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Kuratowski MNC method on a generalized fractional Caputo Sturm–Liouville–Langevin q -difference problem with generalized Ulam–Hyers stability

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Abstract

In this work, we consider a generalized quantum fractional Sturm–Liouville–Langevin difference problem with terminal boundary conditions. The relevant results rely on Mönch's fixed point theorem along with a theoretical method by terms of Kuratowski measure of noncompactness (MNC) and the Banach contraction principle (BCP). Furthermore, two dynamical notions of Ulam–Hyers (UH) and generalized Ulam–Hyers (GUH) stability are addressed for solutions of the supposed Sturm–Liouville–Langevin quantum boundary value problem (q -FBVP). Two examples are presented to show the validity and also the effectiveness of theoretical results. In the last part of the paper, we conclude our exposition with some final remarks and observations.

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1 Introduction

The topics of fractional calculus and (quantum) q -calculus in general and fractional differential equations, especially, have appeared extensively, and they are one of the applied branches in mathematical analysis which have enormous impact in exact description of existing real phenomena. In the meantime, in 1910, q -difference equations were introduced by Jackson [1]. Next, the research on such q -difference equations was implemented in most of the works, more specifically, by Carmichael and Al-Salam in [2, 3]. Regarding several earlier manuscripts on this topic, we cite [4, 5], while the initial ideas on q -FC can be observed in [6]. To review other applications on this field, see for instance [7–9]. Also, some new papers in this regard are [10, 11].

By making use of techniques of nonlinear analysis, many researchers have turned to the existence and uniqueness of solutions to nonlinear fractional differential equations equipped with a variety of boundary conditions as special cases, since they are considered

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as accurate procedures to describe the actual processes [12–34]. On the other hand, some other imperative and extra extremely good studies, which have these days attracted greater interest, have been dedicated to the qualitative investigation of differential equations of non-integer orders [35–39]. The primary effort was made by Ulam in 1940 and later by Hyers [40], and the notion of stability was developed by Rassias. Finally, in comparison to other works, Obloza [41] was the first researcher who investigated the U–H stability for a given differential equation in 1993.

The Langevin equation (designed in 1908 by Langevin for elaborate interpretation of Brownian motions) is introduced to be an applied mathematical model to describe the cases of the evolution of every physical phenomenon in the environments having fluctuations [42]. On the other side, the Sturm–Liouville problems involve various applications in some fields of engineering and science [43]. Recently, Kiataramkul along with a group of mathematicians presented a new structure of Hadamard FBVP in the fractional settings by combining the Langevin and Sturm–Liouville equations [44]. This combination would be an appropriate description in relation to each dynamical process defined in a fractal medium in which both properties of the fractal and memory via a dissipative memory kernel are incorporated.

Regarding the novelty of the present manuscript, no contributions exist, as far as we know, concerning the existence theory on the Caputo q -difference equations with the help of the technique of Kuratowski measure on noncompactness (KMNC-method) combined with Mönch’s fixed point theorem. As a result, the goal of this paper is to enrich this academic area via new techniques based on a special notion of Kuratowski measure. Our proposed method is essentially based on the result given by Banaś et al. [45]. Some authors utilized similar methods via the KMNC technique to different types of FBVPs, including [46–54]. Therefore, it is emphasized that the KMNC technique is implemented for the first time on the generalized fractional q -Caputo Sturm–Liouville–Langevin q -difference problems.

To be more precise, in this paper, we propose the following problem of the generalized fractional q -Caputo Sturm–Liouville–Langevin q -difference equations:

$$\begin{cases} {}_c\mathcal{D}_q^\alpha([\rho(t){}_c\mathcal{D}_q^\beta + r(t)])w(t) = \sigma(t, w(t)), & (t \in \mathbb{I}), \\ w(0) = 0, & {}_c\mathcal{D}_q^\beta w(T) + \frac{r(T)}{\rho(T)}w(T) = 0, \end{cases} \tag{1}$$

where $\mathbb{I} := [0, T]$, $0 < \alpha, \beta \leq 1$, and ${}_c\mathcal{D}_q^\varepsilon$ is a q -derivative of q -Caputo type of order $\varepsilon \in \{\alpha, \beta\}$, $\sigma : \mathbb{I} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, $\rho \in C(\mathbb{I}, \mathbb{R} \setminus \{0\})$ and $r \in C(\mathbb{I}, \mathbb{R})$.

Our suggested model in the context of quantum operators is not only new in the existing structure but also is equivalent to some known physical models pertinent to the specific values of the functions and parameters involved in q -FBVP (1). In spite of some similar research implemented by Berhail et al. [55] based on Hadamard operators and also by Kiataramkul et al. [44] based on anti-periodic boundary conditions, more precisely, our q -FBVP (1) is formulated in a generalized form which combines both Langevin equations and Sturm–Liouville problems in the context of q -operators for the first time. Indeed, we have:

- By choosing $r(t) \equiv 0$ and $q \rightarrow 1$, the nonlinear generalized fractional q -Caputo Sturm–Liouville–Langevin q -difference BVP (1) is converted to the standard

nonlinear Sturm–Liouville equation

$${}_c\mathcal{D}^\alpha [\rho(t) {}_c\mathcal{D}^\beta]w(t) = \sigma(t, w(t)).$$

- By choosing $\rho(t) \equiv 1$ and $r(t) = -\mu$, $\mu \in \mathbb{R}$ and $q \rightarrow 1$, the nonlinear generalized fractional q -Caputo Sturm–Liouville–Langevin q -difference BVP (1) is converted to the standard nonlinear Langevin equation

$${}_c\mathcal{D}^\alpha [{}_c\mathcal{D}^\beta - \mu]w(t) = \sigma(t, w(t)).$$

In fact, in this paper, we aim to show that one can model some known equations with respect to q -operators, because these operators have discrete structures and give accurate numerical results in different simulations. Here is a brief outline of the arrangement of the paper. The next section provides definitions and preliminary lemmas that will be needed to prove the main theorems. In Sect. 3, we establish existence and uniqueness of solutions to the given problem of the generalized Sturm–Liouville–Langevin q -difference equation by following the KMNC-method. In Sect. 4, we discuss some types of fractional Ulam stability. In Sect. 5, we give an example to illustrate numerical findings. In the last part of the paper, we conclude our exposition with some final remarks and observations.

2 Preliminary notions

We follow the present section by recalling and assembling some required notions for further arguments and developments.

Consider the Banach space of all real-valued continuous functions $\mathcal{U} = C(\mathbb{I}, E)$ (here E is assumed to be the space of real numbers) with the supremum norm

$$\|w\|_\infty = \sup\{|w(t)| : t \in \mathbb{I}\},$$

and $\mathfrak{M}_{\mathcal{U}}$ represents the class of all bounded mappings in \mathcal{U} .

Consider $L^1(\mathbb{I}, E)$ as a Banach space of all measurable Bochner integrable mappings like $w : \mathbb{I} \rightarrow E$ which are furnished with the integral norm

$$\|w\|_{L^1} = \int_{\mathbb{I}} |w(s)| \, ds.$$

In what follows, we recollect some elementary definitions and properties related to fractional q -calculus. Refer to [1, 2]. Let $q \in (0, 1)$ be a real number. For each $c \in \mathbb{R}$, we define

$$[c]_q = \frac{1 - q^c}{1 - q}.$$

The q -power function $(c - d)^m$ is defined by

$$(c - d)^{(0)} = 1, \quad (c - d)^{(m)} = \prod_{k=0}^{m-1} (c - dq^k), \quad c, d \in \mathbb{R}, m \in \mathbb{N},$$

and

$$(c - d)^{(\delta)} = c^\delta \prod_{k=0}^{\infty} \left(\frac{c - dq^k}{c - dq^{k+\delta}} \right), \quad c, d, \delta \in \mathbb{R}.$$

Definition 1 ([1, 2]) The q -gamma function is given by

$$\Gamma_q(\delta) = \frac{(1 - q)^{(\delta-1)}}{(1 - q^{\delta-1})}, \quad \delta \in \mathbb{R} - (\mathbb{Z}_- \cup \{0\}),$$

and

$$\Gamma_q(1 + \delta) = [\delta]_q \Gamma_q(\delta).$$

Definition 2 ([1, 2, 6]) Let $\sigma : \mathbb{I} \rightarrow \mathbb{R}$ be a suitable mapping. We define the q -derivative of integer-order $m \in \mathbb{N}$ for σ by $\mathcal{D}_q^0 \sigma(t) = \sigma(t)$,

$$\mathcal{D}_q \sigma(t) := \mathcal{D}_q^1 \sigma(t) = \frac{\sigma(t) - \sigma(qt)}{(1 - q)t}, \quad (t \neq 0), \quad \mathcal{D}_q \sigma(0) = \lim_{t \rightarrow 0} \mathcal{D}_q \sigma(t),$$

and

$$\mathcal{D}_q^m \sigma(t) = \mathcal{D}_q \mathcal{D}_q^{m-1} \sigma(t), \quad t \in \mathbb{I}, m \in \mathbb{N}.$$

Definition 3 ([1, 2, 6]) For a given mapping $\sigma : \mathbb{I}_t \rightarrow \mathbb{R}$, the expression defined by

$$\mathcal{I}_q \sigma(t) = \int_0^t \sigma(s) d_q s = \sum_{m=0}^{\infty} t(1 - q)q^m \sigma(tq^m)$$

is called q -integral if the series is convergent, in which $\mathbb{I}_t := \{tq^m : m \in \mathbb{N}\} \cup \{0\}$.

Definition 4 ([5, 6]) The integral of a function $\sigma : \mathbb{I} \rightarrow \mathbb{R}$ given as

$${}_{RL}\mathcal{I}_q^0 \sigma(t) = \sigma(t),$$

and

$${}_{RL}\mathcal{I}_q^\delta \sigma(t) = \frac{1}{\Gamma_q(\delta)} \int_0^t (t - qs)^{(\delta-1)} \sigma(s) d_q s, \quad t \in \mathbb{I},$$

is named the Riemann–Liouville q -integral of order $\delta \in \mathbb{R}_+$, if the integral exists.

Lemma 5 ([6, 9]) Let $\delta \in \mathbb{R}_+$ and $\beta \in (-1, \infty)$. One has

$${}_{RL}\mathcal{I}_q^\delta t^\beta = \frac{\Gamma_q(\beta + 1)}{\Gamma_q(\delta + \beta + 1)} t^{\delta+\beta}, \quad \delta \geq 0, t > 0.$$

In particular, if $\sigma \equiv 1$, then

$${}_{RL}\mathcal{I}_q^\delta 1(t) = \frac{1}{\Gamma_q(1 + \delta)} t^{(\delta)} \quad \text{for all } t > 0.$$

Definition 6 ([6, 9]) The Riemann–Liouville q -derivative of order $\delta \in \mathbb{R}_+$ of the mapping $\sigma : \mathbb{I} \rightarrow \mathbb{R}$ is defined by

$${}_{RL}\mathcal{D}_q^\delta \sigma(t) = \mathcal{D}_q^{[\delta]} {}_{RL}\mathcal{I}_q^{[\delta]-\delta} \sigma(t),$$

where $[\delta]$ is the integer part of δ .

Definition 7 ([56]) The δ^{th} - q -Caputo derivative for an absolutely continuous mapping σ is given by

$${}_c\mathcal{D}_q^\delta \sigma(t) = {}_{RL}\mathcal{I}_q^{[\delta]-\delta} \mathcal{D}_q^{[\delta]} \sigma(t),$$

where $[\delta]$ is the integer part of δ and the integral exists.

Remark 1 Note that if $q \rightarrow 1$, then both Definition 6 and Definition 7 are converted to the standard Riemann–Liouville and Caputo fractional derivatives.

Lemma 8 ([9, 57]) Let $m - 1 < \delta < m$. Then

$${}_{RL}\mathcal{I}_q^\delta \mathcal{D}_q^\delta \sigma(t) = \sigma(t) - \sum_{i=0}^{m-1} \frac{t^i}{\Gamma_q(i+1)} \mathcal{D}_q^i \sigma(0).$$

Lemma 9 ([57]) Let σ be a function defined on \mathbb{I} and suppose that δ, β are two real non-negative numbers. Then the following hold:

$$\begin{aligned} {}_{RL}\mathcal{I}_q^\delta {}_{RL}\mathcal{I}_q^\beta \sigma(t) &= {}_{RL}\mathcal{I}_q^{\delta+\beta} \sigma(t) = {}_{RL}\mathcal{I}_q^\beta {}_{RL}\mathcal{I}_q^\delta \sigma(t), \\ {}_c\mathcal{D}_q^\delta {}_{RL}\mathcal{I}_q^\delta \sigma(t) &= \sigma(t). \end{aligned}$$

Now we review some properties of the concept of Kuratowski measure of noncompactness (KMNC).

Definition 10 ([45, 46]) The mapping $\kappa : \mathfrak{M}_{\mathcal{U}} \rightarrow [0, \infty)$ denoted by $\kappa(C)$ for $C \in M_{\mathcal{U}}$ is named the Kuratowski MNC if

$$\kappa(C) := \inf \left\{ r > 0 : \exists \text{ finitely many sets } C_i \text{ s.t. } C = \bigcup_{i=1}^m C_i \text{ and } D(C_i) \leq r \right\},$$

where $D(C_i) = \sup\{|w - \hat{w}| : w, \hat{w} \in C_i\}$.

Proposition 11 ([45, 46]) The Kuratowski MNC satisfies the following:

- 1 $C \subset G \Rightarrow \kappa(C) \leq \kappa(G)$,
- 2 $\kappa(C) = 0$ if and only if A is relatively compact,
- 3 $\kappa(C) = \kappa(\overline{C}) = \kappa(\text{conv}(C))$, where \overline{C} and $\text{conv}(C)$ are the closure and the convex hull of C , respectively,
- 4 $\kappa(C + G) \leq \kappa(C) + \kappa(G)$,
- 5 $\kappa(pC) = |p|\kappa(C)$, $p \in \mathbb{R}$.

Notation 12 Let Σ be the set of functions $w : \mathbb{I} \rightarrow E$. Set

$$\Sigma(t) = \{w(t) : w \in \Sigma\}, \quad (\forall t \in \mathbb{I}), \quad \Sigma(\mathbb{I}) = \{w(t) : w \in \Sigma, t \in \mathbb{I}\}.$$

Theorem 13 ([48, 54]) Let the subset $W \neq \emptyset$ be convex and bounded, and in the Banach space \mathcal{U} with $0 \in W$, $\sigma : W \rightarrow W$ be continuous. If $\forall \Sigma \subset W$,

$$\Sigma = \overline{\text{conv}\sigma(\Sigma)} \quad \text{or} \quad \Sigma = \sigma(\Sigma) \cup \{0\} \quad \Rightarrow \quad \kappa(\Sigma) = 0, \tag{2}$$

then σ has a fixed point.

Lemma 14 ([48]) *Let the subset W be convex, bounded, and closed in the Banach space \mathcal{U} , $G \in C(\mathbb{I} \times \mathbb{I})$, $\sigma : \mathbb{I} \times W \rightarrow \mathcal{U}$ be Caratheodory, and there exists $p \in L^1(\mathbb{I}, \mathbb{R}^+)$ such that, for each $t \in \mathbb{I}$ and every bounded set $B \subset W$,*

$$\lim_{h \rightarrow 0^+} \kappa(\sigma(\mathbb{I}_{t,h} \times B)) \leq p(t)\kappa(B); \quad \mathbb{I}_{t,h} = [t - h, t] \cap \mathbb{I}.$$

By assuming Σ as an equicontinuous set of mappings $\mathbb{I} \rightarrow W$, we have

$$\kappa\left(\left\{\int_{\mathbb{I}} G(s, t)\sigma(s, w(s)) ds : w \in \Sigma\right\}\right) \leq \int_{\mathbb{I}} \|G(t, s)\| p(s)\kappa(\Sigma(s)) ds.$$

3 Main theorems

In this section, we are concerned with the existence of solutions of the given generalized Sturm–Liouville–Langevin q -difference FBVP (1).

Definition 15 By a solution of the generalized Sturm–Liouville–Langevin q -difference FBVP (1), we mean a measurable function $w \in \mathcal{U}$ such that $w(0) = 0$, ${}_c\mathcal{D}_q^\beta w(T) + \frac{r(T)}{\rho(T)}w(T) = 0$, and the FDEq

$${}_c\mathcal{D}_q^\alpha([\rho(t){}_c\mathcal{D}_q^\beta + r(t)])w(t) = \sigma(t, w(t))$$

is satisfied on \mathbb{I} .

In what follows, we present the characterization of solutions in relation to suggested generalized Sturm–Liouville–Langevin q -difference FBVP (1).

Lemma 16 *Let $K(t) \in \mathcal{U}$, $0 < \alpha, \beta \leq 1$, $\rho \in C(\mathbb{I}, \mathbb{R} \setminus \{0\})$, and $r \in C(\mathbb{I}, \mathbb{R})$. Then the solution of the following linear generalized Sturm–Liouville–Langevin q -difference FBVP*

$$\begin{cases} {}_c\mathcal{D}_q^\alpha([\rho(t){}_c\mathcal{D}_q^\beta + r(t)])w(t) = K(t), & t \in \mathbb{I}, \\ w(0) = 0, & {}_c\mathcal{D}_q^\beta w(T) + \frac{r(T)}{\rho(T)}w(T) = 0, \end{cases} \tag{3}$$

is given by

$$w(t) = {}_{RL}\mathcal{I}_q^\beta\left(\frac{1}{\rho}{}_{RL}\mathcal{I}_q^\alpha K\right)(t) - {}_{RL}\mathcal{I}_q^\beta\left(\frac{r}{\rho}w\right)(t) - {}_{RL}\mathcal{I}_q^\alpha K(T) {}_{RL}\mathcal{I}_q^\beta\left(\frac{1}{\rho}\right)(t). \tag{4}$$

Proof Taking the α^{th} - q -Riemann–Liouville integral to the FDEq of (3), we get

$${}_c\mathcal{D}_q^\beta w(t) = \frac{{}_{RL}\mathcal{I}_q^\alpha K(t) + c_0 - r(t)w(t)}{\rho(t)}, \tag{5}$$

where $c_0 \in \mathbb{R}$. The second BCs of system (3) gives

$$c_0 = -{}_{RL}\mathcal{I}_q^\alpha K(T).$$

Taking the β^{th} - q -Riemann–Liouville integral to (5), we obtain

$$w(t) = {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho} {}_{RL}\mathcal{I}_q^\alpha K \right) (t) - {}_{RL}\mathcal{I}_q^\beta \left(\frac{r}{\rho} w \right) (t) - {}_{RL}\mathcal{I}_q^\alpha K(T) {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho} \right) (t) + c_1, \tag{6}$$

where $c_1 \in \mathbb{R}$. Using the condition $w(0) = 0$ of (3), we have

$$c_1 = 0.$$

Substituting the obtained value for c_1 , we derive the q -integral equation (4), and the proof is completed. \square

Note that, on the other side, if we apply the Caputo β^{th} - q -derivative and α^{th} - q -derivative to both sides of (4) and use Lemma 9, then the given system (3) immediately is established.

Now, consider the nonlinear generalized Sturm–Liouville–Langevin q -difference FBVP (1). On the basis of Lemma 16, the solutions of (1) correspond to q -integral equation in the following form:

$$w(t) = {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho} {}_{RL}\mathcal{I}_q^\alpha \sigma \right) (t, w(t)) - {}_{RL}\mathcal{I}_q^\beta \left(\frac{r}{\rho} w \right) (t) - {}_{RL}\mathcal{I}_q^\alpha \sigma (T, w(T)) {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho} \right) (t). \tag{7}$$

3.1 Existence result via the KMNC-method

We further will use the following hypotheses.

- (H1) $\sigma : \mathbb{I} \times \mathcal{U} \rightarrow \mathcal{U}$ is Caratheodory;
- (H2) There exists $p \in C(\mathbb{I}, \mathbb{R}^+)$ such that

$$\| \sigma (t, w(t)) \| \leq p(t) \| w \|, \quad \forall t \in \mathbb{I}, \forall w \in \mathcal{U};$$

- (H3) For each $t \in \mathbb{I}$ and each bounded measurable set $B \subset \mathcal{U}$,

$$\lim_{h \rightarrow 0^+} \kappa (\sigma (\mathbb{I}_{t,h} \times B), 0) \leq p(t) \kappa (B),$$

where κ is the Kuratowski MNC and $\mathbb{I}_{t,h} = [t - h, t] \cap \mathbb{I}$.

Set

$$p^* = \sup_{t \in \mathbb{I}} |p(t)|, \quad \rho^* = \inf_{t \in \mathbb{I}} |\rho(t)|, \quad r^* = \sup_{t \in \mathbb{I}} |r(t)|. \tag{8}$$

Theorem 17 *Suppose that conditions (H1)–(H3) hold. If*

$$\Lambda < 1, \tag{9}$$

with

$$\Lambda := \mu p^* + \nu,$$

where

$$\mu = \frac{1}{\rho^*} \left\{ \frac{T^{\alpha+\beta}}{\Gamma_q(\alpha + \beta + 1)} + \frac{T^\beta}{\Gamma_q(1 + \beta)} \frac{T^\alpha}{\Gamma_q(\alpha + 1)} \right\}, \tag{10}$$

$$\nu = \left\{ \frac{r^*}{\rho^*} \frac{T^\beta}{\Gamma_q(\beta + 1)} \right\}, \tag{11}$$

then the nonlinear generalized Sturm–Liouville–Langevin q -difference FBVP (1) has a solution on \mathbb{I} .

Proof Firstly, for $w \in \mathcal{U}$, we consider the operator $\mathcal{G} : \mathcal{U} \rightarrow \mathcal{U}$ defined by

$$\mathcal{G}w = {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho} {}_{RL}\mathcal{I}_q^\alpha \sigma \right) (t, w(t)) - {}_{RL}\mathcal{I}_q^\beta \left(\frac{r}{\rho} w \right) (t) - {}_{RL}\mathcal{I}_q^\alpha \sigma (T, w(T)) {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho} \right) (t). \tag{12}$$

Evidently, the fixed points of \mathcal{G} are solutions of the nonlinear generalized Sturm–Liouville–Langevin q -difference FBVP (1). We take

$$D_R = \{w \in \mathcal{U} : \|w\| \leq R\}.$$

D_R is convex, closed, and bounded. We shall follow the proof in three steps.

STEP 1: \mathcal{G} is sequentially continuous:

Let $\{w_n\}_n$ be a sequence with $w_n \rightarrow w$ in \mathcal{U} . Then, for each $t \in \mathbb{I}$, one may write

$$\begin{aligned} |(\mathcal{G}w_n)(t) - (\mathcal{G}w)(t)| &\leq {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho(t)} {}_{RL}\mathcal{I}_q^\alpha |\sigma(t, w_n) - \sigma(t, w(t))| \right) \\ &\quad + {}_{RL}\mathcal{I}_q^\beta \left(\frac{r}{\rho} |w_n - w| \right) (t) \\ &\quad + {}_{RL}\mathcal{I}_q^\alpha |\sigma(T, w_n(T)) - \sigma(T, w(T))| {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho} \right) (t). \end{aligned}$$

Since the function σ is continuous and satisfies (H1), so $\sigma(t, w_n(t))$ tends uniformly to $\sigma(t, w(t))$. In accordance with Lebesgue’s dominated convergence theorem, $\{\mathcal{G}(w_n)(t)\}$ tends uniformly to $\mathcal{G}(w)(t)$, that is, $\mathcal{G}w_n \rightarrow \mathcal{G}w$. Hence $\mathcal{G} : D_R \rightarrow D_R$ is sequentially continuous.

STEP 2: $\mathcal{G}(D_R) \subseteq D_R$:

Take $w \in D_R$. By (H2) and for each $t \in \mathbb{I}$, let $\mathcal{G}(w)(t) \neq 0$. Then

$$\begin{aligned} |\mathcal{G}w(t)| &\leq \left| {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho} {}_{RL}\mathcal{I}_q^\alpha \sigma \right) (t, w(t)) \right| + \left| {}_{RL}\mathcal{I}_q^\beta \left(\frac{r}{\rho} w \right) (t) \right| \\ &\quad + \left| {}_{RL}\mathcal{I}_q^\alpha \sigma (T, w(T)) {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho} \right) (t) \right| \\ &\leq {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{|\rho|} {}_{RL}\mathcal{I}_q^\alpha |\sigma(t, w(t))| \right) (t) + {}_{RL}\mathcal{I}_q^\beta \left(\frac{|r|}{|\rho|} |w(t)| \right) (t) \\ &\quad + {}_{RL}\mathcal{I}_q^\alpha |\sigma(T, w(T))| {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{|\rho|} \right) (t) \\ &\leq {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho^*} {}_{RL}\mathcal{I}_q^\alpha [\|w\|p(t)] \right) (t) + {}_{RL}\mathcal{I}_q^\beta \left(\frac{r^*}{\rho^*} |w(t)| \right) (t) \end{aligned}$$

$$\begin{aligned}
 & + \frac{t^\beta}{\Gamma_q(1 + \beta)} \left(\frac{1}{\rho^*} {}_{RL}\mathcal{I}_q^\alpha [\|w\|p(t)] \right) (T) \\
 & \leq p^* R \left\{ {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho^*} {}_{RL}\mathcal{I}_q^\alpha (1)(t) \right) + \frac{t^\beta}{\Gamma_q(1 + \beta)} \frac{1}{\rho^*} {}_{RL}\mathcal{I}_q^\alpha (1)(T) \right\} \\
 & \quad + R {}_{RL}\mathcal{I}_q^\beta \left(\frac{r^*}{\rho^*} (1) \right) (t) \\
 & \leq R \frac{p^*}{\rho^*} \left\{ \frac{T^{\alpha+\beta}}{\Gamma_q(\alpha + \beta + 1)} + \frac{T^\beta}{\Gamma_q(1 + \beta)} \frac{T^\alpha}{\Gamma_q(\alpha + 1)} \right\} \\
 & \quad + R \left\{ \frac{r^*}{\rho^*} \frac{T^\beta}{\Gamma_q(\beta + 1)} \right\} \\
 & = R(\mu p^* + \nu).
 \end{aligned}$$

Hence we get

$$\|\mathcal{G}(w)\|_{\mathcal{U}} \leq R(\mu p^* + \nu) = R\Lambda \leq R. \tag{13}$$

STEP 3: $\mathcal{G}(D_R)$ is equicontinuous:

By considering STEP 2, it is known that $\mathcal{G}(D_R) \subset \mathcal{U}$ is bounded uniformly. In relation to the equicontinuity of $\mathcal{G}(D_R)$, we take $t_1, t_2 \in \mathbb{I}$, $t_1 < t_2$, and $w \in D_R$. Then

$$\begin{aligned}
 |\mathcal{G}w(t_2) - \mathcal{G}w(t_1)| & \leq {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho^*} {}_{RL}\mathcal{I}_q^\alpha |\sigma(t_2, w(t_2)) - \sigma(t_1, w(t_1))| \right) \\
 & \quad + {}_{RL}\mathcal{I}_q^\beta \left(\frac{r^*}{\rho^*} |w(t_2) - w(t_1)| \right) \\
 & \quad + {}_{RL}\mathcal{I}_q^\alpha |\sigma(T, w(T))| \left(\frac{1}{\rho^*} \right) |{}_{RL}\mathcal{I}_q^\beta (1)(t_2) - {}_{RL}\mathcal{I}_q^\beta (1)(t_1)| \\
 & \leq p^* R \left\{ {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho^*} |{}_{RL}\mathcal{I}_q^\alpha (1)(t_2) - {}_{RL}\mathcal{I}_q^\alpha (1)(t_1)| \right) \right\} \\
 & \quad + R \left(\frac{r^*}{\rho^*} ({}_{RL}\mathcal{I}_q^\beta (1)(t_2) - {}_{RL}\mathcal{I}_q^\beta (1)(t_1)) \right) \\
 & \quad + \frac{p^*}{\rho^*} R \frac{T^\alpha}{\Gamma_q(\alpha + 1)} |{}_{RL}\mathcal{I}_q^\beta (1)(t_2) - {}_{RL}\mathcal{I}_q^\beta (1)(t_1)| \\
 & \leq \frac{R}{\rho^*} \frac{p^*}{\Gamma_q(\alpha + \beta + 1)} \left\{ (t_2^{\alpha+\beta} - t_1^{\alpha+\beta}) + 2(t_2 - t_1)^{\alpha+\beta} \right\} \\
 & \quad + \frac{r^*}{\rho^*} \frac{R}{\Gamma_q(\beta + 1)} \left\{ (t_2^\beta - t_1^\beta) + 2(t_2 - t_1)^\beta \right\} \\
 & \quad + \frac{p^*}{\rho^*} \frac{RT^\alpha}{\Gamma_q(\alpha + 1)\Gamma_q(\beta + 1)} \left\{ (t_2^\beta - t_1^\beta) + 2(t_2 - t_1)^\beta \right\}. \tag{14}
 \end{aligned}$$

As $t_1 \rightarrow t_2$, the right-hand side of (14) goes to 0 independent of w , and thus $|\mathcal{G}w(t_2) - \mathcal{G}w(t_1)| \rightarrow 0$. The equicontinuity of \mathcal{G} is confirmed.

The implication (2) is proved in the last step:

Let $\Sigma \subset D_R$ be such that $\Sigma = \overline{\text{conv}}(\mathcal{G}(\Sigma) \cup \{0\})$. Since Σ is equicontinuous and bounded, the mapping $t \mapsto w(t) = \kappa(\Sigma(t))$ has the continuity property on \mathbb{I} . From (H2) and some

given properties of the Kuratowski MNC κ , for any $t \in \mathbb{I}$, we get

$$\begin{aligned} w(t) &\leq \kappa(\mathcal{G}(\Sigma)(t) \cup \{0\}) \leq \kappa((\mathcal{G}\Sigma)(t)) \\ &\leq {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho} {}_{RL}\mathcal{I}_q^\alpha p\kappa(\Sigma) \right)(t) + {}_{RL}\mathcal{I}_q^\beta \left(\frac{r}{\rho} \kappa(\Sigma) \right)(t) \\ &\quad + {}_{RL}\mathcal{I}_q^\alpha p(T)\kappa(\Sigma(T)) {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho} \right)(t) \\ &\leq p^* \|w\| \left\{ {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho} {}_{RL}\mathcal{I}_q^\alpha(1) \right)(t) + {}_{RL}\mathcal{I}_q^\alpha(1)(T) {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho} \right)(t) \right\} \\ &\quad + \|w\| \left\{ {}_{RL}\mathcal{I}_q^\beta \left(\frac{r}{\rho}(1) \right)(t) \right\} \\ &\leq p^* \|w\| \left\{ \frac{1}{\rho^*} \left\{ \frac{T^{\alpha+\beta}}{\Gamma_q(\alpha + \beta + 1)} + \frac{T^\beta}{\Gamma_q(1 + \beta)} \frac{T^\alpha}{\Gamma_q(\alpha + 1)} \right\} \right\} \\ &\quad + \|w\| \left\{ \frac{r^*}{\rho^*} \frac{T^\beta}{\Gamma_q(\beta + 1)} \right\} \\ &\leq p^* \|w\| \mu + \|w\| \nu. \end{aligned}$$

This means that

$$\|w\|(1 - p^* \mu - \nu) \leq 0.$$

By (9) it follows that $\|w\| = 0$, that is, $w(t) = 0$ for any $t \in \mathbb{I}$, so $\kappa(\Sigma) = 0$, and then $\Sigma(t)$ is relatively compact in \mathcal{U} . From the Ascoli–Arzela theorem, Σ has the relative compactness in D_R . By Theorem 13, we find out that \mathcal{G} has a fixed point, which is the same solution of the nonlinear generalized Sturm–Liouville–Langevin q -difference FBVP (1). \square

3.2 Uniqueness criterion

Theorem 18 *Let:*

- (G1) $\sigma : \mathbb{I} \times \mathcal{U} \rightarrow \mathcal{U}$ be continuous.
- (G2) There exists the constant $M > 0$ such that

$$|\sigma(t, w) - \sigma(t, v)| \leq M \|w - v\|, \quad \forall t \in \mathbb{I}, \forall w, v \in \mathcal{U}. \tag{15}$$

Then the nonlinear generalized Sturm–Liouville–Langevin q -difference FBVP (1) has a unique solution on \mathbb{I} such that

$$\ell = \mu M + \nu < 1, \tag{16}$$

where μ and ν are given by Equations (10) and (11), respectively.

Proof In the first place, we show $\mathcal{G}B_\omega \subset B_\omega$, where the operator $\mathcal{G} : \mathcal{U} \rightarrow \mathcal{U}$ is defined by Equation (7), and for $\omega > 0$, $B_\omega = \{w \in \mathcal{U}, \|w\| \leq \omega\}$ such that

$$\omega \geq \frac{\mu\sigma_0}{1 - \ell},$$

and $\sigma_0 = \sup_{0 \leq t \leq T} |\sigma(t, 0)|$. For any $w \in B_\omega$, using (G2), we write

$$\begin{aligned}
 |\mathcal{G}w(t)| &\leq \left| {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho} {}_{RL}\mathcal{I}_q^\alpha \sigma \right) (t, w(t)) \right| + \left| {}_{RL}\mathcal{I}_q^\beta \left(\frac{r}{\rho} w \right) (t) \right| \\
 &\quad + \left| {}_{RL}\mathcal{I}_q^\alpha \sigma (T, w(T)) {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho} \right) (t) \right| \\
 &\leq {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho^*} {}_{RL}\mathcal{I}_q^\alpha |\sigma(t, w(t)) - \sigma(t, 0)| + |\sigma(t, 0)| \right) (t) + {}_{RL}\mathcal{I}_q^\beta \left(\frac{r^*}{\rho^*} |w(t)| \right) (t) \\
 &\quad + \frac{t^\beta}{\Gamma_q(1 + \beta)} \left(\frac{1}{\rho^*} {}_{RL}\mathcal{I}_q^\alpha |\sigma(t, w(t)) - \sigma(t, 0)| + |\sigma(t, 0)| \right) (T) \\
 &\leq {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho^*} {}_{RL}\mathcal{I}_q^\alpha (M\|w\| + \sigma_0) \right) (t) + \left(\frac{r^*}{\rho^*} \right) {}_{RL}\mathcal{I}_q^\beta \|w\| (t) \\
 &\quad + \frac{t^\beta}{\rho^* \Gamma_q(1 + \beta)} ({}_{RL}\mathcal{I}_q^\alpha (M\|w\| + \sigma_0))(T) \\
 &\leq \frac{1}{\rho^*} \left\{ \frac{T^{\alpha+\beta}}{\Gamma_q(\alpha + \beta + 1)} + \frac{T^\beta}{\Gamma_q(1 + \beta)} \frac{T^{\alpha+1}}{\Gamma_q(\alpha + 1)} \right\} (M(\|w\|) + \sigma_0) \\
 &\quad + \left\{ \frac{r^*}{\rho^*} \frac{T^\beta}{\Gamma_q(\beta + 1)} \right\} \|w\| \\
 &\leq \mu(M(\|w\|) + \sigma_0) + \nu\|w\| \leq \omega,
 \end{aligned}$$

which implies $\|\mathcal{G}(w)\| \leq \omega$ after taking the supremum on \mathbb{I} . Thus, \mathcal{G} corresponds B_ω to itself.

Next, we investigate that $\mathcal{G}(w)$ is a contraction. For $w, v \in \mathcal{U}$, and by utilizing the notations of (10) and (11), we have

$$\begin{aligned}
 |\mathcal{G}w(t) - \mathcal{G}v(t)| &\leq {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{|\rho|} {}_{RL}\mathcal{I}_q^\alpha |\sigma(t, w(t)) - \sigma(t, v(t))| \right) (t) + {}_{RL}\mathcal{I}_q^\beta \left(\frac{|r|}{|\rho|} |w(t) - v(t)| \right) (t) \\
 &\quad + \frac{t^\beta}{|\rho| \Gamma_q(1 + \beta)} ({}_{RL}\mathcal{I}_q^\alpha |\sigma(t, w(t)) - \sigma(t, v(t))|)(T) \\
 &\leq {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho^*} {}_{RL}\mathcal{I}_q^\alpha (M|w(t) - v(t)|) \right) (t) + \left(\frac{r^*}{\rho^*} \right) {}_{RL}\mathcal{I}_q^\beta (|w(t) - v(t)|)(t) \\
 &\quad + \frac{T^\beta}{\rho^* \Gamma_q(1 + \beta)} ({}_{RL}\mathcal{I}_q^\alpha (M|w(t) - v(t)|))(T) \\
 &\leq \frac{1}{\rho^*} \left\{ \frac{T^{\alpha+\beta}}{\Gamma_q(\alpha + \beta + 1)} + \frac{T^\beta}{\Gamma_q(1 + \beta)} \frac{T^\alpha}{\Gamma_q(\alpha + 1)} \right\} (M|w(t) - v(t)|) \\
 &\quad + \left\{ \frac{r^*}{\rho^*} \frac{T^\beta}{\Gamma_q(\beta + 1)} \right\} |w(t) - v(t)| \leq (\mu M + \nu)\|w - v\|.
 \end{aligned}$$

Consequently, we get

$$\|\mathcal{G}w - \mathcal{G}v\| \leq (\mu M + \nu)\|w - v\| = \ell\|w - v\|,$$

which states that \mathcal{G} is a contraction by (16). By the Banach contraction principle, \mathcal{G} has a unique fixed point, which is the unique solution of the nonlinear generalized Sturm–Liouville–Langevin q -difference FBVP (1) on \mathbb{I} . \square

4 Stability results

In the recent section, we are interested in studying UH and GHR stability of the given nonlinear generalized Sturm–Liouville–Langevin q -difference FBVP (1).

Definition 19 The nonlinear generalized Sturm–Liouville–Langevin q -difference FBVP (1) is UH stable if there is $c_\sigma \in \mathbb{R}^+$ such that, for each $\epsilon \in \mathbb{R}^+$ and for each $w \in \mathcal{U}$ satisfying

$$\begin{cases} |{}_c\mathcal{D}_q^\alpha([\rho(t){}_c\mathcal{D}_q^\beta + r(t)])w(t) - \sigma(t, w(t))| \leq \epsilon, & (t \in \mathbb{I}), \\ w(0) = 0, & {}_c\mathcal{D}_q^\beta w(T) + \frac{r(T)}{\rho(T)}w(T) = 0, \end{cases} \tag{17}$$

a unique solution $\tilde{w} \in \mathcal{U}$ of (1) exists with

$$\|w - \tilde{w}\| \leq c_\sigma \epsilon.$$

Definition 20 The nonlinear generalized Sturm–Liouville–Langevin q -difference FBVP (1) is generalized UH stable (GUH) if there exists $C_\sigma \in C(\mathbb{R}^+, \mathbb{R}^+)$, $C_\sigma(0) = 0$ such that, for each $\epsilon \in \mathbb{R}^+$ and for each $w \in \mathcal{U}$ satisfying (17), a unique solution $\tilde{w} \in \mathcal{U}$ of (1) exists with

$$\|w - \tilde{w}\| \leq C_\sigma(\epsilon).$$

Remark 2 A function $\tilde{w} \in C(\mathbb{I}, \mathbb{R})$ is a solution of (19) if and only if there exists $\psi \in C(\mathbb{I}, \mathbb{R})$ (which depends on \tilde{w}) such that

1. $|\psi(t)| \leq \epsilon, t \in \mathbb{I}$.
2. ${}_c\mathcal{D}_q^\alpha([\rho(t){}_c\mathcal{D}_q^\beta + r(t)])w(t) = \sigma(t, w(t)) + \psi(t), t \in \mathbb{I}$.

Here we review the UH and GUH stability of solutions to the nonlinear generalized Sturm–Liouville–Langevin q -difference FBVP (1).

Theorem 21 Let (G2) and (16) be fulfilled. Then the nonlinear generalized Sturm–Liouville–Langevin q -difference FBVP (1) is UH and GUH stable.

Proof Let $\epsilon > 0$, let $\tilde{w} \in C(\mathbb{I}, \mathbb{R})$ satisfy (17), and let $w \in C(\mathbb{I}, \mathbb{R})$ be the unique solution of the generalized Sturm–Liouville–Langevin q -difference FBVP

$$\begin{cases} {}_c\mathcal{D}_q^\alpha([\rho(t){}_c\mathcal{D}_q^\beta + r(t)])w(t) = \sigma(t, w(t)), & (t \in \mathbb{I}), \\ w(0) = 0, & {}_c\mathcal{D}_q^\beta w(T) + \frac{r(T)}{\rho(T)}w(T) = 0. \end{cases} \tag{18}$$

By Lemma 16, we have

$$\begin{aligned} w(t) = & {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho} {}_{RL}\mathcal{I}^\alpha \sigma \right) (t, w(t)) - {}_{RL}\mathcal{I}^\beta \left(\frac{r}{\rho} w \right) (t) \\ & - {}_{RL}\mathcal{I}^\alpha \sigma (T, w(T)) {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho} \right) (t). \end{aligned} \tag{19}$$

Since we have supposed that \tilde{w} satisfies (19), hence, by Remark 2, we get

$$\begin{cases} {}_c\mathcal{D}_q^\alpha([\rho(t){}_c\mathcal{D}_q^\beta + r(t)])\tilde{w}(t) = \sigma(t, \tilde{w}(t)) + \psi(t), & (t \in \mathbb{I}), \\ \tilde{w}(0) = 0, & {}_c\mathcal{D}_q^\beta \tilde{w}(T) + \frac{r(T)}{\rho(T)}\tilde{w}(T) = 0. \end{cases} \tag{20}$$

Again by Lemma 16, we have

$$\begin{aligned} \tilde{w}(t) &= {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho} {}_{RL}\mathcal{I}^\alpha \sigma \right) (t, \tilde{w}(t)) - {}_{RL}\mathcal{I}^\beta \left(\frac{r}{\rho} \tilde{w} \right) (t) \\ &\quad - {}_{RL}\mathcal{I}^\alpha \sigma (T, \tilde{w}(T)) {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho} \right) (t) \\ &\quad + {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho} {}_{RL}\mathcal{I}^\alpha \psi \right) (t) - {}_{RL}\mathcal{I}^\alpha \psi (T) {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho} \right) (t). \end{aligned} \tag{21}$$

For each $t \in \mathbb{I}$,

$$\begin{aligned} &|\mathcal{G}w(t) - \mathcal{G}\tilde{w}(t)| \\ &\leq {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho^*} {}_{RL}\mathcal{I}_q^\alpha |\sigma(t, w(t)) - \sigma(t, \tilde{w}(t))| \right) (t) + {}_{RL}\mathcal{I}_q^\beta \left(\frac{r^*}{\rho^*} |w(t) - \tilde{w}(t)| \right) (t) \\ &\quad + \frac{t^\beta}{\rho^* \Gamma_q(1 + \beta)} ({}_{RL}\mathcal{I}_q^\alpha |\sigma(t, w(t)) - \sigma(t, \tilde{w}(t))|) (T) \\ &\quad + {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho} {}_{RL}\mathcal{I}^\alpha g \right) (t) + {}_{RL}\mathcal{I}^\alpha g(T) {}_{RL}\mathcal{I}_q^\beta \left(\frac{1}{\rho} \right) (t). \end{aligned}$$

In view of part 1 of Remark 2 and (G2), we obtain

$$\begin{aligned} \|\tilde{w} - w\| &\leq \frac{1}{\rho^*} \left\{ \frac{T^{\alpha+\beta}}{\Gamma_q(\alpha + \beta + 1)} + \frac{T^\beta}{\Gamma_q(1 + \beta)} \frac{T^\alpha}{\Gamma_q(\alpha + 1)} \right\} \varepsilon + \ell \|w - \tilde{w}\| \\ &\leq \mu \varepsilon + \ell \|w - \tilde{w}\|, \end{aligned}$$

in which ℓ is illustrated in (16). In accordance with the above, it gives

$$\|\tilde{w} - w\| \leq \frac{\mu}{1 - \ell} \varepsilon.$$

If we set $c_\sigma = \frac{\mu}{1 - \ell} > 0$, then the UH stability of the nonlinear generalized Sturm–Liouville–Langevin q -difference FBVP (1) is fulfilled. In addition, for $C_\sigma(\varepsilon) = \frac{\mu}{1 - \ell} \varepsilon$, $C_\sigma(0) = 0$, the nonlinear generalized Sturm–Liouville–Langevin q -difference FBVP (1) is GUH stable. This completes the proof. \square

5 Example

In this part, two examples are presented to show the validity and also the effectiveness of theoretical results.

Example 1 Based on the nonlinear generalized Sturm–Liouville–Langevin q -difference FBVP (1), we fix $\rho(t) \equiv 1$ and $r(t) = -\lambda$, $\lambda \in \mathbb{R}$. In this case, (1) is reduced to Langevin

q -difference FBVP

$$\begin{cases} {}_c\mathcal{D}_{1/4}^{1/4}({}_c\mathcal{D}_{1/4}^{1/3} - \frac{1}{27})w(t) = \frac{\sqrt{3}|w|\cos^2(2\pi t)}{3(27-t)}, \\ t \in \mathbb{I} = [0, 1], \\ w(0) = 0, \quad {}_c\mathcal{D}_{1/4}^{1/3}w(1) - \frac{1}{27}w(1) = 0, \end{cases} \tag{22}$$

where

$$\alpha = 1/4, \quad \beta = 1/3, \quad q = 1/4, \quad \lambda = -1/27, \quad q = 1/4, \quad T = 1,$$

and

$$\sigma(t, w) = \frac{\sqrt{3}|w|\cos^2(2\pi t)}{3(27-t)}.$$

The function σ is continuous on \mathbb{I} . On the other hand, $\forall w \in \mathbb{R}$ and $t \in \mathbb{I}$, we get

$$|\sigma(t, w)| \leq \frac{\sqrt{3}}{81}|w|.$$

Accordingly, (H2) is satisfied for $p^* = \frac{\sqrt{3}}{81}$. To check condition (9), we simply obtain

$$\Lambda := \mu p^* + \nu \simeq 0.8226 < 1.$$

All of the above results confirm that Theorem 17 is fulfilled, and it gives that the Langevin q -difference FBVP (22) possesses at least a solution formulated on \mathbb{I} .

Example 2 Consider the nonlinear generalized Sturm–Liouville–Langevin q -difference FBVP

$$\begin{cases} {}_c\mathcal{D}_{1/4}^{1/2}((1+t){}_c\mathcal{D}_{1/4}^{4/5} + \frac{t^3}{100})w(t) = \frac{1}{20} + \frac{t}{16} \frac{|w|}{1+|w|}, \quad t \in \mathbb{I} = [0, 1] \\ w(0) = 0, \quad {}_c\mathcal{D}_{1/4}^{4/5}w(1) + \frac{1}{200}w(1) = 0, \end{cases}$$

where

$$\alpha = 1/2, \quad \beta = 4/5, \quad q = 1/4, \quad T = 1,$$

and

$$\rho(t) = t + 1, \quad r(t) = \frac{t^3}{100}, \quad \sigma(t, w(t)) = \frac{1}{20} + \frac{t}{16} \frac{|w|}{1+|w|}.$$

Using the given data, we find that

$$|\sigma(t, w) - \sigma(t, v)| \leq \frac{1}{16}(|w - v|) \tag{23}$$

for any $t \in [0, 1]$. Then σ satisfies (H1) and (H2) with $M = \frac{1}{16}$. Now, we find that

$$\mu = 10.597, \quad \nu = 0.0189.$$

Hence

$$\ell = \mu M + \nu \simeq 0.68121 < 1.$$

All conditions of Theorem 18 are satisfied. Then there exists a unique solution to the nonlinear generalized Sturm–Liouville–Langevin q -difference FBVP (23) in \mathbb{I} . Moreover, Theorem 21 guarantees the UH and GUH stability for mentioned q -FBVP (23).

6 Conclusion

In this paper, we provided required criteria for the existence/uniqueness of solutions to a new category of nonlinear generalized Sturm–Liouville–Langevin q -difference FBVP (1). To arrive at such an aim, we dealt with a technique involving the Kuratowski measure of noncompactness (KMNC) along with a fixed point theorem of Mönch type. Although the used method based on KMNC–Mönch is considered as a standard method, its application in the current context is new yet, while the uniqueness property was derived with the help of BCP. Subsequently, the Ulam–Hyers (UH) and generalized Ulam–Hyers (GUH) stability were established for the proposed nonlinear generalized Sturm–Liouville–Langevin q -difference FBVP (1). Moreover, two examples were presented for the illustration of the obtained theory. Regarding next research projects, we are going to continue the analysis of such combined structures of physical and mathematical models by using nonsingular fractional operators which give more accurate numerical results. In particular, the Caputo–Fabrizio derivative is clearly well known in the field of fractional differential equations. The appearance of this derivative helps us to deal with some complicated phenomena. We will follow this study on the Caputo–Fabrizio derivative.

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The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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