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Lacunary statistical boundedness on time scales

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Abstract

In this paper, we introduce the concept of lacunary statistical boundedness of Δ-measurable real-valued functions on an arbitrary time scale. We also give the relations between statistical boundedness and lacunary statistical boundedness on time scales.

Keywords: Lacunary statistical convergence; Statistical boundedness; Lacunary statistical boundedness; Time scales

1 Introduction

The idea of statistical convergence was formally introduced by Fast [1] and Steinhaus [2], independently. This concept is a generalization of the classical convergence, and it depends on the density of subsets of the natural numbers $\mathbb{N}$. Let $K \subseteq \mathbb{N}$ and $K_n = \{k \leq n : k \in K\}$. Then the natural density of K is defined by $\delta(K) = \lim_n n^{-1}|K_n|$ if the limit exists, where $|K_n|$ indicates the cardinality of K_n. A sequence $x = (x_k)$ is said to be statistically convergent to L if, for every $\varepsilon > 0$, the set $K_\varepsilon := \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ has natural density zero, i.e., for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n}\left|\left\{k \leq n : |x_k - L| \geq \varepsilon\right\}\right| = 0,$$

and written as $st - \lim x = L$.

Statistical convergence has become very active in different fields of mathematics over the years and has been studied by many researchers, see [3–21].

Before moving on to the main results of the study, we need to give a brief introduction to time scale theory. A time scale is an arbitrary closed subset of the real numbers $\mathbb{R}$ in the usual topology which is denoted by $\mathbb{T}$. The calculus of time scales has been constructed by Hilger [22]. This theory is an efficient tool to unify continuous and discrete analyses in one theory as it allows integration and differentiation of the independent domain used. Therefore, it has received much attention in different branches of science and engineering. The readers can find basic calculus of time scales in [23–25]. Moreover, the idea of statistical convergence was first studied on time scales in [26] and [27], independently. Since then, many concepts related to statistical convergence and summability theory have been applied to time scales by various researchers in the literature [28–36].

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The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$$

for $t \in \mathbb{T}$, and also the graininess function $\mu : \mathbb{T} \to [0,\infty)$ is defined by $\mu(t) = \sigma(t) - t$. A closed interval on a time scale $\mathbb{T}$ is given by $[a,b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$. Open intervals or half-open intervals are defined accordingly.

In this paper, we use the Lebesgue Δ-measure by μ_Δ introduced by Guseinov [24]. In this case, it is known that if $a \in \mathbb{T}\backslash\{\max \mathbb{T}\}$, then the single point set $\{a\}$ is Δ-measurable and $\mu_\Delta(\{a\}) = \sigma(a) - a$. If $a, b \in \mathbb{T}$ and $a \leq b$, then $\mu_\Delta([a,b)_{\mathbb{T}}) = b - a$ and $\mu_\Delta((a,b)_{\mathbb{T}}) = b - \sigma(a)$; if $a, b \in \mathbb{T}\backslash\{\max \mathbb{T}\}$ and $a \leq b$, then $\mu_\Delta((a,b]_{\mathbb{T}}) = \sigma(b) - \sigma(a)$ and $\mu_\Delta([a,b]_{\mathbb{T}}) = \sigma(b) - a$, see [24].

We now recall some of the concepts defined using the time scale calculus on the summability theory:

Throughout this paper, we consider that $\mathbb{T}$ is a time scale satisfying $\inf \mathbb{T} = t_0 > 0$ and $\sup \mathbb{T} = \infty$.

Definition 1.1 ([27]) A Δ-measurable function $f : \mathbb{T} \to \mathbb{R}$ is statistically convergent to a number L on $\mathbb{T}$ if, for every $\varepsilon > 0$,

$$\lim_{t\to\infty} \frac{\mu_\Delta(\{s \in [t_0,t]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\})}{\mu_\Delta([t_0,t]_{\mathbb{T}})} = 0,$$

which is denoted by $st_{\mathbb{T}} - \lim_{t\to\infty} f(t) = L$.

Let $\theta = (k_r)$ be an increasing sequence of nonnegative integers with $k_0 = 0$ and $\sigma(k_r) - \sigma(k_{r-1}) \to \infty$ as $r \to \infty$. Then θ is called a lacunary sequence with respect to $\mathbb{T}$ [29].

Definition 1.2 ([29]) Let θ be a lacunary sequence on $\mathbb{T}$. A Δ-measurable function $f : \mathbb{T} \to \mathbb{R}$ is said to be lacunary statistically convergent to a number L if, for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{\mu_\Delta(\{s \in (k_{r-1},k_r]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\})}{\mu_\Delta((k_{r-1},k_r]_{\mathbb{T}})} = 0,$$

which is denoted by $st_{\theta-\mathbb{T}} - \lim_{t\to\infty} f(t) = L$.

If $f : \mathbb{T} \to \mathbb{R}$ is a function such that $f(t)$ satisfies a property P for all t except a set which has zero lacunary density on time scale, then it is said that $f(t)$ has the property P almost all t with respect to θ.

Definition 1.3 ([28]) Let $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function. Then f is said to be statistically bounded on $\mathbb{T}$ if there exists a number $M > 0$ such that

$$\lim_{t\to\infty} \frac{\mu_\Delta(\{s \in [t_0,t]_{\mathbb{T}} : |f(s)| > M\})}{\mu_\Delta([t_0,t]_{\mathbb{T}})} = 0.$$

The set of all statistically bounded functions on $\mathbb{T}$ is denoted by $S_{\mathbb{T}}(B)$.

The aim of this study is to introduce and examine the concept of lacunary statistical boundedness on time scales.

2 Main results

In this part, we begin by presenting a new definition, namely lacunary statistical boundedness on time scale. We then give some results related to this concept.

Definition 2.1 Let $\theta = (k_r)$ be a lacunary sequence on $\mathbb{T}$ and $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function. Then f is said to be lacunary statistically bounded on $\mathbb{T}$ if there exists a number $M > 0$ such that

$$\lim_{r\to\infty} \frac{\mu_\Delta(\{s \in (k_{r-1}, k_r]_\mathbb{T} : |f(s)| > M\})}{\mu_\Delta((k_{r-1}, k_r]_\mathbb{T})} = 0,$$

i.e., $|f(s)| \le M$ (almost all s with respect to θ). The set of all lacunary statistically bounded functions on $\mathbb{T}$ is denoted by $S_{\theta-\mathbb{T}}(B)$.

Remark 2.1

(i) If we take $\mathbb{T} = \mathbb{N}$ in Definition 2.1, then lacunary statistical boundedness on $\mathbb{T}$ reduces to lacunary statistical boundedness of sequences which is introduced in [19].

(ii) If we choose $\mathbb{T} = [a, \infty)$ $(a > 1)$ in Definition 2.1, then lacunary statistical boundedness on $\mathbb{T}$ reduces to lacunary statistical boundedness of measurable functions which is introduced in [21].

Theorem 2.1 *Every lacunary statistically convergent function on $\mathbb{T}$ is lacunary statistically bounded on $\mathbb{T}$, but the converse does not need to be true.*

Proof Let $f : \mathbb{T} \to \mathbb{R}$ be lacunary statistically convergent to M. Then, for each $\varepsilon > 0$, we have

$$\lim_{r\to\infty} \frac{\mu_\Delta(\{s \in (k_{r-1}, k_r]_\mathbb{T} : |f(s) - M| > \varepsilon\})}{\mu_\Delta((k_{r-1}, k_r]_\mathbb{T})} = 0.$$

From the fact that

$$\left\{s \in (k_{r-1}, k_r]_\mathbb{T} : \left|f(s)\right| > \varepsilon + M\right\} \subseteq \left\{s \in (k_{r-1}, k_r]_\mathbb{T} : \left|f(s) - M\right| > \varepsilon\right\},$$

we obtain that $|f(s)| \le \varepsilon + M$ (almost all s with respect to θ) which is the desired result. For the converse, we consider the following example: Let $f(s) = (-1)^s$ be a function where $s \in \mathbb{T} = \mathbb{N}$. Then f is lacunary statistically bounded, but it is not a lacunary statistically convergent function. □

Theorem 2.2 *Let $\theta = (k_r)$ be a lacunary sequence on $\mathbb{T}$ and $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function. Then f is lacunary statistically bounded if and only if there exists a bounded function $g : \mathbb{T} \to \mathbb{R}$ such that $f(s) = g(s)$ almost all s with respect to θ.*

Proof First assume that f is lacunary statistically bounded. Then there exists $M \ge 0$ such that $\delta_{\theta-\mathbb{T}}(K) = 0$, where $K = \{s \in \mathbb{T} : |f(s)| > M\}$. Let us consider the function $g : \mathbb{T} \to \mathbb{R}$ defined as follows:

$$g(s) = \begin{cases} f(s), & \text{if } s \notin K; \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that g is a Δ-measurable bounded function, and $f(s) = g(s)$ almost all s with respect to θ. Conversely, since g is bounded, there exists $L \geq 0$ such that $|g(s)| \leq L$ for all $s \in \mathbb{T}$. Let $D = \{s \in \mathbb{T} : f(s) \neq g(s)\}$. As $\delta_{\theta-\mathbb{T}}(D) = 0$, so $|f(s)| \leq L$ almost all s with respect to θ. This means that f is lacunary statistically bounded. □

Theorem 2.3 *Let $\theta = (k_r)$ be a lacunary sequence on $\mathbb{T}$. Then we have*

$$S_{\mathbb{T}}(B) \subset S_{\theta-\mathbb{T}}(B) \quad \Leftrightarrow \quad \liminf_{r\to\infty} \frac{\sigma(k_r)}{\sigma(k_{r-1})} > 1.$$

Proof *Sufficiency*. The sufficiency part of this theorem can be proved using a similar technique to Lemma 3.1 of [30].

Necessity. Conversely, assume that $\liminf_{r\to\infty} \frac{\sigma(k_r)}{\sigma(k_{r-1})} = 1$. Let us now select a subsequence $(k_{r(j)})$ of $\theta = (k_r)$ satisfying

$$\frac{\sigma(k_{r(j)}) - t_0}{\sigma(k_{r(j)-1}) - t_0} < 1 + \frac{1}{j} \quad \text{and} \quad \frac{\sigma(k_{r(j)-1}) - t_0}{\sigma(k_{r(j-1)}) - t_0} > j,$$

where $r(j) > r(j-1) + 1$.

Let us define Δ-measurable function $f : \mathbb{T} \to \mathbb{R}$ by

$$f(s) = \begin{cases} s, & s \in (k_{r(j)-1}, k_{r(j)}]_{\mathbb{T}} \quad \text{for } j = 1, 2, 3, \ldots; \\ 0, & \text{otherwise.} \end{cases}$$

Now, for any $M > 0$, there exists $j_0 \in \mathbb{N}$ such that $k_{r(j_0)-1} > M$. If $r = r(j)$, we have

$$\begin{aligned} &\frac{1}{\mu_\Delta((k_{r(j_0)-1}, k_{r(j_0)}]_{\mathbb{T}})} \mu_\Delta\big(\big\{s \in \big(k_{r(j_0)-1}, k_{r(j_0)}\big]_{\mathbb{T}} : \big|f(s)\big| > M\big\}\big) \\ &\quad \geq \frac{1}{\mu_\Delta((k_{r(j_0)-1}, k_{r(j_0)}]_{\mathbb{T}})} \mu_\Delta\big(\big\{s \in \big(k_{r(j_0)-1}, k_{r(j_0)}\big]_{\mathbb{T}} : \big|f(s)\big| > k_{r(j_0)-1}\big\}\big) = 1, \end{aligned}$$

and therefore

$$\frac{1}{\mu_\Delta((k_{r(j)-1}, k_{r(j)}]_{\mathbb{T}})} \mu_\Delta\big(\big\{s \in \big(k_{r(j)-1}, k_{r(j)}\big]_{\mathbb{T}} : \big|f(s)\big| > M\big\}\big) = 1$$

for all $j \geq j_0$. Also, if $r \neq r(j)$, then we get

$$\frac{\mu_\Delta(\{s \in (k_{r-1}, k_r]_{\mathbb{T}} : |f(s)| > M\})}{\mu_\Delta((k_{r-1}, k_r]_{\mathbb{T}})} = 0.$$

Hence, $f \notin S_{\theta-\mathbb{T}}(B)$.

Indeed, for any sufficiently $t \in \mathbb{T}$, we can find a unique $j \in \mathbb{N}$ for which $k_{r(j)-1} < t \leq k_{r(j+1)-1}$, and we can write

$$\begin{aligned} &\frac{1}{\mu_\Delta([t_0, t]_{\mathbb{T}})} \mu_\Delta\left(\left\{s \in [t_0, t]_{\mathbb{T}} : \big|f(s)\big| > \frac{t_0}{2}\right\}\right) \\ &\quad \leq \frac{1}{\mu_\Delta([t_0, k_{r(j)-1}]_{\mathbb{T}})} \mu_\Delta\left(\left\{s \in [t_0, k_{r(j-1)}]_{\mathbb{T}} : \big|f(s)\big| > \frac{t_0}{2}\right\}\right) \end{aligned}$$

$$\begin{aligned}
&+ \frac{1}{\mu_\Delta([t_0, k_{r(j)-1}]_{\mathbb{T}})}\mu_\Delta\left(\left\{s \in (k_{r(j)-1}, k_{r(j)}]_{\mathbb{T}} : \left|f(s)\right| > \frac{t_0}{2}\right\}\right) \\
&= \frac{\sigma(k_{r(j-1)}) - t_0}{\sigma(k_{r(j)-1}) - t_0} + \frac{\sigma(k_{r(j)}) - \sigma(k_{r(j)-1})}{\sigma(k_{r(j)-1}) - t_0} \\
&\le \frac{1}{j} + \frac{\sigma(k_{r(j)}) - t_0}{\sigma(k_{r(j)-1}) - t_0} - 1 \\
&< \frac{1}{j} + \frac{1}{j} = \frac{2}{j}.
\end{aligned}$$

Since $t \to \infty$ implies $j \to \infty$, we have $f \in S_{\mathbb{T}}(B)$. Therefore, we find $S_{\mathbb{T}}(B) \not\subset S_{\theta-\mathbb{T}}(B)$, which is a contradiction. □

Remark 2.2 The function $f : \mathbb{T} \to \mathbb{R}$ given in the necessity part of Theorem 2.3 is an example of a statistically bounded function which is not lacunary statistically bounded.

Theorem 2.4 *Let $\theta = (k_r)$ be a lacunary sequence on $\mathbb{T}$ such that $\mu(t) \le Kt$ for some $K > 0$ and for all $t \in \mathbb{T}$. Then we have*

$$S_{\theta-\mathbb{T}}(B) \subset S_{\mathbb{T}}(B) \quad \Leftrightarrow \quad \limsup_{r\to\infty} \frac{\sigma(k_r)}{\sigma(k_{r-1})} < \infty.$$

Proof Sufficiency. The sufficiency part of this theorem can be proved using a similar technique to Lemma 3.2 of [30].

Necessity. Conversely, assume that $\limsup_{r\to\infty} \frac{\sigma(k_r)}{\sigma(k_{r-1})} = \infty$. By the hypothesis, we can get

$$\frac{k_r}{\sigma(k_{r-1})} = \frac{k_r}{\sigma(k_r)} \frac{\sigma(k_r)}{\sigma(k_{r-1})} \ge \frac{1}{(K+1)} \frac{\sigma(k_r)}{\sigma(k_{r-1})},$$

and so

$$\limsup_{r\to\infty} \frac{k_r}{\sigma(k_{r-1})} = \infty.$$

Let us select a subsequence $(k_{r(j)})$ of $\theta = (k_r)$ such that $\frac{k_{r(j)}}{\sigma(k_{r(j)-1})} > j$.

Now define Δ-measurable function $f : \mathbb{T} \to \mathbb{R}$ by

$$f(s) = \begin{cases} s, & s \in (k_{r(j)-1}, 2\sigma(k_{r(j)-1}))_{\mathbb{T}} \text{ for } j = 1, 2, 3, \ldots; \\ 0, & \text{otherwise.} \end{cases}$$

Letting

$$\tau_r = \frac{1}{\mu_\Delta((k_{r-1}, k_r]_{\mathbb{T}})}\mu_\Delta\left(\left\{s \in (k_{r-1}, k_r]_{\mathbb{T}} : \left|f(s)\right| > \frac{t_0}{2}\right\}\right).$$

If $r \ne r(j)$, then we can easily see that $\tau_r = 0$. If $r = r(j)$, then we get

$$\tau_r = \frac{1}{\mu_\Delta((k_{r-1}, k_r]_{\mathbb{T}})}\mu_\Delta\left(\left\{s \in (k_{r-1}, k_r]_{\mathbb{T}} : \left|f(s)\right| > \frac{t_0}{2}\right\}\right)$$

$$= \frac{1}{\mu_\Delta((k_{r(j)-1}, k_{r(j)}]_\mathbb{T})}\mu_\Delta\left(\left\{s \in \left(k_{r(j)-1}, 2\sigma(k_{r(j)-1})\right)_\mathbb{T} : \left|f(s)\right| > \frac{t_0}{2}\right\}\right)$$
$$= \frac{\mu_\Delta((k_{r(j)-1}, 2\sigma(k_{r(j)-1}))_\mathbb{T})}{\mu_\Delta((k_{r(j)-1}, k_{r(j)}]_\mathbb{T})}.$$

Here, there are two possible cases: $2\sigma(k_{r(j)-1}) \in \mathbb{T}$ and $2\sigma(k_{r(j)-1}) \notin \mathbb{T}$. Let us now examine these: If $2\sigma(k_{r(j)-1}) \in \mathbb{T}$, then we find

$$\mu_\Delta\left(\left(k_{r(j)-1}, 2\sigma(k_{r(j)-1})\right)_\mathbb{T}\right) = \sigma(k_{r(j)-1}),$$

and so

$$\begin{aligned}\tau_r &= \frac{\mu_\Delta((k_{r(j)-1}, 2\sigma(k_{r(j)-1}))_\mathbb{T})}{\mu_\Delta((k_{r(j)-1}, k_{r(j)}]_\mathbb{T})}\\ &= \frac{\sigma(k_{r(j)-1})}{\sigma(k_{r(j)}) - \sigma(k_{r(j)-1})}\\ &\le \frac{\sigma(k_{r(j)-1})}{k_{r(j)} - \sigma(k_{r(j)-1})} < \frac{1}{j-1} \to 0 \quad (\text{as } j \to \infty).\end{aligned}$$

If $2\sigma(k_{r(j)-1}) \notin \mathbb{T}$, then we can write

$$\left(k_{r(j)-1}, 2\sigma(k_{r(j)-1})\right)_\mathbb{T} = (k_{r(j)-1}, \alpha_j]_\mathbb{T},$$

where $\alpha_j := \max\{s \in \mathbb{T} : s < 2\sigma(k_{r(j)-1})\}$. Therefore, using the hypothesis, we get

$$\begin{aligned}\mu_\Delta\left(\left(k_{r(j)-1}, 2\sigma(k_{r(j)-1})\right)_\mathbb{T}\right) &= \mu_\Delta((k_{r(j)-1}, \alpha_j]_\mathbb{T})\\ &= \sigma(\alpha_j) - \sigma(k_{r(j)-1})\\ &\le (K+1)\alpha_j - \sigma(k_{r(j)-1})\\ &\le 2(K+1)\sigma(k_{r(j)-1}) - \sigma(k_{r(j)-1})\\ &= (2K+1)\sigma(k_{r(j)-1}),\end{aligned}$$

and so

$$\tau_r = \frac{\mu_\Delta((k_{r(j)-1}, 2\sigma(k_{r(j)-1}))_\mathbb{T})}{\mu_\Delta((k_{r(j)-1}, k_{r(j)}]_\mathbb{T})} \le \frac{(2K+1)\sigma(k_{r(j)-1})}{\sigma(k_{r(j)}) - \sigma(k_{r(j)-1})} < \frac{2K+1}{j-1} \to 0 \quad (\text{as } j \to \infty).$$

Thus, we get that $f \in S_{\theta-\mathbb{T}}(B)$.

On the other hand, for any real $M > 0$, there exists some $j_0 \in \mathbb{N}$ such that $k_{r(j)-1} > M$ for all $j \ge j_0$. Then we have

$$\begin{aligned}&\frac{1}{\mu_\Delta([t_0, 2\sigma(k_{r(j)-1})]_\mathbb{T})}\mu_\Delta\left(\left\{s \in \left[t_0, 2\sigma(k_{r(j)-1})\right]_\mathbb{T} : \left|f(s)\right| > M\right\}\right)\\ &\quad\ge \frac{1}{\mu_\Delta([t_0, 2\sigma(k_{r(j)-1})]_\mathbb{T})}\mu_\Delta\left(\left\{s \in \left(k_{r(j)-1}, 2\sigma(k_{r(j)-1})\right)_\mathbb{T} : \left|f(s)\right| > k_{r(j)-1}\right\}\right)\\ &\quad= \frac{\mu_\Delta((k_{r(j)-1}, 2\sigma(k_{r(j)-1}))_\mathbb{T})}{\mu_\Delta([t_0, 2\sigma(k_{r(j)-1})]_\mathbb{T})}\end{aligned}$$

for all $j \geq j_0$. Here, if $2\sigma(k_{r(j)-1}) \in \mathbb{T}$, then

$$\begin{aligned}
&\frac{1}{\mu_\Delta([t_0, 2\sigma(k_{r(j)-1})]_{\mathbb{T}})}\mu_\Delta\bigl(\bigl\{s \in \bigl[t_0, 2\sigma(k_{r(j)-1})\bigr]_{\mathbb{T}} : \bigl|f(s)\bigr| > M\bigr\}\bigr) \\
&\quad = \frac{\mu_\Delta((k_{r(j)-1}, 2\sigma(k_{r(j)-1}))_{\mathbb{T}})}{\mu_\Delta([t_0, 2\sigma(k_{r(j)-1})]_{\mathbb{T}})} \\
&\quad = \frac{\sigma(k_{r(j)-1})}{\sigma(2\sigma(k_{r(j)-1})) - t_0} \\
&\quad \geq \frac{\sigma(k_{r(j)-1})}{2(K+1)\sigma(k_{r(j)-1})} \\
&\quad = \frac{1}{2(K+1)}.
\end{aligned}$$

If $2\sigma(k_{r(j)-1}) \notin \mathbb{T}$, then we can write

$$\mu_\Delta\bigl((k_{r(j)-1}, 2\sigma(k_{r(j)-1}))_{\mathbb{T}}\bigr) = \mu_\Delta\bigl((k_{r(j)-1}, \beta_j)_{\mathbb{T}}\bigr)$$

and

$$\mu_\Delta\bigl(\bigl[t_0, 2\sigma(k_{r(j)-1})\bigr]_{\mathbb{T}}\bigr) = \mu_\Delta\bigl([t_0, \alpha_j]_{\mathbb{T}}\bigr),$$

where α_j is the same as in the above and $\beta_j := \min\{s \in \mathbb{T} : s > 2\sigma(k_{r(j)-1})\}$. Hence, if $2\sigma(k_{r(j)-1}) \notin \mathbb{T}$, then we have

$$\begin{aligned}
&\frac{1}{\mu_\Delta([t_0, 2\sigma(k_{r(j)-1})]_{\mathbb{T}})}\mu_\Delta\bigl(\bigl\{s \in \bigl[t_0, 2\sigma(k_{r(j)-1})\bigr]_{\mathbb{T}} : \bigl|f(s)\bigr| > M\bigr\}\bigr) \\
&\quad = \frac{\mu_\Delta((k_{r(j)-1}, 2\sigma(k_{r(j)-1}))_{\mathbb{T}})}{\mu_\Delta([t_0, 2\sigma(k_{r(j)-1})]_{\mathbb{T}})} \\
&\quad = \frac{\mu_\Delta((k_{r(j)-1}, \beta_j)_{\mathbb{T}})}{\mu_\Delta([t_0, \alpha_j]_{\mathbb{T}})} \\
&\quad = \frac{\beta_j - \sigma(k_{r(j)-1})}{\sigma(\alpha_j) - t_0} \\
&\quad \geq \frac{2\sigma(k_{r(j)-1}) - \sigma(k_{r(j)-1})}{(K+1)\alpha_j} \\
&\quad \geq \frac{\sigma(k_{r(j)-1})}{2(K+1)\sigma(k_{r(j)-1})} \\
&\quad = \frac{1}{2(K+1)}.
\end{aligned}$$

Therefore, we get that $f \notin S_{\mathbb{T}}(B)$. Consequently, we find that $S_{\theta-\mathbb{T}}(B) \not\subset S_{\mathbb{T}}(B)$, which is a contradiction. □

Remark 2.3 The function $f : \mathbb{T} \to \mathbb{R}$ given in the necessity part of Theorem 2.4 is an example of a lacunary statistically bounded function which is not statistically bounded.

Theorem 2.5 *Let $\theta = (k_r)$ and $\theta' = (l_r)$ be two lacunary sequences on $\mathbb{T}$ such that $(k_{r-1}, k_r]_{\mathbb{T}} \subset (l_{r-1}, l_r]_{\mathbb{T}}$ for all $r \in \mathbb{N}$. Then we have the following*:

(i) *If* $\liminf_{r\to\infty} \frac{\mu_\Delta((k_{r-1},k_r]_{\mathbb{T}})}{\mu_\Delta((l_{r-1},l_r]_{\mathbb{T}})} > 0$, *then* $S_{\theta'-\mathbb{T}}(B) \subseteq S_{\theta-\mathbb{T}}(B)$.

(ii) *If* $\lim_{r\to\infty} \frac{\mu_\Delta((k_{r-1},k_r]_{\mathbb{T}})}{\mu_\Delta((l_{r-1},l_r]_{\mathbb{T}})} = 1$, *then* $S_{\theta-\mathbb{T}}(B) \subseteq S_{\theta'-\mathbb{T}}(B)$.

Proof (i) Suppose that $(k_{r-1}, k_r]_{\mathbb{T}} \subset (l_{r-1}, l_r]_{\mathbb{T}}$ for all $r \in \mathbb{N}$ and $\liminf_{r\to\infty} \frac{\mu_\Delta((k_{r-1},k_r]_{\mathbb{T}})}{\mu_\Delta((l_{r-1},l_r]_{\mathbb{T}})} > 0$. For $M > 0$, we have

$$\bigl\{s \in (k_{r-1}, k_r]_{\mathbb{T}} : \bigl|f(s)\bigr| > M\bigr\} \subseteq \bigl\{s \in (l_{r-1}, l_r]_{\mathbb{T}} : \bigl|f(s)\bigr| > M\bigr\},$$

and so

$$\begin{aligned}
&\frac{1}{\mu_\Delta((k_{r-1},k_r]_{\mathbb{T}})}\mu_\Delta\bigl(\bigl\{s \in \bigl(k_{r-1},k_r\bigr]_{\mathbb{T}} : \bigl|f(s)\bigr| > M\bigr\}\bigr) \\
&\quad \leq \frac{\mu_\Delta((l_{r-1},l_r]_{\mathbb{T}})}{\mu_\Delta((k_{r-1},k_r]_{\mathbb{T}})} \frac{1}{\mu_\Delta((l_{r-1},l_r]_{\mathbb{T}})} \bigl\{s \in (l_{r-1}, l_r]_{\mathbb{T}} : \bigl|f(s)\bigr| > M\bigr\}
\end{aligned}$$

for all $r \in \mathbb{N}$. Now, taking the limit as $r \to \infty$, we get $S_{\theta'-\mathbb{T}}(B) \subseteq S_{\theta-\mathbb{T}}(B)$.

(ii) Suppose that $(k_{r-1}, k_r]_{\mathbb{T}} \subset (l_{r-1}, l_r]_{\mathbb{T}}$ for all $r \in \mathbb{N}$ and $\lim_{r\to\infty} \frac{\mu_\Delta((k_{r-1},k_r]_{\mathbb{T}})}{\mu_\Delta((l_{r-1},l_r]_{\mathbb{T}})} = 1$. For $M > 0$, we may write

$$\begin{aligned}
&\frac{1}{\mu_\Delta((l_{r-1},l_r]_{\mathbb{T}})}\mu_\Delta\bigl(\bigl\{s \in \bigl(l_{r-1},l_r\bigr]_{\mathbb{T}} : \bigl|f(s)\bigr| > M\bigr\}\bigr) \\
&\quad = \frac{1}{\mu_\Delta((l_{r-1},l_r]_{\mathbb{T}})}\mu_\Delta\bigl(\bigl\{s \in \bigl(l_{r-1},k_{r-1}\bigr]_{\mathbb{T}} : \bigl|f(s)\bigr| > M\bigr\}\bigr) \\
&\qquad + \frac{1}{\mu_\Delta((l_{r-1},l_r]_{\mathbb{T}})}\mu_\Delta\bigl(\bigl\{s \in \bigl(k_{r-1},k_r\bigr]_{\mathbb{T}} : \bigl|f(s)\bigr| > M\bigr\}\bigr) \\
&\qquad + \frac{1}{\mu_\Delta((l_{r-1},l_r]_{\mathbb{T}})}\mu_\Delta\bigl(\bigl\{s \in \bigl(k_r,l_r\bigr]_{\mathbb{T}} : \bigl|f(s)\bigr| > M\bigr\}\bigr) \\
&\quad \leq \frac{\mu_\Delta((l_{r-1},k_{r-1}]_{\mathbb{T}})}{\mu_\Delta((l_{r-1},l_r]_{\mathbb{T}})} \\
&\qquad + \frac{1}{\mu_\Delta((l_{r-1},l_r]_{\mathbb{T}})}\mu_\Delta\bigl(\bigl\{s \in \bigl(k_{r-1},k_r\bigr]_{\mathbb{T}} : \bigl|f(s)\bigr| > M\bigr\}\bigr) + \frac{\mu_\Delta((k_r,l_r]_{\mathbb{T}})}{\mu_\Delta((l_{r-1},l_r]_{\mathbb{T}})} \\
&\quad = \frac{\mu_\Delta((l_{r-1},l_r]_{\mathbb{T}}) - \mu_\Delta((k_{r-1},k_r]_{\mathbb{T}})}{\mu_\Delta((l_{r-1},l_r]_{\mathbb{T}})} + \frac{1}{\mu_\Delta((l_{r-1},l_r]_{\mathbb{T}})}\mu_\Delta\bigl(\bigl\{s \in \bigl(k_{r-1},k_r\bigr]_{\mathbb{T}} : \bigl|f(s)\bigr| > M\bigr\}\bigr) \\
&\quad \leq \biggl(1 - \frac{\mu_\Delta((k_{r-1},k_r]_{\mathbb{T}})}{\mu_\Delta((l_{r-1},l_r]_{\mathbb{T}})}\biggr) + \frac{1}{\mu_\Delta((k_{r-1},k_r]_{\mathbb{T}})}\mu_\Delta\bigl(\bigl\{s \in \bigl(k_{r-1},k_r\bigr]_{\mathbb{T}} : \bigl|f(s)\bigr| > M\bigr\}\bigr)
\end{aligned}$$

for all $r \in \mathbb{N}$. Since $\lim_{r\to\infty} \frac{\mu_\Delta((k_{r-1},k_r]_{\mathbb{T}})}{\mu_\Delta((l_{r-1},l_r]_{\mathbb{T}})} = 1$, if $f \in S_{\theta-\mathbb{T}}(B)$, then we get $f \in S_{\theta'-\mathbb{T}}(B)$. Thus, this implies that $S_{\theta-\mathbb{T}}(B) \subseteq S_{\theta'-\mathbb{T}}(B)$. □

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Declarations

Competing interests
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References

1. Fast, H.: Sur la convergence statistique. Colloq. Math. **2**, 241–244 (1951)
2. Steinhaus, H.: Sur la convergence ordinaire et la convergence asymptotique. Colloq. Math. **2**(1), 73–74 (1951)
3. Schoenberg, I.J.: The integrability of certain functions and related summability methods. Am. Math. Mon. **66**, 361–375 (1959)
4. Freedman, A.R., Sember, J.J., Raphael, M.: Some Cesàro-type summability spaces. Proc. Lond. Math. Soc. **37**(3), 508–520 (1978)
5. Fridy, J.A.: On statistical convergence. Analysis **5**, 301–313 (1985)
6. Fridy, J.A., Orhan, C.: Lacunary statistical convergence. Pac. J. Math. **160**, 43–51 (1993)
7. Connor, J.S.: The statistical and strong p-Cesàro convergence of sequences. Analysis **8**, 47–63 (1988)
8. Mursaleen, M.: λ-statistical convergence. Math. Slovaca **50**(1), 111–115 (2000)
9. Móricz, F.: Statistical limits of measurable functions. Analysis **24**(1), 1–18 (2004)
10. Srivastava, H.M., Et, M.: Lacunary statistical convergence and strongly lacunary summable functions of order α. Filomat **31**(6), 1573–1582 (2017)
11. Belen, C., Mohiuddine, S.A.: Generalized weighted statistical convergence and application. Appl. Math. Comput. **219**(18), 9821–9826 (2013)
12. Braha, N.L., Srivastava, H.M., Mohiuddine, S.A.: A Korovkin's type approximation theorem for periodic functions via the statistical summability of the generalized de la Vallée Poussin mean. Appl. Math. Comput. **228**, 162–169 (2014)
13. Kadak, U., Mohiuddine, S.A.: Generalized statistically almost convergence based on the difference operator which includes the (p,q)-gamma function and related approximation theorems. Results Math. **73**(9), 1–31 (2018)
14. Mohiuddine, S.A., Asiri, A., Hazarika, B.: Weighted statistical convergence through difference operator of sequences of fuzzy numbers with application to fuzzy approximation theorems. Int. J. Gen. Syst. **48**(5), 492–506 (2019)
15. Mohiuddine, S.A., Alamri, B.A.S.: Generalization of equi-statistical convergence via weighted lacunary sequence with associated Korovkin and Voronovskaya type approximation theorems. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **113**(3), 1955–1973 (2019)
16. Fridy, J.A., Orhan, C.: Statistical limit superior and limit inferior. Proc. Am. Math. Soc. **125**(12), 3625–3631 (1997)
17. Tripathy, B.C.: On statistically convergent and statistically bounded sequences. Bull. Malays. Math. Soc. **20**(1), 31–33 (1997)
18. Bhardwaj, V.K., Gupta, S.: On some generalizations of statistical boundedness. J. Inequal. Appl. **2014**(12), 1 (2014)
19. Bhardwaj, V.K., Gupta, S., Mohiuddine, S.A., Kılıçman, A.: On lacunary statistical boundedness. J. Inequal. Appl. **2014**(311), 1 (2014)
20. Et, M., Mohiuddine, S.A., Şengül, H.: On lacunary statistical boundedness of order α. Facta Univ., Ser. Math. Inform. **31**(3), 707–716 (2016)
21. Savaş, R.: Lacunary statistical boundedness of measurable functions. Math. Nat. Sci. **6**, 15–19 (2020)
22. Hilger, S.: Analysis on measure chains-a unified approach to continuous and discrete calculus. Results Math. **18**(1–2), 18–56 (1990)
23. Bohner, M., Peterson, A.: Dynamic Equations on Time Scales: An Introduction with Applications. Birkhäuser, Boston (2001)
24. Guseinov, G.S.: Integration on time scales. J. Math. Anal. Appl. **285**(1), 107–127 (2003)
25. Cabada, A., Vivero, D.R.: Expression of the Lebesgue Δ-integral on time scales as a usual Lebesgue integral: application to the calculus of Δ-antiderivatives. Math. Comput. Model. **43**(1–2), 194–207 (2006)
26. Seyyidoglu, M.S., Tan, N.O.: A note on statistical convergence on time scale. J. Inequal. Appl. **2012**(219), 1 (2012)
27. Turan, C., Duman, O.: Statistical convergence on time scales and its characterizations. Springer Proc. Math. Stat. **41**, 57–71 (2013)
28. Seyyidoglu, M.S., Tan, N.O.: On a generalization of statistical cluster and limit points. Adv. Differ. Equ. **2015**(55), 1 (2015)
29. Turan, C., Duman, O.: Convergence methods on time scales. AIP Conf. Proc. **1558**(1), 1120–1123 (2013)
30. Turan, C., Duman, O.: Fundamental properties of statistical convergence and lacunary statistical convergence on time scales. Filomat **31**(14), 4455–4467 (2017)
31. Altin, Y., Koyunbakan, H., Yilmaz, E.: Uniform statistical convergence on time scales. J. Appl. Math. **2014**(471437), 1–6 (2014)
32. Yilmaz, E., Altin, Y., Koyunbakan, H.: λ-statistical convergence on time scales. Dyn. Contin. Discrete Impuls. Syst., Ser. A Math. Anal. **23**(1), 69–78 (2016)
33. Altin, Y., Er, B.N., Yilmaz, E.: Δ_λ-statistical boundedness on time scales. Commun. Stat., Theory Methods **50**(3), 738–746 (2021)
34. Sozbir, B., Altundag, S.: Weighted statistical convergence on time scale. Dyn. Contin. Discrete Impuls. Syst., Ser. A Math. Anal. **26**(2), 137–143 (2019)
35. Sozbir, B., Altundag, S.: $\alpha\beta$–statistical convergence on time scales. Facta Univ., Ser. Math. Inform. **35**(1), 141–150 (2020)
36. Sözbir, B., Altundağ, S.: On asymptotically statistical equivalent functions on time scales. Math. Methods Appl. Sci. (2021). https://doi.org/10.1002/mma.7587