Sadat et al. *Advances in Difference Equations* (2021) 2021:486
https://doi.org/10.1186/s13662-021-03637-w

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Lie symmetry analysis and invariant solutions of 3D Euler equations for axisymmetric, incompressible, and inviscid flow in the cylindrical coordinates

R. Sadat[1*], Praveen Agarwal[2], R. Saleh[1] and Mohamed R. Ali[3]

*Correspondence:
r.mosa@zu.edu.eg
[1]Department of Mathematics, Zagazig Faculty of Engineering, Zagazig University, Zagazig, Egypt
Full list of author information is available at the end of the article

Abstract

Through the Lie symmetry analysis method, the axisymmetric, incompressible, and inviscid fluid is studied. The governing equations that describe the flow are the Euler equations. Under intensive observation, these equations do not have a certain solution localized in all directions (r,t,z) due to the presence of the term $\frac{1}{r}$, which leads to the singularity cases. The researchers avoid this problem by truncating this term or solving the equations in the Cartesian plane. However, the Euler equations have an infinite number of Lie infinitesimals; we utilize the commutative product between these Lie vectors. The specialization process procures a nonlinear system of ODEs. Manual calculations have been done to solve this system. The investigated Lie vectors have been used to generate new solutions for the Euler equations. Some solutions are selected and plotted as two-dimensional plots.

Keywords: Euler equations; Axisymmetric flow; Lie point symmetries; Analytical solutions

1 Introduction

Suppose that the Euler equations have the form [1–4]

$$
\begin{aligned}
&\frac{\partial w}{\partial t}+w\frac{\partial w}{\partial r}+u\frac{\partial w}{\partial z}-\frac{v^2}{r}+\frac{\partial p}{\partial r}=0,\\
&\frac{\partial v}{\partial t}+w\frac{\partial v}{\partial r}+u\frac{\partial v}{\partial z}-\frac{vw}{r}=0,\\
&\frac{\partial u}{\partial t}+w\frac{\partial u}{\partial r}+u\frac{\partial u}{\partial z}+\frac{\partial p}{\partial z}=0,\\
&\frac{\partial w}{\partial r}+\frac{w}{r}+\frac{\partial u}{\partial z}=0.
\end{aligned}
\tag{1}
$$

That describes the dynamics of incompressible, axisymmetric flow with swirl [3], where $w(r,t,z)$, $u(r,t,z)$, and $v(r,t,z)$ are the components of the velocity in the cylindrical coordinates $(r,\phi,$ and $z)$, and $p(r,t,z)$ is the pressure. The flow is called axisymmetric flow if the velocity component and the pressure are independent of ϕ. Navier–Stokes and Euler equations in the cylindrical coordinates can describe any pipe fluid flow that has more applications, especially in the medical field. For example, blood flow in stenoses narrow artery [5–8]. System (1) had been solved using numerical methods in [1, 2, 9]. Manipulation of the results in most applications needs explicit solutions. The Lie symmetry analysis is one of the most important and powerful methods for obtaining closed-form solutions [10, 11]. The method proves its dependence in the fluid mechanics, turbulence field, and turbulent plane jet model [12–18]. Other researchers apply the method to other applications [19–25]. In (2007), Oberlack et al. [3] deduced five Lie point symmetries for Euler equations. Here, we use the commutative product to explore new Lie infinitesimals for system (1), then we use the investigated Lie vectors to reduce system (1) to the system of ODEs. By solving these ODEs, we explore new analytical solutions for Euler equations.

2 Investigation of Lie infinitesimals for Euler equations

System (1) possesses Lie infinitesimals as follows:

$$\begin{cases} X_1 = \frac{\partial}{\partial t} + f_1(t)\frac{\partial}{\partial z} + f_1'(t)\frac{\partial}{\partial u} + (-f_1''(t)z + f_2(t))\frac{\partial}{\partial p}, \\ X_2 = f_3(t)\frac{\partial}{\partial z} + f_3'(t)\frac{\partial}{\partial u} + \frac{1}{r^2 v}\frac{\partial}{\partial v} + (\frac{-1}{r^2} - f_3''(t)z + f_4(t))\frac{\partial}{\partial p}, \\ X_3 = t\frac{\partial}{\partial t} + f_5(t)\frac{\partial}{\partial z} + (f_5'(t) - u)\frac{\partial}{\partial u} - w\frac{\partial}{\partial w} - v\frac{\partial}{\partial v} + (-2p - f_5''(t)z + f_6(t))\frac{\partial}{\partial p}, \\ X_4 = r\frac{\partial}{\partial r} + (z + f_7(t))\frac{\partial}{\partial z} + (u + f_7'(t))\frac{\partial}{\partial u} + w\frac{\partial}{\partial w} + v\frac{\partial}{\partial v}(2p - f_7''(t)z + f_8(t))\frac{\partial}{\partial p}. \end{cases} \tag{2}$$

There are an infinite number of possibilities for these vectors as the presence of arbitrary functions $f_i(t)$, $i = 1 \ldots 8$. Using the commutative product between these infinitesimals listed in Table 1 authorizes us to specialize these vectors through the same procedure as in [10, 26]. Firstly, we generate the commutator table as follows in Table 1, where

$$\begin{aligned} a_1 &= -zf_3''' + f_4' - f_1 f_3'' + f_3 f_1'', \\ a_2 &= f_5' - tf_1', \\ a_3 &= f_5'' - f_1' - tf_1'', \\ a_4 &= -zf_5''' + f_6' - f_1 f_5'' + f_5 f_1'' + 2zf_1'' - 2f_2 + tzf_1''' - tf_2', \\ a_5 &= f_7' + f_1, \\ a_6 &= f_7'' + f_1', \\ a_7 &= -zf_7''' + f_8' - f_1 f_7'' - zf_1'' + 2f_2 + f_7 f_1'', \\ a_8 &= tf_3'' - f_3', \\ a_9 &= -f_3 f_5'' + 2zf_3'' - 2f_4 + \frac{2}{r^2} + tzf_3''' - tf_4', \\ a_{10} &= \frac{-4}{r^2} + f_7 f_3'' - f_3 f_7'' - zf_3'' + 2f_4, \\ a_{11} &= tf_7' + f_5, \\ a_{12} &= tf_7'' + f_7' + f_5', \\ a_{13} &= -tzf_7''' + tf_8' + f_7 f_5'' - f_5 f_7'' - zf_5'' + 2f_6 + 2f_8 - 2zf_7''. \end{aligned} \tag{3}$$

Table 1 Commutator table

$[V_1, V_2]$	X_1	X_2	X_3	X_4
X_1	0	$f_3'\frac{\partial}{\partial z}+f_3''\frac{\partial}{\partial u}+a_1\frac{\partial}{\partial p}$	$\frac{\partial}{\partial t}+a_2\frac{\partial}{\partial z}+a_3\frac{\partial}{\partial u}+a_4\frac{\partial}{\partial p}$	$a_5\frac{\partial}{\partial z}+a_6\frac{\partial}{\partial u}+a_7\frac{\partial}{\partial p}$
X_2	$-(f_3'\frac{\partial}{\partial z}+f_3''\frac{\partial}{\partial u}+a_1\frac{\partial}{\partial p})$	0	$-tf_3'\frac{\partial}{\partial z}+a_8\frac{\partial}{\partial u}-\frac{2}{r^2v}\frac{\partial}{\partial v}+a_9\frac{\partial}{\partial p}$	$f_3\frac{\partial}{\partial z}+f_3'\frac{\partial}{\partial u}+\frac{4}{r^2v}\frac{\partial}{\partial v}+a_{10}\frac{\partial}{\partial p}$
X_3	$-(\frac{\partial}{\partial t}+a_2\frac{\partial}{\partial z}+a_3\frac{\partial}{\partial u}+a_4\frac{\partial}{\partial p})$	$-(-tf_3'\frac{\partial}{\partial z}+a_8\frac{\partial}{\partial u}-\frac{2}{r^2v}\frac{\partial}{\partial v}+a_9\frac{\partial}{\partial p})$	0	$a_{11}\frac{\partial}{\partial z}+a_{12}\frac{\partial}{\partial u}+a_{13}\frac{\partial}{\partial p}$
X_4	$-(a_5\frac{\partial}{\partial z}+a_6\frac{\partial}{\partial u}+a_7\frac{\partial}{\partial p})$	$-(f_3\frac{\partial}{\partial z}+f_3'\frac{\partial}{\partial u}+\frac{4}{r^2v}\frac{\partial}{\partial v}+a_{10}\frac{\partial}{\partial p})$	$-(a_{11}\frac{\partial}{\partial z}+a_{12}\frac{\partial}{\partial u}+a_{13}\frac{\partial}{\partial p})$	0

Table 2 Commutator table after optimization

$[V_1, V_2]$	X_1	X_2	X_3	X_4
X_1	0	0	X_1	0
X_2	0	0	$-2X_2$	$4X_2$
X_3	$-X_1$	$2X_2$	0	0
X_4	0	$-4X_2$	0	0

The specialization process generates a nonlinear system of ODEs:

$$\begin{aligned}
& tf_1'' + 2f_1' = f_5'', \\
& -zf_5'''' + 3zf_1'' + tzf_1'''' - tf_2' - 3f_2 + f_6' - f_1f_5'' + f_5f_1'' = 0, \\
& f_7'' + f_1' = 0, \qquad -zf_7''' + f_8' - f_1f_7'' - zf_1'' + 2f_2 + f_7f_1'' = 0, \\
& tf_3'' - f_3' = 0, \\
& -f_3f_5'' + tzf_3''' - tf_5' = 0, \qquad f_7f_3'' - f_3f_7'' + 3zf_3'' - 2f_4 = 0, \\
& -tzf_7''' + tf_8' + f_7f_5'' - f_5f_7'' - zf_5'' + 2f_6 + 2f_8 - 2zf_7'' = 0, \\
& tf_7'' + f_5 - f_7' = 0.
\end{aligned} \tag{4}$$

Through manual calculations this system has been solved, and the results are

$$\begin{aligned}
& f_1 = \frac{1}{t}, \qquad f_2 = \frac{1}{t^3}, \qquad f_3 = f_4 = 0, \\
& f_5 = 1, \qquad f_6 = \frac{1}{t^2}, \qquad f_7 = -\ln(t), \qquad f_8 = \frac{-\ln(t)}{t^2}.
\end{aligned} \tag{5}$$

Substituting from (5) into (2), we obtain

$$\begin{cases}
X_1 = \frac{\partial}{\partial t} + \frac{1}{t}\frac{\partial}{\partial z} - \frac{1}{t^2}\frac{\partial}{\partial u} + (-\frac{2}{t^3}z + \frac{1}{t^3})\frac{\partial}{\partial p}, \\
X_2 = \frac{1}{r^2v}\frac{\partial}{\partial v} + (\frac{-1}{r^2})\frac{\partial}{\partial p}, \\
X_3 = t\frac{\partial}{\partial t} + \frac{\partial}{\partial z} + -u\frac{\partial}{\partial u} - w\frac{\partial}{\partial w} - v\frac{\partial}{\partial v} + (-2p + \frac{1}{t^2})\frac{\partial}{\partial p}, \\
X_4 = r\frac{\partial}{\partial r} + (z - \ln(t))\frac{\partial}{\partial z} + (u - \frac{1}{t})\frac{\partial}{\partial u} + w\frac{\partial}{\partial w} + v\frac{\partial}{\partial v}(2p - \frac{Z}{t^2} - \frac{\ln(t)}{t^2})\frac{\partial}{\partial p}.
\end{cases} \tag{6}$$

We use these vectors (6) to reproduce the commutator table (Table 2).

3 Reduction of the independent variables in Euler equations

3.1 Using Lie vector X_1

To snaffle the similarity variables, we solve the associated Lagrange system

$$\frac{dt}{1} = \frac{dz}{\frac{1}{t}} = -\frac{du}{\frac{1}{t^2}} = \frac{dp}{(-\frac{2}{t^3}z + \frac{1}{t^3})}. \tag{7}$$

The similarity variables of system (1) are

$$\begin{aligned} &u(r,t,z) = R(y,x) + \frac{1}{t}, \qquad w(r,t,z) = F(y,x), \qquad v(r,t,z) = G(y,x), \\ &p(r,t,z) = H(y,x) + \frac{z}{t^2}, \\ &\text{where, } y = r, x = z - \ln(t). \end{aligned} \tag{8}$$

Substituting from (8) into (1), we get the following system with two independent variables:

$$\begin{aligned} &y\frac{\partial F}{\partial y} + y\frac{\partial R}{\partial x} + F = 0, \\ &F\frac{\partial G}{\partial y}y + R\frac{\partial G}{\partial x}y + FG = 0, \\ &-F\frac{\partial F}{\partial y}y - R\frac{\partial F}{\partial x}y + G^2 - y\frac{\partial H}{\partial y} = 0, \\ &F\frac{\partial R}{\partial y} + R\frac{\partial R}{\partial x} + \frac{\partial H}{\partial x} = 0. \end{aligned} \tag{9}$$

System (9) has five Lie vectors as follows:

$$\begin{aligned} &V_1 = \frac{\partial}{\partial x}, \qquad V_2 = \frac{\partial}{\partial H}, \qquad V_3 = y\frac{\partial}{\partial y} + x\frac{\partial}{\partial x}, \\ &V_4 = \frac{1}{y^2 G}\frac{\partial}{\partial G} - \frac{1}{y^2}\frac{\partial}{\partial H}, \qquad V_5 = F\frac{\partial}{\partial F} + G\frac{\partial}{\partial G} + 2H\frac{\partial}{\partial H} + R\frac{\partial}{\partial R}. \end{aligned} \tag{10}$$

3.1.1 Using vector V_3

This Lie vector will reduce system (9) to

$$\begin{aligned} &-\eta T\frac{d\theta}{d\eta} + \theta\frac{d\theta}{d\eta} + \frac{d\beta}{d\eta} = 0, \\ &-\eta T\frac{dE}{d\eta} + \theta\frac{dE}{d\eta} + ET = 0, \\ &\eta\frac{dT}{d\eta} - T - \frac{d\theta}{d\eta} = 0, \\ &\eta T\frac{dT}{d\eta} - \theta\frac{dT}{d\eta} + E^2 + \eta\frac{d\beta}{d\eta} = 0, \end{aligned} \tag{11}$$

where the new dependent variables have been obtained from solving the characteristic equation that the V_3 was generated.

$$\begin{aligned} &E(\eta) = G(y,x), \qquad T(\eta) = F(y,x), \qquad \beta(\eta) = H(y,x), \\ &\theta(\eta) = R(y,x), \qquad \eta = \frac{x}{y}. \end{aligned} \tag{12}$$

The solutions for system (11) are as follows:

$$\begin{aligned} T(\eta) &= c_3\eta + c_4\sqrt{1+\eta^2}, \\ \theta(\eta) &= -c_4 \sinh^{-1}(\eta), \\ E(\eta) &= \mp\sqrt{\frac{-c_3(c_4\eta^3 + c_3\eta^2\sqrt{1+\eta^2} + c_4\eta + c_4\sinh^{-1}(\eta)\sqrt{1+\eta^2} - c_2\sqrt{1+\eta^2})}{\sqrt{1+\eta^2}}}, \\ \beta(\eta) &= \frac{-1}{2}\big(c_4\sinh^{-1}(\eta)\big)^2 \\ &\quad - c_4\left(c_3\left(\frac{1}{2}\eta\sqrt{1+\eta^2} - \frac{1}{2}\sinh^{-1}(\eta)\right) - c_2\sinh^{-1}(\eta) + \frac{1}{2}c_4\eta\right) + c_1. \end{aligned} \tag{13}$$

Back substitution to the original variables using similarity variables in (8) and (12) leads to

$$\begin{aligned} w(r,t,z) &= c_3\frac{(z-\ln(t))}{r} + c_4\sqrt{1+\left(\frac{(z-\ln(t))}{r}\right)^2}, \\ u(r,t,z) &= -c_4\sinh^{-1}\left(\frac{(z-\ln(t))}{r}\right), \\ v(r,t,z)& \\ &= \mp\sqrt{\frac{-c_3(c_4(\delta)^3 + c_3(\delta)^2\sqrt{1+(\delta)^2} + c_4(\delta) + c_4\sinh^{-1}(\delta)\sqrt{1+(\delta)^2} - c_2\sqrt{1+(\delta)^2})}{\sqrt{1+(\delta)^2}}}, \\ p(r,t,z) &= \frac{-1}{2}\big(c_4\sinh^{-1}(\delta)\big)^2 \\ &\quad - c_4\left(c_3\left(\frac{1}{2}(\delta)\sqrt{1+(\delta)^2} - \frac{1}{2}\sinh^{-1}(\delta)\right) - c_2\sinh^{-1}(\delta) + \frac{1}{2}c_4(\delta)\right) + c_1, \end{aligned} \tag{14}$$

where $\boldsymbol{\delta} = \frac{(z-\ln(t))}{r}$.

The solutions have been plotted for different values of time as depicted in Figs. 1–4.

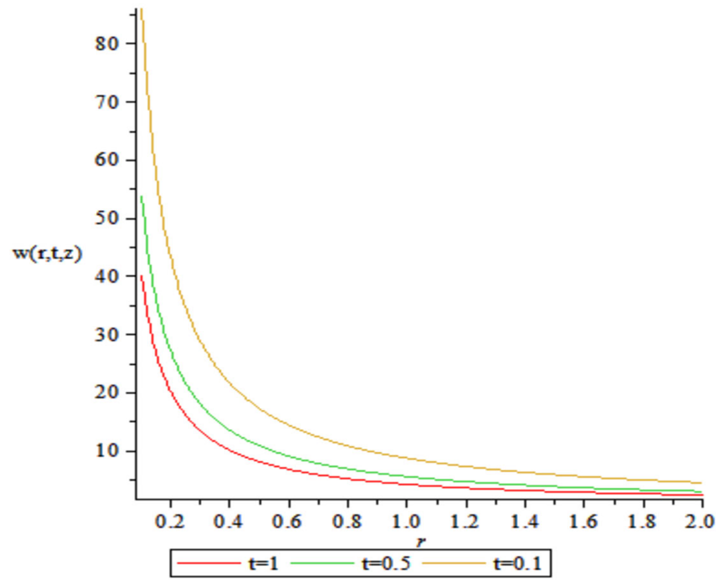

Figure 1 Velocity component $w(r,t,z)$ at $z = 2$, $c_3 = 1$, and $c_4 = 1$

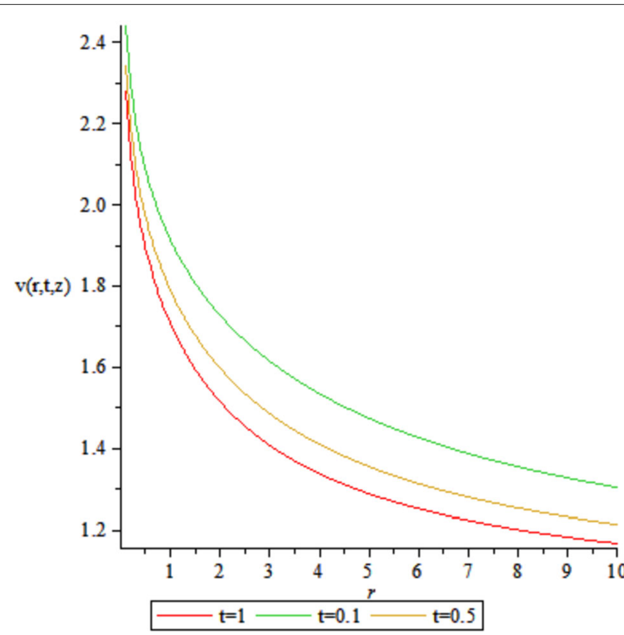

Figure 2 Positive case of velocity component $v(r,t,z)$ at $z = 2$ and $c_4 = -1$

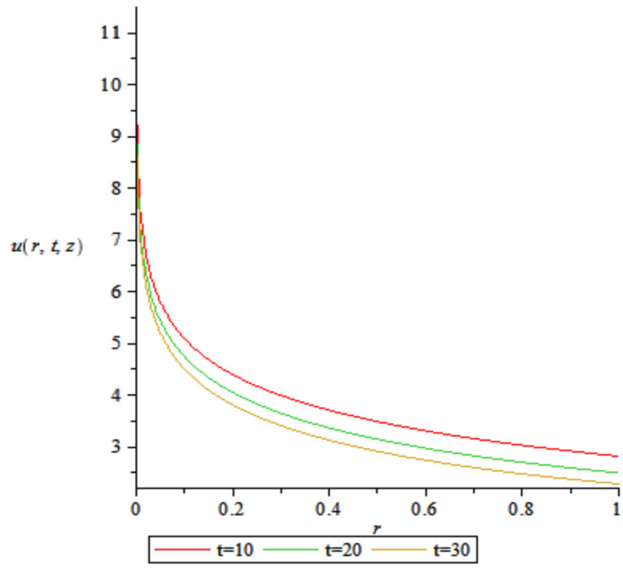

Figure 3 Velocity component $u(r,t,z)$ at $z = 5$, $c_2 = 1$, and $c_4 = -1$

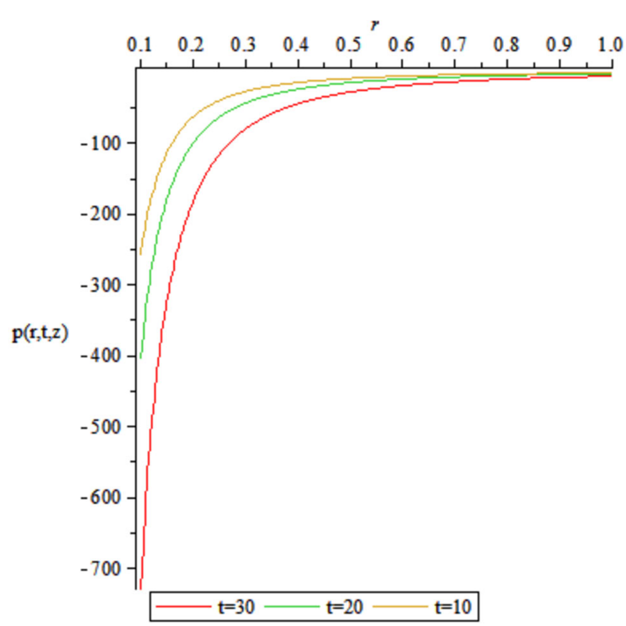

Figure 4 The pressure $p(r,t,z)$ at $z = 5$, $c_1 = 1$, $c_2 = 1$, $c_3 = 1$, and $c_4 = 1$

3.1.2 Using $\boldsymbol{V} = \boldsymbol{V_1} + \boldsymbol{V_4}$

This vector produces a system of nonlinear ODEs as follows:

$$\begin{aligned}
&\eta \frac{dT}{d\eta} + T = 0,\\
&\eta^2 T \frac{d\theta}{d\eta} - 1 = 0,\\
&-\eta T \frac{dT}{d\eta} + E - \eta \frac{d\beta}{d\eta} = 0,\\
&\eta^2 T \frac{dE}{d\eta} + 2\eta T E + 2\theta = 0,
\end{aligned} \tag{15}$$

where the new dependent variables are

$$\begin{aligned}
&E(\eta) = \frac{-2x + y^2 G(y,x)^2}{y^2}, \qquad T(\eta) = F(y,x), \qquad \beta(\eta) = H(y,x) + \frac{x}{y^2},\\
&\theta(\eta) = R(y,x) \quad \text{where } \eta = y.
\end{aligned} \tag{16}$$

By solving system (15), new solutions for Euler equations have been produced:

$$\begin{aligned}
&T(\eta) = \frac{c_4}{\eta},\\
&\theta(\eta) = \frac{\ln(\eta)}{c_4} + c_3,\\
&E(\eta) = \frac{-\eta^2 \ln(\eta) + 0.5\eta^2 - c_3 c_4 \eta^2 + c_2 c_4^2}{(c_4 \eta)^2},\\
&\beta(\eta) = -0.5\left(\frac{c_4^2}{\eta^2} + \frac{(\ln(\eta))^2}{c_4^2} - \frac{\ln(\eta)}{c_4^2} + 2\frac{c_3 \ln(\eta)}{c_4} + \frac{c_2}{\eta^2} - 2c_1\right).
\end{aligned} \tag{17}$$

Using the similarity variables in (8) and (16) leads to back substitution to the original variables:

$$\begin{aligned}
&w(r,t,z) = \frac{c_4}{r},\\
&u(r,t,z) = \frac{\ln(r)}{c_4} + c_3 + t^{-1},\\
&v(r,t,z) = \sqrt{\frac{-r^2 \ln(r) + 0.5r^2 - c_3 c_4 r^2 + c_2 c_4^2}{(c_4 r)^2} + 2\left(\frac{(z - \ln(t))}{r^2}\right)},\\
&p(r,t,z) = -0.5\left(\frac{c_4^2}{r^2} + \frac{(\ln(r))^2}{c_4^2} - \frac{\ln(r)}{c_4^2} + 2\frac{c_3 \ln(r)}{c_4} + \frac{c_2}{r^2} - 2c_1\right)\\
&\qquad - \left(\frac{(z - \ln(t))}{r^2}\right) + z t^{-2}.
\end{aligned} \tag{18}$$

3.1.3 Using Lie vector $\boldsymbol{V} = \boldsymbol{V_1} + \boldsymbol{V_5}$

Through the same previous procedure system (9) has been reduced to

$$\begin{aligned}
&T\frac{d\theta}{d\eta} + \theta^2 + 2\beta = 0,\\
&\eta\frac{dT}{d\eta} + T + \eta\theta = 0,\\
&\eta T\frac{dE}{d\eta} + \eta\theta E + ET = 0,\\
&-\eta T\frac{dT}{d\eta} + E^2 - \eta T\theta - \eta\frac{d\beta}{d\eta} = 0,
\end{aligned} \tag{19}$$

where the similarity variables are

$$\begin{aligned}
&E(\eta) = G(y,x), \qquad e^{-x}, \qquad T(\eta) = F(y,x), \qquad e^{-x},\\
&\beta(\eta) = H(y,x), \qquad e^{-2x}, \qquad \theta(\eta) = R(y,x), \qquad e^{-x}, \qquad \eta = y.
\end{aligned} \tag{20}$$

System (19) has closed form solutions as follows:

$$\begin{aligned}
&T(\eta) = \frac{-c_3 e^{-\frac{0.5I\eta^2}{c_1}} + c_3 e^{\frac{0.5I\eta^2}{c_1}} + c_4 e^{-\frac{0.5I\eta^2}{c_1}}}{\eta},\\
&\theta(\eta) = -\frac{I(c_3 e^{-\frac{0.5I\eta^2}{c_1}} + c_3 e^{\frac{0.5I\eta^2}{c_1}} - c_4 e^{-\frac{0.5I\eta^2}{c_1}})}{c_1},\\
&E(\eta) = \pm\frac{\sqrt{2c_3^2 + 2c_3c_4 e^{-\frac{I\eta^2}{c_1}} - 2c_3c_4 - c_3^2 e^{-\frac{I\eta^2}{c_1}} - c_3^2 e^{\frac{I\eta^2}{c_1}} - c_4^2 e^{-\frac{I\eta^2}{c_1}}}}{\eta},\\
&\beta(\eta) = \frac{2c_3(c_3 - c_4)}{c_1^2}.
\end{aligned} \tag{21}$$

Back substitution using the similarity variables in (20) and (8) is as follows:

$$\begin{aligned}
&w(r,t,z) = \frac{-c_3 e^{-\frac{0.5Ir^2}{c_1}} + c_3 e^{\frac{0.5Ir^2}{c_1}} + c_4 e^{-\frac{0.5Ir^2}{c_1}}}{r} e^{(z-\ln(t))},\\
&u(r,t,z) = -\frac{I(c_3 e^{-\frac{0.5I\eta^2}{c_1}} + c_3 e^{\frac{0.5I\eta^2}{c_1}} - c_4 e^{-\frac{0.5I\eta^2}{c_1}})}{c_1} e^{(z-\ln(t))} + t^{-1},\\
&v(r,t,z) = \pm\frac{\sqrt{2c_3^2 + 2c_3c_4 e^{-\frac{Ir^2}{c_1}} - 2c_3c_4 - c_3^2 e^{-\frac{Ir^2}{c_1}} - c_3^2 e^{\frac{Ir^2}{c_1}} - c_4^2 e^{-\frac{Ir^2}{c_1}}}}{re^{-(z-\ln(t))}},\\
&p(r,t,z) = \frac{2c_3(c_3 - c_4)}{c_1^2} e^{(z-\ln(t))} + zt^{-2}.
\end{aligned} \tag{22}$$

The solutions have been plotted in Figs. 5–8.

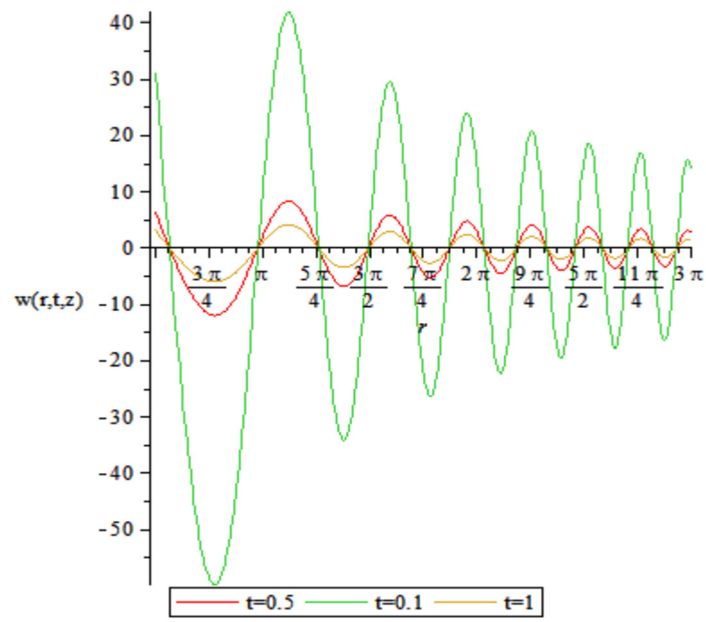

Figure 5 Velocity component $w(r,t,z)$ at $z = 2$, $c_1 = 1$, $c_3 = 1$, and $c_4 = 2$

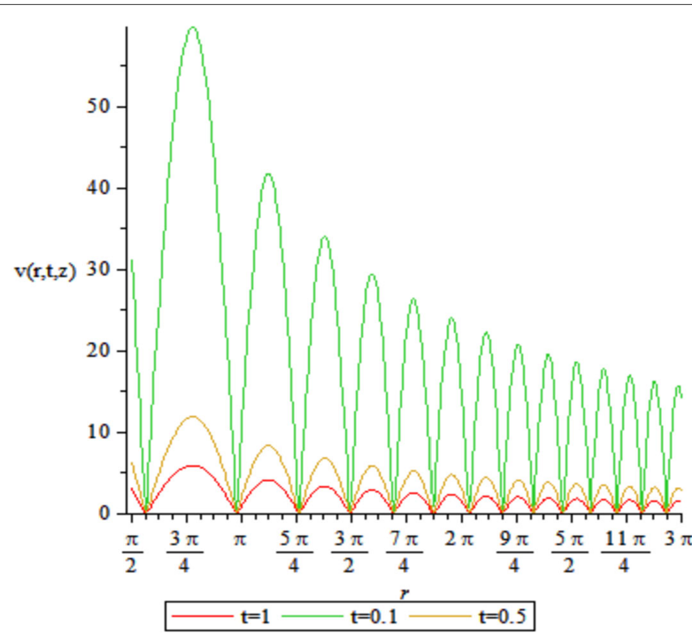

Figure 6 Positive case velocity component $v(r,t,z)$ at $z = 2$, $c_1 = 1$, $c_3 = I$, and $c_4 = 2I$

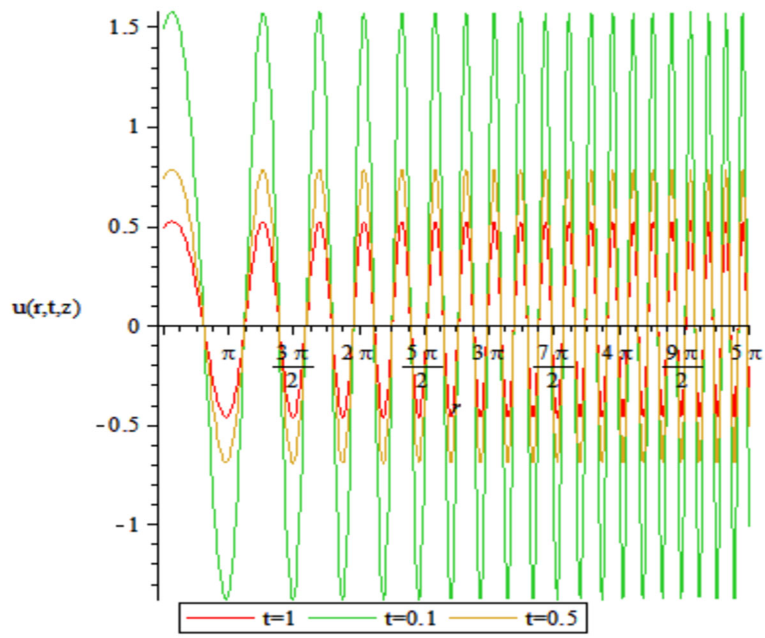

Figure 7 Velocity component $u(r,t,z)$ at $z = 2$, $c_1 = 1$, $c_3 = 1$, and $c_4 = 2$

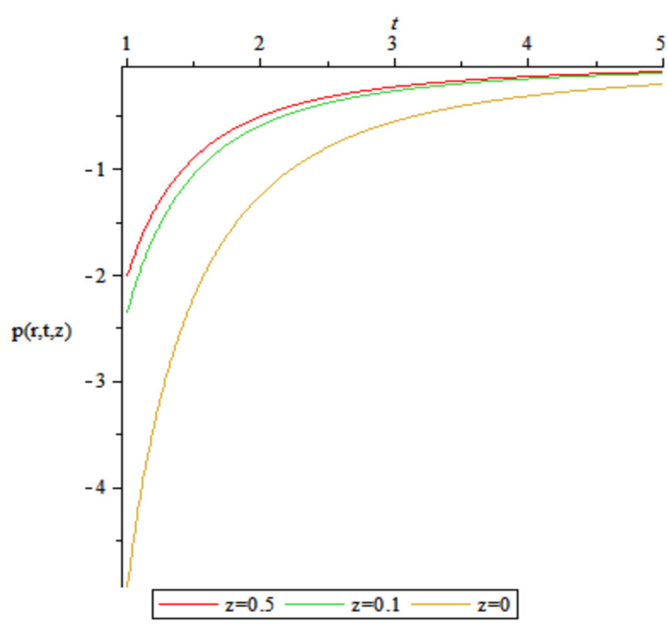

Figure 8 The pressure $p(r,t,z)$ at $c_1 = 1$, $c_3 = 1$, and $c_4 = 2$

3.2 Using Lie vector $X = X_3 + X_4$

By solving the subsidiary equation, we explore the similarity variables

$$
\begin{aligned}
&u(r,t,z) = R(y,x) + \frac{1}{t}, \qquad w(r,t,z) = F(y,x), \qquad v(r,t,z) = G(y,x), \\
&p(r,t,z) = H(y,x) + \frac{z}{t^2}, \\
&\text{where } y = \frac{t}{r}, x = \frac{z - \ln(t)}{r},
\end{aligned}
\tag{23}
$$

which reduce system (1) to

$$
\begin{aligned}
&-\frac{\partial G}{\partial y} + xF\frac{\partial G}{\partial x} + yF\frac{\partial G}{\partial y} - R\frac{\partial G}{\partial x} - FG = 0, \\
&x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} - F + \frac{\partial R}{\partial x} = 0, \\
&-\frac{\partial R}{\partial y} + xF\frac{\partial R}{\partial x} + yF\frac{\partial R}{\partial y} - R\frac{\partial R}{\partial x} - \frac{\partial H}{\partial x} = 0, \\
&-\frac{\partial F}{\partial y} + xF\frac{\partial F}{\partial x} + yF\frac{\partial F}{\partial y} - R\frac{\partial F}{\partial x} + G^2 + \frac{\partial H}{\partial x}x + \frac{\partial H}{\partial y}y = 0.
\end{aligned}
\tag{24}
$$

This system possesses three Lie vectors as follows:

$$
V_1 = \frac{\partial}{\partial H}, \qquad V_2 = y\frac{\partial}{\partial x} + \frac{\partial}{\partial R}, \qquad V_3 = y\frac{\partial}{\partial y} - F\frac{\partial}{\partial F} - G\frac{\partial}{\partial G} - 2H\frac{\partial}{\partial H} - R\frac{\partial}{\partial R}. \tag{25}
$$

- *Using* $\boldsymbol{V} = \boldsymbol{V_1} + \boldsymbol{V_2}$

Following the same procedure system (24) will be reduced to

$$
\begin{aligned}
&-\frac{dE}{d\eta} + \eta T\frac{dE}{d\eta} - ET = 0, \\
&-\frac{dT}{d\eta} + \eta T\frac{dT}{d\eta} + E^2 + \eta\frac{d\beta}{d\eta} = 0, \\
&-\eta\frac{d\theta}{d\eta} - \theta + \eta^2 T\frac{d\theta}{d\eta} - 1 = 0, \\
&\eta^2\frac{dT}{d\eta} - \eta T - 1 = 0
\end{aligned}
\tag{26}
$$

with new variables

$$
\begin{aligned}
&E(\eta) = G(y,x), \qquad T(\eta) = F(y,x), \qquad \beta(\eta) = -H(y,x) + \frac{x}{y}, \\
&\theta(\eta) = R(y,x) - \frac{x}{y}, \qquad \eta = y.
\end{aligned}
\tag{27}
$$

By solving system (26), we have

$$\begin{aligned}
T(\eta) &= \frac{-1}{2\eta},\\
\theta(\eta) &= -1 + \frac{c_3}{\eta^{2/3}},\\
E(\eta) &= c_2\eta^{2/3},\\
\beta(\eta) &= \frac{-3c_2^2}{2}\eta^{2/3} - \frac{3}{8\eta^2} + c_1.
\end{aligned} \tag{28}$$

Using the similarity variables in (23) and (27) authorizes us to back substitution to the original variables

$$\begin{aligned}
w(r,t,z) &= \frac{-r}{2t},\\
u(r,t,z) &= -1 + \frac{c_3}{(\frac{t}{r})^{\frac{2}{3}}} - \frac{z-\ln(t)}{t} + t^{-1},\\
v(r,t,z) &= c_2\left(\frac{t}{r}\right)^{2/3},\\
p(r,t,z) &= \frac{-3c_2^2}{2}\left(\frac{t}{r}\right)^{2/3} - \frac{3}{8(\frac{t}{r})^2} - \frac{z-\ln(t)}{t} + c_1 + zt^{-2}.
\end{aligned} \tag{29}$$

The results have been plotted as shown in Figs. 9–12.

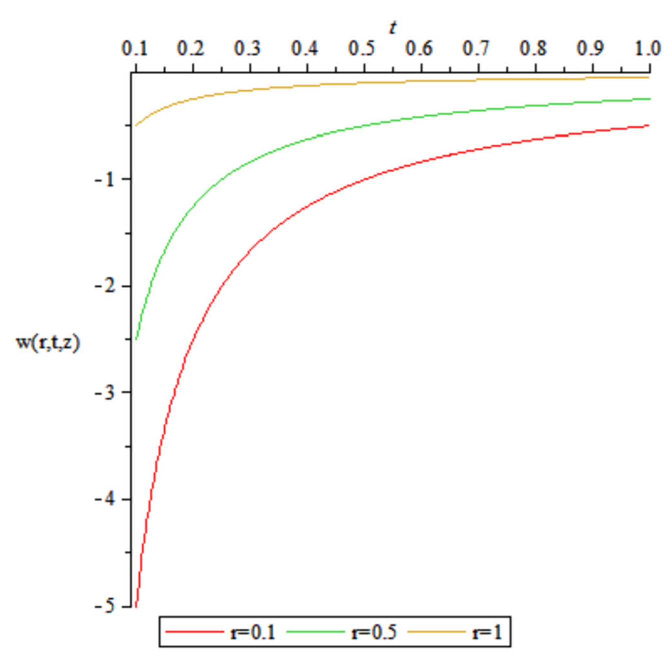

Figure 9 Velocity component $w(r,t,z)$

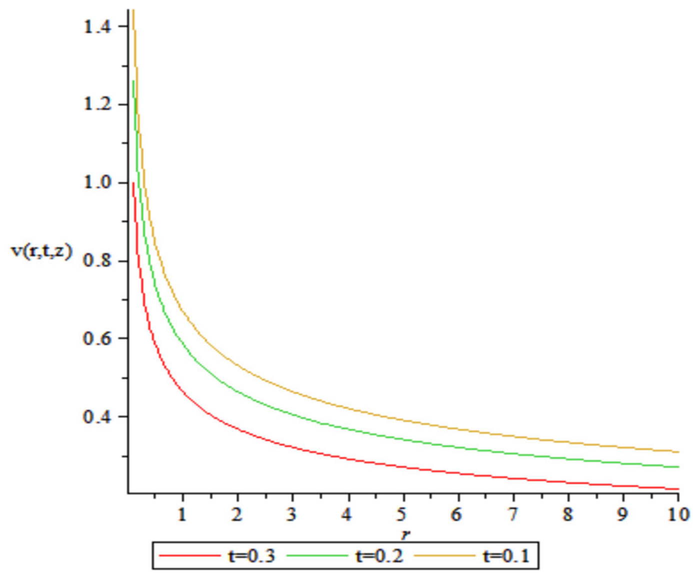

Figure 10 Velocity component $v(r,t,z)$ at $c_2 = 1$

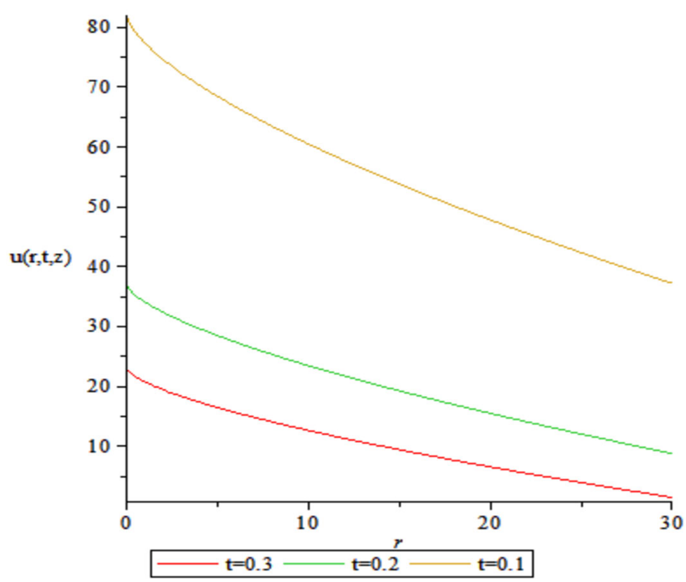

Figure 11 Velocity component $u(r,t,z)$ at $z = 5$ and $c_3 = -1$

4 Conclusions

We deduce an infinite number of Lie infinitesimals, and through commutative product properties, we minimize these vectors to four Lie vectors. Through some combinations between these vectors, we explore exact solutions for Euler equations. The results illustrate that the velocity components decrease with increasing the spatial or temporal coordinates. The pressure may be appearing as a negative value, and this is reasonable according to the human pressure in the case of the tapered artery [6].

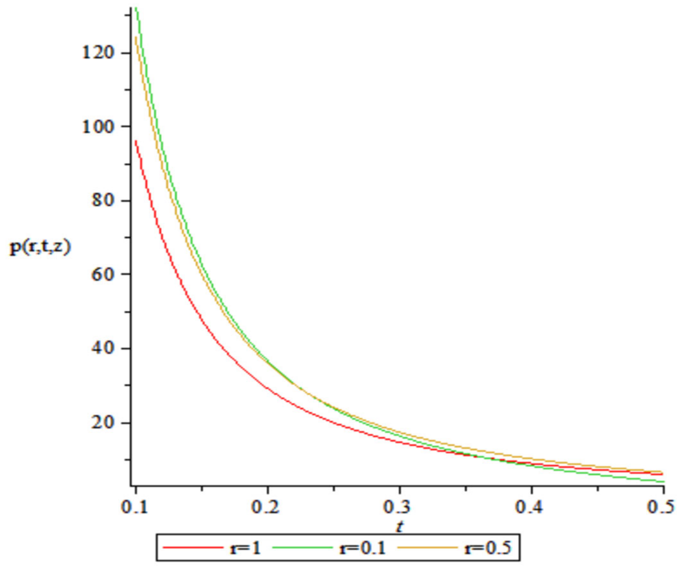

Figure 12 The pressure $p(r,t,z)$ at $z = 1$, $c_1 = 1$, and $c_4 = 1$

Acknowledgements
The authors thank the reviewers.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Zagazig Faculty of Engineering, Zagazig University, Zagazig, Egypt. [2]Department of Mathematics, Anand International College of Engineering, Jaipur, 302012, India. [3]Department of Basic Science, Faculty of Engineering at Benha, Benha University, Benha, 13512, Egypt.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 8 August 2021 Accepted: 19 October 2021 Published online: 06 November 2021

References

1. Verzicco, R., Orlandi, P.: A finite-difference scheme for three-dimensional incompressible flows in cylindrical coordinates. J. Comput. Phys. **123**(2), 402–414 (1996)
2. Saiac, J.H.: Finite element solutions of axisymmetric Euler equations for an incompressible and inviscid fluid. Int. J. Numer. Methods Fluids, **10**(2) 141–160 (1990)
3. Frewer, M., Oberlack, M., Guenther, S.: Symmetry investigations on the incompressible stationary axisymmetric Euler equations with swirl. Fluid Dyn. Res., **39**(8), 647 (2007)
4. Leprovost, N., Dubrulle, B., Chavanis, P.-H.: Dynamics and thermodynamics of axisymmetric flows: theory. Phys. Rev. E, **73**(4), 046308 (2006)
5. Chakravarty, S., Sen, S.: A mathematical model of blood flow in a catheterized artery with a stenosis. J. Mech. Med. Biol. **9**(3), 377–410 (2009)
6. Chakravarty, S., Mandal, P.: Mathematical modelling of blood flow through an overlapping arterial stenosis. Math. Comput. Model. **19**(1), 59–70 (1994)

7. Prasad, K.M., Thulluri, S., Phanikumari, M.: Investigation of blood flow through an artery in the presence of overlapping stenosis. J. Nav. Archit. Mar. Eng. **14**(1), 39–46 (2017)
8. Akbar, N.S.: Blood flow analysis of Prandtl fluid model in tapered stenosed arteries. Ain Shams Eng. J. **5**(4), 1267–1275 (2014)
9. Barbosa, E., Daube, O.: A finite difference method for 3D incompressible flows in cylindrical coordinates. Comput. Fluids **34**(8), 950–971 (2005)
10. Ali, M.R., Sadat, R., Ma, W.X.: Investigation of new solutions for an extended (2 + 1)-dimensional Calogero-Bogoyavlenskii-Schif equation. Front. Math. China **16**, 925–936 (2021). https://doi.org/10.1007/s11464-021-0952-3
11. Sadat, R., et al.: Investigation of Lie symmetry and new solutions for highly dimensional non-elastic and elastic interactions between internal waves. Chaos Solitons Fractals **140**, 110134 (2020)
12. Sadeghi, H., Oberlack, M., Gauding, M.: On new scaling laws in a temporally evolving turbulent plane jet using Lie symmetry analysis and direct numerical simulation. J. Fluid Mech. **854**, 233–260 (2018)
13. Wacławczyk, M., Grebenev, V., Oberlack, M.: Lie symmetry analysis of the Lundgren–Monin–Novikov equations for multi-point probability density functions of turbulent flow. J. Phys. A, Math. Theor. **50**(17), 175501 (2017)
14. Wacławczyk, M., Oberlack, M.: Symmetry analysis and invariant solutions of the multipoint infinite systems describing turbulence. In: Journal of Physics: Conference Series. IOP Publishing, Bristol (2016)
15. Sahoo, S., Saha Ray, S.: On the conservation laws and invariant analysis for time-fractional coupled Fitzhugh-Nagumo equations using the Lie symmetry analysis. Eur. Phys. J. Plus **134**, 83 (2019)
16. Jyoti, D., Kumar, S., Gupta, R.K.: Exact solutions of Einstein field equations in perfect fluid distribution using Lie symmetry method. Eur. Phys. J. Plus **135**, 604 (2020)
17. Zhao, Z., Zhang, Y., Han, Z.: Symmetry analysis and conservation laws of the Drinfeld-Sokolov-Wilson system. Eur. Phys. J. Plus **129**, 143 (2014)
18. Kumar, D., Kumar, S.: Solitary wave solutions of pZK equation using Lie point symmetries. Eur. Phys. J. Plus **135**, 162 (2020). https://doi.org/10.1140/epjp/s13360-020-00218-w
19. Ali, M.R., Sadat, R.: Lie symmetry analysis, new group invariant for the (3 + 1)-dimensional and variable coefficients for liquids with gas bubbles models. Chin. J. Phys. **71**, 539–547 (2021), ISSN 0577-9073
20. Jadaun, V., Kumar, S.: Symmetry analysis and invariant solutions of (3 + 1)-dimensional Kadomtsev–Petviashvili equation. Int. J. Geom. Methods Mod. Phys. **15**(8), 1850125 (2018)
21. Ali, M.R., Sadat, R.: Construction of Lump and optical solitons solutions for (3 + 1) model for the propagation of nonlinear dispersive waves in inhomogeneous media. Opt. Quantum Electron. **53**, 279 (2021). https://doi.org/10.1007/s11082-021-02916-w
22. Agarwal, P., Deniz, S., Jain, S., Alderremy, A.A., Aly, S.: A new analysis of a partial differential equation arising in biology and population genetics via semi analytical techniques. Phys. A, Stat. Mech. Appl. **542**, 122769 (2020) ISSN 0378-4371. https://doi.org/10.1016/j.physa.2019.122769
23. Salahshour, S., Ahmadian, A., Senu, N., Baleanu, D., Agarwal, P.: On analytical solutions of the fractional differential equation with uncertainty: application to the Basset problem. Entropy **17**, 885–902 (2015). https://doi.org/10.3390/e17020885
24. Zhang, Y., Agarwal, P., Bhatnagar, V., Balochian, S., Yan, J.: Swarm intelligence and its applications. Sci. World J. **2013**, Article ID 528069 (2013). https://doi.org/10.1155/2013/528069
25. Zhou, S.-S., Areshi, M., Agarwal, P., Shah, N.A., Chung, J.D., Nonlaopon, K.: Analytical analysis of fractional-order multi-dimensional dispersive partial differential equations. Symmetry **13**, 939 (2021). https://doi.org/10.3390/sym13060939
26. Zhang, Y., Agarwal, P., Bhatnagar, V., Balochian, S., Zhang, X.: Swarm intelligence and its applications 2014. Sci. World J. **2014**, Article ID 204294 (2014). https://doi.org/10.1155/2014/204294