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New version of fractional Simpson type inequalities for twice differentiable functions

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Abstract

Simpson inequalities for differentiable convex functions and their fractional versions have been studied extensively. Simpson type inequalities for twice differentiable functions are also investigated. More precisely, Budak et al. established the first result on fractional Simpson inequality for twice differentiable functions. In the present article, we prove a new identity for twice differentiable functions. In addition to this, we establish several fractional Simpson type inequalities for functions whose second derivatives in absolute value are convex. This paper is a new version of fractional Simpson type inequalities for twice differentiable functions.

MSC: 26D07; 26D10; 26D15

Keywords: Simpson type inequalities; Convex function; Fractional integrals

1 Introduction

Simpson's inequality plays a considerable role in several branches of mathematics. For four times continuously differentiable functions, the classical Simpson's inequality is expressed as follows.

Theorem 1 *Suppose that $\mathcal{F}:[\rho_1,\rho_2]\to\mathbb{R}$ is a four times continuously differentiable mapping on (ρ_1,ρ_2), and let $\|\mathcal{F}^{(4)}\|_\infty=\sup_{\kappa\in(\rho_1,\rho_2)}|\mathcal{F}^{(4)}(\kappa)|<\infty$. Then one has the inequality*

$$\left|\frac{1}{3}\left[\frac{\mathcal{F}(\rho_1)+\mathcal{F}(\rho_2)}{2}+2\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right)\right]-\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\mathcal{F}(\kappa)\,d\kappa\right|$$
$$\leq\frac{1}{2880}\left\|\mathcal{F}^{(4)}\right\|_\infty(\rho_2-\rho_1)^4.$$

Since the convex theory is an effective way to solve a large number of problems from different branches of mathematics, many authors have studied the results of Simpson type for convex mapping. To be more precise, some inequalities of Simpson type for s-convex functions are proved by using differentiable functions [4]. In the papers [34, 36], the new variants of Simpson type inequalities are established based on differentiable convex mapping. Moreover, some papers were devoted to Simpson type inequalities for various convex classes [11, 18, 27, 30, 31].

The Simpson inequalities for differentiable functions are extended to Riemann–Liouville fractional integrals in the papers [8] and [17]. Hence, several paper focused on fractional Simpson inequalities for various fractional integral operators [1–3, 7, 9, 12, 15, 19, 21, 25, 26, 28, 32, 33, 37, 39]. For further information and unexplained subjects about Simpson type inequalities, we refer the reader to [5, 10, 14, 16, 22–24, 38] and the references therein. Besides, Sarikaya et al. established several Simpson type inequalities for functions whose second derivatives are convex [35].

The purpose of this paper is to extend the results given in [35] for twice differentiable functions to Riemann–Liouville fractional integrals. The general structure of the paper consists of four chapters including an introduction. The remaining part of the paper proceeds as follows: In Sect. 2, after giving a general literature survey and the definition of Riemann–Liouville fractional integral operators, we prove an equality for twice differentiable functions. In the next section, for utilizing this equality, we establish several Simpson type inequalities for a mapping whose second derivatives are convex. In the last section, some conclusions and further directions of research are discussed.

Definition 1 Consider $\mathcal{F} \in L_1[\rho_1, \rho_2]$. The Riemann–Liouville integrals $J^{\alpha}_{\rho_1+}\mathcal{F}$ and $J^{\alpha}_{\rho_2-}\mathcal{F}$ of order $\alpha > 0$ with $\rho_1 \geq 0$ are defined by

$$J^{\alpha}_{\rho_1+}\mathcal{F}(\kappa) = \frac{1}{\Gamma(\alpha)} \int_{\rho_1}^{\kappa} (\kappa - \tau)^{\alpha-1} \mathcal{F}(\tau)\, d\tau, \quad \kappa > \rho_1$$

and

$$J^{\alpha}_{\rho_2-}\mathcal{F}(\kappa) = \frac{1}{\Gamma(\alpha)} \int_{\kappa}^{\rho_2} (\tau - \kappa)^{\alpha-1} \mathcal{F}(\tau)\, d\tau, \quad \kappa < \rho_2,$$

respectively. Here, $\Gamma(\alpha)$ is the gamma function and $J^{0}_{\rho_1+}\mathcal{F}(\kappa) = J^{0}_{\rho_2-}\mathcal{F}(\kappa) = \mathcal{F}(\kappa)$.

For more information and several properties of Riemann–Liouville fractional integrals, please refer to [13, 20, 29].

The first result on fractional Simpson inequality for twice differentiable functions was proved by Budak et al. in [6] as follows.

Theorem 2 *Assume that the assumptions of Lemma 1 hold. Assume also that the mapping $|\mathcal{F}''|$ is convex on $[\rho_1, \rho_2]$. Then we have the following inequality*:

$$\begin{aligned} &\left| \frac{1}{6}\left[\mathcal{F}(\rho_1) + 4\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right) + \mathcal{F}(\rho_2) \right] \right. \\ &\quad \left. - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\rho_2-\rho_1)^{\alpha}} \left[J^{\alpha}_{(\frac{\rho_1+\rho_2}{2})+}\mathcal{F}(\rho_2) + J^{\alpha}_{(\frac{\rho_1+\rho_2}{2})-}\mathcal{F}(\rho_1) \right] \right| \\ &\quad \leq \frac{(\rho_2-\rho_1)^2}{6} \mathcal{A}(\alpha) \left[\left|\mathcal{F}''(\rho_1)\right| + \left|\mathcal{F}''(\rho_2)\right| \right], \end{aligned} \tag{1.1}$$

where

$$\mathcal{A}(\alpha) = \frac{1}{4(\alpha+2)} \left(\alpha \left(\frac{\alpha+1}{3}\right)^{\frac{2}{\alpha}} + \frac{3}{\alpha+1} \right) - \frac{1}{8}. \tag{1.2}$$

In this paper, we prove a new version of inequality (1.1).

2 Some equalities

In this section, we give equalities on twice differentiable functions for using the main results.

Lemma 1 *If $\mathcal{F}:[\rho_1,\rho_2]\to\mathbb{R}$ is an absolutely continuous mapping (ρ_1,ρ_2) such that $\mathcal{F}''\in L_1([\rho_1,\rho_2])$, then the following equality*

$$\begin{aligned}
&\frac{1}{6}\left[\mathcal{F}(\rho_1)+4\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right)+\mathcal{F}(\rho_2)\right]\\
&\quad-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\rho_2-\rho_1)^{\alpha}}\left[J^{\alpha}_{\rho_2-}\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right)+J^{\alpha}_{\rho_1+}\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right)\right]\\
&\quad=\frac{(\rho_2-\rho_1)^2}{8(\alpha+1)}\int_0^1\left(\frac{1-2\alpha}{3}+\frac{2(\alpha+1)}{3}\tau-\tau^{\alpha+1}\right)\\
&\quad\quad\times\left[\mathcal{F}''\left(\frac{1+\tau}{2}\rho_2+\frac{1-\tau}{2}\rho_1\right)+\mathcal{F}''\left(\frac{1+\tau}{2}\rho_1+\frac{1-\tau}{2}\rho_2\right)\right]d\tau
\end{aligned}\tag{2.1}$$

is valid.

Proof By using integration by parts, we obtain

$$\begin{aligned}
\Upsilon_1&=\int_0^1\left(\frac{1-2\alpha}{3}+\frac{2(\alpha+1)}{3}\tau-\tau^{\alpha+1}\right)\mathcal{F}''\left(\frac{1+\tau}{2}\rho_2+\frac{1-\tau}{2}\rho_1\right)d\tau\\
&=-2\frac{(1-2\alpha)}{3(\rho_2-\rho_1)}\mathcal{F}'\left(\frac{\rho_1+\rho_2}{2}\right)\\
&\quad-\frac{2}{(\rho_2-\rho_1)}\int_0^1\left(\frac{2(\alpha+1)}{3}-(\alpha+1)\tau^{\alpha}\right)\mathcal{F}'\left(\frac{1+\tau}{2}\rho_2+\frac{1-\tau}{2}\rho_1\right)d\tau\\
&=-2\frac{(1-2\alpha)}{3(\rho_2-\rho_1)}\mathcal{F}'\left(\frac{\rho_1+\rho_2}{2}\right)+\frac{4(\alpha+1)}{3(\rho_2-\rho_1)^2}\mathcal{F}(\rho_2)+\frac{8(\alpha+1)}{3(\rho_2-\rho_1)^2}\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right)\\
&\quad-\frac{4\alpha(\alpha+1)}{\rho_2-\rho_1}\int_0^1\tau^{\alpha-1}\mathcal{F}\left(\frac{1+\tau}{2}\rho_2+\frac{1-\tau}{2}\rho_1\right)d\tau.
\end{aligned}\tag{2.2}$$

By using equation (2.2), the change of the variable $\kappa=\frac{1+\tau}{2}\rho_2+\frac{1-\tau}{2}\rho_1$ for $\tau\in[0,1]$ can be rewritten as follows:

$$\begin{aligned}
\Upsilon_1&=-2\frac{(1-2\alpha)}{3(\rho_2-\rho_1)}\mathcal{F}'\left(\frac{\rho_1+\rho_2}{2}\right)+\frac{4(\alpha+1)}{3(\rho_2-\rho_1)^2}\mathcal{F}(\rho_2)+\frac{8(\alpha+1)}{3(\rho_2-\rho_1)^2}\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right)\\
&\quad-\frac{2^{\alpha+2}(\alpha+1)\Gamma(\alpha+1)}{(\rho_2-\rho_1)^{\alpha+2}}J^{\alpha}_{\rho_2-}\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right).
\end{aligned}\tag{2.3}$$

Similarly, we have

$$\begin{aligned}
\Upsilon_2&=\int_0^1\left(\frac{1-2\alpha}{3}+\frac{2(\alpha+1)}{3}\tau-\tau^{\alpha+1}\right)\mathcal{F}''\left(\frac{1+\tau}{2}\rho_1+\frac{1-\tau}{2}\rho_2\right)d\tau\\
&=2\frac{(1-2\alpha)}{3(\rho_2-\rho_1)}\mathcal{F}'\left(\frac{\rho_1+\rho_2}{2}\right)+\frac{4(\alpha+1)}{3(\rho_2-\rho_1)^2}\mathcal{F}(\rho_1)+\frac{8(\alpha+1)}{3(\rho_2-\rho_1)^2}\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right)\\
&\quad-\frac{2^{\alpha+2}(\alpha+1)\Gamma(\alpha+1)}{(\rho_2-\rho_1)^{\alpha+2}}J^{\alpha}_{\rho_1+}\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right).
\end{aligned}\tag{2.4}$$

From equations (2.3) and (2.4), we get

$$\Upsilon_1+\Upsilon_2=\frac{4(\alpha+1)}{3(\rho_2-\rho_1)^2}\left[\mathcal{F}(\rho_1)+4\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right)+\mathcal{F}(\rho_2)\right] \\ -\frac{2^{\alpha+2}(\alpha+1)\Gamma(\alpha+1)}{(\rho_2-\rho_1)^{\alpha+2}}\left[J^{\alpha}_{\rho_2-}\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right)+J^{\alpha}_{\rho_1+}\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right)\right]. \tag{2.5}$$

Multiplying both sides of (2.5) by $\frac{(\rho_2-\rho_1)^2}{8(\alpha+1)}$, we obtain equation (2.1). This ends the proof of Lemma 1. □

Lemma 2 *Let us consider the function* $\varpi:[0,1]\to\mathbb{R}$ *by* $\varpi(\tau)=\frac{1-2\alpha}{3}+\frac{2(\alpha+1)}{3}\tau-\tau^{\alpha+1}$ *with* $\alpha>0$.

(1) *If* $0<\alpha\le\frac{1}{2}$, *then we have*

$$\int_0^1\left|\varpi(\tau)\right|d\tau=\frac{1-\alpha^2}{3(\alpha+2)}.$$

(2) *If* $\alpha>\frac{1}{2}$, *then there exists a real number* ς_α *such that* $0<\varsigma_\alpha<1$, *and we have*

$$\int_0^1\left|\varpi(\tau)\right|d\tau=2\left(\frac{(\varsigma_\alpha)^{\alpha+2}}{\alpha+2}-\frac{(1-2\alpha)\varsigma_\alpha+(\alpha+1)(\varsigma_\alpha)^2}{3}\right)+\frac{1-\alpha^2}{3(\alpha+2)}.$$

Proof Let us note that $0<\alpha\le\frac{1}{2}$. Then $\varpi(\tau)\ge 0$ for all $\tau\in[0,1]$. Thus, it can be easily seen that

$$\int_0^1\left|\varpi(\tau)\right|d\tau=\int_0^1\varpi(\tau)\,d\tau=\frac{1-\alpha^2}{3(\alpha+2)}.$$

If $\alpha>\frac{1}{2}$, then there exists a real number $\varsigma_\alpha\in(0,1)$ such that $\varpi(\tau)\le 0$ for $0\le\tau\le\varsigma_\alpha$ and $\varpi(\tau)\ge 0$ for $\varsigma_\alpha\le\tau\le 1$. Therefore, we obtain

$$\begin{aligned}\int_0^1\left|\varpi(\tau)\right|d\tau&=\int_0^{\varsigma_\alpha}\left(-\varpi(\tau)\right)d\tau+\int_{\varsigma_\alpha}^1\varpi(\tau)\,d\tau\\&=2\left(\frac{(\varsigma_\alpha)^{\alpha+2}}{\alpha+2}-\frac{(1-2\alpha)\varsigma_\alpha+(\alpha+1)(\varsigma_\alpha)^2}{3}\right)+\frac{1-\alpha^2}{3(\alpha+2)}.\end{aligned}$$ □

Lemma 3 *Define the function* $\varpi:[0,1]\to\mathbb{R}$ *by* $\varpi(\tau)=\frac{1-2\alpha}{3}+\frac{2(\alpha+1)}{3}\tau-\tau^{\alpha+1}$ *with* $\alpha>0$.

(1) *Let us consider* $0<\alpha\le\frac{1}{2}$. *Then we have*

$$\int_0^1\left|\varpi(\tau)\right|\tau\,d\tau=\frac{3-\alpha-2\alpha^2}{18(\alpha+3)}.$$

(2) *If we take* $\alpha>\frac{1}{2}$, *then there exists a real number* ς_α *so that* $0<\varsigma_\alpha<1$, *and we get*

$$\int_0^1\left|\varpi(\tau)\right|\tau\,d\tau=2\left(\frac{(\varsigma_\alpha)^{\alpha+3}}{\alpha+3}-\frac{3(1-2\alpha)(\varsigma_\alpha)^2+4(\alpha+1)(\varsigma_\alpha)^3}{18}\right)+\frac{3+\alpha-2\alpha^2}{18(\alpha+3)}.$$

Proof The proof can be done similar to the proof of Lemma 2. □

3 New Simpson type inequalities for twice differentiable functions

In this section, we prove several Simpson type inequalities for a mapping whose second derivatives are convex.

Theorem 3 *Let us note that the assumptions of Lemma 1 are valid. Let us also note that the mapping $|\mathcal{F}''|$ is convex on $[\rho_1,\rho_2]$. Then we have the following inequality*:

$$\begin{aligned}&\left|\frac{1}{6}\left[\mathcal{F}(\rho_1)+4\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right)+\mathcal{F}(\rho_2)\right]\right.\\&\left.\quad-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\rho_2-\rho_1)^\alpha}\left[J^\alpha_{\rho_2-}\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right)+J^\alpha_{\rho_1+}\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right)\right]\right|\\&\quad\le\frac{(\rho_2-\rho_1)^2}{8(\alpha+1)}\Omega_1(\alpha)\left[\left|\mathcal{F}''(\rho_1)\right|+\left|\mathcal{F}''(\rho_2)\right|\right],\end{aligned}$$

where Ω_1 is defined by

$$\Omega_1(\alpha)=\begin{cases}\frac{1-\alpha^2}{3(\alpha+2)}, & if\, 0<\alpha\le\frac{1}{2},\\ 2\left(\frac{(\varsigma_\alpha)^{\alpha+2}}{\alpha+2}-\frac{(1-2\alpha)\varsigma_\alpha+(\alpha+1)(\varsigma_\alpha)^2}{3}\right)+\frac{1-\alpha^2}{3(\alpha+2)}, & if\, \alpha>\frac{1}{2}.\end{cases}$$

Proof By taking modulus in Lemma 1, we have

$$\begin{aligned}&\left|\frac{1}{6}\left[\mathcal{F}(\rho_1)+4\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right)+\mathcal{F}(\rho_2)\right]\right.\\&\left.\quad-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\rho_2-\rho_1)^\alpha}\left[J^\alpha_{\rho_2-}\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right)+J^\alpha_{\rho_1+}\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right)\right]\right|\\&\quad\le\frac{(\rho_2-\rho_1)^2}{8(\alpha+1)}\int_0^1\left|\frac{1-2\alpha}{3}+\frac{2(\alpha+1)}{3}\tau-\tau^{\alpha+1}\right|d\tau\\&\qquad\times\left[\left|\mathcal{F}''\left(\frac{1+\tau}{2}\rho_2+\frac{1-\tau}{2}\rho_1\right)\right|+\left|\mathcal{F}''\left(\frac{1+\tau}{2}\rho_1+\frac{1-\tau}{2}\rho_2\right)\right|\right]d\tau.\end{aligned}\tag{3.1}$$

By using the convexity of $|\mathcal{F}''|$, we obtain

$$\begin{aligned}&\left|\frac{1}{6}\left[\mathcal{F}(\rho_1)+4\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right)+\mathcal{F}(\rho_2)\right]\right.\\&\left.\quad-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\rho_2-\rho_1)^\alpha}\left[J^\alpha_{\rho_2-}\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right)+J^\alpha_{\rho_1+}\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right)\right]\right|\\&\quad\le\frac{(\rho_2-\rho_1)^2}{8(\alpha+1)}\left[\int_0^1\left|\frac{1-2\alpha}{3}+\frac{2(\alpha+1)}{3}\tau-\tau^{\alpha+1}\right|\right.\\&\qquad\times\left[\left(\frac{1+\tau}{2}\right)\left|\mathcal{F}''(\rho_2)\right|+\left(\frac{1-\tau}{2}\right)\left|\mathcal{F}''(\rho_1)\right|\right]d\tau\\&\qquad\left.+\left(\frac{1+\tau}{2}\right)\left|\mathcal{F}''(\rho_1)\right|+\left(\frac{1-\tau}{2}\right)\left|\mathcal{F}''(\rho_2)\right|\right]d\tau\end{aligned}$$

$$
\begin{aligned}
&= \frac{(\rho_2-\rho_1)^2}{8(\alpha+1)} \int_0^1 \left| \frac{1-2\alpha}{3} + \frac{2(\alpha+1)}{3}\tau - \tau^{\alpha+1} \right| d\tau \bigl[\left|\mathcal{F}''(\rho_2)\right| + \left|\mathcal{F}''(\rho_1)\right| \bigr] \\
&= \frac{(\rho_2-\rho_1)^2}{8(\alpha+1)} \Omega_1(\alpha) \bigl[\left|\mathcal{F}''(\rho_1)\right| + \left|\mathcal{F}''(\rho_2)\right| \bigr].
\end{aligned}
$$

This completes the proof of Theorem 3. □

Theorem 4 *Let us consider that the assumptions of Lemma 1 hold. If the mapping $|\mathcal{F}''|^q$, $q>1$ is convex on $[\rho_1,\rho_2]$, then we have the following inequality*:

$$
\begin{aligned}
&\left| \frac{1}{6}\left[\mathcal{F}(\rho_1) + 4\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right) + \mathcal{F}(\rho_2) \right] \right. \\
&\quad \left. - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\rho_2-\rho_1)^\alpha} \left[J^\alpha_{\rho_2-}\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right) + J^\alpha_{\rho_1+}\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right) \right] \right| \\
&\le \frac{(\rho_2-\rho_1)^2}{8(\alpha+1)} \Psi(\alpha,p) \bigl[\left|\mathcal{F}''(\rho_1)\right|^q + \left|\mathcal{F}''(\rho_2)\right|^q \bigr]^{\frac{1}{q}}.
\end{aligned}
$$

Here, $\frac{1}{p}+\frac{1}{q}=1$ *and* Ψ *is defined by*

$$
\Psi(\alpha,p) = \left(\int_0^1 \left| \frac{1-2\alpha}{3} + \frac{2(\alpha+1)}{3}\tau - \tau^{\alpha+1} \right|^p d\tau \right)^{\frac{1}{p}}.
$$

Proof With the help of Hölder's inequality in inequality (3.1), we get

$$
\begin{aligned}
&\left| \frac{1}{6}\left[\mathcal{F}(\rho_1) + 4\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right) + \mathcal{F}(\rho_2) \right] \right. \\
&\quad \left. - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\rho_2-\rho_1)^\alpha} \left[J^\alpha_{\rho_2-}\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right) + J^\alpha_{\rho_1+}\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right) \right] \right| \\
&\le \frac{(\rho_2-\rho_1)^2}{8(\alpha+1)} \left\{ \left(\int_0^1 \left| \frac{1-2\alpha}{3} + \frac{2(\alpha+1)}{3}\tau - \tau^{\alpha+1} \right|^p d\tau \right)^{\frac{1}{p}} \right. \\
&\quad \times \left(\int_0^1 \left| \mathcal{F}''\left(\frac{1+\tau}{2}\rho_2 + \frac{1-\tau}{2}\rho_1 \right) \right|^q d\tau \right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 \left| \frac{1-2\alpha}{3} + \frac{2(\alpha+1)}{3}\tau - \tau^{\alpha+1} \right|^p d\tau \right)^{\frac{1}{p}} \\
&\quad \left. \times \left(\int_0^1 \left| \mathcal{F}''\left(\frac{1+\tau}{2}\rho_1 + \frac{1-\tau}{2}\rho_2 \right) \right|^q d\tau \right)^{\frac{1}{q}} \right\}.
\end{aligned}
$$

By using the convexity of $|\mathcal{F}''|^q$, we obtain

$$
\begin{aligned}
&\left| \frac{1}{6}\left[\mathcal{F}(\rho_1) + 4\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right) + \mathcal{F}(\rho_2) \right] \right. \\
&\quad \left. - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\rho_2-\rho_1)^\alpha} \left[J^\alpha_{\rho_2-}\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right) + J^\alpha_{\rho_1+}\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right) \right] \right| \\
&\le \frac{(\rho_2-\rho_1)^2}{8(\alpha+1)} \left(\int_0^1 \left| \frac{1-2\alpha}{3} + \frac{2(\alpha+1)}{3}\tau - \tau^{\alpha+1} \right|^p d\tau \right)^{\frac{1}{p}}
\end{aligned}
$$

$$\times\left[\left(\int_0^1\left[\left(\frac{1+\tau}{2}\right)\left|\mathcal{F}''(\rho_2)\right|^q+\left(\frac{1-\tau}{2}\right)\left|\mathcal{F}''(\rho_1)\right|^q\right]d\tau\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left[\left(\frac{1+\tau}{2}\right)\left|\mathcal{F}''(\rho_1)\right|^q+\left(\frac{1-\tau}{2}\right)\left|\mathcal{F}''(\rho_2)\right|^q\right]d\tau\right)^{\frac{1}{q}}\right]$$
$$=\frac{(\rho_2-\rho_1)^2}{8(\alpha+1)}\left(\int_0^1\left|\frac{1-2\alpha}{3}+\frac{2(\alpha+1)}{3}\tau-\tau^{\alpha+1}\right|^p d\tau\right)^{\frac{1}{p}}$$
$$\times\left[\left(\frac{3|\mathcal{F}''(\rho_2)|^q+|\mathcal{F}''(\rho_1)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{F}''(\rho_2)|^q+3|\mathcal{F}''(\rho_1)|^q}{4}\right)^{\frac{1}{q}}\right].$$

This finishes the proof of Theorem 4. □

Theorem 5 *Suppose that the assumptions of Lemma 1 hold. If the mapping $|\mathcal{F}''|^q$, $q\geq 1$ is convex on $[\rho_1,\rho_2]$, then we have the following inequality*:

$$\left|\frac{1}{6}\left[\mathcal{F}(\rho_1)+4\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right)+\mathcal{F}(\rho_2)\right]\right.$$
$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\rho_2-\rho_1)^\alpha}\left[J^\alpha_{\rho_2-}\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right)+J^\alpha_{\rho_1+}\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right)\right]\right|$$
$$\leq\frac{(\rho_2-\rho_1)^2}{8(\alpha+1)}\left(\Omega_1(\alpha)\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left(\frac{(\Omega_1(\alpha)+\Omega_2(\alpha))|\mathcal{F}''(\rho_2)|^q+(\Omega_1(\alpha)-\Omega_2(\alpha))|\mathcal{F}''(\rho_1)|^q}{2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{(\Omega_1(\alpha)+\Omega_2(\alpha))|\mathcal{F}''(\rho_1)|^q+(\Omega_1(\alpha)-\Omega_2(\alpha))|\mathcal{F}''(\rho_2)|^q}{2}\right)^{\frac{1}{q}}\right\},$$

where Ω_1 is defined as in Theorem 3 and Ω_2 is defined by

$$\Omega_2(\alpha)=\begin{cases}\frac{3-\alpha-2\alpha^2}{18(\alpha+3)}, & \text{if } 0<\alpha\leq\frac{1}{2},\\ 2\left(\frac{(\varsigma_\alpha)^{\alpha+3}}{\alpha+3}-\frac{3(1-2\alpha)(\varsigma_\alpha)^2+4(\alpha+1)(\varsigma_\alpha)^3}{18}\right)+\frac{3+\alpha-2\alpha^2}{18(\alpha+3)}, & \text{if } \alpha>\frac{1}{2}.\end{cases}$$

Proof By applying the power-mean inequality in (3.1), we get

$$\left|\frac{1}{6}\left[\mathcal{F}(\rho_1)+4\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right)+\mathcal{F}(\rho_2)\right]\right. \tag{3.2}$$
$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\rho_2-\rho_1)^\alpha}\left[J^\alpha_{\rho_2-}\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right)+J^\alpha_{\rho_1+}\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right)\right]\right|$$
$$\leq\frac{(\rho_2-\rho_1)^2}{8(\alpha+1)}\left[\left(\int_0^1\left|\frac{1-2\alpha}{3}+\frac{2(\alpha+1)}{3}\tau-\tau^{\alpha+1}\right|d\tau\right)^{1-\frac{1}{q}}\right.$$
$$\times\left(\int_0^1\left|\frac{1-2\alpha}{3}+\frac{2(\alpha+1)}{3}\tau-\tau^{\alpha+1}\right|\left|\mathcal{F}''\left(\frac{1+\tau}{2}\rho_2+\frac{1-\tau}{2}\rho_1\right)\right|^q d\tau\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1\left|\frac{1-2\alpha}{3}+\frac{2(\alpha+1)}{3}\tau-\tau^{\alpha+1}\right|d\tau\right)^{1-\frac{1}{q}}$$
$$\left.\times\left(\int_0^1\left|\frac{1-2\alpha}{3}+\frac{2(\alpha+1)}{3}\tau-\tau^{\alpha+1}\right|\left|\mathcal{F}''\left(\frac{1+\tau}{2}\rho_1+\frac{1-\tau}{2}\rho_2\right)\right|^q d\tau\right)^{\frac{1}{q}}\right].$$

Since $|\mathcal{F}''|^q$ is convex, we obtain

$$\begin{aligned}
&\int_0^1 \left|\frac{1-2\alpha}{3}+\frac{2(\alpha+1)}{3}\tau-\tau^{\alpha+1}\right|\left|\mathcal{F}''\left(\frac{1+\tau}{2}\rho_2+\frac{1-\tau}{2}\rho_1\right)\right|^q d\tau \\
&\quad\leq \int_0^1 \left|\frac{1-2\alpha}{3}+\frac{2(\alpha+1)}{3}\tau-\tau^{\alpha+1}\right|\left[\frac{1+\tau}{2}\left|\mathcal{F}''(\rho_2)\right|^q+\frac{1-\tau}{2}\left|\mathcal{F}''(\rho_1)\right|^q\right] d\tau \\
&\quad= \frac{(\Omega_1(\alpha)+\Omega_2(\alpha))|\mathcal{F}''(\rho_2)|^q+(\Omega_1(\alpha)-\Omega_2(\alpha))|\mathcal{F}''(\rho_1)|^q}{2},
\end{aligned}$$

and similarly

$$\begin{aligned}
&\int_0^1 \left|\frac{1-2\alpha}{3}+\frac{2(\alpha+1)}{3}\tau-\tau^{\alpha+1}\right|\left|\mathcal{F}''\left(\frac{1+\tau}{2}\rho_1+\frac{1-\tau}{2}\rho_2\right)\right|^q d\tau \\
&\quad\leq \frac{(\Omega_1(\alpha)+\Omega_2(\alpha))|\mathcal{F}''(\rho_1)|^q+(\Omega_1(\alpha)-\Omega_2(\alpha))|\mathcal{F}''(\rho_2)|^q}{2}.
\end{aligned}$$

Then we obtain the desired result Theorem 5. □

4 Special cases

In this section, we present special cases of the main findings in the paper.

Remark 1 If we choose $\alpha = 1$ in Theorem 3, then $\varsigma_\alpha = \frac{1}{3}$, and we have the inequality

$$\begin{aligned}
&\left|\frac{1}{6}\left[\mathcal{F}(\rho_1)+4\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right)+\mathcal{F}(\rho_2)\right]-\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\mathcal{F}(\kappa)\,d\kappa\right| \\
&\quad\leq \frac{(\rho_2-\rho_1)^2}{162}\left[\left|\mathcal{F}''(\rho_1)\right|+\left|\mathcal{F}''(\rho_2)\right|\right],
\end{aligned}$$

which is proved by Sarikaya et al. in [35].

Corollary 1 *In Theorem 4, if we assign $\alpha = 1$, then $\varsigma_\alpha = \frac{1}{3}$, and the following inequality*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\mathcal{F}(\rho_1)+4\mathcal{F}\left(\frac{\rho_1+\rho_2}{2}\right)+\mathcal{F}(\rho_2)\right]-\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\mathcal{F}(\kappa)\,d\kappa\right| \\
&\quad\leq \frac{(\rho_2-\rho_1)^2}{16}\Psi(1,p)\left[\left|\mathcal{F}''(\rho_1)\right|^q+\left|\mathcal{F}''(\rho_2)\right|^q\right]^{\frac{1}{q}}
\end{aligned}$$

is valid.

Remark 2 If we take $\alpha = 1$ in Theorem 5, then Theorem 5 reduces to [35, Theorem 2.5].

5 Conclusion

In the present article, fractional version of Simpson type inequality for twice differentiable functions are established. Moreover, we show that our results generalize the inequalities obtained by Sarikaya et al. [35]. This work is a new version of fractional Simpson type inequalities for twice differentiable functions. In the future studies, authors can try to generalize our results by utilizing a different kind of convex function classes or another type fractional integral operators. In addition to this, the authors can give some applications of special cases with the help of our results.

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Declarations

Competing interests
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