + \iint_{Q_T} \bigl(a(x)+\varepsilon\bigr)\frac{d}{dt}\int_0^{|\nabla u_\varepsilon|^2+\varepsilon} s^{\frac{p-2}{2}}\,ds\,dx\,dt \\
&\qquad + \iint_{Q_T} b(x)\frac{d}{dt}\int_0^{|\nabla u_\varepsilon|^2+\varepsilon} s^{\frac{q-2}{2}}\,ds\,dx\,dt \\
&\quad \le c,
\end{aligned}$$

and so

$$\begin{aligned}
&\iint_{Q_T} |u_{\varepsilon t}|^2 \,dx\,dt \\
&\quad \le c + c\int_\Omega \bigl(a(x)+\varepsilon\bigr)\int_0^{|\nabla u_{\varepsilon 0}|^2+\varepsilon} s^{\frac{p-2}{2}}\,ds\,dx \\
&\qquad + c\int_\Omega \bigl(b(x)+\varepsilon\bigr)\int_0^{|\nabla u_{\varepsilon 0}|^2+\varepsilon} s^{\frac{q-2}{2}}\,ds\,dx \\
&\quad \le c.
\end{aligned} \tag{3.29}$$

According to the weak convergence theory, by (3.20), (3.21), (3.22), and (3.29), there exist a function u and two N-dimensional vector functions $\overrightarrow{\zeta} = (\zeta_1,\dots,\zeta_N)$ and $\overrightarrow{\xi} = (\xi_1,\dots,\xi_N)$ such that

$$\begin{aligned}
&u \in L^\infty(Q_T), \qquad |\zeta_i| \in L^{\frac{p}{p-1}}(Q_T), \qquad |\xi_i| \in L^{\frac{q}{q-1}}(Q_T), \\
&u_\varepsilon \to u, \quad \text{a.e. in } Q_T, \\
&u_\varepsilon \rightharpoonup *u, \quad \text{in } L^\infty(Q_T), \\
&f_i(u_\varepsilon,x,t) \to f_i(u,x,t), \quad \text{a.e. in } Q_T, \\
&a(x)|\nabla u_\varepsilon|^{p-2}u_{\varepsilon x_i} \rightharpoonup \zeta_i, \quad \text{in } L^{\frac{p}{p-1}}(Q_T), \\
&b(x)|\nabla u_\varepsilon|^{q-2}u_{\varepsilon x_i} \rightharpoonup \xi_i, \quad \text{in } L^{\frac{q}{q-1}}(Q_T).
\end{aligned}$$

At last, it is not difficult to show that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\iint_{Q_T} \bigl(a(x)+\varepsilon\bigr)\bigl(a(x)+\varepsilon\bigr)\bigl(|\nabla u_\varepsilon|^2+\varepsilon\bigr)^{\frac{p-2}{2}}|\nabla u_\varepsilon|^2\nabla u_\varepsilon\nabla\varphi\,dx\,dt \\
&\qquad + \lim_{\varepsilon\to 0}\iint_{Q_T} \bigl(b(x)+\varepsilon\bigr)\bigl(a(x)+\varepsilon\bigr)\bigl(|\nabla u_\varepsilon|^2+\varepsilon\bigr)^{\frac{q-2}{2}}|\nabla u_\varepsilon|^2\nabla u_\varepsilon\nabla\varphi\,dx\,dt \\
&\quad = \iint_{Q_T} (\overrightarrow{\zeta} + \overline{\xi})\cdot\nabla\varphi\,dx\,dt \\
&\quad = \iint_{Q_T} \bigl[a(x)|\nabla u|^{p-2}\nabla u + b(x)|\nabla u|^{q-2}\nabla u\bigr]\nabla\varphi\,dx\,dt
\end{aligned}$$

for any given $\varphi \in C_0^1(Q_T)$. So $u \in L^p(0,T;W_{\mathrm{loc}}^{1,p}(\Omega)) \cap L^q(0,T;W_{\mathrm{loc}}^{1,q}(\Omega))$, and (2.6) is true.

In addition, we can choose the test function $\varphi(x,t) = \chi_{[t_1,t_2]}\phi(x)$ in which $\phi(x) \in C_0^\infty(\Omega)$ and $\chi_{[t_1,t_2]}$ is the characteristic function of $[t_1,t_2] \subset (0,T)$. Then

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_\Omega \Biggl[\bigl(a(x)|\nabla u|^{p-2}\nabla u + b(x)|\nabla u|^{q-2}\nabla u\bigr)\nabla\phi + \sum_{i=1}^N f_i(x,t,u)\phi(x)\Biggr]dx\,dt \\
&\quad = \int_\Omega \bigl(u(x,t_2) - u(x,t_1)\bigr)\phi(x)\,dx.
\end{aligned}$$

Let $t = t_2$ and $t_1 \to 0$. Then we have (2.7). Moreover, by the following proposition, u can be defined as the trace on the boundary $\partial\Omega$, u is a solution of equation (1.1) with the initial-boundary value conditions (1.7)–(1.8). Theorem 2.4 is proved. □

Proposition 3.3 *If $u(x,t)$ is a weak solution of equation* (1.1) *with the initial value condition* (1.4) *and one of the following conditions is true*:

(i)

$$\int_\Omega a(x)^{-\frac{1}{p-1}}\,dx < \infty \tag{3.30}$$

(ii)

$$\int_\Omega b(x)^{-\frac{1}{q-1}}\,dx < \infty, \tag{3.31}$$

then

$$\int_\Omega |\nabla u|\,dx \le c(T). \tag{3.32}$$

Proof If (i) is true, then

$$\begin{aligned}\int_\Omega |\nabla u|\,dx &= \int_{\{x\in\Omega: a(x)^{\frac{1}{p-1}}|\nabla u|\le 1\}} |\nabla u|\,dx \\ &\quad + \int_{\{x\in\Omega: a^{\frac{1}{p-1}}|\nabla u|>1\}} |\nabla u|\,dx \\ &\le c\int_\Omega a(x)^{-\frac{1}{p-1}}\,dx + \int_\Omega a(x)|\nabla u|^p\,dx + c \\ &\le c.\end{aligned}$$

Similarly, if (ii) is true, we also have (3.32). □

4 The stability of the initial-boundary value problem

For small $\eta > 0$, we introduce the following functions:

$$S_\eta(s) = \int_0^s h_\eta(\tau)\,d\tau, \qquad h_\eta(s) = \frac{2}{\eta}\left(1 - \frac{|s|}{\eta}\right)_+, \qquad H_\eta(s) = \int_0^s S_\eta(\tau)\,d\tau.$$

Obviously, we have $|sh_\eta(s)| \le 1$ and

$$\lim_{\eta\to 0} S_\eta(s) = \operatorname{sgn} s, \qquad \lim_{\eta\to 0} sh_\eta(s) = 0, \qquad \lim_{\eta\to 0} H_\eta(s) = |s|, \quad s \in (-\infty, +\infty). \tag{4.1}$$

Proposition 4.1 *Let $u(x,t)$ and $v(x,t)$ be two solutions of equation* (1.1) *with the homogeneous value condition*

$$u(x,t) = v(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \tag{4.2}$$

and with different initial values $u_0(x)$ and $v_0(x)$ respectively. If $p > 1$, (3.30) *or* (3.31) *is true, and*

$$\bigl|f_i(u,x,t) - f_i(v,x,t)\bigr| \le c\bigl[a(x)^{\frac{1}{p}} + b(x)^{\frac{1}{q}}\bigr]|u - v|, \quad i = 1,2,\ldots,N, \tag{4.3}$$

then

$$\int_\Omega \left|u(x,t)-v(x,t)\right| dx \le \int_\Omega \left|u_0(x)-v_0(x)\right| dx, \quad t\in[0,T).$$

Proof Since (3.30) or (3.31) is true, Proposition 3.3 implies that the boundary value condition (4.2) is true in the sense of trace. Choose $\varphi = \chi_{[\tau,s]}S_\eta(u-v)$ as the test function, where $\chi_{[\tau,s]}$ is the characteristic function of $[\tau,s]\subset(0,T)$. Then

$$\begin{aligned}
&\int_\tau^s\int_\Omega S_\eta(u-v)\frac{\partial(u-v)}{\partial t}\,dx\,dt \\
&\quad+\int_\tau^s\int_\Omega a(x)\bigl(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\bigr)\nabla(u-v)h_\eta(u-v)\,dx\,dt \\
&\quad+\int_\tau^s\int_\Omega b(x)\bigl(|\nabla u|^{q-2}\nabla u-|\nabla v|^{q-2}\nabla v\bigr)\nabla(u-v)h_\eta(u-v)\,dx\,dt \\
&\quad+\sum_{i=1}^N\int_\tau^s\int_\Omega\bigl[f_i(u,x,t))-f_i(v,x,t)\bigr](u-v)_{x_i}h_\eta(u-v)\,dx\,dt \\
&=0.
\end{aligned}\tag{4.4}$$

By that $\iint_{Q_T}|u_t|\,dx\,dt\le c$, $\iint_{Q_T}|v_t|\,dx\,dt\le c$, we can use the dominated convergence theorem to obtain

$$\begin{aligned}
&\lim_{\eta\to0}\int_\tau^s\int_\Omega S_\eta(u-v)\frac{\partial(u-v)}{\partial t}\,dx\,dt \\
&=\lim_{\eta\to0}\int_\Omega\bigl[H_\eta(u-v)(x,s)-H_\eta(u-v)(x,\tau)\bigr]\,dx \\
&=\int_\Omega|u-v|(x,s)\,dx-\int_\Omega|u-v|(x,\tau)\,dx.
\end{aligned}\tag{4.5}$$

Also, since a p-Laplacian operator is a monotone operator, then we have

$$\int_\tau^s\int_\Omega a(x)\bigl(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\bigr)\nabla(u-v)h_\eta(u-v)\,dx\,dt\ge0\tag{4.6}$$

and

$$\int_\tau^s\int_\Omega b(x)\bigl(|\nabla u|^{q-2}\nabla u-|\nabla v|^{q-2}\nabla v\bigr)\nabla(u-v)h_\eta(u-v)\,dx\,dt\ge0.\tag{4.7}$$

Moreover, since $f_i(s,x,t)$ satisfies (4.3), we have

$$\begin{aligned}
&\lim_{\eta\to0}\sum_{i=1}^N\left|\int_\tau^s\int_\Omega\bigl[f_i(u,x,t)-f_i(v,x,t)\bigr](u-v)_{x_i}h_\eta(u-v)\,dx\,dt\right| \\
&\le c\lim_{\eta\to0}\int_\tau^s\int_\Omega\bigl|h_\eta(u-v)(u-v)\bigr|\bigl[a(x)^{\frac1p}+b(x)^{\frac1q}\bigr]\bigl|\nabla(u-v)\bigr|\,dx\,dt \\
&\le c\lim_{\eta\to0}\int_\tau^s\int_\Omega\bigl|h_\eta(u-v)(u-v)\bigr|a(x)^{\frac1p}\bigl|\nabla(u-v)\bigr|\,dx\,dt
\end{aligned}$$

$$\begin{aligned}
&+ c \lim_{\eta\to 0} \int_\tau^s \int_\Omega \left|h_\eta(u-v)(u-v)\right| b(x)^{\frac{1}{q}} \left|\nabla(u-v)\right| \, dx\, dt \\
&\leq c \lim_{\eta\to 0} \left(\int_\tau^s \int_\Omega a(x)\left(|\nabla u|^p + |\nabla v|^p\right) dx\, dt \right)^{\frac{1}{p}} \\
&\quad \cdot \left(\int_\tau^s \int_\Omega \left|(u-v)h_\eta(u-v)\right|^{\frac{p}{p-1}} dx\, dt \right)^{\frac{p-1}{p}} \\
&\quad + c \lim_{\eta\to 0} \left(\int_\tau^s \int_\Omega b(x)\left(|\nabla u|^q + |\nabla v|^q\right) dx\, dt \right)^{\frac{1}{q}} \\
&\quad \cdot \left(\int_\tau^s \int_\Omega \left|(u-v)h_\eta(u-v)\right|^{\frac{q}{q-1}} dx\, dt \right)^{\frac{q-1}{q}} \\
&= 0.
\end{aligned} \tag{4.8}$$

Finally, let $\eta \to 0$ in (4.4). By (4.5)–(4.8), we have

$$\int_\Omega \left|u(x,s) - v(x,s)\right| dx \leq \int_\Omega \left|u(x,\tau) - v(x,\tau)\right| dx.$$

Let $\tau \to 0$. Then

$$\int_\Omega \left|u(x,s) - v(x,s)\right| dx \leq \int_\Omega \left|u_0(x) - v_0(x)\right| dx.$$

Proposition 4.1 is proved. □

Proof of Theorem 2.5 Comparing with Proposition 4.1, Theorem 2.5 is not with condition (4.3). By checking the proof of Proposition 4.1, we only need to show

$$\lim_{\eta\to 0} \left| \int_\Omega \left[f_i(u,x,t) - f_i(v,x,t)\right](u-v)_{x_i} h_\eta(u-v)\, dx \right| = 0 \tag{4.9}$$

without condition (4.3). We give the details below

$$\begin{aligned}
&\int_\Omega \left[f_i(u,x,t) - f_i(v,x,t)\right](u-v)_{x_i} h_\eta(u-v)\, dx \\
&\quad = \int_{\{\Omega:|u-v|<\eta\}} \left[f_i(u,x,t) - f_i(v,x,t)\right](u-v)_{x_i} h_\eta(u-v)\, dx.
\end{aligned} \tag{4.10}$$

When the set $\{\Omega : |u-v| = 0\}$ is with zero measure, since $|f_i(u,x,t) - f_i(v,x,t)| \leq c$, we have

$$\begin{aligned}
&\lim_{\eta\to 0} \left| \int_{\{\Omega:|u-v|<\eta\}} \left[f_i(u,x,t) - f_i(v,x,t)\right](u-v)_{x_i} h_\eta(u-v)\, dx \right| \\
&\quad = \frac{1}{2} \lim_{\eta\to 0} \left| \int_{\{\Omega:|u-v|<\eta\}} \left[f_i(u,x,t) - f_i(v,x,t)\right] \right. \\
&\qquad \left. \cdot \left[a(x)^{\frac{1}{p-1}} a(x)^{-\frac{1}{p-1}} + b(x)^{\frac{1}{q-1}} b(x)^{-\frac{1}{q-1}}\right](u-v)_{x_i} h_\eta(u-v)\, dx \right| \\
&\quad \leq c \left(\int_{\{\Omega:|u-v|=0\}} \left(a(x)\left(|\nabla u|^p + |\nabla v^p)\right)\right)^p dx \right)^{\frac{1}{p}} \left(\int_\Omega a(x)^{-\frac{1}{p-1}} dx \right)^{\frac{p-1}{p}}
\end{aligned}$$

$$+c\left(\int_{\{\Omega:|u-v|=0\}}\left(b(x)\left(|\nabla u|^q+|\nabla v|^q\right)\right)^q dx\right)^{\frac{1}{q}}\left(\int_\Omega b(x)^{-\frac{1}{q-1}}dx\right)^{\frac{q-1}{q}}$$
$$=0.$$

When the set $\{\Omega:|u-v|=0\}$ is with a positive measure, by (2.13), $[a(x)^{-\frac{1}{p-1}}+b(x)^{-\frac{1}{p-1}}]\in L^1(\Omega)$, we have

$$\begin{aligned}
&\lim_{\eta\to 0}\left|\int_{\{\Omega:|u-v|<\eta\}}\left[f_i(u,x,t)-f_i(v,x,t)\right](u-v)_{x_i}h_\eta(u-v)\,dx\right|\\
&\le c\left(\int_{\{\Omega:|u-v|=0\}}\left(a(x)^{\frac{1}{p}}|\nabla u-\nabla v|\right)^p dx\right)^{\frac{1}{p}}\left(\int_\Omega a(x)^{-\frac{1}{p-1}}dx\right)^{\frac{p-1}{p}}\\
&\quad+c\left(\int_{\{\Omega:|u-v|=0\}}\left(b(x)^{\frac{1}{q}}|\nabla u-\nabla v|\right)^q dx\right)^{\frac{1}{q}}\left(\int_\Omega b(x)^{-\frac{1}{q-1}}dx\right)^{\frac{q-1}{q}}\\
&=0.
\end{aligned}$$

Thus, we have the conclusion. □

5 Proof of Theorems 2.6

Proof of Theorems 2.6 Since $a(x)b(x)=0, x\in\partial\Omega$, we can define $\Omega_\eta=\{x\in\Omega:a(x)b(x)>\eta\}$,

$$\phi_\eta(x)=\begin{cases}1, & \text{if } x\in\Omega_\eta,\\ \frac{a(x)b(x)}{\eta}, & \text{if } x\in\Omega\setminus\Omega_\eta,\end{cases}\tag{5.1}$$

and choose $\chi_{[\tau,s]}\phi_\eta(x)S_\eta(u-v)$ as the test function. Thus

$$\begin{aligned}
&\int_\tau^s\int_\Omega\phi_\eta S_\eta(u-v)\frac{\partial(u-v)}{\partial t}\,dx\,dt\\
&\quad+\int_\tau^s\int_\Omega a(x)\left(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\right)\nabla(u-v)h_\eta(u-v)\phi_\eta(x)\,dx\,dt\\
&\quad+\int_\tau^s\int_\Omega a(x)\left(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\right)(u-v)S_\eta(u-v)\nabla\phi_\eta\,dx\,dt\\
&\quad+\int_\tau^s\int_\Omega b(x)\left(|\nabla u|^{q-2}\nabla u-|\nabla v|^{q-2}\nabla v\right)\nabla(u-v)h_\eta(u-v)\phi_\eta(x)\,dx\,dt\\
&\quad+\int_\tau^s\int_\Omega b(x)\left(|\nabla u|^{q-2}\nabla u-|\nabla v|^{q-2}\nabla v\right)(u-v)S_\eta(u-v)\nabla\phi_\eta\,dx\,dt\\
&\quad+\sum_{i=1}^N\int_\tau^s\int_\Omega\left[f_i(u,x,t)-f_i(v,x,t)\right]\phi_{\eta x_i}S_\eta(u-v)\,dx\,dt\\
&\quad+\sum_{i=1}^N\int_\tau^s\int_\Omega\left[f_i(u,x,t)-f_i(v,x,t)\right](u-v)_{x_i}\phi_\eta h_\eta(u-v)\,dx\,dt\\
&=0.
\end{aligned}\tag{5.2}$$

At first, we have

$$\begin{aligned} &\lim_{\eta\to 0}\int_\tau^s\int_\Omega \phi_\eta(x)S_\eta(u-v)\frac{\partial(u-v)}{\partial t}\,dx\,dt \\ &\quad=\lim_{\eta\to 0}\int_\tau^s\int_\Omega \frac{\partial[\phi_\eta(x)H_\eta(u-v)]}{\partial t}\,dx\,dt \\ &\quad=\int_\Omega |u-v|(x,s)\,dx-\int_\Omega |u-v|(x,\tau)\,dx. \end{aligned} \tag{5.3}$$

Secondly, we have

$$\int_\Omega a(x)\bigl(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\bigr)\nabla(u-v)h_\eta(u-v)\phi_\eta(x)\,dx\geq 0 \tag{5.4}$$

and

$$\int_\Omega b(x)\bigl(|\nabla u|^{q-2}\nabla u-|\nabla v|^{q-2}\nabla v\bigr)\nabla(u-v)h_\eta(u-v)\phi_\eta(x)\,dx\geq 0. \tag{5.5}$$

Thirdly, we have

$$\nabla\phi_\eta(x)=\begin{cases} 0, & \text{if } x\in\Omega_\eta, \\ \frac{1}{\eta}\nabla(a(x)b(x)), & \text{if } x\in\Omega\setminus\Omega_\eta, \end{cases}$$

condition (2.17) yields

$$\begin{aligned} &\biggl|\int_\Omega a(x)\bigl(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\bigr)\nabla\phi_\eta S_\eta(u-v)\,dx\biggr| \\ &\quad=\biggl|\int_{\Omega\setminus\Omega_\eta} a(x)\bigl(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\bigr)\nabla\phi_\eta S_\eta(u-v)\,dx\biggr| \\ &\quad\leq\frac{1}{\eta}\int_{\Omega\setminus\Omega_\eta} a(x)\bigl(|\nabla u|^{p-1}+|\nabla v|^{p-1}\bigr)\nabla\bigl(a(x)b(x)\bigr)S_\eta(u-v)|\,dx \\ &\quad\leq\frac{c}{\eta}\biggl(\int_{\Omega\setminus\Omega_\eta}\bigl[a(x)\bigl(|\nabla u|^p+|\nabla v|^p\bigr)\bigr]\,dx\biggr)^{\frac{p-1}{p}} \\ &\qquad\cdot\biggl(\int_{\Omega\setminus\Omega_\eta} a(x)\bigl|\nabla\bigl(a(x)b(x)\bigr)\bigr|^p\,dx\biggr)^{\frac{1}{p}} \\ &\quad\leq c\biggl[\biggl(\int_{\Omega\setminus\Omega_\eta} a(x)|\nabla u|^p\,dx\biggr)^{\frac{p-1}{p}}+\biggl(\int_{\Omega\setminus\Omega_\eta} a(x)|\nabla v|^p\,dx\biggr)^{\frac{p-1}{p}}\biggr] \\ &\qquad\cdot\biggl[\biggl(\int_{\Omega\setminus\Omega_\eta} a(x)^{1-p}|\nabla a|^p\,dx\biggr)^{\frac{1}{p}}+\biggl(\int_{\Omega\setminus\Omega_\eta} a(x)b(x)^{-p}|\nabla b|^p\,dx\biggr)^{\frac{1}{p}}\biggr] \\ &\quad\leq c\biggl[\biggl(\int_{\Omega\setminus\Omega_\eta} a(x)|\nabla u|^p\,dx\biggr)^{\frac{p-1}{p}}+\biggl(\int_{\Omega\setminus\Omega_\eta} a(x)|\nabla v|^p\,dx\biggr)^{\frac{p-1}{p}}\biggr], \end{aligned} \tag{5.6}$$

which goes to zero as $\eta \to 0$. Similarly, by (2.18), we have

$$\begin{aligned}
&\left|\int_\Omega b(x)\big(|\nabla u|^{q-2}\nabla u-|\nabla v|^{q-2}\nabla v\big)\nabla\phi_\eta S_\eta(u-v)\,dx\right| \\
&\quad=\left|\int_{\Omega\setminus\Omega_\eta} b(x)\big(|\nabla u|^{q-2}\nabla u-|\nabla v|^{q-2}\nabla v\big)\nabla\phi_\eta S_\eta(u-v)\,dx\right| \\
&\quad\le\frac{1}{\eta}\int_{\Omega\setminus\Omega_\eta} b(x)\big(|\nabla u|^{q-1}+|\nabla v|^{q-1}\big)\nabla\big(a(x)b(x)\big)S_\eta(u-v)|\,dx \\
&\quad\le\frac{c}{\eta}\left(\int_{\Omega\setminus\Omega_\eta}\big[b(x)\big(|\nabla u|^q+|\nabla v|^q\big)\big]\,dx\right)^{\frac{q-1}{q}}\left(\int_{\Omega\setminus\Omega_\eta} b(x)\big|\nabla\big(a(x)b(x)\big)\big|^q\,dx\right)^{\frac{1}{q}} \\
&\quad\le c\left[\left(\int_{\Omega\setminus\Omega_\eta} b(x)|\nabla u|^q\,dx\right)^{\frac{q-1}{q}}+\left(\int_{\Omega\setminus\Omega_\eta} b(x)|\nabla v|^q\,dx\right)^{\frac{q-1}{q}}\right] \\
&\qquad\cdot\left[\left(\int_{\Omega\setminus\Omega_\eta} b(x)^{1-q}|\nabla b|^q\,dx\right)^{\frac{1}{q}}+\left(\int_{\Omega\setminus\Omega_\eta} b(x)a(x)^{-q}|\nabla a|^q\,dx\right)^{\frac{1}{q}}\right] \\
&\quad\le c\left[\left(\int_{\Omega\setminus\Omega_\eta} b(x)|\nabla u|^q\,dx\right)^{\frac{q-1}{q}}+\left(\int_{\Omega\setminus\Omega_\eta} b(x)|\nabla v|^q\,dx\right)^{\frac{q-1}{q}}\right],
\end{aligned}\tag{5.7}$$

which goes to zero as $\eta \to 0$.

Fourthly, since $u(x,t), v(x,t) \in L^\infty(Q_T)$, condition (2.17) yields

$$\begin{aligned}
&\lim_{\eta\to0}\left|\int_\Omega\big[f_i(u,x,t)-f_i(v,x,t)\big]\phi_{\eta x_i}S_\eta(u-v)\,dx\right| \\
&\quad=\lim_{\eta\to0}\left|\int_{\Omega\setminus\Omega_\eta}\big[f_i(u,x,t)-f_i(v,x,t)\big]\phi_{\eta x_i}S_\eta(u-v)\,dx\right| \\
&\quad\le\lim_{\eta\to0}\frac{1}{\eta}\int_{\Omega\setminus\Omega_\eta}\big[a(x)^{\frac{1}{p}}+b(x)^{\frac{1}{q}}\big]\big|\big(a(x)b(x)\big)_{x_i}\big|\big|S_\eta(u-v)(u-v)\big|\,dx \\
&\quad\le\lim_{\eta\to0}\left(\int_{\Omega\setminus\Omega_\eta}\big(a(x)^{1-p}|\nabla a|^p+a(x)b(x)^{-p}|\nabla b|^p\big)\,dx\right)^{\frac{1}{p}} \\
&\qquad\times\left(\int_\Omega\big|S_\eta(u-v)(u-v)\big|^{\frac{p}{p-1}}\,dx\right)^{\frac{p-1}{p}} \\
&\qquad+\lim_{\eta\to0}\left(\int_{\Omega\setminus\Omega_\eta}\big(b(x)^{1-q}|\nabla b|^q+b(x)a(x)^{-q}|\nabla a|^q\big)\,dx\right)^{\frac{1}{q}} \\
&\qquad\times\left(\int_\Omega\big|S_\eta(u-v)(u-v)\big|^{\frac{q}{q-1}}\,dx\right)^{\frac{q-1}{q}} \\
&\quad\le c\left(\int_\Omega|u-v|\,dx\right)^{\frac{p-1}{p}}+c\left(\int_\Omega|u-v|\,dx\right)^{\frac{q-1}{q}}.
\end{aligned}\tag{5.8}$$

Moreover, by condition (2.16), we have

$$\lim_{\eta\to0}\sum_{i=1}^N\int_\tau^s\int_\Omega\big[f_i(u,x,t)-f_i(v,x,t)\big](u-v)_{x_i}\phi_\eta h_\eta(u-v)\,dx\,dt=0.\tag{5.9}$$

Finally, let $\eta \to 0$ in (5.2). Then

$$\int_\Omega \left|u(x,s)-v(x,s)\right|dx \le \int_\Omega \left|u(x,\tau)-v(x,\tau)\right|dx + c\left(\int_0^t\int_\Omega |u-v|\,dx\,dt\right)^k.$$

where $k<1$.

By the generalized Gronwall inequality [28], we can extrapolate

$$\int_\Omega \left|u(x,s)-v(x,s)\right|dx \le c\int_\Omega \left|u(x,\tau)-v(x,\tau)\right|dx.$$

Letting $\tau \to 0$, we have the stability (2.15). □

Corollary 5.1 *Suppose that $a(x)$ and $b(x)$ satisfy condition* (a) *or condition* (b), *and satisfy* (2.17)(2.18). *Let $q \ge p > 1$, $a(x), b(x) \in C^1(\overline{\Omega})$ satisfy* (1.5), *and when $|s_1|, |s_2| \le c$, there is a nonnegative continuous function $g_i(x)$ such that*

$$\left|f_i(x)(s_1,x,t)-f_i(s_2,x,t)\right| \le cg_i(x), \quad i=1,2,\ldots,N. \tag{5.10}$$

If $u(x,t)$ and $v(x,t)$ are two solutions of (1.1) *with the initial values $u_0(x)$ and $v_0(x)$ respectively and with the homogeneous boundary value condition*

$$u(x,t)=v(x,t)=0, \qquad (x,t)=\Sigma_{pq}\times(0,T), \tag{5.11}$$

then stability (2.15) *is true, where*

$$\Sigma_{pq}=\left\{x\in\partial\Omega : \sum_{i=1}^N \left(a(x)b(x)\right)_{x_i} g_i(x)=0\right\}.$$

Proof Similar to the proof of Theorem 2.6, we have (5.2)–(5.7). Since $a(x)$ and $b(x)$ satisfy condition (a) or condition (b), Proposition 3.3 means that the partial boundary value condition (5.11) is true in the classical sense of the trace. Then by condition (2.17) it yields

$$\begin{aligned}
&\lim_{\eta\to 0}\left|\int_\Omega \sum_{i=1}^N \left[f_i(u,x,t)-f_i(v,x,t)\right]\phi_{\eta x_i} S_\eta(u-v)\,dx\right| \\
&= \lim_{\eta\to 0}\left|\int_{\Omega\setminus\Omega_\eta} \sum_{i=1}^N \left[f_i(u,x,t)-f_i(v,x,t)\right]\phi_{\eta x_i} S_\eta(u-v)\,dx\right| \\
&\le c\lim_{\eta\to 0}\frac{1}{\eta}\int_{\Omega\setminus\Omega_\eta} \sum_{i=1}^N \left|\left(a(x)b(x)\right)_{x_i} g_i(x)\right|\left|S_\eta(u-v)(u-v)\right|dx \\
&= c\int_{\partial\Omega}\sum_{i=1}^N \left|\left(a(x)b(x)\right)_{x_i} g_i(x)\right||u-v|\,dx \\
&= 0.
\end{aligned} \tag{5.12}$$

The remaining process of the proof can be completed as that of Theorem 2.6. □

6 Proof of Theorems 2.7

In this section, we use a similar method as that used in the proof of Theorem 2.6 to prove Theorem 2.7.

Proof of Theorem 2.7 Let $u(x,t)$ and $v(x,t)$ be two weak solutions of equation (1.1) with the initial values $u_0(x)$ and $v_0(x)$ respectively. Different from the proof of Theorem 2.6), $a(x)$ and $b(x)$ may satisfy (2.22).

Since $a(x) = b(x) = 0$ when $x \in \partial\Omega$, but $a(x) + b(x) > 0$ when $x \in \Omega$, we define $D_\eta = \{x \in \Omega : a(x) + b(x) > \eta\}$ and let

$$\varphi_\eta(x) = \begin{cases} 1, & \text{if } x \in D_\eta, \\ \frac{a(x)+b(x)}{\eta}, & \text{if } x \in \Omega \setminus D_\eta. \end{cases} \tag{6.1}$$

By choosing $\chi_{[\tau,s]}\varphi_\eta(x)S_\eta(u-v)$ as the test function, we have

$$\begin{aligned} &\int_\tau^s \int_\Omega \varphi_\eta S_\eta(u-v)\frac{\partial(u-v)}{\partial t}\,dx\,dt \\ &\quad + \int_\tau^s \int_\Omega a(x)\bigl(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\bigr)\nabla(u-v)h_\eta(u-v)\varphi_\eta(x)\,dx\,dt \\ &\quad + \int_\tau^s \int_\Omega a(x)\bigl(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\bigr)(u-v)S_\eta(u-v)\nabla\varphi_\eta\,dx\,dt \\ &\quad + \int_\tau^s \int_\Omega b(x)\bigl(|\nabla u|^{q-2}\nabla u - |\nabla v|^{q-2}\nabla v\bigr)\nabla(u-v)h_\eta(u-v)\varphi_\eta(x)\,dx\,dt \\ &\quad + \int_\tau^s \int_\Omega b(x)\bigl(|\nabla u|^{q-2}\nabla u - |\nabla v|^{q-2}\nabla v\bigr)(u-v)S_\eta(u-v)\nabla\varphi_\eta\,dx\,dt \\ &\quad + \sum_{i=1}^N \int_\tau^s \int_\Omega \bigl[f_i(u,x,t) - f_i(v,x,t)\bigr]\varphi_{\eta x_i}S_\eta(u-v)\,dx\,dt \\ &\quad + \sum_{i=1}^N \int_\tau^s \int_\Omega \bigl[f_i(u,x,t) - f_i(v,x,t)\bigr](u-v)_{x_i}\varphi_\eta h_\eta(u-v)\,dx\,dt \\ &= 0. \end{aligned} \tag{6.2}$$

Directly, we have the following three formulas similar to (5.3)–(5.5):

$$\begin{aligned} &\lim_{\eta\to 0}\int_\tau^s \int_\Omega \varphi_\eta(x)S_\eta(u-v)\frac{\partial(u-v)}{\partial t}\,dx\,dt \\ &= \lim_{\eta\to 0}\int_\tau^s \int_\Omega \frac{\partial[\varphi_\eta(x)H_\eta(u-v)]}{\partial t}\,dx\,dt \\ &= \int_\Omega |u-v|(x,s)\,dx - \int_\Omega |u-v|(x,\tau)\,dx, \end{aligned} \tag{6.3}$$

$$\int_\Omega a(x)\bigl(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\bigr)\nabla(u-v)h_\eta(u-v)\varphi_\eta(x)\,dx \ge 0 \tag{6.4}$$

and

$$\int_\Omega b(x)\bigl(|\nabla u|^{q-2}\nabla u - |\nabla v|^{q-2}\nabla v\bigr)\nabla(u-v)h_\eta(u-v)\varphi_\eta(x)\,dx \ge 0. \tag{6.5}$$

Moreover, condition (2.20) yields

$$\begin{aligned}
&\left|\int_\Omega a(x)\big(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\big)\nabla\varphi_\eta S_\eta(u-v)\,dx\right| \\
&\quad=\left|\int_{\Omega\setminus D_\eta} a(x)\big(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\big)\nabla\varphi_\eta S_\eta(u-v)\,dx\right| \\
&\quad\le\frac{1}{\eta}\int_{\Omega\setminus D_\eta} a(x)\big(|\nabla u|^{p-1}+|\nabla v|^{p-1}\big)\nabla\big(a(x)+b(x)\big)S_\eta(u-v)|\,dx \\
&\quad\le\frac{c}{\eta}\left(\int_{\Omega\setminus D_\eta}\big[a(x)\big(|\nabla u|^{p}+|\nabla v|^{p}\big)\big]\,dx\right)^{\frac{p-1}{p}}\left(\int_{\Omega\setminus D_\eta} a(x)\big|\nabla\big(a(x)+b(x)\big)\big|^p\,dx\right)^{\frac{1}{p}} \\
&\quad\le c\left[\left(\int_{\Omega\setminus D_\eta} a(x)|\nabla u|^p\,dx\right)^{\frac{p-1}{p}}+\left(\int_{\Omega\setminus D_\eta} a(x)|\nabla v|^p\,dx\right)^{\frac{p-1}{p}}\right],
\end{aligned}\tag{6.6}$$

which goes to zero as $\eta\to 0$. Similarly, by (2.14), we can show that

$$\begin{aligned}
&\left|\int_\Omega b(x)\big(|\nabla u|^{q-2}\nabla u-|\nabla v|^{q-2}\nabla v\big)\nabla\varphi_\eta S_\eta(u-v)\,dx\right| \\
&\quad\le c\left[\left(\int_{\Omega\setminus D_\eta} b(x)|\nabla u|^q\,dx\right)^{\frac{q-1}{q}}+\left(\int_{\Omega\setminus D_\eta} b(x)|\nabla v|^q\,dx\right)^{\frac{q-1}{q}}\right],
\end{aligned}\tag{6.7}$$

which goes to zero as $\eta\to 0$.

At the same time, since $u(x,t),v(x,t)\in L^\infty(Q_T)$, condition (2.19) yields

$$\begin{aligned}
&\lim_{\eta\to 0}\left|\int_\Omega\big[f_i(u,x,t)-f_i(v,x,t)\big]\varphi_{\eta x_i}S_\eta(u-v)\,dx\right| \\
&\quad=\lim_{\eta\to 0}\left|\int_{\Omega\setminus D_\eta}\big[f_i(u,x,t)-f_i(v,x,t)\big]\varphi_{\eta x_i}S_\eta(u-v)\,dx\right| \\
&\quad\le\lim_{\eta\to 0}\frac{1}{\eta}\int_{\Omega\setminus D_\eta}\left[\big(a(x)+b(x)\big)^{\frac{1}{p}}+\big(a(x)+b(x)\big)^{\frac{1}{q}}\right] \\
&\qquad\times\left|\big(a(x)+b(x)\big)_{x_i}\right|\left|S_\eta(u-v)(u-v)\right|\,dx \\
&\quad\le\lim_{\eta\to 0}\left(\int_{\Omega\setminus D_\eta}\left(\left|\nabla(a+b)\right|^{1+\frac{1}{p}}\right)dx\right)^{\frac{1}{p}}\left(\int_\Omega\left|S_\eta(u-v)(u-v)\right|^{\frac{p}{p-1}}\,dx\right)^{\frac{p-1}{p}} \\
&\qquad+\lim_{\eta\to 0}\left(\int_{\Omega\setminus D_\eta}\left(\left|\nabla(a+b)\right|^{1+\frac{1}{p}}\right)dx\right)^{\frac{1}{q}}\left(\int_\Omega\left|S_\eta(u-v)(u-v)\right|^{\frac{q}{q-1}}\,dx\right)^{\frac{q-1}{q}} \\
&\quad\le c\left(\int_\Omega|u-v|\,dx\right)^{\frac{p-1}{p}}+c\left(\int_\Omega|u-v|\,dx\right)^{\frac{q-1}{q}}.
\end{aligned}\tag{6.8}$$

For another term on the left-hand side of (6.2), conditions (2.19)–(2.21) yield

$$\lim_{\eta\to 0}\sum_{i=1}^N\int_\tau^s\int_\Omega\big[f_i(u,x,t)-f_i(v,x,t)\big](u-v)_{x_i}\varphi_\eta h_\eta(u-v)\,dx\,dt=0.\tag{6.9}$$

Finally, let $\eta \to 0$ in (6.2). Then

$$\int_\Omega \bigl|u(x,s)-v(x,s)\bigr|\,dx \le \int_\Omega \bigl|u(x,\tau)-v(x,\tau)\bigr|\,dx + c\biggl(\int_0^t\int_\Omega |u-v|\,dx\,dt\biggr)^k.$$

where $k<1$.

Similar to the proof of Theorem 2.6, we can deduce conclusion (2.15). □

Corollary 6.1 *Suppose that $a(x)$ and $b(x)$ satisfy condition (a) or condition (b), and satisfy (2.20), (2.21). Let $q \ge p > 1$, $a(x), b(x) \in C^1(\overline{\Omega})$ satisfy (1.4), and when $|s_1|, |s_2| \le c$, $f_i(s,x,t)$ satisfies (5.10). If $u(x,t)$ and $v(x)$ are two solutions of (1.1) with the initial values $u_0(x)$ and $v_0(x)$ respectively and with the homogeneous boundary value condition*

$$u(x,t)=v(x,t)=0, \qquad (x,t)=\Sigma_{pq}\times(0,T), \tag{6.10}$$

then stability (2.15) is true, where

$$\Sigma_{pq}=\Biggl\{x\in\partial\Omega: \sum_{i=1}^N \bigl(a(x)+b(x)\bigr)_{x_i} g_i(x)=0\Biggr\}.$$

The proof is similar to that of Corollary 5.1, we omit the details here.

7 A generalization of trace

Let $BV(\Omega)$ be the BV function space, i.e., $|\frac{\partial f}{\partial x_i}|$ is a regular measure, and

$$BV(\Omega)=\biggl\{f(x): \int_\Omega \biggl|\frac{\partial f}{\partial x_i}\biggr| < c, i=1,2,\ldots,N\biggr\}.$$

Then $C_0^\infty(\Omega)$ is dense in $BV(\Omega)$, and so the trace of $f(x)\in BV(\Omega)$ on the boundary $\partial\Omega$ is defined as the limit of a sequence $f_\varepsilon(x)$ as

$$f(x)|_{x\in\partial\Omega}=\lim_{\varepsilon\to 0} f_\varepsilon(x)|_{x\in\partial\Omega}. \tag{7.1}$$

It is well known that a BV function space is the weakest space such that integration by parts is still true.

For a degenerate parabolic equation, how to impose a suitable boundary condition has been an important and difficult problem for a long time. For example, if we consider the reaction-diffusion equation

$$\frac{\partial u}{\partial t}=\operatorname{div}\bigl(a(u,x,t)\nabla u\bigr)+\operatorname{div}\bigl(b(u)\bigr), \quad (x,t)\in Q_T, \tag{7.2}$$

if $a(u,x,t)$ is smooth enough, then the weak solution $u(x,t)\in BV(Q_T)$ can be proved, and so one can impose the boundary value condition (1.8) in the sense of trace in the classical way [26, 30, 32]. However, if $a(u,x,t)$ is just a continuous function or just a integral function, then one only can prove that there is a weak solution $u(x,t)\in L^\infty(Q_T)$, but $u(x,t)$ may not be a BV function. Equation (7.2) is of hyperbolic-parabolic mixed type. When $a\equiv 0$, equation (7.1) becomes a first-order hyperbolic equation, if the solution is merely in L^∞, the author of [23] first extended the trace in a weaker sense by introducing an integral formulation of the boundary condition. [23]'s idea was generalized to deal with well-posedness of weak solutions to the strongly degenerate parabolic equations (7.2) in [1, 2, 7, 11, 14, 15, 18, 22].

In this paper, we first consider the evolutionary p-Laplacian equations of the form

$$\frac{\partial u}{\partial t} - \operatorname{div}\bigl(\alpha(x)|\nabla u|^{p-2}\nabla u\bigr) - b_i(x)D_i u + c(x,t)u = f(x,t), \quad (x,t) \in Q_T, \tag{7.3}$$

where $D_i = \frac{\partial}{\partial x_i}$, $\alpha(x) \in C(\Omega)$, $\alpha(x) > 0$ in Ω but may be equal to 0 on the boundary $\partial\Omega$. The author of [25] classified the boundary $\partial\Omega$ into three parts: the nondegenerate boundary Σ_3, the weakly degenerate boundary Σ_4, and the strongly degenerate boundary Σ^0. In detail, the author of [25] denoted that

$$\Sigma_3 = \bigl\{x \in \partial\Omega : \alpha(x) > 0\bigr\},$$
$$\Sigma_4 = \left\{x \in \partial\Omega : \alpha(x) = 0, \text{there exists } r > 0, \text{such that } \int_{\Omega\cap B_r(x)} a(y)^{-\frac{1}{p-1}}\,dy < +\infty\right\},$$
$$\Sigma^0 = \partial\Omega \setminus (\Sigma_3 \cup \Sigma_4) = \left\{x \in \partial\Omega : \text{for any small } r > 0, \int_{\Omega\cap B_r(x)} a(y)^{-\frac{1}{p-1}}\,dy = +\infty\right\},$$

where $B_r(x) = \{y : d(x,y) < r\}$. Meanwhile, they defined

$$\Sigma_0 = \left\{x \in \Sigma^0 : \sum_{i=1}^N b_i(x)n_i(x) = 0\right\},$$
$$\Sigma_1 = \left\{x \in \Sigma^0 : \sum_{i=1}^N b_i(x)n_i(x) > 0\right\},$$
$$\Sigma_2 = \left\{x \in \Sigma^0 : \sum_{i=1}^N b_i(x)n_i(x) < 0\right\},$$

where $\vec{n} = \{n_i(x)\}$ is the inner normal vector of $\partial\Omega$. In order to study the well-posedness of weak solutions to equation (7.3), they imposed a partial boundary value condition as

$$u(x,t) = g(x,t), (x,t) \in (\Sigma_2 \cup \Sigma_3 \cup \Sigma_4) \times (0,T), \tag{7.4}$$

where $g(x,t)$ is an appropriately smooth function.

According to Proposition 3.3, it is obvious that on $(\Sigma_3 \cup \Sigma_4) \times (0,T)$ the boundary value condition is true in the classical trace sense. So, the trouble lies in that the classical trace of u on the strongly degenerate boundary Σ^0 cannot be defined.

Denote that $\Omega_\lambda = \{x \in \Omega : d(x) > \lambda\}$ when λ is a positive infinite variable, and denote by $\mathbf{B}$ the closure of the set $C_0^\infty(Q_T)$ with respect to the norm

$$\|u\|_{\mathbf{B}} = \iint_{Q_T} a(x)\bigl(\bigl|u(x,t)\bigr|^p + \bigl|\nabla u(x,t)\bigr|^p\bigr)\,dx\,dt, \quad u \in \mathbf{B}.$$

The author of [25] defined the trace of $u \in \mathbf{B}$, $u(x,t)=0$ on Σ_2 as

$$\operatorname{ess}\lim_{\lambda\to 0}\int_{\{x\in\partial\Omega_\lambda:\sum_{i=1}^N b_i(x)n_i(x)<0\}} u^2\sum_{i=1}^N b_i(x)n_i(x)\,d\sigma = 0. \tag{7.5}$$

Remark 2.2 in [25] pointed out that the usual trace of $u \in \mathbf{B}$, $u(x,t)=0$ on $\Sigma_3\cup\Sigma_4$ also satisfies (7.5). So, (7.5) is a generalization of the usual trace of $u\in BV(Q_T)$ to that of $u\in\mathbf{B}$.

Moreover, we can generalize the trace of $u\in BV(Q_T)$ to that of $u\in\mathbf{B}$ by a more general way. Let $\phi(x)$ be a weak characteristic function of Ω [33], i.e., $\phi(x)\in C(\overline{\Omega})\cap C^1(\Omega\setminus\Omega_\mu)$ and

$$\phi(x)>0,\quad x\in\Omega,$$

where

$$\Omega_\mu=\bigl\{x\in\Omega:\phi(x)>\mu\bigr\}.$$

In a very recent paper [34], using some idea of [25], we defined the trace of $u\in\mathbf{B}$, $u(x,t)=0$ on Σ^0 as

$$\operatorname{ess}\lim_{\mu\to 0^+}\int_{\{x\in\partial\Omega_\mu:\sum_{i=1}^N b_i(x)\phi_i(x)<0\}} u^2\sum_{i=1}^N b_i(x)\phi_{x_i}(x)\,d\sigma = 0, \tag{7.6}$$

and the partial boundary value condition matching up with equation (7.3) is

$$u(x,t)=v(x,t)=0,\qquad (x,t)=\Sigma_p\times(0,T), \tag{7.7}$$

in the sense of (7.5), where

$$\Sigma_p=\Biggl\{x\in\partial\Omega:\sum_{i=1}^N b_i(x)\phi_{x_i}(x)<0\Biggr\}.$$

Secondly, let us come back to our main equation (1.1). Denote that

$$\|u\|_{\mathbf{B_p}}=\iint_{Q_T} a(x)\bigl(\bigl|u(x,t)\bigr|^p+\bigl|\nabla u(x,t)\bigr|^p\bigr)\,dx\,dt,\quad u\in\mathbf{B_p},$$

and

$$\|u\|_{\mathbf{B_q}}=\iint_{Q_T} b(x)\bigl(\bigl|u(x,t)\bigr|^q+\bigl|\nabla u(x,t)\bigr|^q\bigr)\,dx\,dt,\quad u\in\mathbf{B_q}.$$

If $a(x)$, $b(x)$ satisfy (c), we cannot impose the boundary value condition (7.7) in the sense of the classical trace generally. However, inspired by [25, 34], if f_i satisfies (5.10), by checking the proof of Corollary 5.1, then we may generalize the trace of $u\in BV(Q_T)$ to that of $u\in\mathbf{B_p}\cap\mathbf{B_q}$, $u(x,t)=0$ as

$$\operatorname{ess}\lim_{\lambda\to 0}\int_{\{x\in\partial\Omega_\mu:\sum_{i=1}^N g_i(x)\phi_{x_i}\neq 0\}} |u|\left|\sum_{i=1}^N g_i(x)\phi_{x_i}\right|d\sigma = 0. \tag{7.8}$$

Accordingly, if $a(x)$, $b(x)$ satisfy (c), in order to study the uniqueness of weak solution to equation (1.1), we can impose the partial boundary value condition

$$u(x,t) = 0, \qquad (x,t) \in (\Sigma_p \cup \Sigma_q) \times (0,T), \tag{7.9}$$

in the sense of (7.8), where

$$\Sigma_p = \left\{ x \in \partial\Omega : \text{for any small } r > 0, \int_{\Omega\cap B_r(x)} a(y)^{-\frac{1}{p-1}}\,dy = +\infty \right\}$$

and

$$\Sigma_q = \left\{ x \in \partial\Omega : \text{for any small } r > 0, \int_{\Omega\cap B_r(x)} b(y)^{-\frac{1}{q-1}}\,dy = +\infty \right\}.$$

Naturally, there are other ways to generalize the trace. For example, similar to [25, 34], one also can generalize the trace of $u \in BV(Q_T)$ to that of $u \in \mathbf{B_p} \cap \mathbf{B_q}$, $u(x,t) = 0$ as

$$\operatorname{ess}\lim_{\lambda\to 0} \int_{\{x\in\partial\Omega_\mu : \sum_{i=1}^N g_i(x)\phi_{x_i} \neq 0\}} |u|^2 \left| \sum_{i=1}^N g_i(x)\phi_{x_i} \right| d\sigma = 0.$$

In this weak sense of trace, one also can study the stability of weak solution to equation (1.1) with the partial boundary value condition (7.9), provided f_i satisfies

$$\left| f_i(u,x,t) - f_i(v,x,t) \right| \leq g_i(x)|u-v|^2.$$

The details are omitted here.

8 About the regularity

The following parabolic equation with p, q-growth

$$u_t = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u\right), \quad (x,t) \in Q_T, \tag{8.1}$$

was studied in [8]. Actually, the main equation considered in [8] has a more general sense. The following definitions and theorems are deduced from [8] directly.

Definition 8.1 We identify a function $u \in L^q_{\mathrm{loc}}(0,T; W^{1,q}_{\mathrm{loc}}(\Omega))$ as a weak solution of equation (8.1) if and only if

$$\iint_{Q_T} \left[u\varphi_t - \left(|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u\right)\nabla\varphi \right] dx\,dt = 0, \quad \forall \varphi \in C_0^\infty(Q_T). \tag{8.2}$$

Theorem 8.2 (A priori estimate) *Let $u \in L^q_{\mathrm{loc}}(0,T; W^{1,q}_{\mathrm{loc}}(\Omega))$ be a weak solution of equation (8.1). Assume that*

$$2 \leq p \leq q < p + \frac{4}{N}. \tag{8.3}$$

Then we have $\nabla u \in L^\infty(Q_T, \mathbb{R}^N)$, *and for any parabolic cylinder* $Q_\rho(z_0) \subset\subset Q_T$ *and* $s \in (0,1)$, *there holds*

$$\sup_{Q_{s\rho}(z_0)} |\nabla u| \le c\left[\frac{1}{[(1-s)\rho]^{\hat{n}+2}} \iint_{Q_\rho(z_0)} \left(1+|\nabla u|^2\right)^{\frac{q}{2}}\,dz\right]^{\frac{1}{q}\frac{2q}{4-\hat{n}q-p}} \tag{8.4}$$

for a constant c *which goes to* ∞ *as* $q \to p + \frac{4}{N}$, *where* $\hat{n} = n$ *if* $n \ge 3$, $\hat{n}$ = *any number* $\in (2, \frac{4}{q-p})$ *if* $n = 2$.

Definition 8.3 We identify a function

$$u \in L^p\left(0,T; W_0^{1,p}(\Omega)\right) \cap L^q_{\mathrm{loc}}\left(0,T; W^{1,q}_{\mathrm{loc}}(\Omega)\right)$$

as a weak solution of the Cauchy–Dirichlet problem to equation (8.1) if and only if (8.2) holds and, moreover, the homogeneous Dirichlet boundary value condition is true in the sense of trace, the initial value condition is true in the sense

$$\lim_{h\to 0} \frac{1}{h} \int_0^h \int_\Omega \left|u(x,t) - u_0(x)\right|^2 \,dx\,dt = 0.$$

By considering the Cauchy–Dirichlet problem to the following equation related to the q-Laplacian

$$u_t = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \left(|\nabla u|^2 + \varepsilon\right)^{\frac{q-2}{2}}\nabla u\right), \quad (x,t) \in Q_T, \tag{8.5}$$

according to [17], there is a unique weak solution $u_\varepsilon \in C^0([0,T]; L^2(\Omega)) \cap L^q(0,T; W_0^{1,q}(\Omega))$ with $\partial_t u_\varepsilon \in L^{q'}(0,T; W^{-1,q'}(\Omega))$. Based on this fact, using the Morse iterative technique and by the Stekov mean value method, the author of [8] proved the following theorem.

Theorem 8.4 (Existence of weak solutions) *Suppose that*

$$2 \le p \le q < p + \frac{4}{N+2} \tag{8.6}$$

holds. Then there exists a weak solution u *of the Cauchy–Dirichlet problem to equation* (8.1). *Moreover, its* $L^p(0,T; W_0^{1,p}(\Omega))$ *norm bounded by a constant depends only on* p, q, N, $|u_0|_{L^\infty(\Omega)}$, *and* $|\nabla u_0|_{L^r(\Omega)}$. *Further, the solution* u *satisfies* $\nabla u \in L^\infty_{\mathrm{loc}}(Q_T, \mathbb{R}^N)$ *and* $u_t \in L^{\frac{q}{q-1}}(0,T; W^{-1,\frac{p}{q-1}}(\Omega))$. *Once more, for any parabolic cylinder* $Q_\rho(z_0) \subset\subset Q_T$ *and* $s \in (0,1)$, *there holds*

$$\sup_{Q_{s\rho}(z_0)} |\nabla u| \le c\left[\frac{1}{[(1-s)\rho]^{\hat{n}+2}} \iint_{Q_\rho(z_0)} \left(1+|\nabla u|^2\right)^{\frac{q}{2}}\,dz\right]^{\frac{1}{p}\frac{2q}{4-(\hat{n}+2)q-p}}, \tag{8.7}$$

where $\hat{n} = n$ *if* $n \ge 3$, $\hat{n}$ = *any number* $\in (2, \frac{4}{q-p} - 2)$ *if* $n = 2$.

Recalling the main equation considered in this paper

$$u_t = \operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u + b(x)|\nabla u|^{q-2}\nabla u\right) + \sum_{i=1}^N \frac{\partial f_i(u,x,t)}{\partial x_i}, \quad (x,t) \in Q_T, \tag{8.8}$$

with that $a(x) > 0$ and $b(x) > 0$ when $x \in \Omega$. Since $f_i(s,x,t)$ is a C^1 function on $\mathbb{R} \times \overline{Q}_T$, $i = 1,2,\ldots,N$, if we notice that the estimates about $|\nabla u|$ are local in Theorems 8.2 and 8.4, besides the existence theorem (Theorem 2.4 in Sect. 2), under conditions (8.3) and (8.6), then we conjecture that estimates (8.4) and (8.7) are true correspondingly. We are ready to discuss this problem thoroughly in the future; in particular, we are concerned with the boundary estimates about the weak solution $u(x,t)$ and the estimate of its gradient $|\nabla u|$ near the boundary.

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