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Positive solutions for eigenvalue problems of fractional q-difference equation with ϕ-Laplacian

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Abstract

The aim of this paper is to investigate the boundary value problem of a fractional q-difference equation with ϕ-Laplacian, where ϕ-Laplacian is a generalized p-Laplacian operator. We obtain the existence and nonexistence of positive solutions in terms of different eigenvalue intervals for this problem by means of the Green function and Guo–Krasnoselskii fixed point theorem on cones. Finally, we give some examples to illustrate the use of our results.

Keywords: Fractional q-difference equations; Positive solutions; Boundary value problem; Guo–Krasnoselskii fixed point theorem

1 Introduction

The theory of q-calculus has been developed for more than 100 years; see [1]. As a branch of q-calculus, fractional q-calculus was first proposed by Al-Salam and Agarwal in the 1960s; see [2, 3]. Fractional q-calculus has a wide range of applications in many fields, such as cosmic strings and black holes, quantum theory, aerospace dynamics, biology, economics, control theory, medicine, electricity, signal processing, image processing, biophysics, blood flow phenomenon, and so on; see [4–10] and the references therein. The fractional q-difference equations are very important, and their basic theory has been continuously developed. Recently, as a new research direction, the solvability of boundary value problems (BVPs) of fractional q-difference equations have been widely concerned by scholars at home and abroad, and some conclusions have been obtained; see [11–14]. However, there are a few studies of eigenvalue problems for fractional q-difference equations with ϕ-Laplacian operator, and lots of work should be done.

In 2013, Li et al. [15] studied some positive solutions for a class of nonlinear fractional q-difference equations with parameters involving the Riemann–Liouville fractional derivative by means of a fixed theorem in cones,

$$\begin{cases} (D_q^\alpha u)(x) + \lambda f(u(x)) = 0, & 0 < x < 1, \\ u(0) = (D_q u)(0) = (D_q u)(1) = 0, \end{cases}$$

where $2 < \alpha \le 3$, and $f : [0,1] \times R \to R$ is a nonnegative continuous function.

In 2015, Wang et al. [16] obtained the existence and uniqueness of solutions for a class of singular BVPs of nonlinear fractional q-difference equations by a fixed point theorem in partially ordered sets,

$$\begin{cases} (D_q^\alpha u)(t) + f(t,u(t)) = 0, & 0 < t < 1, \\ u(0) = u(1) = 0, & (D_q u)(0) = 0, \end{cases}$$

where $2 < \alpha \le 3$, and $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous with $\lim_{t\to 0^+} f(t,\cdot) = \infty$.

In 2015, Han et al. [17] used the Green function and Guo–Krasnoselskii fixed-point theorem on cones to study solutions for eigenvalue problems of fractional differential equations with generalized p-Laplacian

$$\begin{cases} D_{0^+}^\beta(\phi(D_{0^+}^\alpha u(t))) = \lambda f(u(t)), & 0 < t < 1, \\ u(0) = u'(0) = u'(1) = 0, & \phi(D_{0^+}^\alpha u(0)) = (\phi(D_{0^+}^\alpha u(1)))' = 0, \end{cases} \tag{1.1}$$

where $0 < q < 1$, $2 < \alpha \le 3$, $1 < \beta \le 2$, $\lambda > 0$ is a parameter, and $D_{0^+}^\beta$ and $D_{0^+}^\alpha$ are the standard Riemann–Liouville fractional derivatives.

Motivated by the work above, in this paper, we investigate the following BVP of fractional q-difference equation with ϕ-Laplacian:

$$\begin{cases} D_q^\beta(\phi(D_q^\alpha u(t))) = \lambda f(u(t)), & 0 < t < 1, \\ u(0) = D_q u(0) = D_q u(1) = 0, & \phi(D_q^\alpha u(0)) = D_q(\phi(D_q^\alpha u(1))) = 0, \end{cases} \tag{1.2}$$

where $0 < q < 1$, $2 < \alpha \le 3$, $1 < \beta \le 2$, $\lambda > 0$ is a parameter, and D_q^β, D_q^α are the standard Riemann–Liouville fractional q-derivatives. As $q \to 1^-$, problem (1.2) reduces to problem (1.1).

In this paper, we always assume that

(A1) $\phi : R \to R$ is an odd increasing homeomorphism, and there exist two increasing homeomorphisms ψ_1, $\psi_2 : (0,\infty) \to (0,\infty)$ such that

$$\psi_1(x)\phi(y) \le \phi(xy) \le \psi_2(x)\phi(y), \quad x, y > 0;$$

(A2) $f : (0,+\infty) \to (0,+\infty)$ is a continuous function.

A function ϕ satisfying (A1) is called a generalized p-Laplacian operator. Two important cases are $\phi(u) = u$ and $\phi(u) = |u|^{p-2}u$ $(p > 1)$; see [18] and the references therein.

We aim to obtain the existence of at least one or two positive solutions in terms of different eigenvalue intervals using the Green function and Guo–Krasnoselskii fixed point theorem on cones. We also consider the nonexistence of positive solutions in terms of the parameter λ. Finally, we give some examples to illustrate our main results.

2 Preliminary results

In this section, we cite some definitions and fundamental results of the q-calculus and fractional q-calculus.

Definition 2.1 ([1]) For $0 < q < 1$, we define the q-derivative of a real-valued function f as

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \qquad (D_q f)(0) = \lim_{x \to 0}(D_q f)(x).$$

Note that $\lim_{q \to 1^-} D_q f(x) = f'(x)$.

Definition 2.2 ([1]) The q-integral of a function f in the interval $[0, b]$ is given by

$$(I_q f)(x) = \int_0^x f(t)\, d_q t = x(1-q) \sum_{n=0}^{\infty} f\bigl(xq^n\bigr) q^n, \quad x \in [0, b].$$

If $a \in [0, b]$ and f is defined in the interval $[0, b]$, then its integral from a to b is defined as

$$\int_a^b f(t)\, d_q t = \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t.$$

More definitions and properties of q-calculus can be found in [1]. In recent years, some results of q-calculus have been obtained; see [18–20] and the references therein.

Definition 2.3 ([10]) Let $\alpha \geq 0$, and let f be a function on $[0, 1]$. The fractional q-integral of the Riemann–Liouville type is defined by $(I_q^0 f)(x) = f(x)$ and

$$\bigl(I_q^{\alpha} f\bigr)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x - qt)^{(\alpha - 1)} f(t)\, d_q t, \quad \alpha > 0, x \in [0, 1].$$

Definition 2.4 ([21]) The fractional q-derivative of the Riemann–Liouville type of order $\alpha \geq 0$ is defined by $(D_q^0 f)(x) = f(x)$ and

$$\bigl(D_q^{\alpha} f\bigr)(x) = \bigl(D_q^m I_q^{m-\alpha} f\bigr)(x), \quad \alpha > 0,$$

where m is the smallest integer greater than or equal to α.

Lemma 2.1 ([10, 21]) *Let $\alpha, \beta \geq 0$, and let f be a continuous differentiable function on $[0, 1]$. Then we have the following formulas*:

1. $(I_q^{\beta} I_q^{\alpha} f)(x) = (I_q^{\alpha+\beta} f)(x)$,
2. $(D_q^{\alpha} I_q^{\alpha} f)(x) = f(x)$.

Lemma 2.2 ([22]) *Let $f : [0, 1] \to R$ be differentiable, let p be a positive integer, and let $\alpha > p - 1$. Then*

$$\bigl(I_q^{\alpha} D_q^{p} f\bigr)(x) = \bigl(D_q^{p} I_q^{\alpha} f\bigr)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha - p + k}}{\Gamma_q(\alpha + k - p + 1)} \bigl(D_q^{k} f\bigr)(0).$$

Lemma 2.3 ([23]) *Let $\alpha \in R^+$, $n := \lceil \alpha \rceil$. Then*

$$\bigl(I_q^{\alpha} D_q^{\alpha} f\bigr)(x) = f(x) - \sum_{j=1}^{n} D_q^{\alpha - j} f\bigl(0^+\bigr) \frac{x^{\alpha - j}}{\Gamma_q(\alpha - j + 1)}.$$

Lemma 2.4 ([22]) *Let $\alpha \geq 0$. Then we have the following three formulas*:

$$\begin{aligned}
&\big[a(t-s)\big]^{(\alpha)} = a^{\alpha}(t-s)^{(\alpha)},\\
&{}_tD_q(t-s)^{(\alpha)} = [\alpha]_q(t-s)^{(\alpha-1)},\\
&{}_xD_q\int_0^x f(x,t)\,d_qt = \int_0^x {}_xD_qf(x,t)\,d_qt + f(qx,x).
\end{aligned}$$

Remark 2.1 ([22]) Note that if $\alpha \geq 0$ and $a \leq b \leq t$, then $(t-a)^{(\alpha)} \geq (t-b)^{(\alpha)}$.

Lemma 2.5 *Let $y \in C_q[0,1]$, $2 < \alpha \leq 3$, $1 < \beta \leq 2$. Then the BVP*

$$\begin{cases} D_q^{\beta}(\phi(D_q^{\alpha}u(t))) = \lambda y(t), \quad 0 < t < 1, \\ u(0) = (D_qu)(0) = D_qu(1) = 0, \qquad \phi(D_q^{\alpha}u(0)) = D_q(\phi(D_q^{\alpha}u(1))) = 0, \end{cases} \tag{2.1}$$

has a unique solution

$$u(t) = \int_0^1 G(t,qs)\phi^{-1}\left(\lambda\int_0^1 H(s,q\tau)y(\tau)\,d_q\tau\right)d_qs,$$

where

$$H(s,\tau) = \frac{1}{\Gamma_q(\beta)}\begin{cases} s^{\beta-1}(1-\tau)^{(\beta-2)} - (s-\tau)^{(\beta-1)}, & \tau \leq s, \\ s^{\beta-1}(1-\tau)^{(\beta-2)}, & \tau \geq s, \end{cases} \tag{2.2}$$

$$G(t,s) = \frac{1}{\Gamma_q(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{(\alpha-2)} - (t-s)^{(\alpha-1)}, & s \leq t, \\ t^{\alpha-1}(1-s)^{(\alpha-2)}, & s \geq t. \end{cases} \tag{2.3}$$

Proof By Lemma 2.3 we have

$$\phi\big(D_q^{\alpha}u(t)\big) = C_1t^{\beta-1} + C_2t^{\beta-2} + \lambda\int_0^t \frac{(t-q\tau)^{(\beta-1)}}{\Gamma_q(\beta)}y(\tau)\,d_q\tau.$$

Using Lemma 2.4 and the boundary conditions $\phi(D_q^{\alpha}u(0)) = D_q(\phi(D_q^{\alpha}u(1))) = 0$, we get that

$$C_2 = 0,\ C_1 = -\lambda\int_0^1 \frac{(1-q\tau)^{(\beta-2)}}{\Gamma_q(\beta)}y(\tau)\,d_q\tau.$$

So we can obtain

$$\begin{aligned}
\phi\big(D_q^{\alpha}u(t)\big) &= -\lambda\int_0^1 \frac{t^{\beta-1}(1-q\tau)^{(\beta-2)}}{\Gamma_q(\beta)}y(\tau)\,d_q\tau + \lambda\int_0^t \frac{(t-q\tau)^{(\beta-1)}}{\Gamma_q(\beta)}y(\tau)\,d_q\tau \\
&= -\lambda\int_0^1 H(t,q\tau)y(\tau)\,d_q\tau.
\end{aligned}$$

Further, from

$$D_q^{\alpha}u(t) = -\phi^{-1}\left(\lambda\int_0^1 H(t,q\tau)y(\tau)\,d_q\tau\right)$$

by Lemma 2.3 we have

$$u(t) = C_3 t^{\alpha-1} + C_4 t^{\alpha-2} + C_5 t^{\alpha-3} - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \phi^{-1}\left(\lambda \int_0^1 H(s,q\tau) y(\tau)\, d_q\tau\right) d_q s.$$

Using the boundary condition $u(0) = (D_q u)(0) = D_q u(1) = 0$, we get

$$C_5 = 0, \qquad C_4 = 0, \qquad C_3 = \int_0^1 \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha)} \phi^{-1}\left(\lambda \int_0^1 H(s,q\tau) y(\tau)\, d_q\tau\right) d_q s,$$

so we have

$$\begin{aligned} u(t) &= \int_0^1 \frac{t^{\alpha-1}(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha)} \phi^{-1}\left(\lambda \int_0^1 H(s,q\tau) y(\tau)\, d_q\tau\right) d_q s \\ &\quad - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \phi^{-1}\left(\lambda \int_0^1 H(s,q\tau) y(\tau)\, d_q\tau\right) d_q s \\ &= \int_0^1 G(t,qs) \phi^{-1}\left(\lambda \int_0^1 H(s,q\tau) y(\tau)\, d_q\tau\right) d_q s. \end{aligned}$$

□

Lemma 2.6 *Let $2 < \alpha \le 3$. The functions $G(t,qs)$ and $H(s,q\tau)$ defined by (2.2) and (2.3), respectively, are continuous on $[0,1] \times [0,1]$ and satisfy*

(i) $G(t,qs) \ge 0$, $H(s,q\tau) \ge 0$ *for* $t,s,\tau \in [0,1]$;

(ii) $G(t,qs) \le G(1,qs)$, $H(s,q\tau) \le H(q\tau,q\tau)$ *for* $t,s,\tau \in [0,1]$;

(iii) $G(t,qs) \ge k(t)G(1,qs)$, $H(s,q\tau) \ge s^{\beta-1}H(1,q\tau)$, *where* $k(t) = t^{\alpha-1}$, *for* $t,s,\tau \in [0,1]$.

Proof (i) Let

$$\begin{aligned} g_1(t,qs) &= t^{\alpha-1}(1-qs)^{(\alpha-2)} - (t-qs)^{(\alpha-1)}, \quad s \le t, \\ g_2(t,qs) &= t^{\alpha-1}(1-qs)^{(\alpha-2)}, \quad s \ge t. \end{aligned}$$

It is clear that $g_2(t,qs) \ge 0$ for $t,s \in [0,1]$. If $s \le t$, then in view of Remark 2.1, for $t \ne 0$,

$$\begin{aligned} g_1(t,qs) &= t^{\alpha-1}(1-qs)^{(\alpha-2)} - (t-qs)^{(\alpha-1)} \\ &= t^{\alpha-1}(1-qs)^{(\alpha-2)} - t^{\alpha-1}(1-qs/t)^{(\alpha-1)} \\ &\ge t^{\alpha-1}\left[(1-qs)^{(\alpha-2)} - (1-qs)^{(\alpha-1)}\right] \\ &\ge 0. \end{aligned}$$

Therefore $G(t,qs) \ge 0$. In the same way, we can obtain that $H(s,q\tau) \ge 0$.

(ii) Fix $s \in [0,1]$. For $t \ne 0$, we have

$$\begin{aligned} {}_tD_q g_1(t,qs) &= [\alpha-1]_q t^{\alpha-2}(1-qs)^{(\alpha-2)} - [\alpha-1]_q (t-qs)^{(\alpha-2)} \\ &= [\alpha-1]_q t^{\alpha-2}(1-qs)^{(\alpha-2)} - [\alpha-1]_q t^{\alpha-2}\left(1-\frac{qs}{t}\right)^{(\alpha-2)} \\ &= [\alpha-1]_q t^{\alpha-2}\left[(1-qs)^{(\alpha-2)} - \left(1-\frac{qs}{t}\right)^{(\alpha-2)}\right] \ge 0 \end{aligned}$$

and

$$ {}_tD_q g_2(t,qs) = [\alpha-1]_q t^{\alpha-2}(1-qs)^{(\alpha-2)} \geq 0. $$

Therefore $g_1(t,qs)$, $g_2(t,qs)$ are increasing functions of t for $s \in [0,1]$. Thus $G(t,qs) \leq G(1,qs)$. In the same way, we get that $H(s,q\tau) \leq H(q\tau,q\tau)$.

(iii) Suppose that $s \leq t$. Then

$$ \begin{aligned} \frac{G(t,qs)}{G(1,qs)} &= \frac{t^{\alpha-1}(1-qs)^{(\alpha-2)} - (t-qs)^{(\alpha-1)}}{(1-qs)^{(\alpha-2)} - (1-qs)^{(\alpha-1)}} \\ &= \frac{t^{\alpha-1}(1-qs)^{(\alpha-2)} - t^{\alpha-1}(1-\frac{qs}{t})^{(\alpha-1)}}{(1-qs)^{(\alpha-2)} - (1-qs)^{(\alpha-1)}} \\ &\geq t^{\alpha-1}. \end{aligned} $$

For the other circumstance, we also get $G(t,qs) \geq t^{\alpha-1}G(1,qs)$. In the same way, we get that $H(s,q\tau) \geq s^{\beta-1}H(1,q\tau)$. The proof is completed. □

Lemma 2.7 ([17]) *Assume that* $(A1)$ *holds. Then*

$$ \psi_2^{-1}(x)y \leq \phi^{-1}\big(x\phi(y)\big) \leq \psi_1^{-1}(x)y, \quad x,y \in (0,\infty). $$

Theorem 2.1 ([24] (Krasnoselskii)) *Let E be a Banach space, and let $K \in E$ be a cone in E. Let Ω_1 and Ω_2 be open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $T: K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$ be a completely continuous operator. In addition, suppose that either*

(H_1) $\|Tu\| \leq \|u\|$, $\forall u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$, $\forall u \in K \cap \partial\Omega_2$ *or*

(H_2) $\|Tu\| \leq \|u\|$, $\forall u \in K \cap \partial\Omega_2$ *and* $\|Tu\| \geq \|u\|$, $\forall u \in K \cap \partial\Omega_1$.

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3 Main results

In this section, we consider the existence of at least one or two positive solutions or no positive solution for the BVP (1.1).

Let the Banach space $E = C_q[0,1]$ be endowed with norm $\|u\| = \max_{0\leq t\leq 1}|u(t)|$. Define the cone

$$ P = \big\{u \in E \mid u(t) \geq k(t)\|u\|, t \in [0,1]\big\} \subset E. $$

Let $T_\lambda : P \to P$ be the operator defined by

$$ T_\lambda u(t) := \int_0^1 G(t,qs)\phi^{-1}\left(\lambda \int_0^1 H(s,q\tau) f\big(u(\tau)\big)\, d_q\tau\right) d_q s. $$

Lemma 3.1 *Assume that* $(A2)$ *holds. Then* $T_\lambda : P \to P$ *is completely continuous.*

Proof By Lemma 2.6 we have

$$ (T_\lambda u)(t) \geq t^{\alpha-1} \int_0^1 G(1,qs)\phi^{-1}\left(\lambda \int_0^1 H(s,q\tau) f\big(u(\tau)\big)\, d_q\tau\right) d_q s = k(t)\big\|T_\lambda u(t)\big\|. $$

Thus $T_\lambda(P) \subset P$. In view of the nonnegativeness and continuity of $G(t,qs)$, $H(s,q\tau)$, and $f(u(\tau))$, we have that $T_\lambda : P \to P$ is continuous.

Next, we prove that T_λ is uniformly bounded.

Let $\Omega \subset P$ be bounded, that is, there exists a positive constant $M > 0$ such that $\|u\| \le M$ for all $u \in \Omega$. Set $L = \max_{0\le u\le M} |f(u)| + 1$. Then, for $u \in \Omega$ and all $t \in [0,1]$, we have

$$\begin{aligned}
\left|T_\lambda u(t)\right| &= \left|\int_0^1 G(t,qs)\phi^{-1}\left(\lambda \int_0^1 H(s,q\tau)f\big(u(\tau)\big)\,d_q\tau\right)d_qs\right| \\
&\le \psi_1^{-1}(\lambda L)\int_0^1 G(t,qs)\phi^{-1}\left(\int_0^1 H(s,q\tau)\,d_q\tau\right)d_qs \\
&\le \psi_1^{-1}(\lambda L)\int_0^1 G(1,qs)\phi^{-1}\left(\int_0^1 H(q\tau,q\tau)\,d_q\tau\right)d_qs \\
&< +\infty.
\end{aligned}$$

Hence $T_\lambda(\Omega)$ is uniformly bounded.

On the other hand, we prove that T_λ is equicontinuous.

Since $G(t,qs)$ is continuous on $[0,1]\times[0,1]$, it is uniformly continuous on $[0,1]\times[0,1]$. Thus, for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta$ imply

$$\left|G(t_2,qs) - G(t_1,qs)\right| < \frac{\varepsilon}{\psi_1^{-1}(\lambda L)\phi^{-1}(\int_0^1 H(q\tau,q\tau)\,d_q\tau)}.$$

Then, for all $u \in \Omega$,

$$\begin{aligned}
\left|T_\lambda u(t_2) - T_\lambda u(t_1)\right| &\le \int_0^1 \left|G(t_2,qs) - G(t_1,qs)\right|\phi^{-1}\left(\lambda\int_0^1 H(s,q\tau)f\big(u(\tau)\big)\,d_q\tau\right)d_qs \\
&\le \psi_1^{-1}(\lambda L)\int_0^1 \left|G(t_2,qs) - G(t_1,qs)\right|\phi^{-1}\left(\int_0^1 H(q\tau,q\tau)\,d_q\tau\right)d_qs \\
&= \psi_1^{-1}(\lambda L)\phi^{-1}\left(\int_0^1 H(q\tau,q\tau)\,d_q\tau\right)\int_0^1 \left|G(t_2,qs) - G(t_1,qs)\right|d_qs \\
&< \varepsilon.
\end{aligned}$$

Hence $T_\lambda(\Omega)$ is equicontinuous. By the Arzelà–Ascoli theorem we have that $T_\lambda : P \to P$ is completely continuous. The proof is completed. □

For convenience, we denote

$$\begin{aligned}
&F_0 = \lim_{u\to 0^+}\sup\frac{f(u)}{\phi(u)}, \qquad F_\infty = \lim_{u\to+\infty}\sup\frac{f(u)}{\phi(u)}, \\
&f_0 = \lim_{u\to 0^+}\inf\frac{f(u)}{\phi(u)}, \qquad f_\infty = \lim_{u\to+\infty}\inf\frac{f(u)}{\phi(u)}, \\
&A_1 = \int_0^1 G(1,qs)\psi_1^{-1}\left(\int_0^1 H(q\tau,q\tau)\,d_q\tau\right)d_qs, \\
&A_2 = k(\delta)\int_0^1 \psi_2^{-1}\big(s^{\beta-1}\big)G(1,qs)\psi_2^{-1}\left(\int_0^1 \psi_1\big(\tau^{\alpha-1}\big)H(1,q\tau)\,d_q\tau\right)d_qs,
\end{aligned}$$

$$A_3 = k(\delta)\int_0^1 \psi_2^{-1}\bigl(s^{\beta-1}\bigr)G(1,qs)\psi_2^{-1}\left(\int_0^1 H(1,q\tau)\,d_q\tau\right)d_qs.$$

Theorem 3.1 *Assume that* $(A1)$, $(A2)$, *and* $f_\infty\psi_1(A_1^{-1}) > F_0\psi_2(A_2^{-1})$ *hold. Then for each*

$$\lambda \in \bigl(\psi_2\bigl(A_2^{-1}\bigr)f_\infty^{-1}, \psi_1\bigl(A_1^{-1}\bigr)F_0^{-1}\bigr), \tag{3.1}$$

the BVP of fractional q-difference Eq. (1.1) *has at least one positive solution. Here we impose that* $f_\infty^{-1} = 0$ *if* $f_\infty = +\infty$ *and* $F_0^{-1} = +\infty$ *if* $F_0 = 0$.

Proof Let λ satisfy (3.1), and let $\varepsilon > 0$ be such that

$$\psi_2\bigl(A_2^{-1}\bigr)(f_\infty - \varepsilon)^{-1} \le \lambda \le \psi_1\bigl(A_1^{-1}\bigr)(F_0 + \varepsilon)^{-1}. \tag{3.2}$$

We separate the proof into two steps.

(1) By the definition of F_0 there exists $r_1 > 0$ such that

$$f(u) \le (F_0 + \varepsilon)\phi(u), \quad 0 < u < r_1. \tag{3.3}$$

If $u \in P$ with $\|u\| = r_1$, then from (3.2) and (3.3) we obtain

$$\begin{aligned}
\bigl\|T_\lambda u(t)\bigr\| &\le \int_0^1 G(1,qs)\phi^{-1}\left(\lambda\int_0^1 H(q\tau,q\tau)f\bigl(u(\tau)\bigr)\,d_q\tau\right)d_qs \\
&\le \int_0^1 G(1,qs)\phi^{-1}\left(\lambda\int_0^1 H(q\tau,q\tau)(F_0+\varepsilon)\phi(r_1)\right)d_q\tau)\,d_qs \\
&\le \psi_1^{-1}\bigl(\lambda(F_0+\varepsilon)\bigr)\int_0^1 G(1,qs)\phi^{-1}\left(\int_0^1 H(q\tau,q\tau)\phi(r_1)\right)d_q\tau)\,d_qs \\
&\le \psi_1^{-1}\bigl(\lambda(F_0+\varepsilon)\bigr)r_1\int_0^1 G(1,qs)\psi_1^{-1}\left(\int_0^1 H(q\tau,q\tau)\,d_q\tau\right)d_qs \\
&= \psi_1^{-1}\bigl(\lambda(F_0+\varepsilon)\bigr)A_1r_1 \le r_1 = \|u\|.
\end{aligned}$$

Let $\Omega_1 = \{u \in E \mid \|u\| < r_1\}$. Then

$$\|T_\lambda u\| \le \|u\|, \quad u \in P \cap \partial\Omega_1. \tag{3.4}$$

(2) By the definition of f_∞ there exists $r_3 > 0$ such that

$$f(u) \ge (f_\infty - \varepsilon)\phi(u), \quad u > r_3. \tag{3.5}$$

If $u \in P$ with $\|u\| = r_2 = \max\{2r_1, r_3\}$, then from (3.2) and (3.5) we obtain

$$\begin{aligned}
\bigl\|T_\lambda u(t)\bigr\| &\ge \int_0^1 k(\delta)G(1,qs)\phi^{-1}\left(\lambda\int_0^1 H(s,q\tau)f\bigl(u(\tau)\bigr)\,d_q\tau\right)d_qs \\
&\ge \int_0^1 k(\delta)G(1,qs)\phi^{-1}\left(\lambda\int_0^1 s^{\beta-1}H(1,q\tau)(f_\infty-\varepsilon)\phi\bigl(\tau^{\alpha-1}\|u\|\bigr)\,d_q\tau\right)d_qs \\
&\ge \int_0^1 k(\delta)G(1,qs)\phi^{-1}\left(\lambda\int_0^1 s^{\beta-1}H(1,q\tau)(f_\infty-\varepsilon)\psi_1\bigl(\tau^{\alpha-1}\bigr)\phi(r_2)\,d_q\tau\right)d_qs
\end{aligned}$$

$$
\begin{aligned}
&\geq \psi_2^{-1}\big(\lambda(f_\infty-\varepsilon)\big)r_2k(\delta) \\
&\quad\times \int_0^1 \psi_2^{-1}\big(s^{\beta-1}\big)G(1,qs)\psi_2^{-1}\left(\int_0^1 H(1,q\tau)\psi_1\big(\tau^{\alpha-1}\big)\,d_q\tau\right)d_qs \\
&= \psi_2^{-1}\big(\lambda(f_\infty-\varepsilon)\big)A_2r_2 \geq r_2 = \|u\|.
\end{aligned}
$$

Let $\Omega_2 = \{u \in E \mid \|u\| < r_2\}$. Then

$$
\|T_\lambda u\| \geq \|u\|, \quad u \in P \cap \partial\Omega_2. \tag{3.6}
$$

From (3.4) and (3.6) and from Theorem 2.1 we have that T_λ has a fixed point $u \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ with $r_1 \leq \|u\| \leq r_2$ and that u is a positive solution for the BVP of fractional q-difference Eq. (1.1). The proof is completed. □

Theorem 3.2 *Assume that (A1), (A2), and $f_0\psi_1(A_1^{-1}) > F_\infty\psi_2(A_2^{-1})$ hold. Then for each*

$$
\lambda \in \big(\psi_2\big(A_2^{-1}\big)f_0^{-1}, \psi_1\big(A_1^{-1}\big)F_\infty^{-1}\big), \tag{3.7}
$$

the BVP of fractional q-difference Eq. (1.1) *has at least one positive solution. Here we impose that $f_0^{-1} = 0$ if $f_0 = +\infty$ and $F_\infty^{-1} = +\infty$ if $F_\infty = 0$.*

Proof Let λ satisfy (3.7), and let $\varepsilon > 0$ be such that

$$
\psi_2\big(A_2^{-1}\big)(f_0-\varepsilon)^{-1} \leq \lambda \leq \psi_1\big(A_1^{-1}\big)(F_\infty+\varepsilon)^{-1}. \tag{3.8}
$$

We separate the proof into two steps.

(1) By the definition of f_0 there exists $r_1 > 0$ such that

$$
f(u) \geq (f_0-\varepsilon)\phi(u), \quad 0 < u \leq r_1. \tag{3.9}
$$

If $u \in P$ with $\|u\| = r_1$, then similarly to the second part of the proof of Theorem 3.1, let $\Omega_1 = \{u \in E \mid \|u\| < r_1\}$. Then

$$
\|T_\lambda u\| \geq \|u\|, \quad u \in P \cap \partial\Omega_1. \tag{3.10}
$$

(2) By the definition F_∞ there exists $R_1 > 0$ such that

$$
f(u) \leq (F_\infty+\varepsilon)\phi(u), \quad u > R_1. \tag{3.11}
$$

We consider two cases:

Case 1: When f is bounded, then there exists $N > 0$, such that $|f(u)| \leq N$ for $u \in (0,+\infty)$. If $u \in P$ with $\|u\| = r_3$, where $r_3 = \max\{2r_1, \phi^{-1}(\lambda N)A_1\}$, then

$$
\begin{aligned}
\big\|T_\lambda u(t)\big\| &\leq \int_0^1 G(1,qs)\phi^{-1}\left(\lambda\int_0^1 H(q\tau,q\tau)f\big(u(\tau)\big)\,d_q\tau\right)d_qs \\
&\leq \phi^{-1}(\lambda N)\int_0^1 G(1,qs)\psi_1^{-1}\left(\int_0^1 H(q\tau,q\tau)\,d_q\tau\right)d_qs
\end{aligned}
$$

$$
\begin{aligned}
&\le \phi^{-1}(\lambda N)A_1 \\
&\le r_3 = \|u\|.
\end{aligned}
$$

So let $\Omega_3 = \{u \in E \mid \|u\| < r_3\}$. Then

$$
\|T_\lambda u\| \le \|u\|, \quad u \in P \cap \partial\Omega_3. \tag{3.12}
$$

Case 2: Suppose f is unbounded. Then there exists $r_4 > \max\{2r_1, R_1\}$ such that $f(u) \le f(r_4)$ for $0 < u < r_4$. If $u \in P$ with $\|u\| = r_4$, then by (3.7) and (3.11) we have

$$
\begin{aligned}
\|T_\lambda u(t)\| &\le \int_0^1 G(1,qs)\phi^{-1}\left(\lambda \int_0^1 H(q\tau,q\tau)f\big(u(\tau)\big)\,d_q\tau\right)d_qs \\
&\le \int_0^1 G(1,qs)\phi^{-1}\left(\lambda \int_0^1 H(q\tau,q\tau)(F_\infty+\varepsilon)\phi(r_4)\,d_q\tau\right)d_qs \\
&\le \psi_1^{-1}\big(\lambda(F_\infty+\varepsilon)\big)A_1 r_4 \\
&\le r_4 = \|u\|.
\end{aligned}
$$

Thus we suppose $\Omega_4 = \{u \in E \mid \|u\| < r_4\}$. Then

$$
\|T_\lambda u\| \le \|u\|, \quad u \in P \cap \partial\Omega_4. \tag{3.13}
$$

In view of Cases 1 and 2, we let $\Omega_2 = \{u \in E \mid \|u\| < r_2\}$, where $r_2 = \max\{r_3, r_4\}$. Then

$$
\|T_\lambda u\| \le \|u\|, \quad u \in P \cap \partial\Omega_2. \tag{3.14}
$$

From (3.10) and (3.14) and from Theorem 2.1 we obtain that T_λ has a fixed point $u \in P \cap (\overline{\Omega}_2 \backslash \Omega_1)$ with $r_1 \le \|u\| \le r_2$. Obviously, u is a positive solution of the BVP of fractional q-difference Eq. (1.1). The proof is completed. □

Theorem 3.3 *Assume that (A1) and (A2) hold and there exist $r_2 > r_1 > 0$ such that*

$$
\lambda \min_{k(\delta)r_1 \le u \le r_1} f(u) \ge \phi\left(\frac{r_1}{A_3}\right), \qquad \lambda \max_{0 \le u \le r_2} f(u) \le \phi\left(\frac{r_2}{A_1}\right).
$$

Then the BVP of fractional q-difference Eq. (1.1) has a positive solution $u \in P$ with $r_1 \le \|u\| \le r_2$.

Proof Let $\Omega_1 = \{u \in E | \|u\| < r_1\}$. Then for $u \in P \cap \partial\Omega_1$, we obtain

$$
\begin{aligned}
\|T_\lambda u(t)\| &\ge T_\lambda u(\delta) \\
&= \int_0^1 G(\delta,qs)\phi^{-1}\left(\lambda \int_0^1 H(s,q\tau)f\big(u(\tau)\big)\,d_q\tau\right)d_qs \\
&\ge \int_0^1 k(\delta)G(1,qs)\phi^{-1}\left(\lambda \int_0^1 s^{\beta-1}H(1,q\tau) \min_{k(\delta)r_1 \le u \le r_1} f\big(u(\tau)\big)\,d_q\tau\right)d_qs \\
&\ge \int_0^1 k(\delta)G(1,qs)\psi_2^{-1}\big(s^{\beta-1}\big)\phi^{-1}\left(\lambda \int_0^1 H(1,q\tau) \min_{k(\delta)r_1 \le u \le r_1} f\big(u(\tau)\big)\,d_q\tau\right)d_qs
\end{aligned}
$$

$$\begin{aligned}&\geq \phi^{-1}\Bigl(\lambda \min_{k(\delta)r_1\leq u\leq r_1} f(u)\Bigr)A_3\\&\geq r_1 = \|u\|.\end{aligned}$$

Suppose $\Omega_2 = \{u \in E \mid \|u\| < r_2\}$. Then for $u \in P \cap \partial\Omega_2$, we have

$$\begin{aligned}\bigl\|T_\lambda u(t)\bigr\| &\leq \int_0^1 G(1,qs)\phi^{-1}\biggl(\lambda \int_0^1 H(q\tau,q\tau)f\bigl(u(\tau)\bigr)\,d_q\tau\biggr)\,d_qs\\&\leq \int_0^1 G(1,qs)\phi^{-1}\biggl(\lambda \int_0^1 H(q\tau,q\tau)\max_{0\leq u\leq r_2} f\bigl(u(\tau)\bigr)\,d_q\tau\biggr)\,d_qs\\&\leq \phi^{-1}\Bigl(\lambda \max_{0\leq u\leq r_2} f\bigl(u(\tau)\bigr)\Bigr)A_1\\&\leq r_2 = \|u\|.\end{aligned}$$

Thus by Theorem 2.1 the BVP of fractional q-difference Eq. (1.1) has a positive solution $u \in P$ with $r_1 \leq \|u\| \leq r_2$. The proof is completed. □

Theorem 3.4 *Assume that* $(A1)$ *and* $(A2)$ *hold. Let* $\lambda_1 = \sup_{r>0} \frac{\phi(r)}{\phi(A_1)\max_{0\leq u\leq r} f(u)}$. *If* $f_0 = +\infty$ *and* $f_\infty = +\infty$, *then the BVP of fractional q-difference Eq.* (1.1) *has at least two positive solutions for each* $\lambda \in (0,\lambda_1)$.

Proof We define $x(r) = \frac{\phi(r)}{\psi_2(A_1)\max_{0\leq u\leq r} f(u)}$. In view of the continuity of f, $f_0 = +\infty$, and $f_\infty = +\infty$, we obtain that $x(r) : (0,+\infty) \to (0,+\infty)$ is continuous and $\lim_{r\to 0^+} x(r) = \lim_{r\to+\infty} x(r) = 0$. So there exists $r_0 \in (0,+\infty)$ such that $x(r_0) = \sup_{r>0} x(r) = \lambda_1$. For all $\lambda \in (0,\lambda_1)$, there exist constants a_1, $a_2 > 0$ such that $x(a_1) = x(a_2) = \lambda$, where $0 < a_1 < r_0 < a_2 < +\infty$. Thus

$$\lambda f(u) \leq \frac{\phi(a_1)}{\psi_2(A_1)} \leq \phi\biggl(\frac{a_1}{A_1}\biggr), \quad u \in [0,a_1], \tag{3.15}$$

$$\lambda f(u) \leq \frac{\phi(a_2)}{\psi_2(A_1)} \leq \phi\biggl(\frac{a_2}{A_1}\biggr), \quad u \in [0,a_2], \tag{3.16}$$

By the conditions $f_0 = +\infty$ and $f_\infty = +\infty$ there exist constants b_1, $b_2 > 0$, where $0 < b_1 < a_1 < r_0 < a_2 < b_2 < +\infty$, such that

$$\frac{f(u)}{\phi(u)} \geq \frac{1}{\lambda\psi_1(k(\delta))\phi(A_3)}, \quad u \in (0,b_1) \cup \bigl(k(\delta)b_2,+\infty\bigr),$$

so that

$$\lambda \min_{k(\delta)b_1\leq u\leq b_1} f(u) \geq \phi\biggl(\frac{b_1}{A_3}\biggr), \tag{3.17}$$

$$\lambda \min_{k(\delta)b_2\leq u\leq b_2} f(u) \geq \phi\biggl(\frac{b_2}{A_3}\biggr). \tag{3.18}$$

By (3.15), (3.17), (3.16), and (3.18), combined with Theorems 2.1 and 3.3, the BVP of fractional q-difference Eq. (1.1) has at least two positive solutions for each $\lambda \in (0,\lambda_1)$. The proof is completed. □

Theorem 3.5 *Assume that* $(A1)$ *and* $(A2)$ *hold. If* $F_0 < +\infty$ *and* $F_\infty < +\infty$, *then there exists* $\lambda_0 > 0$ *such that for all* $0 < \lambda < \lambda_0$, *the BVP of fractional* q-*difference Eq.* (1.1) *has no positive solution.*

Proof Since $F_0 < +\infty$ and $F_\infty < +\infty$, there exist $M_1, M_2, r_1, r_2 > 0$, such that $r_1 < r_2$ and

$$f(u) \le M_1\phi(u), \quad u \in [0, r_1],$$
$$f(u) \le M_2\phi(u), \quad u \in [r_2, +\infty).$$

Let $M_0 = \max\{M_1, M_2, \max_{r_1 \le u \le r_2} \frac{f(u)}{\phi(u)}\}$. Then we have

$$f(u) \le M_0\phi(u), \quad u \in [0, +\infty).$$

Let v be a positive solution of the fractional q-difference equation boundary value problem (1.1). We will show that this leads to a contradiction for $0 < \lambda < \lambda_0$, where $\lambda_0 := M_0^{-1}\psi_1(A_1^{-1})$. Indeed, since $T_\lambda v(t) = v(t)$ for $t \in [0, 1]$, we have

$$\begin{aligned} \|v\| = \|T_\lambda v\| &\le \int_0^1 G(1, qs)\phi^{-1}\left(\lambda \int_0^1 H(q\tau, q\tau) f\bigl(v(\tau)\bigr)\, d_q\tau\right) d_qs \\ &\le \int_0^1 G(1, qs)\phi^{-1}\left(\lambda \int_0^1 H(q\tau, q\tau) M_0\phi\bigl(v(\tau)\bigr)\, d_q\tau\right) d_qs \\ &\le \psi_1^{-1}(\lambda M_0)\|v\|A_1 < \|v\|, \end{aligned}$$

a contradiction. Therefore the BVP of fractional q-difference Eq. (1.1) has no positive solution. The proof is completed. □

Theorem 3.6 *Assume that* $(A1)$ *and* $(A2)$ *hold. If* $f_0 > 0$ *and* $f_\infty > 0$, *then there exists* $\lambda_0' > 0$ *such that for all* $\lambda > \lambda_0'$, *the BVP of fractional* q-*difference Eq.* (1.1) *has no positive solution.*

Proof Since $f_0 > 0$ and $f_\infty > 0$, there exist $m_1, m_2, r_3, r_4 > 0$ such that $r_3 < r_4$ and

$$f(u) \ge m_1\phi(u), \quad u \in [0, r_3],$$
$$f(u) \ge m_2\phi(u), \quad u \in [r_4, +\infty).$$

Let $m_0 = \max\{m_1, m_2, \max_{r_3 \le u \le r_4} \frac{f(u)}{\phi(u)}\}$. Then we have

$$f(u) \ge m_0\phi(u), \quad u \in [0, +\infty).$$

Let v be a positive solution of the fractional q-difference equation BVP (1.1). We will show that this leads to a contradiction for $\lambda > \lambda_0'$, where $\lambda_0' := m_0^{-1}\psi_2(A_2^{-1})$. Indeed, since $T_\lambda v(t) = v(t)$ for $t \in [0, 1]$, we have

$$\begin{aligned} \|v\| = \|T_\lambda v\| &\ge \int_0^1 k(\delta)G(1, qs)\phi^{-1}\left(\lambda \int_0^1 s^{\beta-1}H(1, q\tau) f\bigl(v(\tau)\bigr)\, d_q\tau\right) d_qs \\ &\ge \int_0^1 k(\delta)G(1, qs)\phi^{-1}\left(\lambda \int_0^1 s^{\beta-1}H(1, q\tau) m_0\phi\bigl(v(\tau)\bigr)\, d_q\tau\right) d_qs \end{aligned}$$

$$\geq \int_0^1 k(\delta)\psi_2^{-1}(s^{\beta-1})G(1,qs)\phi^{-1}\left(\lambda\int_0^1 H(1,q\tau)m_0\phi(\tau^{\alpha-1}\|v\|)\,d_q\tau\right)d_qs$$
$$\geq \psi_2^{-1}(\lambda m_0)\|v\|A_2 > \|v\|,$$

a contradiction. Therefore the BVP of fractional q-difference Eq. (1.1) has no positive solution. The proof is completed. □

4 Some examples of application

Example 4.1 Consider the following fractional q-difference equation BVP:

$$\begin{cases} D_q^{\frac{3}{2}}(D_q^{\frac{5}{2}}u(t)) = \lambda(7u(t) - 6\sin(u(t))), \quad 0 < t < 1, \\ u(0) = D_q u(0) = D^q u(1) = 0, \qquad D_q^{\frac{5}{2}}u(0) = D_q(D_q^{\frac{5}{2}}u(1)) = 0. \end{cases} \tag{4.1}$$

Here $q = \frac{1}{2}$, $\alpha = \frac{5}{2}$, $\beta = \frac{3}{2}$, $\phi(u) = u$, $f(u) = 7u - 6\sin u$. Take $\psi_1(x) = \psi_2(x) = x$, $\delta = 0.9$.

By a simple calculation we obtain $\Gamma_{\frac{1}{2}}(\frac{5}{2}) \approx 1.1906$, $\Gamma_{\frac{1}{2}}(\frac{3}{2}) \approx 0.9209$, $A_1 \approx 0.05523$, $A_2 \approx 0.00811$, $F_0 = 1$, $f_\infty = 7$, $f_\infty\psi_1(A_1^{-1}) \approx 126.74271$, $F_0\psi_2(A_2^{-1}) \approx 123.30456$, $\psi_2(A_2^{-1})f_\infty^{-1} \approx 17.6194$ and $\psi_1(A_1^{-1})F_0^{-1}) \approx 18.10610$.

Obviously, $f_\infty\psi_1(A_1^{-1}) > F_0\psi_2(A_2^{-1})$. By Theorem 3.1 we obtain that BVP (4.1) has at least one positive solution for each $\lambda \in (17.61494, 18.10610)$.

Example 4.2 Consider the following fractional q-difference equation BVP with ϕ-Laplacian:

$$\begin{cases} D_q^{\frac{3}{2}}(\phi(D_q^{\frac{5}{2}}u(t))) = \lambda\frac{(u^3(t)+u^2(t))(\sin(u(t))+9)}{25u(t)+1}, \quad 0 < t < 1, \\ u(0) = D_q u(0) = D^q u(1) = 0, \qquad \phi(D_q^{\frac{5}{2}}u(0)) = D_q(\phi(D_q^{\frac{5}{2}}u(1))) = 0. \end{cases} \tag{4.2}$$

Here $q = \frac{1}{2}$, $\alpha = \frac{5}{2}$, $\beta = \frac{3}{2}$, $\phi(u) = |u|u$, and $f(u) = \frac{(u^3+u^2)(\sin u+9)}{25u+1}$.

Take $\psi_1(x) = \psi_2(x) = x^2$, $\delta = 0.9$. By calculating we get $A_1 \approx 0.07088$, $A_2 \approx 0.01550$, $F_\infty = 0.4$ and $f_0 = 9$. Thus, $f_0\psi_1(A_1^{-1}) > F_\infty\psi_2(A_2^{-1})$. By Theorem 3.1 we obtain that BVP (4.2) has at least one positive solution for each $\lambda \in (462.24260, 497.6280425)$.

Example 4.3 Consider the following fractional q-difference equation BVP:

$$\begin{cases} D_q^{\frac{3}{2}}(\phi(D_q^{\frac{5}{2}}u(t))) = \lambda\frac{(20u^2(t)+u(t))(\sin(u(t))+2)}{u(t)+1}, \quad 0 < t < 1, \\ u(0) = D_q u(0) = D^q u(1) = 0, \qquad D_q^{\frac{5}{2}}u(0) = D_q(D_q^{\frac{5}{2}}u(1)) = 0. \end{cases} \tag{4.3}$$

Here $q = \frac{1}{2}$, $\alpha = \frac{5}{2}$, $\beta = \frac{3}{2}$, $\phi(u) = u$, and $f(u) = \frac{(20u^2+u)(\sin u+2)}{u+1}$. Take $\psi_1(x) = \psi_2(x) = x$, $\delta = 0.9$. By calculating we have $A_1 \approx 0.05523$, $A_2 \approx 0.00811$, $F_0 = f_0 = 2$, $F_\infty = 60$, $f_\infty = 20$, and $u < f(u) < 60u$ for $u > 0$.

(i) By Theorem 3.1 we obtain that BVP (4.3) has at least one positive solution for each $\lambda \in (6.1653, 9.05305)$.

(ii) By Theorem 3.5 we obtain that BVP (4.3) has no positive solution for each $\lambda \in (0, 0.30177)$.

(iii) By Theorem 3.6 we obtain that BVP (4.3) has no positive solution for each $\lambda \in (123.30456, +\infty)$.

Example 4.4 Consider the following fractional q-difference equation BVP with ϕ-Laplacian:

$$\begin{cases} D_q^{\frac{3}{2}}(\phi(D_q^{\frac{5}{2}}u(t))) = \lambda \frac{(u^3(t)+u^2(t))(\arctan(u(t))+8)}{40u(t)+1}, & 0 < t < 1, \\ u(0) = D_q u(0) = D^q u(1) = 0, & D_q^{\frac{5}{2}}u(0) = D_q(D_q^{\frac{5}{2}}u(1)) = 0. \end{cases} \tag{4.4}$$

Here $q = \frac{1}{2}$, $\alpha = \frac{5}{2}$, $\beta = \frac{3}{2}$, $\phi(u) = |u|u$, and $f(u) = \frac{(u^3+u^2)(\arctan u+8))}{40u+1}$. Take $\psi_1(x) = \psi_2(x) = x^2$, $\delta = 0.9$. Then we get $A_1 \approx 0.07088$, $A_2 \approx 0.01550$, $f_0 = 8$, $F_\infty = 0.23927$, and $\frac{1}{5}\phi(u) < f(u) < 9.57080\phi(u)$ for $u > 0$.

(i) By Theorem 3.2 we obtain that BVP (4.4) has at least one positive solution for each $\lambda \in (520.02293, 831.91078)$.

(ii) By Theorem 3.5 we obtain that BVP (4.3) has no positive solution for each $\lambda \in (0, 20.79777)$.

(iii) By Theorem 3.6 we obtain that BVP (4.3) has no positive solution for each $\lambda \in (20800.91719, +\infty)$.

5 Conclusions

This research establishes the existence of at least one or two positive solutions in terms of different eigenvalue intervals for the BVP of ϕ-Laplacian fractional q-difference equation, by applying the Green function and Guo–Krasnoselskii fixed point theorem on cones. This enriches the theories for fractional q-difference equations and provides the theoretical guarantee for the application of fractional q-difference equations in such fields as aerodynamics, electrodynamics of complex medium, capacitor theory, electrical circuits, control theory, and so on. At the same time, we also consider the nonexistence of a positive solution in terms of the parameter λ. In the future, we will use bifurcation theory, critical point theory, variational method, and other methods to continue our works in this area.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Conceptualization, CY and JW; methodology, CY; data curation, JW and SW; original draft preparation, BZ; review and editing, JW. All authors have read and agreed with the published version of the manuscript. All authors read and approved the final manuscript.

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