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Numerical solvability of generalized Bagley–Torvik fractional models under Caputo–Fabrizio derivative

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Abstract

This paper deals with the generalized Bagley–Torvik equation based on the concept of the Caputo–Fabrizio fractional derivative using a modified reproducing kernel Hilbert space treatment. The generalized Bagley–Torvik equation is studied along with initial and boundary conditions to investigate numerical solution in the Caputo–Fabrizio sense. Regarding the generalized Bagley–Torvik equation with initial conditions, in order to have a better approach and lower cost, we reformulate the issue as a system of fractional differential equations while preserving the second type of these equations. Reproducing kernel functions are established to construct an orthogonal system used to formulate the analytical and approximate solutions of both equations in the appropriate Hilbert spaces. The feasibility of the proposed method and the effect of the novel derivative with the nonsingular kernel were verified by listing and treating several numerical examples with the required accuracy and speed. From a numerical point of view, the results obtained indicate the accuracy, efficiency, and reliability of the proposed method in solving various real life problems.

Keywords: Generalized Bagley–Torvik equations; Caputo–Fabrizio fractional derivative; Modified reproducing kernel Hilbert spaces

1 Introduction

In the last two decades, fractional calculus has become one of the most effective tools used in modeling of dynamical systems, to name a few, quantum, quantum physics, liquids, mechanics, optimization of biological models [1–6], where these phenomena are modeled and updated in equations that are closest to actually describing their condition. The first application for fractional calculus was probably what is referred to as a tautochrone problem that was given by Niels Henrik Abel in 1823 [2]. The great expansion in this field caused the attention and concern of researchers from various sciences, who in turn rushed to improve and correct some of the deficiencies and gaps in modeling.

However, there is no generally accepted definition of fractional operator, but some papers suggested mathematical properties that a fractional derivative must have. Hence, we can find many definitions each has advantages and disadvantages, see for example [5–8].

In 2015, the scientists Caputo and Fabrizio introduced a novel operator without a singular kernel [9], called Caputo–Fabrizio fractional derivative (CFFD). The advantage of this operator is that it provides an accurate update and description of some typical phenomena in which the most famous definitions often complain, similar to the Caputo, Riemann–Liouville, Riesz fractional operators, of limitations and deficiencies in the description [10–15], since all of these definitions have a singular kernel. The interest of the nonsingular kernel is due to the necessity of describing material heterogeneitics and structures with distinct scales which local theory fails to describe. For more information, the reader is kindly requested to read the papers [16–21].

Anyhow, Caputo–Fabrizio derivative has a nice characteristic; it has an equivalent representation with singular kernel which allows to study more general materials such as visco-plastic materials rather than only visco-elastic ones. The reader can refer to [22] in which the equivalence of CFFD with a model of singular kernel is presented. The Bagley–Torvik equation is one of the most important equations that occupies a leading status in the study of applied sciences and engineering applications. It was originally formulated in the eighties of the last century, where both Bagley and Torvik presented a study model viscoelasticity damped structure in the following equation [23]:

$$a_1 z''(\xi) + a_2 \mathcal{D}_0^{\gamma} z(\xi) + a_3 z(\xi) = h(\xi), \quad \gamma = 3/2, \tag{1.1}$$

along with the constants $a_i \in R$, $i = 1, 2, 3$, $h(\xi) \in C[0, X]$ which is the space of continuous functions over the interval $[0, X]$, and $z(\xi)$ is an unknown function to be determined. Moreover, the Bagley–Torvik equation (1.1) has been generalized as $\gamma \in (0, 2)$, and it can be modeled from a generalized fractional vibration equation [24]. Most previously prepared works are mostly related to Riemann–Liouville and Caputo derivatives.

Furthermore, the coefficients a_i may also change parallel with the changes in fluid density and viscosity. That is, a_i may be functions with respect to time. Exclusively, we generally consider the following generalized Bagley–Torvic equation (GBT) within CFFD:

$$a_1(\xi) z''(\xi) + a_2(\xi)\, {}^{CF}\mathcal{D}_0^{\frac{3}{2}} z(\xi) + a_3(\xi) z'(\xi) + a_4(\xi)\, {}^{CF}\mathcal{D}_0^{\frac{1}{2}} z(\xi) + a_5(\xi) z(\xi) = h(\xi), \tag{1.2}$$

with initial conditions given by

$$z(0) = \alpha_0, \qquad z'(0) = \alpha_1, \tag{1.3}$$

whereas the boundary conditions are of the form

$$z(0) = \beta_0, \qquad z(X) = \beta_1, \tag{1.4}$$

whereas $0 \le \xi \le X$, $a_i(\xi), h(\xi) \in C[0, X]$, $\alpha_i, \beta_i \in R$, while $z(\xi)$ are the solutions to the equation under initial or boundary conditions that are given in the Hilbert spaces $\mathcal{H}_2[0, X]$ and $\mathcal{H}_3[0, X]$ which will be defined later in Sect. 3.

Physically, the fractional 1/2-order and 3/2-order derivatives are common to predict the frequency-dependent damping material. These derivatives are often used to model the motion of physical systems, as a study of an immersed plate in a Newtonian fluid was associated with the derivative 3/2 and gas in a fluid with the derivative 1/2. For more information, we refer to [25–30] and the references therein.

The value and importance of the Bagley–Torvic equation attracted the interest of researchers in finding approximate solutions using a range of iterative methods, among them, Adomian decomposition method, the generalized Taylor collocation method, homotopy analysis method, Chebyshev wavelet method, multistep methods, and predictor–corrector method of Adams type [31–33]. In this present paper, we are interested in studying and developing the modified reproducing kernel Hilbert space (MRKHS) under the influence of the CFFD, and making it suitable for solving the GBT under initial or boundary conditions, and that with greater accuracy and a lower time effort. The beginnings of the MRKHS method dates back to 1907, when Stanisław Zaremba presented research treating the boundary value problem of the harmonic functions. Thereafter, the process of developing this method proceeded until it has reached its current form. The MRKHS method occupies an important position among other numerical methods, as it is a very effective tool in many fields, such as machine learning, statistics, probability theory, economics, and the theory of integral operators [34–38]. In addition to that, this method is not limited to treating the well-posed problems, but also allows treating ill-posed problems.

The aim of this research paper can be summarized in several key points:

- We undertake adjustments and improvements at the level of the MRKHS method, under the influence of the novel operator CFFD.
- We use the CFFD properties to formulate GBT under initial conditions and convert it into an equivalent system of fractional differential equations. Then we apply the method given to solve this system. Finally, we get a relationship between the system solution and our equation.
- We solve the GBT equation under the boundary conditions using the proposed method.
- We prove some theorems related to the RKHS solution and its convergence to the exact solution.
- We add mathematical simulations to determine the appropriateness and effectiveness of the accounts created.

2 Basic definitions and concepts

In this part, we present some basic definitions and theories for the Caputo–Fabrizio fractional derivative (CFFD) used in our study. It should be noted that rewriting Eq. (1.1) in the sense of this operator is due to its property of having a nonsingular kernel that enables it to represent some phenomena that well brief the functions of kernels. This data is not presented in other definitions, which creates difficulties in modeling. The use of exponentially based kernel and the equivalence of CFFD with a model of singular kernel give high precision in realistic representation phenomena, especially for physical problems.

Definition 2.1 ([9]) The nonlocal fractional derivative of order $0 \le \alpha \le 1$ of a smooth function $z : [a, \infty) \to \Re$ is given as

$$ {}^{CF}\mathcal{D}_{\xi}^{\alpha} z(\xi) = \begin{cases} \frac{M(\alpha)}{1-\alpha} \int_a^{\xi} z'(\tau) e^{\frac{-\alpha(\xi-\tau)}{1-\alpha}}\, d\tau, & 0 \le \alpha < 1, \\ \frac{dz(\xi)}{d\xi}, & \alpha = 1, \end{cases} \tag{2.1} $$

such that $a \leq 0, \xi \geq 0$, and $M(\alpha)$ is a normalization function that satisfies $M(0) = M(1) = 1$. This operator is called Caputo–Fabrizio fractional derivative and is denoted by (CFFD).

Definition 2.2 ([9]) Let z be a smooth function over $[0, X]$ and $0 \leq \alpha \leq 1$. Then the corresponding fractional integral of CFFD operator with order α of a function z is given by

$$^{CF}\mathcal{I}_0^\alpha z(\xi) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} z(\xi) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^\xi z(\tau)\,d\tau. \tag{2.2}$$

Proposition 2.3 ([10]) *For $0 < \alpha < 1$ and $z \in \mathcal{H}_1[0, X]$, we can conclude that*

$$\left(^{CF}\mathcal{I}_0^\alpha\right)\left(^{CF}\mathcal{D}_0^\alpha\right) z(\xi) = z(\xi) - z(0), \tag{2.3}$$

where $H_1[0, X]$ is the usual Sobolev space over $[0, X]$.

The following proposition will be used as an important hint for the higher-order derivations.

Proposition 2.4 ([10]) *If $\alpha \in [0, 1]$ and $n \in N$, then the CFFD of order $n + \alpha$ is defined as*

$$^{CF}\mathcal{D}_\xi^{\alpha+n} z(\xi) = {}^{CF}\mathcal{D}_\xi^\alpha \left(\mathcal{D}^n z(\xi)\right).$$

Remark 2.5 In this present work, we take $M(\alpha) = 1$ and $a = 0$, so we can reformulate Definition 2.1 as follows:

$$\begin{aligned} ^{CF}\mathcal{D}_\xi^\alpha z(\xi) &= \frac{1}{1-\alpha} \int_0^\xi z'(\tau) e^{\frac{-\alpha(\xi-\tau)}{1-\alpha}}\,d\tau \\ &= \frac{1}{1-\alpha} z'(\xi) * e^{\frac{-\alpha\xi}{1-\alpha}}, \end{aligned}$$

where $(*)$ is the convolution operator.

In addition to that, the CFFD of any constant is zero, that is, for all $c \geq 0$, we have $^{CF}\mathcal{D}_\xi^\alpha c = 0$. Anyhow, we end this section with the equivalent representation of the CFFD which is

$$^{CF}\mathcal{D}_\xi^\alpha z(\xi) = \frac{1}{(1-\alpha)} \int_0^\xi \left(\frac{g_0'(\tau)\delta(\xi-\tau)}{z'(\xi)} + e^{\frac{-\alpha(\xi-\tau)}{1-\alpha}} \right) z'(\tau)\,d\tau,$$

where $g_0'(\xi) = \int_a^0 e^{\frac{-\alpha(\xi-\tau)}{1-\alpha}} z'(\tau)\,d\tau$.

3 Preliminaries of MRKHS method

In this section, some of the essential facts of reproducing kernel theory are presented to construct Hilbert spaces associated with the reproducing function of our method. For more details, please read the papers [39–47]. During this study, $AC[0, T]$ denotes the absolutely continuous real functions.

Definition 3.1 Let Λ be a nonempty abstract set, and let $\mathcal{W}$ be a Hilbert space of functions $v : \Lambda \longrightarrow \mathcal{W}$. Then any function $B : \Lambda \times \Lambda \longrightarrow C$ that attains both

(i) $B(\cdot,\xi) \in \mathcal{W}$ for each $\xi \in \Lambda$ and
(ii) $\langle v(\cdot), B(\cdot,\xi)\rangle = v(\xi)$ for each $v \in \mathcal{W}$

is called reproducing-kernel function, whereas the property in (ii) is called "the reproducing property".

This function possesses some important properties such as being of unique representation, conjugate symmetric, and positive-definite.

Definition 3.2 Any Hilbert space $\mathcal{W}$ defined on a nonempty abstract Λ which possesses a reproducing kernel function is called a reproducing kernel Hilbert space.

Definition 3.3 For $m = 1,2,3$, the Hilbert spaces $\mathcal{H}_m[0,X]$ are described by

$$\begin{aligned}\mathcal{H}_m[0,X] = \bigl\{z(\xi) | z : [0,X] \longrightarrow R, z^{(m-1)}(\xi) \in AC[0,X] \\ \text{and } z^{(m)}(\xi) \in L^2[0,X], m = 1,2,3\bigr\}.\end{aligned} \tag{3.1}$$

The inner product and the norm corresponding in $\mathcal{H}_m[0,X]$ for $m = 1,2,3$ are given as follows:

$$\begin{cases} \langle z,w\rangle_{\mathcal{H}_m} = \sum_{i=0}^{m-1} z^{(i)}(0)w^{(i)}(0) + \int_0^X z^{(m)}(\xi)w^{(m)}(\xi)\,d\xi, \quad z,w \in \mathcal{H}_m[0,X], \\ \|z\|_{\mathcal{H}_m} = \langle z,z\rangle^{1/2}, \quad z \in \mathcal{H}_m[0,X]. \end{cases} \tag{3.2}$$

Theorem 3.4 ([34]) *The unique representation of the reproducing-kernel function associated with the Hilbert space $\mathcal{H}_1[0,X]$ is given by*

$$V_\xi^{\{1\}}(\tau) = \frac{1}{2\sinh(X)}\bigl[\cosh(\xi + \tau - X) + \cosh\bigl(|\xi - \tau| - X\bigr)\bigr]. \tag{3.3}$$

Theorem 3.5 ([35]) *The unique representation of the reproducing-kernel function associated with $\mathcal{H}_2[0,X]$ can be written as*

$$V_\xi^{\{2\}}(\tau) = \begin{cases} \frac{1}{6}\tau(-\tau^2 + 3\xi(\tau + 2)), & 0 \le \tau < \xi, \\ \frac{1}{6}\xi(-\xi^2 + 3\tau(\xi + 2)), & \xi < \tau \le X. \end{cases} \tag{3.4}$$

Theorem 3.6 ([35]) *The unique representation of the reproducing-kernel function associated with $\mathcal{H}_3[0,X]$ can be written as*

$$V_\xi^{\{3\}}(\tau) = \begin{cases} \frac{1}{120}(120 + \tau^5 + 10\xi^2\tau^2(3 + \tau) - 5\xi\tau(-24 + \tau^3)), & 0 \le \tau < \xi, \\ \frac{1}{120}(120 + \xi^5 + 10\xi^2\tau^2(3 + \xi) - 5\xi\tau(-24 + \xi^3)), & \xi < \tau \le X. \end{cases} \tag{3.5}$$

Definition 3.7 We describe the inner product Hilbert space $\mathcal{W}_m[0,X]$, $m = 1,2$, of $\mathcal{H}_m[0,X]$ and $\mathcal{H}_m[0,X]$ by

$$\mathcal{W}_m[0,X] = \bigl\{z(\xi) = \bigl(z_1(\xi), z_2(\xi)\bigr)^T | z_1, z_2 \in \mathcal{H}_m[0,X], m = 1,2\bigr\},$$

as well as the inner product and the norm associated with $\mathcal{W}_m[0,X]$ are built as follows:

$$\begin{cases} \langle z,w\rangle_{\mathcal{W}_m} = \sum_{i=1}^{2}\langle z_i,w_i\rangle_{\mathcal{W}_m}, & z_i,w_i \in \mathcal{H}_m[0,X], m=1,2, \\ \|z\|_{\mathcal{W}_m} = \sqrt{\sum_{i=1}^{2}\langle z_i,z_i\rangle}, & z_i \in \mathcal{H}_m[0,X], m=1,2. \end{cases} \tag{3.6}$$

4 Structure of analytical solution

4.1 MRKHS solution of GBT equation along with boundary conditions

In this section, we present a brief description for the notations and preliminary definitions of the MRKHS theory. Additionally, we explain how to solve GBT with boundary conditions (1.2)–(1.4) using the MRKHS method. Accordingly, we construct orthonormal function systems of the space based on the process orthogonalization of Gram–Schmidt.

Consider the GBT equation within CFFD

$$\begin{cases} a_1(\xi)\omega''(\xi) + a_2(\xi)^{CF}\mathcal{D}_0^{\frac{3}{2}}\omega(\xi) + a_3(\xi)\omega'(\xi) \\ \quad + a_4(\xi)^{CF}\mathcal{D}_0^{\frac{1}{2}}\omega(\xi) + a_5(\xi)\omega(\xi) = h(\xi), \\ \omega(0) = \mu_1; \qquad \omega(X) = \mu_2. \end{cases} \tag{4.1}$$

Now, to apply our technique to GBT equation (4.1) on the Hilbert space $\mathcal{H}_2[0,X]$, we consider a linear differential operator defined as follows:

$$\begin{cases} \mathcal{L} : \mathcal{H}_3[0,X] \longrightarrow \mathcal{H}_1[0,X], \\ \mathcal{L}\omega(\xi) = a_1(\xi)\omega''(\xi) + a_2(\xi)^{CF}\mathcal{D}_0^{\frac{3}{2}}\omega(\xi) + a_3(\xi)\omega'(\xi) \\ \qquad + a_4(\xi)^{CF}\mathcal{D}_0^{\frac{1}{2}}\omega(\xi) + a_5(\xi)\omega(\xi). \end{cases} \tag{4.2}$$

Thus, using the simple transform $z(\xi) := (\omega(\xi) - (\mu_2 - \mu_1)\xi + \mu_1)$, the GBT with the boundary condition equation (4.1) can be equivalently converted to the form

$$\begin{cases} \mathcal{L}z(\xi) = h(\xi), & \xi \in [0,X], \\ z(0) = 0, & z(X) = 0. \end{cases} \tag{4.3}$$

Theorem 4.1 *The differential operator $\mathcal{L}$ from $\mathcal{H}_3[0,X]$ into $\mathcal{H}_1[0,X]$ is bounded and linear. Hence, $\mathcal{L}$ is continuous.*

Proof In order to prove that $\mathcal{L}$ is a bounded operator, it is enough to find $M > 0$ such that $\frac{\|\mathcal{L}z(\xi)\|_{\mathcal{H}_1}}{\|z(\xi)\|_{\mathcal{H}_3}} \leq M$. By the definition of the inner product for $m = 1$ in (3.2) on the Hilbert space $\mathcal{H}_1[0,X]$, we have

$$\left\|\mathcal{L}z(\xi)\right\|_{\mathcal{H}_1}^2 = \left\langle \mathcal{L}z(\xi),\mathcal{L}z(\xi)\right\rangle_{\mathcal{H}_1} = \left[\mathcal{L}z(0)\right]^2 + \int_0^X \left|(\mathcal{L}z)'(\xi)\right|^2 d\xi.$$

On the other hand, using the reproducing property of the MRKHS and by the Cauchy–Schwarz inequality, we can write

$$\begin{aligned} \left|(\mathcal{L}z)^{(i)}(\xi)\right| &= \left|\left\langle z(\xi), \left(\mathcal{L}V_\xi^{\{3\}}\right)^{(i)}(\xi)\right\rangle_{\mathcal{H}_3}\right| \\ &\leq \left\|\left(\mathcal{L}V_\xi^{\{3\}}\right)^{(i)}(\xi)\right\|_{\mathcal{H}_3}\left\|z(\xi)\right\|_{\mathcal{H}_3} \\ &\leq M_{\{i\}}\left\|z(\xi)\right\|_{\mathcal{H}_3}, \quad i = 0,1. \end{aligned} \tag{4.4}$$

Hence,

$$\begin{aligned}\left\|\mathcal{L}z(\xi)\right\|_{\mathcal{H}_1}^2 &\leq \left[M_{\{0\}}^2 + \int_0^X M_{\{1\}}^2\,d\xi\right]\left\|z(\xi)\right\|_{\mathcal{H}_3}^2 \\ &\leq \left(M_{\{0\}}^2 + XM_{\{1\}}^2\right)\left\|z(\xi)\right\|_{\mathcal{H}_3}^2 \\ &\leq M^2\left\|z(\xi)\right\|_{\mathcal{H}_3}^2,\end{aligned} \tag{4.5}$$

where $M = \sqrt{(M_{\{0\}}^2 + XM_{\{1\}}^2)}$.

Next, for all $z, \eta \in \mathcal{H}_3[0, X]$, we have $\|\mathcal{L}(z+\eta) - \mathcal{L}(z)\|_{\mathcal{H}_1} = \|\mathcal{L}(\eta)\|_{\mathcal{H}_1} \leq M\|\eta\|_{\mathcal{H}_3}$. Letting $\eta \to 0$ implies that $\mathcal{L}$ is continuous. □

We construct an orthonormal function system of $\mathcal{H}_3[0, X]$ as follows: Put $\Phi_i(\cdot) = V_{\xi_i}^{\{1\}}(\cdot)$ and $\Psi_i(\cdot) = \mathcal{L}^\star\Phi_i(\cdot)$, where $\mathcal{L}^\star$ is the adjoint operator of $\mathcal{L}$ and $\{\xi\}_{i=1}^\infty$ is a dense set on $[0, X]$.

The orthonormal system $\{\widehat{\Psi}_i(\xi)\}_{i=1}^\infty$ of the space $\mathcal{H}_3[0, X]$ can be generated from the well-known Gram–Schmidt orthogonalization process as follows:

$$\widehat{\Psi}_i(\xi) = \sum_{k=1}^{i}\sigma_{ik}\Psi_{i1}(\xi), \quad i = 1, 2, \ldots, \tag{4.6}$$

where $\sigma_{ik} > 0$ are the orthogonalization coefficients.

Theorem 4.2 *For* (4.3), *if* $\{\xi_i\}_{i=1}^\infty$ *is a dense set on* $[0, X]$, *then the orthogonal function system* $\{\Psi_i(\xi)\}_{i=1}^\infty$ *is complete in* $\mathcal{H}_3[0, X]$.

Proof Note that

$$\Psi_i(\xi) = \left\langle \mathcal{L}^\star\Phi_i(\tau), V_{\xi_i}^{\{3\}}(\tau)\right\rangle = \left\langle \Phi_i(\tau), \mathcal{L}_\tau V_{\xi_i}^{\{3\}}(\tau)\right\rangle = \mathcal{L}_\tau V_{\xi_i}^{\{3\}}(\tau) \in \mathcal{H}_1[0, X]. \tag{4.7}$$

Now, for each $z(\xi) \in \mathcal{H}_3[0, X]$, suppose $\langle z(\xi), \Psi_i(\xi)\rangle = 0$, thus we have

$$\begin{aligned}\left\langle z(\xi), \Psi_i(\xi)\right\rangle &= \left\langle z(\xi), \mathcal{L}^\star\Phi_i(\xi)\right\rangle \\ &= \left\langle \mathcal{L}z(\xi), \Phi_i(\xi)\right\rangle \\ &= \mathcal{L}z(\xi_i) = 0.\end{aligned} \tag{4.8}$$

Because $\{\xi_i\}_{i=1}^\infty$ is dense in [0, X] and $\mathcal{L}$ is continuous, we get $\mathcal{L}z(\xi_i) = 0$. Using the existence of the inverse operator $\mathcal{L}^{-1}$, we conclude that $z(\xi) = 0$. So the proof is complete. □

Theorem 4.3 *If* $\{\xi_i\}_{i=1}^\infty$ *is a dense subset on* $[0, X]$ *and the analytic solution* $z(\xi)$ *of* (4.1) *is unique, then the solution* $z(\xi)$ *can be represented in the following form*:

$$z(\xi) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\sigma_{ik}h(\xi_k))\widehat{\Psi}_i(\xi). \tag{4.9}$$

Proof For each $z(\xi) \in \mathcal{H}_3[0, X]$, $z(\xi)$ can be extended in the Fourier series $\sum_{i=1}^\infty\langle z(\xi), \widehat{\Psi}_i(\xi)\rangle\widehat{\Psi}_i(\xi)$ about the orthonormal function system $\{\Psi_i(\xi)\}_{i=1}^\infty$ of $\mathcal{H}_3[0, X]$. Moreover, the series $\sum_{i=1}^{\infty}\langle z(\xi),\widehat{\Psi}_i(\xi)\rangle$ is uniformly convergent in the Hilbert space $\mathcal{W}_2[0,X]$. So, we have

$$\begin{aligned} z(\xi) &= \sum_{i=1}^{\infty}\bigl\langle z(\xi),\widehat{\Psi}_i(\xi)\bigr\rangle\widehat{\Psi}_i(\xi) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\sigma_{ik}\bigl\langle z(\xi),\Psi_k(\xi)\bigr\rangle\widehat{\Psi}_i(\xi) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\sigma_{ik}\bigl\langle \mathcal{L}z(\xi),\Phi_{k1}(\xi)\bigr\rangle\widehat{\Psi}_i(\xi) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\sigma_{ik}\mathcal{L}z(\xi_k)\widehat{\Psi}_i(\xi) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\sigma_{ik}h(\xi_k)\widehat{\Psi}_i(\xi). \end{aligned} \tag{4.10}$$

□

Now, the approximate solution can be obtained by truncating the series in (4.9) as follows:

$$z(\xi) = \sum_{i=1}^{n}\sum_{k=1}^{i}\sigma_{ik}h(\xi_k))\widehat{\Psi}_i(\xi). \tag{4.11}$$

In the following theorem, we prove that the error that results when approximating the solution in (4.9) by the form (4.11) is decreasing to zero.

Theorem 4.4 *Let $E_n = \|z - z_n\|_{\mathcal{H}_3}$, where z, z_n are respectively the exact and the approximate solution of* (4.1) *represented in* (4.9) *and* (4.11), *then the error E_n decreases monotonously in the sense of $\|\cdot\|_{\mathcal{H}_3}$, and $E_n \to 0$ as $n \to \infty$.*

Proof We have

$$\begin{aligned} E_n^2 &= \|z - z_n\|_{\mathcal{H}_3}^2 \\ &= \left\|\sum_{i=n+1}^{\infty}\sum_{k=1}^{i}\sigma_{ik}h(\xi_k)\widehat{\Psi}_i(\xi)\right\|_{\mathcal{H}_3}^2 \\ &= \left\|\sum_{i=n+1}^{\infty}A_i\widehat{\Psi}_i(\xi)\right\|_{\mathcal{H}_3}^2 \\ &= \sum_{i=n+1}^{\infty}(A_i)^2 \end{aligned}$$

and

$$E_{n-1}^2 = \|z - z_{n-1}\|_{\mathcal{H}_3}^2 = \sum_{i=n-1}^{\infty}(A_i)^2 = (A_{n-1})^2 + \sum_{i=n}^{\infty}(A_i)^2.$$

Note that $E_{n-1}^2 > E_n^2$, so we conclude that the error E_n is monotone decreasing in the sense of $\|\cdot\|_{\mathcal{H}_3}^2$, and because $\sum_{i=1}^{\infty}\langle z(\xi), \widehat{\Psi}_i(\xi)\rangle\widehat{\Psi}_i(\xi)$ is convergent, $E_n \to 0$ as $n \to \infty$. Hence, the proof is complete. □

4.2 MRKHS solution of GBT equation along with initial conditions

Now, we give some notations and preliminary definitions of the MRKHS theory. We then explain how to reformulate the GBT under the initial conditions (1.2)–(1.3) into an equivalent system of first-order fractional differential equations and how to implement our method to solve this system, with highlighting the relationship between the solution of the system and the solution of GBT. Accordingly, we construct an orthonormal function system of the space based on the Gram–Schmidt orthogonalization process.

Consider the following GBT with initial conditions:

$$\begin{cases} a_1(\xi)\omega''(\xi) + a_2(\xi)^{CF}\mathcal{D}_{xi}^{\frac{3}{2}}\omega(\xi) + a_3(\xi)\omega'(\xi) \\ \quad + a_4(\xi)^{CF}\mathcal{D}_{\xi}^{\frac{1}{2}}\omega(\xi) + a_5(\xi)\omega(\xi) = h(\xi), \\ \omega(0) = \mu_1, \qquad \omega'(0) = \mu_2. \end{cases} \tag{4.12}$$

We can obtain an equivalent form of (4.12) by homogenizing the initial conditions using a transformation given by the following formula $z(\xi) := \omega(\xi) - (\mu_2\xi + \mu_1)$

$$\begin{cases} a_1(\xi)z''(\xi) + a_2(\xi)^{CF}\mathcal{D}_{\xi}^{\frac{3}{2}}z(\xi) + a_3(\xi)z'(\xi) \\ \quad + a_4(\xi)^{CF}\mathcal{D}_{\xi}^{\frac{1}{2}}z(\xi) + a_5(\xi)z(\xi) = H(\xi), \\ z(0) = 0, \qquad z'(0) = 0, \end{cases} \tag{4.13}$$

where $H(\xi) = h(\xi) - (a_3(\xi)\mu_2 + 2\mu_2 a_4(\xi)(1 - e^{-\xi}) + a_5(\xi)(\mu_2\xi + \mu_1))$.

Before starting to describe the approximate RKHS scheme, we find it appropriate to rewrite the equivalent GBT equation (4.13) in the form of a system of fractional differential equations (SFDE) of first order by setting $z(\xi) = z_1(\xi)$ and $z_1'(\xi) = z_2(\xi)$. In this sense, the equivalent SFDE that we design has the form

$$\begin{aligned} &z_1'(\xi) = z_2(\xi), \\ &a_1(\xi)z_2'(\xi) + a_2(\xi)^{CF}\mathcal{D}_0^{\frac{1}{2}}z_2(\xi) = H(\xi) - a_3(\xi)z_1(\xi), \end{aligned} \tag{4.14}$$

equipped with the initial conditions

$$z_1(0) = 0; \qquad z_2(0) = 0. \tag{4.15}$$

It is obvious that GBT equation (4.13) with its original initial conditions is equivalent to SFDE (4.14) with the new initial conditions (4.15) in the following sense: whenever $Z = (z_1, z_2)^T$ with $z_1 \in \mathcal{H}_2[0, X]$ is a solution of SFDE (4.14) associated with initial conditions (4.15), the solution $z := z_1$ solves GBT equation (4.13). Also, whenever $z \in \mathcal{H}_2[0, X]$ is a solution to GBT equation (4.13) with original initial conditions, the vector of solutions $Z := (z_1, z_2)^T := (z, z')^T$ solves SFDE (4.14) equipped with initial conditions (4.15).

Now, we begin by applying the MRKHS approach to solve the SFDE by defining differential operators as

$$\mathcal{L}_1, \mathcal{L}_2 : \mathcal{H}_2[0,X] \longrightarrow \mathcal{H}_1[0,X], \tag{4.16}$$

whereas $\mathcal{L}_1 z_1(\xi) = z_1'(\xi)$ and $\mathcal{L}_2 z_2(\xi) = a_1(\xi)z_2'(\xi) + a_2(\xi)^{CF}\mathcal{D}_\xi^{\frac{1}{2}} z_2(\xi)$. Put $\mathcal{L} = \left(\begin{smallmatrix} \mathcal{L}_1 & 0 \\ 0 & \mathcal{L}_2 \end{smallmatrix}\right) \in \mathcal{W}_2[0,X]$, $H_1(\xi, Z(\xi)) = z_2(\xi)$, $H_2(\xi, Z(\xi)) = a_3(\xi)z_1(\xi) + H(\xi)$, and $H(\xi, Z(\xi)) = (H_1(\xi, Z(\xi)), H_2(\xi, Z(\xi)))^T$.

Consequently, the initial value GBT equation can be converted into the form

$$\mathcal{L}Z(\xi) = H\bigl(\xi, Z(\xi)\bigr),$$

equipped with the initial conditions

$$Z(\mathbf{0}) = \mathbf{0}.$$

Lemma 4.5 *The differential operators $\mathcal{L}_1, \mathcal{L}_2 : \mathcal{H}_2[0,X] \longrightarrow \mathcal{H}_1[0,X]$ are linear and bounded operators. Consequently, the operator $\mathcal{L} : \mathcal{W}_2[0,X] \longrightarrow \mathcal{W}_1[0,X]$ is also linear and bounded.*

Proof The proof is divided into two parts; first, we prove that $\mathcal{L}_j : \mathcal{H}_2[0,X] \longrightarrow \mathcal{H}_1[0,X], j = 1,2$, are bounded and linear. The linearity is obvious since both integer order and Caputo–Fabrizio derivatives are linear.

For boundedness, let $z \in \mathcal{H}_2[0,X]$, then

$$\|\mathcal{L}_j z_j\|_{\mathcal{H}_1}^2 = \langle \mathcal{L}_j z_j, \mathcal{L}_j z_j \rangle_{\mathcal{H}_1} = \int_0^X \bigl((\mathcal{L}_j z_j)(\tau)\bigr)^2 + \bigl((\mathcal{L}_j z_j)'(\tau)\bigr)^2 \, d\tau.$$

Using the reproducing property of $V_\xi^{\{2\}}(\tau)$, we can write $z(\xi) = \langle z(\cdot), V_\xi^{\{2\}}(\cdot)\rangle_{\mathcal{H}_2}$ and

$$z'(\xi) = \left\langle z(\cdot), \frac{d}{d\xi} V_\xi^{\{2\}}(\cdot) \right\rangle_{\mathcal{H}_2},$$

and

$$a_1(\xi)z'(\xi) + a_2(\xi)^{CF}\mathcal{D}_\xi^{\frac{1}{2}} z(\xi) = \left\langle z(\cdot), a_1(\xi)\frac{d}{d\xi} V_\xi^{\{2\}}(\cdot) + a_2(\xi)^{CF}\mathcal{D}_\xi^{\frac{1}{2}} V_\xi^{\{2\}}(\cdot) \right\rangle_{\mathcal{H}_2}.$$

Applying the Schwarz inequality and using the fact that z, a_1, a_2 are continuous over $[0,X]$ and ${}^{CF}\mathcal{D}_\xi^{\frac{1}{2}} V_\xi^{\{2\}}$ is continuous and uniformly bounded, we get

$$\begin{aligned}
\bigl|(\mathcal{L}_1 z)(\xi)\bigr| &= \bigl|z'(\xi)\bigr| = \left|\left\langle z(\cdot), \frac{d}{d\xi} V_\xi^{\{2\}}(\cdot)\right\rangle\right|_{\mathcal{H}_2} \le \|z\|_{\mathcal{H}_2} \left\| \frac{d}{d\xi} V_\xi^{\{2\}} \right\|_{\mathcal{H}_2} \le \Upsilon_{11}\|z\|_{\mathcal{H}_2}, \\
\bigl|(\mathcal{L}_1 z)'(\xi)\bigr| &= \left|\left\langle z(\cdot), \frac{d^2}{d\xi^2} V_\xi^{\{2\}}(\cdot)\right\rangle\right|_{\mathcal{H}_2} \le \|z\|_{\mathcal{H}_2} \left\| \frac{d^2}{d\xi^2} V_\xi^{\{2\}} \right\|_{\mathcal{H}_2} \le \Upsilon_{12}\|z\|_{\mathcal{H}_2}, \\
\bigl|\bigl(\mathcal{L}_2 z(\cdot)\bigr)(\xi)\bigr| &= \left\langle z, a_1(\cdot)\frac{d}{d\xi} V_\xi^{\{2\}}(\cdot) + a_2(\cdot)^{CF}\mathcal{D}_\xi^{\frac{1}{2}} V_\xi^{\{2\}}(\cdot) \right\rangle_{\mathcal{H}_2} \\
&\le \|z\|_{\mathcal{H}_2} \left\| a_1 \frac{d}{d\xi} V_\xi^{\{2\}} + a_2{}^{CF}\mathcal{D}_\xi^{\frac{1}{2}} V_\xi^{\{2\}} \right\|_{\mathcal{H}_2} \le \Upsilon_{21}\|z\|_{\mathcal{H}_2},
\end{aligned}$$

and similarly,

$$\left|(\mathcal{L}_2 z)'(\xi)\right| \le \Upsilon_{22}\|z\|_{\mathcal{H}_2},$$

where $\Upsilon_{ij} \in \Re^+$, $\forall i,j = 1,2$.

Hence,

$$\|\mathcal{L}_j z_j\|^2_{\mathcal{H}_1} \le \left(\Upsilon^2_{1j} + \Upsilon^2_{2j}\right) X \|z\|^2_{\mathcal{H}_2} \le M_j^2 \|z\|_{\mathcal{H}_2},$$

where $M_j^2 = (\Upsilon^2_{1j} + \Upsilon^2_{2j})X$.

So, $\|\mathcal{L}_j z_j\|^2_{\mathcal{H}_1} \le M_j \|z\|_{\mathcal{H}_2}$.

Now, for any $Z = (z_1, z_2)^T \in \mathcal{W}_2[0,X]$, we have

$$\begin{aligned} \|\mathcal{L}Z\|_{\mathcal{W}_1} &= \sqrt{\sum_{i=1}^{2} \|\mathcal{L}_i z_i\|^2_{\mathcal{H}_1}} \\ &\le \sqrt{M_1^2 \|z_1\|^2_{\mathcal{H}_2} + M_2^2 \|z_2\|^2_{\mathcal{H}_2}} \\ &\le M \|Z\|_{\mathcal{W}_2}, \end{aligned} \tag{4.17}$$

where $M = \max\{M_1, M_2\}$. So, the proof is complete. □

Next, we create an orthonormal function system of $\mathcal{W}_2[0,X]$ as follows: Let $\Phi_{ij}(\cdot) = V^{\{1\}}_{\xi_i}(\cdot)$ and $\Psi_{ij}(\cdot) = \mathcal{L}^\star \Phi_{ij}(\cdot)$ for each $i = 1,2,\ldots$, and $j = 1,2$, where $\mathcal{L}^\star$ is the adjoint operator of $\mathcal{L}$ and $\{\xi_i\}_{i=1}^\infty$ is a countable dense subset of $[0,X]$. Moreover, using the properties of the reproducing kernel, we find

$$\begin{aligned} \Psi_{ij}(\xi) &= \mathcal{L}^\star \Phi_{ij}(\xi) = \left\langle \mathcal{L}^\star \Phi_{ij}(\tau), V^{\{2\}}_{\xi_i}(\tau)\right\rangle_{\mathcal{W}_2} = \left\langle \Phi_{ij}(\tau), \mathcal{L} V^{\{2\}}_{\xi_i}(\tau)\right\rangle_{\mathcal{W}_1} \\ &= \mathcal{L} V^{\{2\}}_{\xi_i}(\tau) \in \mathcal{W}_2[0,X]. \end{aligned}$$

To build the representative form of the MRKHS solutions of SFDE (4.14) equipped with the initial conditions (4.15) in the space $\mathcal{W}_2[0,X]$, we use the well-known Gram–Schmidt process that outputs an orthonormal function $\{(\widehat{\Psi}_{i1}(\xi), \widehat{\Psi}_{i2}(\xi))^T\}_{i=1}^\infty$ of the space $\mathcal{W}_2[0,X]$ constructed from $\{(\Psi_{i1}(\xi), \Psi_{i2}(\xi))^T\}_{i=1}^\infty$ such that

$$\widehat{\Psi}_i(\xi) = \begin{pmatrix} \widehat{\Psi}_{i1}(\xi) \\ \widehat{\Psi}_{i2}(\xi)) \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{i} \sigma^1_{ik} \Psi_{i1}(\xi) \\ \sum_{k=1}^{i} \sigma^2_{ik} \Psi_{i2}(\xi) \end{pmatrix}, \tag{4.18}$$

where σ^j_{ik}, $j = 1,2$, are the orthogonalization coefficients.

Theorem 4.6 *If* $\{\xi_i\}_{i=1}^\infty$ *is a dense set on* $[0, X]$, *then the orthogonal function system* $\{(\Psi_{i1}(\xi), \Psi_{i2}(\xi))^T\}_{i=1}^\infty$ *is complete in* $\mathcal{W}_2[0,X]$.

Proof For each $Z(\xi) = \binom{z_1(\xi)}{z_2(\xi)} \in \mathcal{W}_2[0,X]$, suppose $\langle Z(\xi), \Psi_{ij}(\xi)\rangle = 0$. On the other hand, we have

$$\begin{aligned}
\langle Z(\xi), \Psi_{ij}(\xi)\rangle &= \langle Z(\xi), \mathcal{L}^\star \Phi_{ij}(\xi)\rangle \\
&= \left\langle \begin{pmatrix} z_1(\xi) \\ z_2(\xi) \end{pmatrix}, \begin{pmatrix} \mathcal{L}_1^\star \Phi_{i1}(\xi) \\ \mathcal{L}_2^\star \Phi_{i2}(\xi) \end{pmatrix} \right\rangle \\
&= \langle z_1(\xi), \mathcal{L}_1^\star \Phi_{i1}(\xi)\rangle + \langle z_2(\xi), \mathcal{L}_2^\star \Phi_{i1}(\xi)\rangle \\
&= \langle \mathcal{L}_1 z_1(\xi), \Phi_{i1}(\xi)\rangle + \langle \mathcal{L}_2 z_2(\xi), \Phi_{i1}(\xi)\rangle \\
&= \mathcal{L}_1 z_1(\xi_i) + \mathcal{L}_2 z_2(\xi_i) \\
&= \mathcal{L} Z(\xi_i) = 0.
\end{aligned} \tag{4.19}$$

Because $\{\xi_i\}_{i=1}^{\infty}$ is dense in [0, X] and $\mathcal{L}$ is continuous, we get $\mathcal{L}Z(\xi_i) = 0$. Using the existence of the inverse operator $\mathcal{L}^{-1}$, we conclude that $Z(\xi) = 0$. So the proof is complete. □

Theorem 4.7 *If* $\{\xi_i\}_{i=1}^{\infty}$ *is a dense subset on* $[0, X]$ *and the analytic solution* $Z(\xi)$ *of SFDE* (4.14) *is unique, then the analytic solution* $Z(\xi)$ *can be represented in the following form*:

$$Z(\xi) = \begin{bmatrix} \sum_{i=1}^{\infty} \sum_{k=1}^{i} [\sigma_{ik}^1 H_1(\xi_k, Z(\xi_k)) + \sigma_{ik}^2 H_2(\xi_k, Z(\xi_k))] \widehat{\Psi}_{i1}(\xi) \\ \sum_{i=1}^{\infty} \sum_{k=1}^{i} [\sigma_{ik}^1 H_1(\xi_k, Z(\xi_k)) + \sigma_{ik}^2 H_2(\xi_k, Z(\xi_k))] \widehat{\Psi}_{i2}(\xi) \end{bmatrix}. \tag{4.20}$$

Proof For each $Z(\xi) \in \mathcal{W}_2[0,X]$, $Z(\xi)$ can be extended in the Fourier series $\sum_{i=1}^{\infty} \langle Z(\xi), \widehat{\Psi}_i(\xi)\rangle \widehat{\Psi}_i(\xi)$ about the orthonormal function system $\{(\Psi_{i1}(\xi), \Psi_{i2}(\xi))^T\}_{i=1}^{\infty}$, as $\mathcal{W}_2[0,X]$. Moreover, the series $\sum_{i=1}^{\infty} \langle Z(\xi), \widehat{\Psi}_i(\xi)\rangle$ is convergent in norm in the Hilbert space $\mathcal{W}_2[0,X]$. So, we have

$$\begin{aligned}
Z(\xi) &= \sum_{i=1}^{\infty} \langle Z(\xi), \widehat{\Psi}_i(\xi)\rangle \widehat{\Psi}_i(\xi) \\
&= \sum_{i=1}^{\infty} \left\langle \begin{pmatrix} z_1(\xi) \\ z_2(\xi) \end{pmatrix}, \begin{pmatrix} \widehat{\Psi}_{i1}(\xi) \\ \widehat{\Psi}_{i2}(\xi) \end{pmatrix} \right\rangle \widehat{\Psi}_{ij}(\xi) \\
&= \sum_{i=1}^{\infty} \left\langle \begin{pmatrix} z_1(\xi) \\ z_2(\xi) \end{pmatrix}, \begin{pmatrix} \sum_{k=1}^{i} \sigma_{ik}^1 \Psi_{k1}(\xi) \\ \sum_{k=1}^{i} \sigma_{ik}^2 \Psi_{k2}(\xi) \end{pmatrix} \right\rangle \widehat{\Psi}_i(\xi) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \sigma_{ik}^1 \langle z_1(\xi), \Psi_{k1}(\xi)\rangle \widehat{\Psi}_i(\xi) + \sum_{i=1}^{\infty} \sum_{k=1}^{i} \sigma_{ik}^2 \langle z_2(\xi), \Psi_{k2}(\xi)\rangle] \widehat{\Psi}_i(\xi) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \sigma_{ik}^1 \langle \mathcal{L}_1 z_1(\xi), \Phi_{k1}(\xi)\rangle \widehat{\Psi}_i(\xi) + \sum_{i=1}^{\infty} \sum_{k=1}^{i} \sigma_{ik}^2 \langle \mathcal{L}_2 z_2(\xi), \Phi_{i2}(\xi)\rangle \widehat{\Psi}_i(\xi) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \bigl[\sigma_{ik}^1 \mathcal{L}_1 z_1(\xi_k) + \sigma_{ik}^2 \mathcal{L}_2 z_2(\xi_k)\bigr] \widehat{\Psi}_i(\xi) \\
&= \begin{bmatrix} \sum_{i=1}^{\infty} \sum_{k=1}^{i} [\sigma_{ik}^1 H_1(\xi_k, Z(\xi_k)) + \sigma_{ik}^2 H_2(\xi_k, Z(\xi_k))] \widehat{\Psi}_{i1}(\xi) \\ \sum_{i=1}^{\infty} \sum_{k=1}^{i} [\sigma_{ik}^1 H_1(\xi_k, Z(\xi_k)) + \sigma_{ik}^2 H_2(\xi_k, Z(\xi_k))] \widehat{\Psi}_{i2}(\xi) \end{bmatrix}.
\end{aligned} \tag{4.21}$$

□

Moreover, if we take finitely many terms in the series representation for the analytic solution $Z(\xi)$, we get directly the approximate solution SFDE (4.14), and it is given as the form

$$Z_n(\xi) = \begin{bmatrix} \sum_{i=1}^{n} \sum_{k=1}^{i} [\sigma_{ik}^1 H_1(\xi_k, Z(\xi_k)) + \sigma_{ik}^2 H_2(\xi_k, Z(\xi_k))] \widehat{\Psi}_{i1}(\xi) \\ \sum_{i=1}^{n} \sum_{k=1}^{i} [\sigma_{ik}^1 H_1(\xi_k, Z(\xi_k)) + \sigma_{ik}^2 H_2(\xi_k, Z(\xi_k))] \widehat{\Psi}_{i2}(\xi) \end{bmatrix}. \tag{4.22}$$

Lemma 4.8 *The analytical solution of FBT equation* (4.13) *is given as*

$$z(\xi) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} [\sigma_{ik}^1 H_1(\xi_k, Z(\xi_k)) + \sigma_{ik}^2 H_2(\xi_k, Z(\xi_k))] \widehat{\Psi}_{i1}(\xi).$$

Proof From Theorem 4.6, the proof is direct. □

Theorem 4.9 *If $E_n = \|Z - Z_n\|_{\mathcal{W}_2}$, where Z, Z_n are the exact and the approximate solutions of SFDE* (4.14), *respectively, represented in* (4.20) *and* (4.22), *then the error E_n is monotonic decreasing in the sense of $\|\cdot\|_{\mathcal{W}_2}$, and $E_n \to 0$ as $n \to \infty$. Consequently, the behavior error of the solution of the GBT equation decreases monotonously in the sense of $\|\cdot\|_{\mathcal{W}_2}$.*

Proof We have

$$\begin{aligned} E_n^2 &= \|Z - Z_n\|_{\mathcal{W}_2}^2 \\ &= \left\| \begin{bmatrix} \sum_{i=n+1}^{\infty} \sum_{k=1}^{i} [\sigma_{ik}^1 H_1(\xi_k, Z(\xi_k)) + \sigma_{ik}^2 H_2(\xi_k, Z(\xi_k))] \widehat{\Psi}_{i1}(\xi) \\ \sum_{i=n+1}^{\infty} \sum_{k=1}^{i} [\sigma_{ik}^1 H_1(\xi_k, Z(\xi_k)) + \sigma_{ik}^2 H_2(\xi_k, Z(\xi_k))] \widehat{\Psi}_{i2}(\xi) \end{bmatrix} \right\|_{\mathcal{W}_2}^2 \\ &= \left\| \begin{bmatrix} \sum_{i=n+1}^{\infty} A_i \widehat{\Psi}_{i1}(\xi) \\ \sum_{i=n+1}^{\infty} A_i \widehat{\Psi}_{i2}(\xi) \end{bmatrix} \right\|_{\mathcal{W}_2}^2 \\ &= \left\| \sum_{i=n+1}^{\infty} A_i \widehat{\Psi}_i(\xi) \right\|_{\mathcal{W}_2}^2 \\ &= \sum_{i=n+1}^{\infty} (A_i)^2 \end{aligned}$$

and

$$E_{n-1}^2 = \|Z - Z_{n-1}\|_{\mathcal{W}_2}^2 = \sum_{i=n-1}^{\infty} (A_i)^2 = (A_{n-1})^2 + \sum_{i=n}^{\infty} (A_i)^2.$$

Note that $E_{n-1}^2 > E_n^2$. So, we conclude that the error E_n is monotone decreasing in the sense of $\|\cdot\|_{\mathcal{W}_2}^2$, and because $\sum_{i=1}^{\infty} \langle Z(\xi), \widehat{\Psi}_i(\xi) \rangle \widehat{\Psi}_i(\xi)$ is convergent, then $E_n \to 0$ as $n \to \infty$. So, the proof is complete. □

5 Numerical experiments

In order to demonstrate the effectiveness and eligibility of the proposed method for solving generalized Bagley–Torvik equations under the effect of CFFD derivative, we provide some illustrative examples and then solve them using MRKHS. In each example, we compare their results with the exact solution. All numerical computations are carried out using Mathematica 12.0 software package.

Table 1 Numerical results for Example 5.1

ξ	Exact solution $z(\xi)$	MRKHS $z(\xi)$	Absolute error	Relative error
0.0	0.0	0.000000	0.000000000	0.00000000
0.1	0.09	0.090424	0.000423726	0.00470807
0.2	0.16	0.160444	0.000443675	0.00277297
0.3	0.21	0.210464	0.000463918	0.00220913
0.4	0.24	0.240485	0.000484604	0.00201918
0.5	0.25	0.250506	0.000505862	0.00202345
0.6	0.24	0.240528	0.000527807	0.00219920
0.7	0.21	0.210551	0.000550541	0.00262162
0.8	0.16	0.160574	0.000574156	0.00358848
0.9	0.09	0.090599	0.000598737	0.00665263
1.0	0.00	0.000624	0.000624362	0.00000000

Example 5.1 Consider the GBT equation equipped with the following initial conditions (ICs):

$$\begin{cases} z''(\xi) + \frac{2}{5}{}^{CF}\mathcal{D}_{\xi}^{\frac{3}{2}} z(\xi) - \frac{1}{4}z(\xi) = \frac{\xi^2}{4} - \frac{\xi}{4} + \frac{8e^{-\xi}}{5} - \frac{18}{5}, \\ z(0) = 0, \qquad z'(0) = 1, \end{cases} \tag{5.1}$$

which can be rewritten along with homogeneous ICs as follows:

$$\begin{cases} \nu''(\xi) + \frac{2}{5}{}^{CF}\mathcal{D}_{\xi}^{\frac{3}{2}} \nu(\xi) - \frac{1}{4}\nu(\xi) = \frac{\xi^2}{4} - \frac{\xi}{2} + \frac{8e^{-\xi}}{5} - \frac{18}{5}, \\ \nu(0) = 0, \qquad \nu'(0) = 0, \end{cases} \tag{5.2}$$

where the exact solution is $z(\xi) = \xi(1 - \xi)$.

In order to apply the MRKHS method with high efficiency, we make the substitution $z_1(\xi) = \nu(\xi)$ and $z_2(\xi) = \nu'(\xi)$ to convert it into the equivalent system:

$$\begin{aligned} &z_1'(\xi) - z_2(\xi) = 0, \\ &z_2'(\xi) + \frac{2}{5}{}^{CF}\mathcal{D}_{\xi}^{\frac{1}{2}} z_2(\xi) - \frac{1}{4}z_1(\xi) = \frac{\xi^2}{4} - \frac{\xi}{2} + \frac{8e^{-\xi}}{5} - \frac{18}{5}, \\ &z_1(0) = 0, \qquad z_2(0) = 0. \end{aligned}$$

Now, we apply the proposed method with $n = 70$ to get some numerical results as shown in Table 1. A comparison between the exact and the approximate curves is displayed in Fig. 1, which shows the accuracy of our method.

Example 5.2 Consider the GBT equation equipped with the following initial conditions:

$$\begin{cases} z''(\xi) + {}^{CF}\mathcal{D}_{\xi}^{\frac{3}{2}} z(\xi) + z(\xi) = \xi + 1, \\ z(0) = 1, \qquad z'(0) = 1, \end{cases} \tag{5.3}$$

which can be rewritten along with homogeneous ICs as follows:

$$\begin{cases} \nu''(\xi) + {}^{CF}\mathcal{D}_{\xi}^{\frac{3}{2}} \nu(\xi) + \nu(\xi) = 0, \\ \nu(0) = 0, \qquad \nu'(0) = 0, \end{cases} \tag{5.4}$$

where the exact solution is $z(\xi) = \xi + 1$.

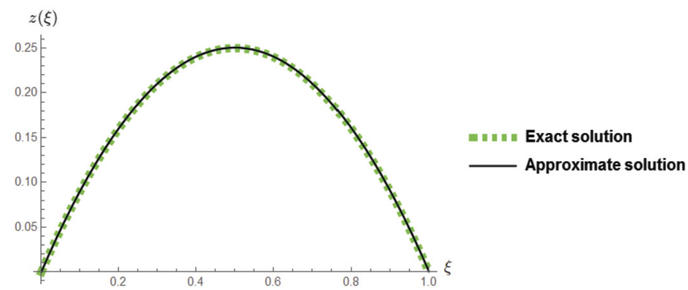

Figure 1 Exact and MRKHS solution curves of Example 5.1

Table 2 Numerical results for Example 5.2

ξ	Exact solution $z(\xi)$	MRKHS $z(\xi)$	Absolute error	Relative error
0.0	1.0	1.0	0.0	0.0
0.1	1.1	1.1	0.0	0.0
0.2	1.2	1.2	0.0	0.0
0.3	1.3	1.3	0.0	0.0
0.4	1.4	1.4	0.0	0.0
0.5	1.5	1.5	0.0	0.0
0.6	1.6	1.6	0.0	0.0
0.7	1.7	1.7	0.0	0.0
0.8	1.8	1.8	0.0	0.0
0.9	1.9	1.9	0.0	0.0
1.0	2.0	2.0	0.0	0.0

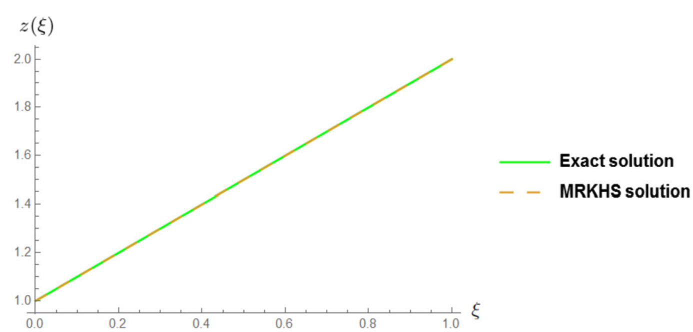

Figure 2 Exact and MRKHS solution curves of Example 5.2

The SFDE related to this GBT equation assuming $z_1(\xi) = v(\xi)$ and $z_2(\xi) = v'(\xi)$ is

$$\begin{cases} z_1'(\xi) - z_2(\xi) = 0, \\ z_2'(\xi) + {}^{CF}\mathcal{D}_\xi^{\frac{1}{2}} z_2(\xi) + z_1(\xi) = 0, \\ z_1(0) = 0, \qquad z_2'(0) = 0. \end{cases} \tag{5.5}$$

To achieve high accuracy, we found it enough to take $n = 5$ as shown in Table 2 and Fig. 2.

Table 3 Numerical results for Example 5.3

ξ	Exact solution $z(\xi)$	MRKHS $z(\xi)$	Absolute error	Relative error
0.0	0.00	0.000000	0.0000000000	0.0000000
0.1	0.01	0.009990	0.0000104827	0.0010483
0.2	0.04	0.039981	0.0000195138	0.0004879
0.3	0.09	0.089973	0.0000270488	0.0003005
0.4	0.16	0.159967	0.0000333901	0.0002087
0.5	0.25	0.249961	0.0000387662	0.0001551
0.6	0.36	0.359957	0.0000433495	0.0001204
0.7	0.49	0.489953	0.0000472704	0.0000965
0.8	0.64	0.639949	0.0000506276	0.0000791
0.9	0.81	0.809947	0.0000534959	0.0000660
1.0	1.00	0.999944	0.0000559322	0.0000559

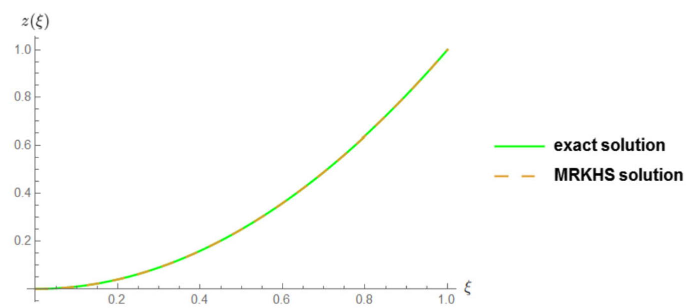

Figure 3 Exact and MRKHS solution curves of Example 5.3

Example 5.3 Consider the following GBT equation:

$$z''(\xi) + {}^{CF}\mathcal{D}_{\xi}^{\frac{3}{2}} z(\xi) + z(\xi) = 6 + \xi^2 - 4e^{-t},$$

subject to

$$z(0) = z'(0) = 0,$$

where the exact solution $z(\xi) = \xi^2$.

Applying the methodology described in this paper, with $n = 90$, some tabulated and graphical results are shown in Table 3 and Fig. 3.

Example 5.4 Consider the GBT equation equipped with the following ICs:

$$z''(t) + D_{\xi}^{3/2} z - D_{\xi}^{1/2} z + 4z'(t) + z(\xi) = -2(\xi + 1) - 4e^{-\xi}; \quad z(0) = 2, z'(0) = -2,$$

whose the exact solution is $z(\xi) = 2(1 - \xi)$.

The comparison between the exact and the approximate curves for $n = 5$ is displayed in Fig. 4, which shows high accuracy of this method.

Example 5.5 Consider the following GBT equation:

$$z''(\xi) + D_{\xi}^{3/2} z(\xi) - D_{\xi}^{\frac{1}{2}} z(\xi) + 4z'(\xi) + z(\xi) = -4e^{-\xi}\left(\xi^2 - 6\xi + 2\right),$$

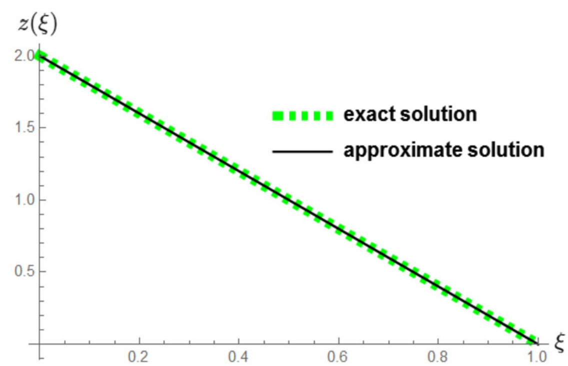

Figure 4 Exact and MRKHS solution curves of Example 5.4

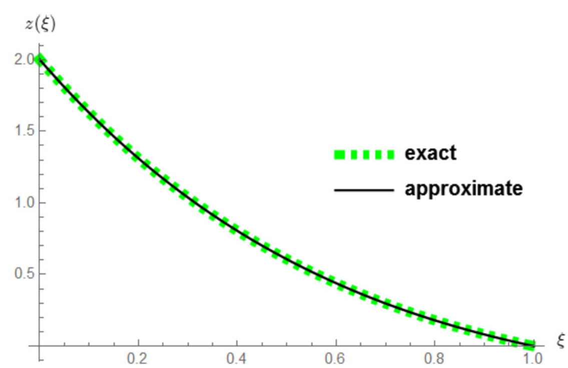

Figure 5 Exact and MRKHS solution curves of Example 5.5

subject to ICs

$$z(0) = 2, \qquad z'(0) = -4,$$

which is equivalent to

$$v''(\xi) + D_\xi^{3/2} v(\xi) - D_\xi^{\frac{1}{2}} v(\xi) + 4v'(\xi) + v(\xi) = e^{-\xi}\left(e^\xi(4\xi + 6) - 4(\xi - 6)\xi\right),$$

subject to the homogeneous ICs

$$v(0) = 0, \qquad v'(0) = 0,$$

where the exact solution is $z(\xi) = 2(1 - \xi)\exp(-\xi)$.

By applying the MRKHS method with $n = 55$, a comparison between the exact and the approximate curves is displayed in Fig. 5, which shows the accuracy of our method.

Example 5.6 Consider the GBT equation with boundary conditions (BCs) within CFFD of the form

$$\begin{cases} z''(\xi) + {}^{CF}\mathcal{D}_0^\alpha z(\xi) + z(\xi) = (\xi + 1)^2 + 2e^{-\xi}, & 1 < \alpha < 2, \\ z(0) = 1, \qquad z(1) = 3. \end{cases} \tag{5.6}$$

Table 4 Comparison of MRKHS solutions for different values of α for Example 5.6

ξ	$\alpha = 2$	$\alpha = 1.9$	$\alpha = 1.8$	$\alpha = 1.7$	$\alpha = 1.6$	$\alpha = 1.5$
0.0	1.000	1.00000	1.00000	1.00000	1.00000	1.00000
0.1	1.101	1.10940	1.11688	1.14628	1.14628	1.12147
0.2	1.208	1.22339	1.23804	1.29026	1.29026	1.24713
0.3	1.327	1.34784	1.36904	1.43430	1.43430	1.38237
0.4	1.464	1.48863	1.51528	1.58439	1.58439	1.53232
0.5	1.625	1.65160	1.68204	1.74770	1.74770	1.70187
0.6	1.816	1.84257	1.87446	1.93145	1.93145	1.89569
0.7	2.043	2.06730	2.09761	2.14251	2.14251	2.11825
0.8	2.312	2.33144	2.35642	2.38718	2.38718	2.37381
0.9	2.629	2.64053	2.65566	2.67124	2.67124	2.66644
1.0	3.000	3.00000	3.00000	3.00000	3.00000	3.00000

Table 5 Absolute errors of Example 5.7

ξ	Absolute error
0.1	2.9361×10^{-9}
0.2	6.9784×10^{-9}
0.3	1.2978×10^{-8}
0.4	2.0742×10^{-8}
0.5	2.9237×10^{-8}
0.6	3.6643×10^{-8}
0.7	4.0493×10^{-8}
0.8	3.7848×10^{-8}
0.9	2.5471×10^{-8}

In this example, we assume different values of the fractional CFFD orders to see the effect of this nonsingular kernel derivative to the GBT equations. Hence, a comparison between the approximate solutions for $\alpha \in \{1, 2, 1.9, 1.8, 1.7, 1.6, 1.5\}$ is shown in Table 4.

Example 5.7 Consider the following GBT equation:

$$z''(\xi) + {}^{CF}\mathcal{D}_{\xi}^{\frac{1}{2}} z(\xi) + z(\xi) = 6 + \xi^2 - 4e^{-t} - 5\xi,$$

subject to the BCs

$$z(0) = z'(1) = 0,$$

in which the exact solution is $z(\xi) = \xi^2 - \xi$.

By applying the methodology described in this paper for $n = 55$, some tabulated and graphical results are shown in Table 5 and Fig. 6.

6 Conclusion

In this paper, solutions of generalized Bagely–Torvik equations under the fractional derivative of a nonsingular kernel, the Caputo–Fabrizio derivative, are meaningfully discussed. These types of differential equations were solved under appropriate initial or boundary conditions. In the case of ICs, we prefer to convert the problem into a system of fractional differential equations seeking more efficiency and simplicity of the RKHS technique. Some theories related to the proposed method solutions, convergence, and error estimation have been proven. Numerical examples have been presented to reveal the

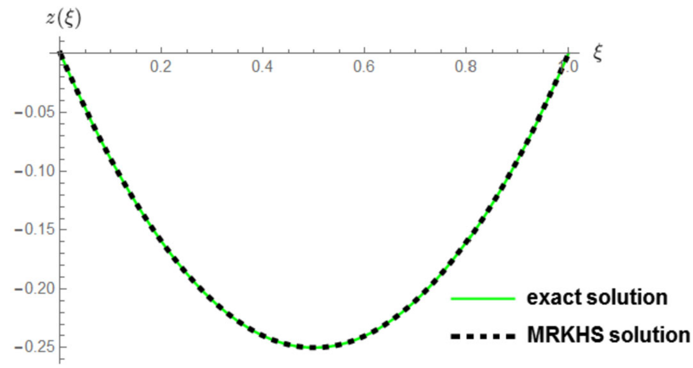

Figure 6 Exact and MRKHS solution curves of Example 5.7

effects of CFFD on solutions of the GBT equation. These examples showed how the proposed method is effective in solving GBT equations and showed how close the fractional solutions are to the solutions of the integer order.

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