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Two hybrid and non-hybrid k-dimensional inclusion systems via sequential fractional derivatives

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Abstract

Some complicated events can be modeled by systems of differential equations. On the other hand, inclusion systems can describe complex phenomena having some shocks better than the system of differential equations. Also, one of the interests of researchers in this field is an investigation of hybrid systems. In this paper, we study the existence of solutions for hybrid and non-hybrid k-dimensional sequential inclusion systems by considering some integral boundary conditions. In this way, we use different methods such as α-ψ contractions and the endpoint technique. Finally, we present two examples to illustrate our main results.

MSC: Primary 34A08; secondary 34A12

Keywords: α-ψ-contraction; Endpoint; Inclusion system; Sequential hybrid inclusion problem; The Caputo derivative

1 Introduction

Nowadays, the need for fractional calculations and differential inclusions to describe the relationships between different phenomena has doubled because the main tool for modern modeling of different events is inclusion technique. In addition, the use of fractional calculus in various sciences such as computer, physics, electronics, etc. is not hidden from anyone. In recent years, many researchers have become interested in the field of fractional calculus. There are various techniques in this way such as fixed point theory to investigate the existence of solutions for fractional differential equations (see, for example, [1–10]). One can find different techniques or ideas in [11–13] or many applied ideas in this area (see, for example, [14–16]). We ask the reader to focus on the works [17–19] to find appropriate ideas for their research.

In 1993, Miller and Ross defined sequential derivatives as combinations of derivative operators [20]. After that time, some researchers always sought to discover the relationship between sequential derivatives and fractional derivatives [21–23]. The efforts of researchers led to the publication of several articles on the issue of boundary value problems of consecutive fractional derivatives (see, for example, [24–30]).

In 2015, Alsaedi et al. reviewed the sequential differential problem

$$\begin{cases} ({}^cD^{\varrho^*} + t\,{}^cD^{\varrho^*-1})j(r) = h(r, j(r)), \quad r \in [0,1], \\ j(0) = 0, \quad j'(0) = 0, \quad j(\lambda) = \frac{a}{\Gamma(\beta)} \int_0^{\hbar} (\hbar - s)^{\beta-1} j(s)\,\mathrm{d}s, \end{cases}$$

where $0 < \hbar < \lambda < 1$, $2 < \varrho^* \leq 3$, $a, t \in \mathbb{R}^+$, and ${}^cD^{\varrho^*}$ denotes the standard Caputo derivative of fractional order ϱ^*, and $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function [31]. Also, some researchers investigated hybrid differential problems with different boundary conditions (see, for example, [32]). In 2010, Dhage and Lakshmikantham started working on hybrid equations [33, 34]. In 2011, Zhao et al. defined studied hybrid differential equations [35]. In 2016, Ahmed et al. reviewed the existence of solution for the fractional inclusion differential equation

$${}^cD^\alpha \left[\frac{q(r) - \sum_{j=1}^k {}^RI^{\varpi_j} h_j(r, q(r))}{h(r, q(r))} \right] \in \mathcal{H}\big(r, q(r)\big) \quad \big(r \in \mathcal{I} = [0,1]\big)$$

with the boundary conditions $q(0) = \beta(s)$ and $q(1) = \varrho \in \mathbb{R}$, where ${}^cD^\alpha$ is the Caputo derivative of order $\alpha \in (1,2]$ and ${}^RI^\varpi$ is the Riemann–Liouville integral of order $\varpi > 0$ so that $\varpi \in \{\varpi_1, \varpi_2, \ldots, \varpi_k\}$, $\varpi_j > 0$, $h_j \in C(\mathcal{I} \times \mathbb{R}, \mathbb{R})$, $j = 1, 2, \ldots, k$, $h \in C(\mathcal{I} \times \mathbb{R}, \mathbb{R} - \{0\})$, $\beta : C(\mathcal{I}, \mathbb{R}) \to \mathbb{R}$, and $\mathcal{H} : \mathcal{I} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multifunction, $\mathcal{P}(\mathbb{R})$ is the set of all subsets of $\mathbb{R}$ [36]. In 2020, Baleanu et al. examined the hybrid inclusion model of the thermostat problem

$${}^cD^\alpha \left[\frac{y(r)}{\rho(r, y(r))} \right] \in \kappa\big(r, y(r)\big) \quad \big(r \in [0,1]\big)$$

with the hybrid boundary conditions

$$\begin{cases} \mathcal{D}[\frac{y(r)}{\rho(r,y(r))}]|_{r=0} = 0, \\ \lambda\,{}^cD^\alpha [\frac{y(r)}{\rho(r,y(r))}]|_{r=1} + [\frac{y(r)}{\rho(r,y(r))}]|_{r=\eta} = 0, \end{cases}$$

where $\alpha \in (2,3]$, $\lambda > 0$, $\eta \in [0,1]$, $\kappa : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $\rho \in \mathcal{C}([0,1] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $D = \frac{\mathrm{d}}{\mathrm{d}r}$. They also studied the thermostatic model

$${}^cD^\zeta \left[\frac{y(r)}{h(r, y(r))} \right] + \Phi\big(r, y(r)\big) = 0 \quad \big(\zeta \in (1,2], r \in [0,1]\big)$$

with hybrid boundary conditions

$$\begin{cases} D[\frac{y(r)}{h(r,y(r))}]|_{r=0} = 0, \\ \xi\,{}^cD^{\zeta-1} [\frac{y(r)}{h(r,y(r))}]|_{r=1} + [\frac{y(r)}{h(r,y(r))}]|_{r=\lambda} = 0, \end{cases}$$

where $\xi > 0$ is a parameter, $\lambda \in [0,1]$, $\zeta - 1 \in (0,1]$, $D = \frac{\mathrm{d}}{\mathrm{d}r}$, ${}^cD^\mu$ is the Caputo derivative of fractional order $\mu \in \{\zeta, \zeta - 1\}$, the function $\Phi : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, and $h \in \mathcal{C}([0,1] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ [37].

By using and mixing the main idea of the works, we first review the k-dimensional hybrid system of fractional differential inclusions

$$\begin{cases} d_{1_1}({}^cD^{\alpha} + d_{2_1}{}^cD^{\alpha-1})[\frac{q_1(t)}{\varrho_1(t,q_1(t),{}^RI^{\rho}q_1(t))}] \in \mathcal{S}_1(t,q_1(t),\ldots,q_k(t),q_1'(t),\ldots,q_k'(t)),\\ d_{1_2}({}^cD^{\alpha} + d_{2_2}{}^cD^{\alpha-1})[\frac{q_2(t)}{\varrho_2(t,q_2(t),{}^RI^{\rho}q_2(t))}] \in \mathcal{S}_2(t,q_1(t),\ldots,q_k(t),q_1'(t),\ldots,q_k'(t)),\\ \vdots\\ d_{1_k}({}^cD^{\alpha} + d_{2_k}{}^cD^{\alpha-1})[\frac{q_k(t)}{\varrho_k(t,q_k(t),{}^RI^{\rho}q_k(t))}] \in \mathcal{S}_k(t,q_1(t),\ldots,q_k(t),q_1'(t),\ldots,q_k'(t)), \end{cases} \tag{1}$$

with three-point hybrid boundary conditions

$$\begin{cases} [\frac{q_i(t)}{\varrho_i(t,q_i(t),{}^RI^{\rho}q_i(t))}]|_{t=0} = 0,\\ {}^cD^1[\frac{q_i(t)}{\varrho_i(t,q_i(t),{}^RI^{\rho}q_i(t))}]|_{t=0} + {}^cD^2[\frac{q_i(t)}{\varrho_i(t,q_i(t),{}^RI^{\rho}q_i(t))}]|_{t=0} = 0,\\ [\frac{q_i(t)}{\varrho_i(t,q_i(t),{}^RI^{\rho}q_i(t))}]|_{t=1} + {}^RI^{\xi}[\frac{q_i(t)}{\varrho_i(t,q_i(t),{}^RI^{\rho}q_i(t))}]|_{t=p} = 0, \end{cases} \tag{2}$$

where $1 \le i \le k$, $t \in [0,1]$, $\alpha \in (2,3]$, $p \in (0,1)$, $d_{1_1},\ldots,d_{1_k},d_{2_1},\ldots,d_{2_k},\rho,\xi > 0$, ${}^cD$ and ${}^RI$ denote the Caputo fractional derivative and the Riemann–Liouville fractional integral, respectively. The nonzero continuous function ϱ_i is defined as $\varrho_i : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $\mathcal{S}_i : [0,1] \times \mathbb{R}^{2k} \to \mathcal{P}(\mathbb{R})$ is a multifunction such that $\mathcal{P}(\mathbb{R})$ is the set of all subsets of $\mathbb{R}$ and any q_i is a real-valued continuous function.

2 Preliminaries

Suppose that $\alpha > 0$, $\alpha \in (k-1,k)$ and $k = [\alpha] + 1$. The Riemann–Liouville integral for a function $q : [0,+\infty) \to \mathbb{R}$ is defined by ${}^RI^{\alpha}q(t) = \int_0^t \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)} q(r)\,\mathrm{d}r$, whenever the integral exists [38, 39]. If $q \in \mathcal{AC}_{\mathbb{R}}^{(k)}([0,+\infty))$, the fractional Caputo derivative is defined by ${}^cD^{\alpha}q(t) = \int_0^t \frac{(t-r)^{k-\alpha-1}}{\Gamma(n-\alpha)} q^{(k)}(r)\,\mathrm{d}r$ provided that the integral is finite-valued [38, 39]. Moreover, for a sufficiently smooth function $q : [0,+\infty) \to \mathbb{R}$, the sequential fractional derivative is defined by

$$D^{\alpha}q(t) = \big(D^{\alpha_1}D^{\alpha_2}\cdots D^{\alpha_k}\big)q(t),$$

where $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_k)$ is a multi-index [20]. Note that the sequential derivative operator D^{α} can be Riemann–Liouville, Caputo, Hadamard, Caputo–Hadamard, or any other version of derivative operators in general. In this research, we employ the sequential derivative of Caputo type which is defined as follows. For $k-1 < \alpha < k$, the Caputo sequential fractional derivative for a sufficiently smooth function $q : [0,+\infty) \to \mathbb{R}$ is given by

$${}^cD^{\alpha}q(t) = D^{-(k-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k q(t),$$

where $D^{-(k-\alpha)}q(t) = {}^RI^{(k-\alpha)}q(t)$ is the Riemann–Liouville fractional integral of order $k-\alpha$ [38]. It has been verified that the general solution for the homogeneous differential equation ${}^cD^{\alpha}_{0^+}q(t) = 0$ is given by $q(t) = \tilde{b}_0 + \tilde{b}_1t + \tilde{b}_2t^2 + \cdots + \tilde{b}_{k-1}t^{k-1}$ and

$${}^RI^{\alpha}\big({}^cD^{\alpha}q(t)\big) = q(t) + \sum_{h=0}^{k-1}\tilde{b}_ht^h = q(t) + \tilde{b}_0 + \tilde{b}_1t + \tilde{b}_2t^2 + \cdots + \tilde{b}_{k-1}t^{k-1},$$

where $\tilde{b}_0,\ldots,\tilde{b}_{k-1} \in \mathbb{R}$ with $k = [\alpha] + 1$ [20].

Let $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ be a normed space. We use the notations $\mathcal{P}(\mathcal{Q})$, $\mathcal{P}_{cls}(\mathcal{Q})$, $\mathcal{P}_{bnd}(\mathcal{Q})$, $\mathcal{P}_{cmp}(\mathcal{Q})$, and $\mathcal{P}_{cvx}(\mathcal{Q})$ for the sets of all subsets of the space $\mathcal{Q}$, all closed subsets of the space $\mathcal{Q}$, all bounded subsets of the space $\mathcal{Q}$, all compact and all convex subsets of $\mathcal{Q}$, respectively. The element $q^* \in \mathcal{Q}$ is a fixed point for given set-valued map $\mathcal{S} : \mathcal{Q} \to \mathcal{P}(\mathcal{Q})$ whenever $q^* \in \mathcal{S}(q^*)$ [40]. We express the family of all fixed points $\mathcal{S}$ with the symbol $\mathcal{FIX}(\mathcal{S})$ [40]. Let $(\mathcal{Q}, d_{\mathcal{Q}})$ be a metric space, then the Pompeiu–Hausdorff metric $\mathrm{PH}_{d_{\mathcal{Q}}} : \mathcal{P}_{cls}(\mathcal{Q}) \times \mathcal{P}_{cls}(\mathcal{Q}) \to \mathbb{R}^* = \mathbb{R} \cup \{\infty\}$ is defined by $\mathrm{PH}_{d_{\mathcal{Q}}}(B_1, B_2) = \max\{\sup_{b_1 \in B_1} d_{\mathcal{Q}}(b_1, B_2), \sup_{b_2 \in B_2} d_{\mathcal{Q}}(B_1, b_2)\}$, where $d_{\mathcal{Q}}(B_1, b_2) = \inf_{b_1 \in B_1} d_{\mathcal{Q}}(b_1, b_2)$ and $d_{\mathcal{Q}}(b_1, B_2) = \inf_{b_2 \in B_2} d_{\mathcal{Q}}(b_1, b_2)$ [40]. A set-valued map $\mathcal{S} : \mathcal{Q} \to \mathcal{P}_{cls}(\mathcal{Q})$ is Lipschitzian with positive constant $\hat{\kappa}$ if the inequality $\mathrm{PH}_{d_{\mathcal{Q}}}(\mathcal{S}(q), \mathcal{S}(q')) \le \hat{\kappa} d_{\mathcal{Q}}(q, q')$ holds for all $q, q' \in \mathcal{Q}$. A Lipschitz map $\mathcal{S}$ is called contraction if $\hat{\kappa} \in (0, 1)$ [40]. In the sequel, $\mathcal{S}$ is said to be completely continuous if $\mathcal{S}(K)$ is relatively compact for each $K \in \mathcal{P}_{bnd}(\mathcal{Q})$, whereas $\mathcal{S} : [0, 1] \to \mathcal{P}_{cls}(\mathbb{R})$ is called measurable if $t \longmapsto d_{\mathcal{Q}}(\upsilon, \mathcal{S}(t)) = \inf\{|\upsilon - z| : z \in \mathcal{S}(t)\}$ is measurable for any $\upsilon \in \mathbb{R}$ [40, 41]. Also, $\mathcal{S}$ is upper semi-continuous whenever for every $q^* \in \mathcal{Q}$, the set $\mathcal{S}(q^*)$ belongs to $\mathcal{P}_{cls}(\mathcal{Q})$ and also, for each open set $\mathcal{U}$ of $\mathcal{Q}$ containing $\mathcal{S}(q^*)$, there is a neighborhood $\mathcal{O}_0^*$ of q^* provided that $\mathcal{S}(\mathcal{O}_0^*) \subseteq \mathcal{U}$ [40].

We construct the graph of the set-valued map $\mathcal{S} : \mathcal{Q} \to \mathcal{P}_{cls}(\mathcal{Z})$ by $\mathrm{Graph}(\mathcal{S}) = \{(q, z) \in \mathcal{Q} \times \mathcal{Z} : z \in \mathcal{S}(q)\}$. The $\mathrm{Graph}(\mathcal{S})$ is closed whenever for two arbitrary convergent sequences $\{q_n\}_{n \ge 1}$ in $\mathcal{Q}$ and $\{y_n\}_{n \ge 1}$ in $\mathcal{Y}$ with $q_n \to q_0$, $y_n \to y_0$, and $y_n \in \mathcal{S}(q_n)$, then if $n \to \infty$ we have $y_0 \in \mathcal{S}(q_0)$ [40, 41]. In view of [40], it is deduced that if the set-valued map $\mathcal{S} : \mathcal{Q} \to \mathcal{P}_{cls}(\mathcal{Y})$ has an upper semi-continuity property, then $\mathrm{Graph}(\mathcal{S})$ is a closed subset of $\mathcal{Q} \times \mathcal{Y}$. If $\mathcal{S}$ has the complete continuity and closed graph property, then $\mathcal{S}$ is upper semi-continuous [40]. In addition, $\mathcal{S}$ has convex values if $\mathcal{S}(k) \in \mathcal{P}_{cvx}(\mathcal{Q})$ for all $q \in \mathcal{Q}$. Furthermore, a collection of selections of $\mathcal{S}$ at point $q \in \mathcal{C}_{\mathbb{R}}([0, 1])$ is represented by $(\mathcal{SEL})_{\mathcal{S}, q} := \{\hat{u} \in \mathcal{L}^1_{\mathbb{R}}([0, 1]) : \hat{u}(t) \in \mathcal{S}(t, q(t))\}$ for almost all $t \in [0, 1]$ [40, 41].

If we assume that $\mathcal{S}$ is an arbitrary set-valued map, then for each $q \in \mathcal{C}_{\mathcal{Q}}([0, 1])$ we have $(\mathcal{SEL})_{\mathcal{S}, q} \neq \emptyset$ whenever $\dim(\mathcal{Q}) < \infty$ [40]. We say that $\mathcal{S} : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is called Caratheodory if $t \mapsto \mathcal{S}(t, q)$ is measurable for every $q \in \mathbb{R}$ and $q \mapsto \mathcal{S}(t, q)$ is upper semi-continuous for almost all $q \in [0, 1]$ [40, 41]. A Caratheodory set-valued map $\mathcal{S} : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is called $\mathcal{L}^1$-Caratheodory whenever, for each $\vartheta > 0$, there is $\varphi_\vartheta \in \mathcal{L}^1_{\mathbb{R}^+}([0, 1])$ such that $\|\mathcal{S}(t, q)\| = \sup_{t \in [0, 1]}\{|w| : w \in \mathcal{S}(t, q)\} \le \varphi_\vartheta(t)$ for all $|q| \le \vartheta$ and for almost any $t \in [0, 1]$ [40, 41].

In 2012, Samet et al. considered the set of all nonnegative and nondecreasing functions $\phi : [0, \infty) \to [0, \infty)$ with $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ [42]. They denoted it by Φ. One can easily see that $\phi(t) < t$ for all $t > 0$ [42]. Let $\phi \in \Phi$ and $\alpha : \mathcal{Q} \times \mathcal{Q} \to \mathbb{R}$ be a map. A multifunction $\mathcal{S} : \mathcal{Q} \to \mathcal{P}_{cls,bnd}(\mathcal{Q})$ is said to be α-ψ-contraction if $\alpha(q, q')\mathrm{PH}_{d_{\mathcal{Q}}}(\mathcal{S}q, \mathcal{S}q') \le \psi(d_{\mathcal{Q}}(q, q'))$ for all $q, q' \in \mathcal{Q}$ [43]. We say that $\mathcal{Q}$ has the property (E_α) whenever, for any sequence $\{q_n\}$ in $\mathcal{Q}$ with $q_n \to q$ and $\alpha(q_n, q_{n+1}) \ge 1$ for all $n \in \mathbb{N}$, there is a subsequence $\{q_{n_i}\}$ of $\{q_n\}$ such that $\alpha(q_{n_i}, q) \ge 1$ for all $i \in \mathbb{N}$. Also, $\mathcal{S}$ is called α-admissible whenever, for every $q \in \mathcal{Q}$ and $q' \in \mathcal{S}(q)$ with $\alpha(q, q') \ge 1$, we have $\alpha(q', q'') \ge 1$ for all $q'' \in \mathcal{S}(q')$ [43]. Finally, $q \in \mathcal{Q}$ is called the endpoint of $\mathcal{W} : \mathcal{Q} \to \mathcal{P}(\mathcal{Q})$ whenever $\mathcal{W}(q) = \{q\}$ [44]. We say that $\mathcal{W}$ has an approximate endpoint property if $\inf_{q \in \mathcal{Q}} \sup_{z \in \mathcal{W}q} d_{\mathcal{Q}}(q, z) = 0$ [44]. To continue, we will need the following theorems.

Theorem 2.1 ([45]) *Let $\mathcal{Q}$ be a separable Banach space, $\mathcal{G} : [0, 1] \times \mathcal{Q} \to \mathcal{P}_{cmp,cvx}(\mathcal{Q})$ be an $\mathcal{L}^1$-Caratheodory multifunction, and $\Omega : \mathcal{L}^1_{\mathcal{Q}}([0, 1]) \to \mathcal{C}_{\mathcal{Q}}([0, 1])$ be a linear continuous* *map. Then the map* $\Omega \circ (\mathcal{SEL})_{\mathcal{G}} : \mathcal{C}_{\mathcal{Q}}([0,1]) \to \mathcal{P}_{cmp,cvx}(\mathcal{C}_{\mathcal{W}}([0,1]))$ *defined by* $q \mapsto (\Omega \circ (\mathcal{SEL})_{\mathcal{G}})(q) = \Omega((\mathcal{SEL})_{\mathcal{G},q})$ *is an operator in* $\mathcal{C}_{\mathcal{Q}}([0,1]) \times \mathcal{C}_{\mathcal{Q}}([0,1])$ *and has the closed graph property.*

Theorem 2.2 ([33]) *Let* $\mathcal{Q}$ *be a Banach algebra,* $\Psi_1^* : \mathcal{Q} \to \mathcal{Q}$ *be a map, and* $\Psi_2^* : \mathcal{Q} \to \mathcal{P}_{cmp,cvx}(\mathcal{Q})$ *be a multifunction. Assume that*

(i) Ψ_1^* *is Lipschitzian with constant* λ^*;
(ii) Ψ_2^* *is compact and upper semi-continuous*;
(iii) $2\lambda^* \hat{\Theta} < 1$ *provided that* $\hat{\Theta} = \|\Psi_2^*(\mathcal{Q})\|$.

Then either the set $\mathcal{S}^* = \{\nu^* \in \mathcal{Q} | \alpha_0 \nu^* \in (\Psi_1^* \nu^*)(\Psi_2^* \nu^*), \alpha_0 > 1\}$ *is not bounded or there exists* $q \in \mathcal{Q}$ *such that* $q \in (\Psi_1^* q)(\Psi_2^* q)$.

Theorem 2.3 ([33]) *Assume that* $\mathcal{Q}$ *is a Banach space and,* $\mathcal{E}$ *is a closed convex subset of* $\mathcal{Q}$, $\mathcal{V}$ *is an open subset of* $\mathcal{E}$, *and* $0 \in \mathcal{V}$. *Let* $G : \bar{\mathcal{V}} \to \mathcal{P}_{cmp,cvx}(\mathcal{E})$ *be an upper semi-continuous compact map, where* $\mathcal{P}_{cmp,cvx}(\mathcal{E})$ *denotes the family of nonempty, compact convex subsets of* $\mathcal{E}$. *Then either* G *has a fixed point in* $\bar{\mathcal{V}}$ *or there exist* $\nu \in \partial\mathcal{V}$ *and* $\lambda \in (0,1)$ *such that* $\nu \in \lambda G(\nu)$.

Lemma 2.4 ([46]) *Let* $\tilde{l} \in \mathcal{Q}$. *Then* q_0 *is a solution for the fractional differential equation*

$$d_1\left({}^cD^\alpha + d_2{}^cD^{\alpha-1}\right)\left[\frac{q(t)}{\varrho(t, q(t), {}^RI^\gamma q(t))}\right] = \tilde{l}(t) \quad \left(t \in [0,1], \alpha \in (2,3], d_1, d_2 > 0\right) \tag{3}$$

with three-point hybrid integro-derivative boundary conditions

$$\begin{cases} [\frac{q(t)}{\varrho(t,q(t),{}^RI^\gamma q(t))}]|_{t=0} = 0, \\ {}^cD^1[\frac{q(t)}{\varrho(t,q(t),{}^RI^\gamma q(t))}]|_{t=0} + {}^cD^2[\frac{q(t)}{\varrho(t,q(t),{}^RI^\gamma q(t))}]|_{t=0} = 0, \\ [\frac{q(t)}{\varrho(t,q(t),{}^RI^\gamma q(t))}]|_{t=1} + {}^RI^\xi[\frac{q(t)}{\varrho(t,q(t),{}^RI^\gamma q(t))}]|_{t=p} = 0, \end{cases} \tag{4}$$

if and only if q_0 *is a solution for the integral equation*

$$\begin{aligned} q(t) = \varrho\left(t, q(t), {}^RI^\gamma q(t)\right)\Bigg(&\frac{1}{d_1}\int_0^t e^{-d_2(t-\rho)}\int_0^r \frac{(\rho - r)^{\alpha-2}}{\Gamma(\alpha-1)}\tilde{l}(r)\,\mathrm{d}r\,\mathrm{d}\rho \\ &+ \frac{1 - e^{-d_2 t} + (d_2^2 - d_2)t}{d_1(\tilde{\Delta}_2 - d_2\Omega^*)}\Bigg[\int_0^1 e^{-d_2(1-\rho)}\int_0^r \frac{(\rho - r)^{\alpha-2}}{\Gamma(\alpha-1)}\tilde{l}(r)\,\mathrm{d}r\,\mathrm{d}\rho \\ &+ \int_0^p \frac{(p-\rho)^{\chi-1}}{\Gamma(\chi)}\int_0^r e^{-d_2(r-\iota)}\int_0^\iota \frac{(\iota - r)^{\alpha-2}}{\Gamma(\alpha-1)}\tilde{l}(r)\,\mathrm{d}m\,\mathrm{d}\iota\,\mathrm{d}\rho\Bigg]\Bigg), \end{aligned} \tag{5}$$

where $\tilde{\Delta}_1 := 1 - e^{-d_2} + \frac{p^\chi}{\Gamma(\chi+1)} - \int_0^p \frac{(p-\rho)^{\chi-1}}{\Gamma(\chi)}e^{-d_2 r}\,\mathrm{d}r \neq 0$, $\tilde{\Delta}_2 := d_2 - 1 + e^{-d_2} + \frac{d_2 p^{\chi+1}}{\Gamma(\chi+2)} - \frac{p^\chi}{\Gamma(\chi+1)} + \int_0^p \frac{(p-\rho)^{\chi-1}}{\Gamma(\chi)}e^{-p_2\rho}\,\mathrm{d}\rho \neq 0$, *and* $\Omega^* := \tilde{\Delta}_1 + \tilde{\Delta}_2 = d_2(1 + \frac{p^{\chi+1}}{\Gamma(\chi+2)}) \neq 0$.

3 Main results

We are now ready to start an investigation of the k-dimensional hybrid inclusions system (1). We say that $(q_1, q_2, \dots, q_k)$ is a solution for system (1) whenever there exist functions $\{l_1, l_2, \dots, l_k\} \in L^1[0,1]$ such that

$$l_i(s) \in \mathcal{S}_i\left(s, q_1(s), q_2(s), \dots, q_i(s), q_1'(s), q_2'(s), \dots, q_i'(s)\right) \tag{6}$$

for all i and almost all $s \in [0,1]$ and

$$
\begin{aligned}
q_i(s) = \varrho_i\big(s, q_i(s), {}^R I^\gamma q_i(s)\big)\bigg(&\frac{1}{d_{1_i}}\int_0^s e^{-d_{2_i}(s-\rho)}\int_0^\rho \frac{(\rho-r)^{\alpha-2}}{\Gamma(\alpha_i-1)}\tilde{l}_i(r)\,\mathrm{d}m\,\mathrm{d}\rho \\
&+ \frac{1-e^{-d_{2_i}s}+(d_{2_i}^2-d_{2_i})s}{d_{1_i}(\tilde{\Delta}_{2_i}-d_{2_i}\Omega_i^*)}\bigg[\int_0^1 e^{-d_{2_i}(1-\rho)}\int_0^\rho \frac{(\rho-r)^{\alpha-2}}{\Gamma(\alpha_i-1)}\tilde{l}_i(r)\,\mathrm{d}r\,\mathrm{d}r \\
&+ \int_0^p \frac{(p-\rho)^{\iota-1}}{\Gamma(\xi_i)}\int_0^\rho e^{-d_{2_i}(\rho-\iota)}\int_0^\iota \frac{(\iota-r)^{\alpha-2}}{\Gamma(\alpha_i-1)}\tilde{l}_i(r)\,\mathrm{d}r\,\mathrm{d}\iota\,\mathrm{d}\rho\bigg]\bigg)
\end{aligned}
\tag{7}
$$

for all i and $s \in [0,1]$. Here, we have $\alpha \in (2,3]$, $p \in (0,1)$, $d_{1_i}, d_{2_i}, \varrho, \iota > 0$, and ${}^cD$ and ${}^RI$ denote the Caputo fractional derivative and the Riemann–Liouville fractional integral, respectively. Note that ${}^cD_{0^+}^1 = \frac{\mathrm{d}}{\mathrm{d}s}$ and ${}^cD_{0^+}^2 = \frac{\mathrm{d}^2}{\mathrm{d}s^2}$. The nonzero continuous real-valued function α_i is supposed to be defined on $[0,1]\times\mathbb{R}$ and $\mathcal{S}:[0,1]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ for all $i = 1,\dots,k$.

Defined the space $\mathcal{Q}_i = \{s : s, q(s), q'(s) \in \mathcal{C}([0,1],\mathbb{R})\}$ endowed with the norm $\|q\|_{\mathcal{Q}_i} = \sup_{s\in[0,1]}|q_i(s)| + \sup_{s\in[0,1]}|q_i'(s)|$ for all $\{i \in 1,2,\dots,k\}$. Also, define the product space $\mathcal{Q} = Q_1 \times Q_2 \times \cdots \times Q_k$ endowed with the norm $\|(q_1,q_2,\dots,q_k)\| = \sum_{i=1}^k \|q_i\|$. Then $(\mathcal{Q}, \|\cdot\|)$ is a Banach space. Consider the set of the selections

$$
\begin{aligned}
\mathcal{Q}_{A_{i,q}} = \big\{&l \in L^1[0,1] : l(s) \in A_i\big(s, q_1(s),\dots,q_k(s), q_1'(s),\dots,q_k'(s)\big) \\
&\text{for all } q = (q_1,\dots,q_k) \in \mathcal{Q}\big\},
\end{aligned}
$$

where $1 \le i \le k$, and we consider the inclusion for almost all $s \in [0,1]$.

Theorem 3.1 *Suppose that $A_1,\dots,A_k : [0,1]\times\mathbb{R}^{3k} \to \mathcal{P}_{cmp,cvx}(\mathbb{R})$ are Caratheodory multifunctions and there exist a nondecreasing, bounded, and continuous map $\psi : [0,\infty) \to (0,\infty)$ and continuous functions $b_1,\dots,b_k : [0,1] \to (0,\infty)$ such that*

$$
\begin{aligned}
&\big\|A_i\big(s, q_1(s), q_2(s),\dots,q_i(s), q_1'(s), q_2'(s),\dots,q_i'(s)\big)\big\| \\
&\quad = \sup\big\{|z| z \in A_i\big(s, q_1(s), q_2(s),\dots,q_i(s), q_1'(s), q_2'(s),\dots,q_i'(s)\big)\big\} \\
&\quad \le b_i(s)\psi\big(\|q_1, q_2,\dots,q_i\|\big)
\end{aligned}
$$

for all $1 \le i \le k$, $(q_1,\dots,q_k) \in \mathcal{Q}$ and almost all $s \in [0,1]$. Assume that there exist constants L_i such that $\frac{L_i}{M_{i_1}+M_{i_2}} \le 1$, where

$$
\begin{aligned}
M_{i_1} &= \bigg[\frac{(1-e^{-d_2})}{d_1 d_2\Gamma(\alpha_i)} + \frac{|1-e^{-d_2}| + |d_2^2-d_2|}{d_1|\tilde{\Delta}_{2_i} - d_2\Omega_i^*|}\bigg(\frac{(1-e^{-d_2})}{d_2\Gamma(\alpha_i)} + \frac{p^{\alpha_i+\xi_i-1}(d_2p + e^{-d_2p}-1)}{d_2^2\Gamma_i(\alpha_i)\Gamma(\xi_i)}\bigg)\bigg], \\
M_{i_2} &= \bigg[\frac{1}{d_1\Gamma(\alpha_i)} + \frac{|d_2e^{-d_2}| + |d_2^2-d_2|}{d_1|\tilde{\Delta}_{2_i} - d_2\Omega_i^*|}\bigg(\frac{(1-e^{-d_2})}{d_2\Gamma(\alpha_i)} + \frac{p^{\alpha_i+\xi_i-1}(d_2p + e^{-d_2p}-1)}{d_2^2\Gamma_i(\alpha_i)\Gamma(\xi_i)}\bigg)\bigg]
\end{aligned}
$$

and $\|\theta_i\| = \sup_{s\in[0,1]}|\theta_i(s)|$ for all $i = 1,\dots,k$. Then the k-dimensional hybrid inclusions system (1) has at least one solution.

Proof Define the operator $T : \mathcal{Q} \to 2^{\mathcal{Q}}$ by

$$
T(q_1,\dots,q_k) = \big(T_1(q_1,\dots,q_k), T_2(q_1,\dots,q_k),\dots,T_k(q_1,\dots,q_k)\big),
$$

where

$$\begin{aligned}T_i(q_1,\dots,q_k) = \bigg\{ & z \in \mathcal{Q}_i : \text{there exists } l \in \mathcal{Q}_{A_{i,(q_1,\dots,q_k)}} \text{ such that} \\ & z(s) = \alpha\big(s, q(s), {}^R I_{0^+}^{\varrho} q(s)\big)\bigg(\frac{1}{d_1}\int_0^s e^{-d_2(s-r)}\int_0^r \frac{(r-m)^{\alpha-2}}{\Gamma(\alpha-1)}\tilde{l}_i(m)\,\mathrm{d}m\,\mathrm{d}r \\ & + \frac{1-e^{-d_2 s}+(d_2^2-d_2)s}{d_1(\tilde{\Delta}_2 - d_2\Delta^*)}\bigg[\int_0^1 e^{-d_2(1-r)}\int_0^r \frac{(r-m)^{\alpha-2}}{\Gamma(\alpha-1)}\tilde{l}_i(m)\,\mathrm{d}m\,\mathrm{d}r \\ & + \int_0^p \frac{(p-r)^{\xi-1}}{\Gamma(\xi)}\int_0^r e^{-d_2(r-\iota)}\int_0^\iota \frac{(\iota-m)^{\alpha-2}}{\Gamma(\alpha-1)}\tilde{l}_i(m)\,\mathrm{d}m\,\mathrm{d}\iota\,\mathrm{d}r\bigg]\bigg)\bigg\}. \quad (8)\end{aligned}$$

We show that the operator T has a fixed point. Consider the maps $\Psi_{1_i}^* : \mathcal{Q} \to \mathcal{Q}$ and $\Psi_{2_i}^* : \mathcal{Q} \to \mathcal{P}(\mathcal{Q})$ defined by $(\Psi_{1_i}^* q)(s) = \alpha_i(s, q_i(s), {}^R I_{0^+}^{\varrho} q_i(s))$ and

$$\big(\Psi_{2_i}^* q\big)(s) = \big\{ l_i \in \mathcal{Q} : l_i(s) = b_{\hat{\vartheta}_i}(s) \text{ for all } s \in [0,1]\big\},$$

where $b_{\hat{\vartheta}_i} \in T_i(q_1,\dots,q_k)$ and

$$\begin{aligned}b_{\hat{\vartheta}_i}(s) = {} & \frac{1}{d_1}\int_0^s e^{-d_2(s-\rho)}\int_0^\rho \frac{(\rho-r)^{\alpha-2}}{\Gamma(\alpha-1)} b_{\hat{\vartheta}_i(r)}\,\mathrm{d}r\,\mathrm{d}\rho \\ & + \frac{1-e^{-d_2 s}+(d_2^2-d_2)s}{d_1(\tilde{\Delta}_2 - d_2\Delta^*)}\bigg[\int_0^1 e^{-d_2(1-\rho)}\int_0^\rho \frac{(\rho-r)^{\alpha-2}}{\Gamma(\alpha-1)} b_{\hat{\vartheta}_i(r)}\,\mathrm{d}r\,\mathrm{d}\rho \\ & + \int_0^p \frac{(p-\rho)^{\chi-1}}{\Gamma(\chi)}\int_0^\rho e^{-d_2(\rho-\iota)}\int_0^\iota \frac{(\iota-r)^{\alpha-2}}{\Gamma(\alpha-1)} b_{\hat{\vartheta}_i(r)}\,\mathrm{d}r\,\mathrm{d}\iota\,\mathrm{d}\rho\bigg].\end{aligned}$$

Put $\mathcal{G}_i(q) = \Psi_{1_i}^* q \Psi_{2_i}^* q$ for $i = 1,\dots,k$. We show that $\Psi_{1_i}^*$ and $\Psi_{2_i}^*$ satisfy the assumptions of Theorem 2.2 for all i. We first prove that the operator $\Psi_{1_i}^*$ is Lipschitzian on $\mathcal{Q}$. Let $q_1, q_2 \in \mathcal{Q}$. Then

$$\begin{aligned}\big|\big(\Psi_{1_i}^* q_1\big)(s) - \big(\Psi_{1_i}^* q_2\big)(s)\big| &= \big|\alpha_i\big(s, q_1(s), {}^R I_{0^+}^{\varrho} q_1(s)\big) - \alpha_i\big(s, q_2(s), {}^R I_{0^+}^{\varrho} q_2(s)\big)\big| \\ &\le \nu_i(s)\bigg(\big|q_1(s)-q_2(s)\big| + \frac{1}{\Gamma(\varrho_i+1)}\big|q_1(s)-q_2(s)\big|\bigg) \\ &= \nu_i(s)\bigg(1 + \frac{1}{\Gamma(\varrho_i+1)}\bigg)\big|q_1(s)-q_2(s)\big|\end{aligned}$$

for all $s \in [0,1]$. Hence, $\|\Psi_{1_i}^* q_1 - \Psi_{1_i}^* q_2\|_{\mathcal{Q}} \le \nu_i^*(1 + \frac{1}{\Gamma(\varrho_i+1)})\|q_1 - q_2\|_{\mathcal{Q}}$, and so $\Psi_{1_i}^*$ is a Lipschitzian map with constant $\nu_i^*(1 + \frac{1}{\Gamma(\varrho+1)})$. Now we show that $T(q_1, q_2, \dots, q_k)$ is convex for all $(q_1, q_2, \dots, q_k) \in \mathcal{Q}$. Let $(z_1,\dots,z_k), (z_{t_1},\dots,z_{t_k}) \in T(q_1, q_2, \dots, q_k)$. Choose $l_i, l_{t_i} \in \mathcal{Q}_{A_{i,(q_1,q_2,\dots,q_k)}}$ such that

$$\begin{aligned}z_{t_i}(s) = {} & \frac{1}{d_{1_i}}\int_0^s e^{-d_{2_i}(s-\rho)}\int_0^\rho \frac{(\rho-r)^{\alpha-2}}{\Gamma(\alpha_i-1)}\tilde{l}_i(r)\,\mathrm{d}r\,\mathrm{d}\rho \\ & + \frac{1-e^{-d_{2_i}s}+(d_{2_i}^2-d_{2_i})s}{d_{1_i}(\tilde{\Delta}_{2_i} - d_{2_i}\Omega_i^*)}\bigg[\int_0^1 e^{-d_{2_i}(1-\rho)}\int_0^\rho \frac{(\rho-r)^{\alpha-2}}{\Gamma(\alpha_i-1)}\tilde{l}_i(r)\,\mathrm{d}r\,\mathrm{d}\rho \\ & + \int_0^p \frac{(p-\rho)^{\chi-1}}{\Gamma(\chi_i)}\int_0^\rho e^{-d_{2_i}(\rho-\iota)}\int_0^\iota \frac{(\iota-r)^{\alpha-2}}{\Gamma(\alpha_i-1)}\tilde{l}_i(r)\,\mathrm{d}r\,\mathrm{d}\iota\,\mathrm{d}\rho\bigg] \quad (9)\end{aligned}$$

and

$$
\begin{aligned}
z_{t_i}(s) &= \frac{1}{d_{1_i}} \int_0^s e^{-d_{2_i}(s-\rho)} \int_0^\rho \frac{(\rho - r)^{\alpha-2}}{\Gamma(\alpha_i - 1)} \tilde{l}_{t_i}(r)\,\mathrm{d}r\,\mathrm{d}\rho \\
&\quad + \frac{1 - e^{-d_{2_i}s} + (d_{2_i}^2 - d_{2_i})s}{d_{1_i}(\tilde{\Delta}_{2_i} - d_{2_i}\Omega_i^*)} \biggl[\int_0^1 e^{-d_{2_i}(1-\rho)} \int_0^\rho \frac{(\rho - r)^{\alpha-2}}{\Gamma(\alpha_i - 1)} \tilde{l}_{t_i}(r)\,\mathrm{d}r\,\mathrm{d}\rho \\
&\quad + \int_0^p \frac{(p-\rho)^{\chi-1}}{\Gamma(\chi_i)} \int_0^\rho e^{-d_{2_i}(\rho-\iota)} \int_0^\iota \frac{(\iota - r)^{\alpha-2}}{\Gamma(\alpha_i - 1)} \tilde{l}_{t_i}(r)\,\mathrm{d}r\,\mathrm{d}\iota\,\mathrm{d}\rho \biggr]
\end{aligned} \tag{10}
$$

for all $s \in [0,1]$ and $1 \le i \le k$. Let $0 \le h \le 1$. Then we have

$$
\begin{aligned}
&\bigl[hz_i + (1-h)z_{t_i}\bigr](s) \\
&\quad = \frac{1}{d_{1_i}} \int_0^s e^{-d_{2_i}(s-\rho)} \int_0^\rho \frac{(\rho - r)^{\alpha-2}}{\Gamma(\alpha_i - 1)} \bigl[hz_i(r) + (1-h)z_{t_i}(r)\bigr]\,\mathrm{d}r\,\mathrm{d}\rho \\
&\quad + \frac{1 - e^{-d_{2_i}s} + (d_{2_i}^2 - d_{2_i})s}{d_{1_i}(\tilde{\Delta}_{2_i} - d_{2_i}\Omega_i^*)} \biggl[\int_0^1 e^{-d_{2_i}(1-\rho)} \int_0^\rho \frac{(\rho - r)^{\alpha-2}}{\Gamma(\alpha_i - 1)} \bigl[hz_i(r) + (1-h)z_{t_i}(r)\bigr]\,\mathrm{d}r\,\mathrm{d}\rho \\
&\quad + \int_0^p \frac{(p-\rho)^{\xi-1}}{\Gamma(\chi_i)} \int_0^\rho e^{-d_{2_i}(\rho-\iota)} \int_0^\iota \frac{(\iota - r)^{\alpha-2}}{\Gamma(\alpha_i - 1)} \bigl[hz_i(r) + (1-h)z_{t_i}(r)\bigr]\,\mathrm{d}r\,\mathrm{d}\iota\,\mathrm{d}\rho \biggr].
\end{aligned} \tag{11}
$$

Since A_i is convex-valued for all $1 \le i \le k$, $[hz_i + (1-h)z_{t_i}](s) \in T_i(q_1, \dots, q_k)$. Thus,

$$
h(z_1, \dots, z_k) + (1-h)(z_{t_1}, \dots, z_{t_k}) = \bigl(hz_1 + (1-h)z_{t_1}, \dots, hz_k + (1-h)z_{t_k}\bigr) \in T(q_1, \dots, q_k).
$$

Now, we show that T maps bounded sets of $\mathcal{Q}$ into bounded sets. Let $\rho > 0$,

$$
B_\rho = \bigl\{(q_1, \dots, q_k) \in \mathcal{Q} : \bigl\|(q_1, \dots, q_k)\bigr\| \le r\bigr\},
$$

$(q_1, \dots, q_k) \in B_\rho$ and $(z_1, \dots, z_k) \in T(q_1, \dots, q_k)$. Choose

$$
(q_1, \dots, q_k) \in \mathcal{Q}_{A_{1,(q_1,\dots,q_k)}} \times \cdots \times \mathcal{Q}_{A_{k,(q_1,\dots,q_k)}}
$$

such that

$$
\begin{aligned}
z_i(s) &= \frac{1}{d_{1_i}} \int_0^s e^{-d_{2_i}(s-\rho)} \int_0^\rho \frac{(\rho - r)^{\alpha-2}}{\Gamma(\alpha_i - 1)} \tilde{l}_i(r)\,\mathrm{d}r\,\mathrm{d}\rho \\
&\quad + \frac{1 - e^{-d_{2_i}s} + (d_{2_i}^2 - d_{2_i})s}{d_{1_i}(\tilde{\Delta}_{2_i} - d_{2_i}\Omega_i^*)} \biggl[\int_0^1 e^{-d_{2_i}(1-\rho)} \int_0^\rho \frac{(\rho - r)^{\alpha-2}}{\Gamma(\alpha_i - 1)} \tilde{l}_i(r)\,\mathrm{d}r\,\mathrm{d}\rho \\
&\quad + \int_0^p \frac{(p-\rho)^{\chi-1}}{\Gamma(\chi_i)} \int_0^\rho e^{-d_{2_i}(\rho-\iota)} \int_0^\iota \frac{(\iota - r)^{\alpha-2}}{\Gamma(\alpha_i - 1)} \tilde{l}_i(r)\,\mathrm{d}r\,\mathrm{d}\iota\,\mathrm{d}\rho \biggr]
\end{aligned} \tag{12}
$$

for all $s \in [0,1]$ and $1 \le i \le k$. Hence,

$$
\begin{aligned}
z_i'(s) &= \frac{1}{d_{1_i}} \int_0^{\rho} \frac{(\rho - r)^{\alpha-2}}{\Gamma(\alpha_i - 1)} \tilde{l}_i(r)\,\mathrm{d}r \\
&\quad + \frac{d_{2_i} e^{-d_{2_i} s} + (d_{2_i}^2 - d_{2_i})}{d_{1_i}(\tilde{\Delta}_{2_i} - d_{2_i}\Omega_i^*)} \biggl[\int_0^1 e^{-d_{2_i}(1-\rho)} \int_0^{\rho} \frac{(\rho - r)^{\alpha-2}}{\Gamma(\alpha_i - 1)} \tilde{l}_i(r)\,\mathrm{d}r\,\mathrm{d}\rho \\
&\quad + \int_0^p \frac{(p-\rho)^{\xi-1}}{\Gamma(\chi_i)} \int_0^{\rho} e^{-d_{2_i}(\rho-\iota)} \int_0^{\iota} \frac{(\iota - r)^{\alpha-2}}{\Gamma(\alpha_i - 1)} \tilde{l}_i(r)\,\mathrm{d}r\,\mathrm{d}\iota\,\mathrm{d}\rho \biggr],
\end{aligned}
\tag{13}
$$

and so

$$
\begin{aligned}
\bigl|z_i(s)\bigr| &\le \frac{1}{d_{1_i}} \int_0^s e^{-d_{2_i}(s-\rho)} \int_0^{\rho} \frac{(\rho - r)^{\alpha-2}}{\Gamma(\alpha_i - 1)} \bigl|\tilde{z}_i(r)\bigr|\,\mathrm{d}r\,\mathrm{d}\rho \\
&\quad + \frac{1 - e^{-d_{2_i} s} + (d_{2_i}^2 - d_{2_i})s}{d_{1_i}(\tilde{\Delta}_{2_i} - d_{2_i}\Omega_i^*)} \biggl[\int_0^1 e^{-d_{2_i}(1-\rho)} \int_0^{\rho} \frac{(\rho - r)^{\alpha-2}}{\Gamma(\alpha_i - 1)} \bigl|\tilde{z}_i(r)\bigr|\,\mathrm{d}r\,\mathrm{d}\rho \\
&\quad + \int_0^p \frac{(p-\rho)^{\xi-1}}{\Gamma(\chi_i)} \int_0^{\rho} e^{-d_{2_i}(\rho-\iota)} \int_0^{\iota} \frac{(\iota - r)^{\alpha-2}}{\Gamma(\alpha_i - 1)} \bigl|\tilde{z}_i(r)\bigr|\,\mathrm{d}r\,\mathrm{d}\iota\,\mathrm{d}\rho \biggr] \\
&\le \frac{1}{d_1} \int_0^s e^{-d_2(s-\rho)} \int_0^{\rho} \frac{(\rho - r)^{\alpha_i-2}}{\Gamma(\alpha_i - 1)} \theta_i(r)\,\mathrm{d}r\,\mathrm{d}\rho \\
&\quad + \frac{|1 - e^{-d_2 s}| + |d_2^2 - d_2|s}{d_1|\tilde{\Delta}_{2_i} - d_2\Omega_i^*|} \biggl[\int_0^1 e^{-d_2(1-\rho)} \int_0^{\rho} \frac{(\rho - r)^{\alpha_i-2}}{\Gamma(\alpha_i - 1)} \theta_i(r)\,\mathrm{d}r\,\mathrm{d}\rho \\
&\quad + \int_0^p \frac{(p-\rho)^{\xi_i-1}}{\Gamma(\chi_i)} \int_0^{\rho} e^{-d_2(\rho-\iota)} \int_0^{\iota} \frac{(\iota - r)^{\alpha_i-2}}{\Gamma(\alpha_i - 1)} \theta_i(r)\,\mathrm{d}r\,\mathrm{d}\iota\,\mathrm{d}\rho \biggr] \\
&\le \biggl[\frac{(1 - e^{-d_2})}{d_1 d_2 \Gamma(\alpha_i)} + \frac{|1 - e^{-d_2}| + |d_2^2 - d_2|}{d_1|\tilde{\Delta}_{2_i} - d_2\Omega_i^*|} \\
&\quad \times \biggl(\frac{(1 - e^{-d_2})}{d_2\Gamma(\alpha_i)} + \frac{p^{\alpha_i+\xi_i-1}(d_2 p + e^{-d_2 p} - 1)}{d_2^2 \Gamma_i(\alpha_i)\Gamma(\chi_i)} \biggr) \biggr] \|\theta_i\|_{\mathcal{L}^1} = M_{i_1}\|\theta_i\|_{\mathcal{L}^1}, \\
\bigl|z_i'(s)\bigr| &= \frac{1}{d_{1_i}} \int_0^{\rho} \frac{(\rho - r)^{\alpha-2}}{\Gamma(\alpha_i - 1)} \bigl|\tilde{z}'_i(r)\bigr|\,\mathrm{d}r \\
&\quad + \frac{d_{2_i} e^{-d_{2_i} s} + (d_{2_i}^2 - d_{2_i})}{d_{1_i}(\tilde{\Delta}_{2_i} - d_{2_i}\Omega_i^*)} \biggl[\int_0^1 e^{-d_{2_i}(1-\rho)} \int_0^{\rho} \frac{(\rho - r)^{\alpha-2}}{\Gamma(\alpha_i - 1)} \bigl|\tilde{z}'_i(r)\bigr|\,\mathrm{d}r\,\mathrm{d}\rho \\
&\quad + \int_0^p \frac{(p-\rho)^{\chi-1}}{\Gamma(\chi_i)} \int_0^{\rho} e^{-d_{2_i}(\rho-\iota)} \int_0^{\iota} \frac{(\iota - r)^{\alpha-2}}{\Gamma(\alpha_i - 1)} \bigl|\tilde{z}'_i(r)\bigr|\,\mathrm{d}r\,\mathrm{d}\iota\,\mathrm{d}\rho \\
&\le \frac{1}{d_1} \int_0^s e^{-d_2(s-\rho)} \int_0^{\rho} \frac{(\rho - r)^{\alpha_i-2}}{\Gamma(\alpha_i - 1)} \theta_i'(r)\,\mathrm{d}r\,\mathrm{d}\rho \\
&\quad + \frac{|1 - e^{-d_2 s}| + |d_2^2 - d_2|s}{d_1|\tilde{\Delta}_{2_i} - d_2\Omega_i^*|} \biggl[\int_0^1 e^{-d_2(1-\rho)} \int_0^{\rho} \frac{(\rho - r)^{\alpha_i-2}}{\Gamma(\alpha_i - 1)} \theta_i'(r)\,\mathrm{d}r\,\mathrm{d}\rho \\
&\quad + \int_0^p \frac{(p-\rho)^{\chi_i-1}}{\Gamma(\chi_i)} \int_0^{\rho} e^{-d_2(\rho-\iota)} \int_0^{\iota} \frac{(\iota - r)^{\alpha_i-2}}{\Gamma(\alpha_i - 1)} \theta_i'(r)\,\mathrm{d}r\,\mathrm{d}\iota\,\mathrm{d}\rho \biggr] \\
&\le \biggl[\frac{1}{d_1 \Gamma(\alpha_i)} + \frac{|d_2 e^{-d_2}| + |d_2^2 - d_2|}{d_1|\tilde{\Delta}_{2_i} - d_2\Omega_i^*|} \\
&\quad \times \biggl(\frac{(1 - e^{-d_2})}{d_2\Gamma(\alpha_i)} + \frac{p^{\alpha_i+\chi_i-1}(d_2 p + e^{-d_2 p} - 1)}{d_2^2 \Gamma_i(\alpha_i)\Gamma(\chi_i)} \biggr) \biggr] \bigl\|\theta_i'\bigr\|_{\mathcal{L}^1} = M_{i_2}\bigl\|\theta_i'\bigr\|_{\mathcal{L}^1} \biggr],
\end{aligned}
\tag{14}
$$

for all $s \in [0,1]$ and $1 \leq i \leq k$. Thus, $\|z_i\|_i \leq (M_{i_1} + M_{i_2})\|\theta_i\|_{\mathcal{L}^1}$. Hence,

$$|z_1,\dots,z_k| = \sum_{i=1}^{k} \|z_i\| \leq \sum_{i=1}^{k} (M_{i_1} + M_{i_2})\|\theta_i\|_{\mathcal{L}^1}.$$

Now, we show that T maps bounded sets to equicontinuous subsets of $\mathcal{Q}$. Assume that $(l_1,\dots,l_k) \in \mathcal{B}_\rho$, $s_1, s_2 \in [0,1]$ with $s_1 \leq s_2$ and $(z_1,\dots,z_k) \in T(l_1,\dots,l_k)$. Then we have

$$\begin{aligned}
\big|l_i(s_2) - l_i(s_1)\big| \leq{}& \bigg|\frac{1}{d_1}\int_0^{s_2} e^{-d_2(s_2-\rho)}\int_0^{\rho}\frac{(\rho-r)^{\alpha_i-2}}{\Gamma(\alpha_i-1)} b_{\hat{\vartheta}_i(r)}\,\mathrm{d}r\,\mathrm{d}\rho \\
&- \frac{1}{d_1}\int_0^{s_1} e^{-d_2(s_1-\rho)}\int_0^{\rho}\frac{(\rho-r)^{\alpha_i-2}}{\Gamma(\alpha_i-1)} b_{\hat{\vartheta}_i(r)}\,\mathrm{d}r\,\mathrm{d}\rho\bigg| \\
&+ \frac{(e^{-d_2 s_1} - e^{-d_2 s_2}) + |d_2^2 - d_2|(s_2 - s_1)}{d_1|\tilde{\Delta}_{2_i} - d_2\Omega_i^*|} \\
&\times \bigg[\int_0^1 e^{-d_2(1-\rho)}\int_0^{\rho}\frac{(\rho-r)^{\alpha_i-2}}{\Gamma(\alpha_i-1)}|b_{\hat{\vartheta}_i(r)}|\,\mathrm{d}r\,\mathrm{d}\rho \\
&+ \int_0^{p}\frac{(p-\rho)^{\xi_i-1}}{\Gamma(\chi)_i}\int_0^{\rho} e^{-d_2(\rho-\iota)}\int_0^{\iota}\frac{(\iota-r)^{\alpha-2}}{\Gamma(\alpha_i-1)}|b_{\hat{\vartheta}_i(r)}|\,\mathrm{d}r\,\mathrm{d}\iota\,\mathrm{d}\rho\bigg] \\
\leq{}& M_{i_1}\|\theta_i\|_{\mathcal{L}^1}, \\
\big|l_i'(s_2) - l_i'(s_1)\big| \leq{}& \bigg|\frac{1}{d_1}\int_0^{\rho}\frac{(\rho-r)^{\alpha_i-2}}{\Gamma(\alpha_i-1)} b_{\hat{\vartheta}_i'(r)}\,\mathrm{d}r\,\mathrm{d}\rho \\
&- \frac{1}{d_1}\int_0^{\rho}\frac{(\rho-r)^{\alpha_i-2}}{\Gamma(\alpha_i-1)} b_{\hat{\vartheta}_i'(r)}\,\mathrm{d}r\,\mathrm{d}\rho\bigg| \\
&+ \frac{(-d_2 e^{-d_2 s_1} - d_2 e^{-d_2 s_2}) + |d_2^2 - d_2|}{d_1|\tilde{\Delta}_{2_i} - d_2\Omega_i^*|} \\
&\times \bigg[\int_0^1 e^{-d_2(1-\rho)}\int_0^{\rho}\frac{(\rho-r)^{\alpha_i-2}}{\Gamma(\alpha_i-1)}|b_{\hat{\vartheta}'_i(r)}|\,\mathrm{d}r\,\mathrm{d}\rho \\
&+ \int_0^{p}\frac{(p-\rho)^{\chi_i-1}}{\Gamma(\chi)_i}\int_0^{\rho} e^{-d_2(\rho-\iota)}\int_0^{\iota}\frac{(\iota-r)^{\alpha-2}}{\Gamma(\alpha_i-1)}|b_{\hat{\vartheta}'_i(r)}|\,\mathrm{d}r\,\mathrm{d}\iota\,\mathrm{d}\rho\bigg] \\
\leq{}& M_{i_2}\big\|\theta_i'\big\|_{\mathcal{L}^1}
\end{aligned}$$

for all $1 \leq i \leq k$. This implies that $\lim_{s_2\to s_1} |l_1(s_2) - l_1(s_1),\dots,l_k(s_2) - l_k(s_1)| = 0$ and

$$\lim_{s_2\to s_1}\big|l_1'(s_2) - l_1'(s_1),\dots,l_k(s_2) - l_k(s_1)\big| = 0.$$

By using the Arzela–Ascoli theorem for each bounded subset $\mathcal{B}_r$ of $\mathcal{Q}$, $T(\mathcal{B}_r)$ is relatively compact. Thus, T is completely continuous. Now, we show that T has a closed graph. Let $(l_1^n,\dots,l_k^n) \in \mathcal{Q}$ and $(z_1^n,\dots,z_k^n) \in T(l_1^0,\dots,l_k^0)$ with $(l_1^n,\dots,l_k^n) \to (l_1^0,\dots,l_k^0)$ and $(z_1^n,\dots,z_k^n) \to (z_1^0,\dots,z_k^0)$. We show that $(z_1^0,\dots,z_k^0) \in T(l_1^0,\dots,l_k^0)$. For each natural number

n, choose $(u_1^n, \dots, u_k^n) \in \mathcal{Q}_{A_{1,q}} \times \dots \times \mathcal{Q}_{A_{k,q}}$ such that

$$\begin{aligned} z_i^n(s) &= \frac{1}{d_{1_i}} \int_0^s e^{-d_{2_i}(s-\rho)} \int_0^\rho \frac{(\rho-r)^{\alpha-2}}{\Gamma(\alpha_i-1)} \tilde{l}_i^n(r)\,\mathrm{d}r\,\mathrm{d}\rho \\ &\quad + \frac{1-e^{-d_{2_i}s} + (d_{2_i}^2 - d_{2_i})s}{d_{1_i}(\tilde{\Delta}_{2_i} - d_{2_i}\Omega_i^*)} \bigg[\int_0^1 e^{-d_{2_i}(1-\rho)} \int_0^\rho \frac{(\rho-r)^{\alpha-2}}{\Gamma(\alpha_i-1)} \tilde{l}_i^n(r)\,\mathrm{d}r\,\mathrm{d}\rho \\ &\quad + \int_0^p \frac{(p-\rho)^{\chi-1}}{\Gamma(\chi_i)} \int_0^\rho e^{-d_{2_i}(\rho-\iota)} \int_0^\iota \frac{(\iota-r)^{\alpha-2}}{\Gamma(\alpha_i-1)} \tilde{l}_i^n(r)\,\mathrm{d}r\,\mathrm{d}\iota\,\mathrm{d}\rho \bigg] \end{aligned} \tag{15}$$

for $t \in [0,1]$ and $1 \le i \le k$. Now, define the continuous linear operator $\theta_i : L^1([0,1],\mathbb{R}) \to \mathcal{Q}_i$ by

$$\begin{aligned} \theta_i(l,s) &= \frac{1}{d_{1_i}} \int_0^s e^{-d_{2_i}(s-\rho)} \int_0^\rho \frac{(\rho-r)^{\alpha-2}}{\Gamma(\alpha_i-1)} \tilde{l}_i(r)\,\mathrm{d}r\,\mathrm{d}\rho \\ &\quad + \frac{1-e^{-d_{2_i}s} + (d_{2_i}^2 - d_{2_i})s}{d_{1_i}(\tilde{\Delta}_{2_i} - d_{2_i}\Omega_i^*)} \bigg[\int_0^1 e^{-d_{2_i}(1-\rho)} \int_0^\rho \frac{(\rho-r)^{\alpha-2}}{\Gamma(\alpha_i-1)} \tilde{l}_i(r)\,\mathrm{d}r\,\mathrm{d}\rho \\ &\quad + \int_0^p \frac{(p-\rho)^{\chi-1}}{\Gamma(\chi_i)} \int_0^\rho e^{-d_{2_i}(\rho-\iota)} \int_0^\iota \frac{(\iota-r)^{\alpha-2}}{\Gamma(\alpha_i-1)} \tilde{l}_i(r)\,\mathrm{d}r\,\mathrm{d}\iota\,\mathrm{d}\rho \bigg]. \end{aligned} \tag{16}$$

By using Theorem 2.1, $\theta_i \circ \mathcal{Q}_{A_{i,q}}$ is a closed graph operator. Since $z_i^n \in \theta_i(\mathcal{Q}_{A_{i,(l_1,\dots,l_k)}})$ for all $n, 1 \le i \le k$ and $(l_1^n, \dots, l_k^n) \to (l_1^0, \dots, l_k^0)$, there exists $u_i^0 \in \mathcal{Q}_{A_{i,(l_1,\dots,l_k)}}$ such that

$$\begin{aligned} z_1^0(s) &= \frac{1}{d_{1_i}} \int_0^s e^{-d_{2_i}(s-\rho)} \int_0^\rho \frac{(\rho-r)^{\alpha-2}}{\Gamma(\alpha_i-1)} \tilde{l}_i^0(r)\,\mathrm{d}r\,\mathrm{d}\rho \\ &\quad + \frac{1-e^{-d_{2_i}s} + (d_{2_i}^2 - d_{2_i})s}{d_{1_i}(\tilde{\Delta}_{2_i} - d_{2_i}\Omega_i^*)} \bigg[\int_0^1 e^{-d_{2_i}(1-\rho)} \int_0^\rho \frac{(\rho-r)^{\alpha-2}}{\Gamma(\alpha_i-1)} \tilde{l}_i^0(r)\,\mathrm{d}r\,\mathrm{d}\rho \\ &\quad + \int_0^p \frac{(p-\rho)^{\chi-1}}{\Gamma(\chi_i)} \int_0^\rho e^{-d_{2_i}(\rho-\iota)} \int_0^\iota \frac{(\iota-r)^{\alpha-2}}{\Gamma(\alpha_i-1)} \tilde{l}_i^0(r)\,\mathrm{d}r\,\mathrm{d}\iota\,\mathrm{d}\rho \bigg]. \end{aligned} \tag{17}$$

Hence, $z_i^0 \in T(l_1^0, \dots, l_k^0)$ for all $1 \le i \le k$. This implies that T_i has a closed graph for all $1 \le i \le k$, and so T has a closed graph. Now, suppose that there exists $\lambda \in (0,1)$ such that $(l_1, \dots, l_n) \in \lambda T(l_1, \dots, l_n)$. Then there exists $(l_1, \dots, l_n) \in \mathcal{Q}_{A_{1,(l_1,\dots,l_k)}} \times \dots \times \mathcal{Q}_{A_{k,(l_1,\dots,l_k)}}$ such that

$$\begin{aligned} l_i(s) &= \frac{1}{d_{1_i}} \int_0^s e^{-d_{2_i}(s-\rho)} \int_0^\rho \frac{(\rho-r)^{\alpha-2}}{\Gamma(\alpha_i-1)} \tilde{l}_i(r)\,\mathrm{d}r\,\mathrm{d}\rho \\ &\quad + \frac{1-e^{-d_{2_i}s} + (d_{2_i}^2 - d_{2_i})s}{d_{1_i}(\tilde{\Delta}_{2_i} - d_{2_i}\Omega_i^*)} \bigg[\int_0^1 e^{-d_{2_i}(1-\rho)} \int_0^\rho \frac{(\rho-r)^{\alpha-2}}{\Gamma(\alpha_i-1)} \tilde{l}_i(r)\,\mathrm{d}r\,\mathrm{d}\rho \\ &\quad + \int_0^p \frac{(p-\rho)^{\chi-1}}{\Gamma(\chi_i)} \int_0^\rho e^{-d_{2_i}(\rho-\iota)} \int_0^\iota \frac{(\iota-r)^{\alpha-2}}{\Gamma(\alpha_i-1)} \tilde{l}_i(r)\,\mathrm{d}r\,\mathrm{d}\iota\,\mathrm{d}\rho \bigg] \end{aligned} \tag{18}$$

for all $s \in [0,1]$ and $1 \le i \le k$. Since $\frac{\|l_i\|}{M_1^i + M_2^i}\|\theta_i\| \le 1$, $\|l_i\|_i \le \theta_i$ for all $i = 1, 2, \dots, k$. Now, put $\mathrm{L} = \{(\nu_1, \dots, \nu_k) \in \mathrm{Q} : \|(l_1, \dots, l_k)\| \le \sum_{i=1}^k \theta_i + 1\}$. Thus, there are no $(l_1, \dots, l_k) \in \partial\mathrm{L}$ and $\lambda \in (0,1)$ such that $(l_1, \dots, l_k) \in \lambda\mathrm{T}(l_1, \dots, l_k)$. Also, the operator $T : \bar{\mathrm{L}} \to P_{cmp,cvx}(\bar{\mathrm{L}})$ is upper semi-continuous because it is completely continuous and has a closed graph. By using the definition of L, there is no $(l_1, \dots, l_k) \in \partial \mathrm{L}$ such that $(l_1, \dots, l_k) \in \lambda T(l_1, \dots, l_k)$ for some $\lambda \in (0,1)$. Now, by using Theorem 2.3, T has a fixed point in $\bar{L}$ which is a solution for the k-dimensional hybrid inclusion system. □

Now, we review the k-dimensional non-hybrid inclusion system

$$\begin{cases} d_{1_1}({}^cD^\alpha + d_{2_1}{}^cD^{\alpha-1})q_1(s) \in \mathcal{S}_1(s, q_1(s), \dots, q_k(s), q_1'(s), \dots, q_k'(s), q_1''(s), \dots, q_k''(s)), \\ d_{1_2}({}^cD^\alpha + d_{2_2}{}^cD^{\alpha-1})q_2(s) \in \mathcal{S}_2(s, q_1(s), \dots, q_k(s), q_1'(s), \dots, q_k'(s), q_1''(s), \dots, q_k''(s)), \\ \vdots \\ d_{1_k}({}^cD^\alpha + d_{2_k}{}^cD^{\alpha-1})q_k(s) \in \mathcal{S}_k(s, q_1(s), \dots, q_k(s), q_1'(s), \dots, q_k'(s), q_1''(s), \dots, q_k''(s)), \end{cases} \tag{19}$$

with three-point integro-derivative boundary conditions

$$q_i(0) = 0, \qquad q_i'(0) + q_i''(0) = 0, \qquad q_i(1) + {}^RI^\xi q_i(p) = 0, \quad (1 \le i \le k), \tag{20}$$

where $s \in [0,1]$, $\alpha \in (2,3]$, $p \in (0,1)$, $d_{1_1}, \dots, d_{1_k}, d_{2_1}, \dots, d_{2_k} \in (0,\infty)$ and ${}^RI^\xi$ denotes the Riemann–Liouville fractional integral of order $\xi > 0$. Define the space

$$\mathcal{Q}_i = \bigl\{s : s, q(s), q'(s), q''(s) \in \mathcal{C}\bigl([0,1], \mathbb{R}\bigr)\bigr\}$$

endowed with the norm $\|q\|_{\mathcal{Q}_i} = \sup_{s\in[0,1]} |q_i(s)| + \sup_{s\in[0,1]} |q_i'(s)| + \sup_{s\in[0,1]} |q_i''(s)|$ for all $\{i \in 1, 2, \dots, k\}$. Also, define the product space $\mathcal{Q} = Q_1 \times Q_2 \times \cdots \times Q_k$ endowed with the norm $\|(q_1, q_2, \dots, q_k)\| = \sum_{i=1}^k \|q_i\|$. Then $(\mathcal{Q}, \|\cdot\|)$ is a Banach space. We need the next result.

Lemma 3.2 ([46]) *A function $q \in \mathcal{AC}_{\mathbb{R}}([0,1])$ is a solution for the k-dimensional non-hybrid inclusion system (19)–(20) whenever there is an integrable function $\hat{u} \in \mathcal{L}^1_{\mathbb{R}}([0,1])$ such that $\hat{u} \in \mathcal{S}(s, q(s))$ for almost all $s \in [0,1]$, $q(0) = 0$, $q'(0) + q''(0) = 0$, $q(1) + {}^RI^\chi q(p) = 0$ and*

$$\begin{aligned} q(s) = {}& \frac{1}{p_1} \int_0^s e^{-d_2(s-\rho)} \int_0^\rho \frac{(\rho - r)^{\alpha-2}}{\Gamma(\alpha-1)} \hat{u}(r)\,\mathrm{d}r\,\mathrm{d}\rho \\ & + \frac{1 - e^{-d_2 s} + (d_2^2 - d_2)s}{d_1(\tilde{\Delta}_2 - d_2\Omega^*)} \biggl[\int_0^1 e^{-d_2(1-\rho)} \int_0^\rho \frac{(\rho - r)^{\alpha-2}}{\Gamma(\alpha-1)} \hat{u}(r)\,\mathrm{d}r\,\mathrm{d}\rho \\ & + \int_0^p \frac{(p-\rho)^{\chi-1}}{\Gamma(\chi)} \int_0^\rho e^{-d_2(\rho-\iota)} \int_0^\iota \frac{(\iota - r)^{\alpha-2}}{\Gamma(\alpha-1)} \hat{u}(r)\,\mathrm{d}r\,\mathrm{d}\iota\,\mathrm{d}\rho \biggr] \end{aligned}$$

for all $s \in [0,1]$.

We say that a function $(q_1, q_2, \dots, q_k) \in \mathcal{Q}$ is a solution for the k-dimensional system of non-hybrid inclusions (19) whenever there exist functions $u_1, u_2, \dots, u_k$ in $L^1[0,1]$ such that

$$u_i(s) \in \mathcal{S}_i\bigl(s, q_1(s), q_2(s), \dots, q_i(s), q_1'(s), \dots, q_i'(s), q_1''(s), \dots, q_i''(s)\bigr)$$

for all $s \in [0,1]$

$$
\begin{aligned}
q_i(s) &= \frac{1}{d_{1_i}} \int_0^s e^{-d_{2_i}(s-\rho)} \int_0^\rho \frac{(\rho-r)^{\alpha_i-2}}{\Gamma_i(\alpha_i-1)} \hat{u}_i(r)\,\mathrm{d}r\,\mathrm{d}\rho \\
&\quad + \frac{1-e^{-d_{2_i}s} + (d_{2_i}^2 - d_{2_i})s}{d_{1_i}(\tilde{\Delta}_{2_i} - d_{2_i}\Omega_i^*)} \biggl[\int_0^1 e^{-d_{2_i}(1-\rho)} \int_0^\rho \frac{(\rho-r)^{\alpha_i-2}}{\Gamma_i(\alpha_2-1)} \hat{u}_i(r)\,\mathrm{d}r\,\mathrm{d}\rho \\
&\quad + \int_0^p \frac{(p-\rho)^{\chi_i-1}}{\Gamma_i(\chi_i)} \int_0^\rho e^{-d_{2_i}(\rho-\iota)} \int_0^\iota \frac{(\iota-r)^{\alpha_i-2}}{\Gamma_i(\alpha_i-1)} \hat{u}_i(r)\,\mathrm{d}r\,\mathrm{d}\iota\,\mathrm{d}\rho \biggr]
\end{aligned}
$$

and $\alpha \in (2,3]$, $p \in (0,1)$, $d_{1_i}, d_{2_i}, \gamma, \chi > 0$, ${}^cD^{(\cdot)}$ and ${}^RI^{(\cdot)}$ denote the Caputo fractional derivative and the Riemann–Liouville fractional integral, respectively. By using the idea of [37], we consider the set of the selections

$$
\mathcal{S}_{G_{i,q}} = \bigl\{ u \in L^1[0,1] : u(s) \in J_i(s) \text{ for all } s \in [0,1], q = (q_1, \ldots, q_k) \in \mathcal{Q} \text{ and } 1 \le i \le k \bigr\},
$$

where $J_i(s) = A_i(s, q_1(s), \ldots, q_k(s), q_1'(s), \ldots, q_k'(s), q_1''(s), \ldots, q_k''(s))$.

Theorem 3.3 *Let $\theta_1, \ldots, \theta_i \in \mathcal{C}([0,1], \mathbb{R})$ be such that $L = \sum_{i=1}^k \|\theta_i\|_\infty (M_{i_1} + M_{i_2} + M_{i_3}) \le 1$, where*

$$
\begin{aligned}
M_{i_1} &= \frac{1}{d_1} \int_0^s e^{-d_2(s-\rho)} \int_0^\rho \frac{(\rho-r)^{\alpha-2}}{\Gamma(\alpha-1)} \bigl| u_i(r) - u_{t_i}(r) \bigr|\,\mathrm{d}r\,\mathrm{d}\rho \\
&\quad + \frac{1-e^{-d_2 s} + (d_2^2 - d_2)s}{d_1(\tilde{\Delta}_2 - d_2\Omega^*)} \biggl[\int_0^1 e^{-d_2(1-\rho)} \int_0^\rho \frac{(\rho-r)^{\alpha-2}}{\Gamma(\alpha-1)} \bigl| u_i(r) - u_{t_i}(r) \bigr|\,\mathrm{d}r\,\mathrm{d}\rho \\
&\quad + \int_0^p \frac{(p-\rho)^{\chi-1}}{\Gamma(\chi)} \int_0^\rho e^{-d_2(\rho-\iota)} \int_0^\iota \frac{(\iota-r)^{\alpha-2}}{\Gamma(\alpha-1)} \bigl| u_i(r) - u_{t_i}(r) \bigr|\,\mathrm{d}r\,\mathrm{d}\iota\,\mathrm{d}\rho \biggr], \\
M_{i_2} &= \biggl| \frac{1}{d_1} \int_0^\rho \frac{(\rho-r)^{\alpha_i-2}}{\Gamma(\alpha_i-1)} \hat{u_i'}(r)\,\mathrm{d}r\,\mathrm{d}\rho - \frac{1}{d_1} \int_0^\rho \frac{(\rho-r)^{\alpha_i-2}}{\Gamma(\alpha_i-1)} \hat{u_i'}(r)\,\mathrm{d}r\,\mathrm{d}\rho \biggr| \\
&\quad + \frac{(-d_2 e^{-d_2 s_1} - d_2 e^{-d_2 s_2}) + |d_2^2 - d_2|}{d_1 |\tilde{\Delta}_{2_i} - d_2\Omega_i^*|} \\
&\quad \times \biggl[\int_0^1 e^{-d_2(1-\rho)} \int_0^\rho \frac{(\rho-r)^{\alpha_i-2}}{\Gamma(\alpha_i-1)} \bigl| \hat{u'}_i(r) \bigr|\,\mathrm{d}r\,\mathrm{d}\rho \\
&\quad + \int_0^p \frac{(p-\rho)^{\chi_i-1}}{\Gamma(\chi)_i} \int_0^\rho e^{-d_2(\rho-\iota)} \int_0^\iota \frac{(\iota-r)^{\alpha-2}}{\Gamma(\alpha_i-1)} \bigl| \hat{u'}_i(r) \bigr|\,\mathrm{d}r\,\mathrm{d}\iota\,\mathrm{d}\rho \biggr]
\end{aligned}
$$

and

$$
\begin{aligned}
M_{i_3} &= \biggl| \frac{1}{d_1} \int_0^\rho \frac{(\rho-r)^{\alpha_i-2}}{\Gamma(\alpha_i-1)} \hat{u_i''}(r)\,\mathrm{d}r\,\mathrm{d}\rho - \frac{1}{d_1} \int_0^\rho \frac{(\rho-r)^{\alpha_i-2}}{\Gamma(\alpha_i-1)} \hat{u_i''}(r)\,\mathrm{d}r\,\mathrm{d}\rho \biggr| \\
&\quad + \frac{(d_2^2 e^{-d_2 s_1} + d_2^2 e^{-d_2 s_2})}{d_1 |\tilde{\Delta}_{2_i} - d_2\Omega_i^*|} \times \biggl[\int_0^1 e^{-d_2(1-\rho)} \int_0^\rho \frac{(\rho-r)^{\alpha_i-2}}{\Gamma(\alpha_i-1)} \bigl| \hat{u''}_i(r) \bigr|\,\mathrm{d}r\,\mathrm{d}\rho \\
&\quad + \int_0^p \frac{(p-\rho)^{\chi_i-1}}{\Gamma(\chi)_i} \int_0^\rho e^{-d_2(r-\iota)} \int_0^\iota \frac{(\iota-r)^{\alpha-2}}{\Gamma(\alpha_i-1)} \bigl| \hat{u''}_i(r) \bigr|\,\mathrm{d}r\,\mathrm{d}\iota\,\mathrm{d}r \biggr]
\end{aligned}
$$

for $i = 1, \dots, k$. Suppose that $G_i : [0,1] \times \mathbb{R}^{3k} \to \mathcal{P}_{cmp}(\mathbb{R})$ is a multifunction such that the map $s \to G_i(s, x_1, \dots, x_k, y_1, \dots, y_k, z_1, \dots, z_k)$ is integrable bounded, measurable and

$$\begin{aligned}
&\mathrm{PH}_{d_{\mathcal{Q}}}\big(G_i(s,x_1,\dots,x_k,y_1,\dots,y_k,z_1,\dots,z_k), G_i(s,x_{i_1},\dots,x_{i_k},y_{i_1},\dots,y_{i_k},z_{i_1},\dots,z_{i_k})\big) \\
&\quad \le \|\theta_i\| \left(\sum_{i=1}^{k} |x_i - x_{i_1}| \right)
\end{aligned}$$

for almost all $s \in [0,1]$, $x_{i_1}, \dots, x_{i_k}, y_{i_1}, \dots, y_{i_k}, z_{i_1}, \dots, z_{i_k} \in \mathbb{R}$ and $i = 1, \dots, k$. Then the non-hybrid k-dimensional inclusion system (19)–(20) has at least one solution.

Proof Note that the multifunction

$$s \to G_i\big(s, q_1(s), \dots, q_k(s), q_1'(s), \dots, q_k'(s), q_1''(s), \dots, q_k''(s)\big)$$

is measurable and closed-valued for all $q_1, \dots, q_k \in \mathcal{Q}$ and $i = 1, \dots, k$. Hence, it has measurable selection, and so the set $\mathcal{S}_{G_{i,(q_1,\dots,q_k)}}$ is nonempty for all $i = 1, \dots, k$. Consider the operator $H : \mathcal{Q} \to 2^{\mathcal{Q}}$ defined by $H(q_1, \dots, q_k) = (H_1(q_1, \dots, q_k), H_2(q_1, \dots, q_k), \dots, H_k(q_1, \dots, q_k))$, where

$$\begin{aligned}
H_i(q_1,\dots,q_k) = \Bigg\{ &z \in \mathcal{Q}_i : \text{there exists } q \in \mathcal{S}_{G_{i,(q_1,\dots,q_k)}} : \\
&z(s) = \frac{1}{d_1} \int_0^s e^{-d_2(s-\rho)} \int_0^\rho \frac{(\rho - r)^{\alpha-2}}{\Gamma(\alpha-1)} q(r)\,\mathrm{d}r\,\mathrm{d}\rho \\
&\quad + \frac{1 - e^{-d_2 s} + (d_2^2 - d_2)s}{d_1(\tilde{\Delta}_2 - d_2\Omega^*)} \Bigg[\int_0^1 e^{-d_2(1-\rho)} \int_0^\rho \frac{(\rho - r)^{\alpha-2}}{\Gamma(\alpha-1)} q(r)\,\mathrm{d}r\,\mathrm{d}\rho \\
&\quad + \int_0^p \frac{(p-\rho)^{\chi-1}}{\Gamma(\chi)} \int_0^\rho e^{-d_2(\rho-\iota)} \int_0^\iota \frac{(\iota - r)^{\alpha-2}}{\Gamma(\alpha-1)} q(r)\,\mathrm{d}r\,\mathrm{d}\iota\,\mathrm{d}\rho \Bigg] \Bigg\}.
\end{aligned}$$

First, we show that $H(q_1, \dots, q_k)$ is a closed subset of $\mathcal{Q}$ for all $(q_1, \dots, q_k) \in \mathcal{Q}$. Let $\{(q_1^n, \dots, q_k^n)\}$ be a sequence in $H(q_1, \dots, q_k)$ such that $(q_1^n, \dots, q_k^n) \to (q_1^0, \dots, q_k^0)$. Choose $(u_1^n, \dots, u_k^n) \in \mathcal{S}_{G_{1,(q_1,\dots,q_k)}} \times \mathcal{S}_{G_{2,(q_1,\dots,q_k)}} \times \cdots \times \mathcal{S}_{G_{k,(q_1,\dots,q_k)}}$ such that

$$\begin{aligned}
q_i^n(s) &= \frac{1}{d_1} \int_0^s e^{-d_2(s-\rho)} \int_0^\rho \frac{(\rho - r)^{\alpha-2}}{\Gamma(\alpha-1)} u_i^n(r)\,\mathrm{d}r\,\mathrm{d}\rho \\
&\quad + \frac{1 - e^{-d_2 s} + (d_2^2 - d_2)s}{d_1(\tilde{\Delta}_2 - d_2\Omega^*)} \Bigg[\int_0^1 e^{-d_2(1-\rho)} \int_0^\rho \frac{(\rho - r)^{\alpha-2}}{\Gamma(\alpha-1)} u_i^n(r)\,\mathrm{d}r\,\mathrm{d}\rho \\
&\quad + \int_0^p \frac{(p-\rho)^{\chi-1}}{\Gamma(\chi)} \int_0^\rho e^{-d_2(\rho-\iota)} \int_0^\iota \frac{(\iota - r)^{\alpha-2}}{\Gamma(\alpha-1)} u_i^n(r)\,\mathrm{d}r\,\mathrm{d}\iota\,\mathrm{d}\rho \Bigg]
\end{aligned}$$

for all $s \in [0,1]$ and $i = 1, \dots, k$. Since G_i is compact-valued for all i, $\{u_i^n\}_{n \le 1}$ has a subsequence which converges to some $u_i^0 \in L^1([0,1], \mathbb{R})$. Denote the subsequence again by

$\{u_i^n\}_{n\leq 1}$. It is easy to check that $u_i^0 \in \mathcal{S}_{G_{i,(q_1,\dots,q_k)}}$ and

$$\begin{aligned} q_i^n(s) \to q_i^0(s) &= \frac{1}{d_1}\int_0^s e^{-d_2(s-\rho)}\int_0^\rho \frac{(\rho-r)^{\alpha-2}}{\Gamma(\alpha-1)}u_i^0(r)\,\mathrm{d}r\,\mathrm{d}\rho \\ &\quad + \frac{1-e^{-d_2 s}+(d_2^2-d_2)s}{d_1(\tilde{\Delta}_2-d_2\Omega^*)}\biggl[\int_0^1 e^{-d_2(1-\rho)}\int_0^\rho \frac{(\rho-r)^{\alpha-2}}{\Gamma(\alpha-1)}u_i^0(r)\,\mathrm{d}r\,\mathrm{d}\rho \\ &\quad + \int_0^p \frac{(p-\rho)^{\chi-1}}{\Gamma(\xi)}\int_0^\rho e^{-d_2(r-\iota)}\int_0^\iota \frac{(\iota-r)^{\alpha-2}}{\Gamma(\alpha-1)}u_i^0(r)\,\mathrm{d}r\,\mathrm{d}\iota\,\mathrm{d}r\biggr] \end{aligned}$$

for all $s\in[0,1]$. This implies that $q_i^0 \in H_i(q_1,\dots,q_k)$ for any $i=1,2,\dots,k$. This concludes that $(q_1^0,\dots,q_k^0)\in H_i(q_1,\dots,q_k)$. Now, we show that H is a contractive multifunction with the constant $L\leq 1$, where $\sum_{i=1}^k (M_{i_1}+M_{i_2}+M_{i_3})\leq 1$. Let $(y_1,\dots,y_k),(z_1,\dots,z_k)\in\mathcal{Q}$ and $(h_1,\dots,h_k)\in H(z_1,\dots,z_k)$ be given. Then we can choose

$$(u_1,\dots,u_k)\in \mathcal{S}_{G_{1,(z_1,\dots,z_k)}}\times \mathcal{S}_{G_{2,(z_1,\dots,z_k)}}\times\cdots\times \mathcal{S}_{G_{k,(z_1,\dots,z_k)}}$$

such that

$$\begin{aligned} h_i(s) &= \frac{1}{d_1}\int_0^s e^{-d_2(s-\rho)}\int_0^\rho \frac{(\rho-r)^{\alpha-2}}{\Gamma(\alpha-1)}u_i(r)\,\mathrm{d}r\,\mathrm{d}\rho \\ &\quad + \frac{1-e^{-d_2 s}+(d_2^2-d_2)s}{d_1(\tilde{\Delta}_2-d_2\Omega^*)}\biggl[\int_0^1 e^{-d_2(1-\rho)}\int_0^\rho \frac{(\rho-r)^{\alpha-2}}{\Gamma(\alpha-1)}u_i(r)\,\mathrm{d}r\,\mathrm{d}\rho \\ &\quad + \int_0^p \frac{(p-\rho)^{\xi-1}}{\Gamma(\chi)}\int_0^\rho e^{-d_2(\rho-\iota)}\int_0^\iota \frac{(\iota-r)^{\alpha-2}}{\Gamma(\alpha-1)}u_i(r)\,\mathrm{d}r\,\mathrm{d}\iota\,\mathrm{d}\rho\biggr] \end{aligned}$$

for all $t\in[0,1]$ and $i=1,\dots,k$. Since

$$\begin{aligned} &\mathrm{PH}_{d_Q}(G_i\bigl(s,y_1(s),\dots,y_k(s),y_1'(s),\dots,y_k'(s),y_1''(s),\dots,y_k''(s),\\ &\qquad G_i\bigl(s,z_1(s),\dots,z_k(s),z_1'(s),\dots,z_k'(s),z_1''(s),\dots,z_k''(s)\bigr)\bigr)\\ &\quad\leq M_i(s)\sum_{i=1}^k\bigl(\bigl|y_i(s)-z_i(s)\bigr|\bigr)+\bigl(\bigl|y_i'(s)-z_i'(s)\bigr|\bigr)+\bigl(\bigl|y_i''(s)-z_i''(s)\bigr|\bigr) \end{aligned}$$

for almost all $s\in[0,1]$ and $i=1,\dots,k$, there exists

$$u_i\in G_i\bigl(s,y_1(s),\dots,y_k(s),y_1'(s),\dots,y_k'(s),y_1''(s),\dots,y_k''(s)\bigr)$$

such that

$$\bigl|u_i(s)-u_i\bigr|\leq M_i(s)\sum_{i=1}^k\bigl(\bigl|y_i(s)-z_i(s)\bigr|\bigr)+\bigl(\bigl|y_i'(s)-z_i'(s)\bigr|\bigr)+\bigl(\bigl|y_i''(s)-z_i''(s)\bigr|\bigr)$$

for almost all $s\in[0,1]$ and $i=1,\dots,k$. Consider the multifunction $\mathcal{U}_i:[0,1]\to 2^{\mathbb{R}}$ by $\mathcal{U}_i(s)=\{u\in\mathbb{R}:|u_i(s)-u_i|\leq M_i(s)f(s)$ for almost all $s\in[0,1]\}$, where

$$f(s)=\sum_{i=1}^k\bigl(\bigl|y_i(s)-z_i(s)\bigr|\bigr)+\bigl(\bigl|y_i'(s)-z_i'(s)\bigr|\bigr)+\bigl(\bigl|y_i''(s)-z_i''(s)\bigr|\bigr).$$

Since u_i and $\phi_i = M_i(s)\sum_{i=1}^{k}(|y_i(s)-z_i(s)|)+(|y_i'(s)-z_i'(s)|)+(|y_i''(s)-z_i''(s)|)$ are measurable for all $i,\mathcal{U}_i(\cdot)\cap G_i(s,y_1(\cdot),\dots,y_k(\cdot),y_1'(\cdot),\dots,y_k'(\cdot),y_1''(\cdot),\dots,y_k''(\cdot))$ is a measurable multifunction. Thus, we can choose

$$u_i'(s)=G_i\bigl(s,y_1(s),\dots,y_k(s),y_1'(s),\dots,y_k'(s),y_1''(s),\dots,y_k''(s)\bigr)$$

such that

$$\begin{aligned}h_i'(s)&=\frac{1}{d_1}\int_0^s e^{-d_2(s-\rho)}\int_0^\rho\frac{(\rho-r)^{\alpha-2}}{\Gamma(\alpha-1)}u_i'(r)\,\mathrm{d}r\,\mathrm{d}\rho\\&\quad+\frac{1-e^{-d_2s}+(d_2^2-d_2)s}{d_1(\tilde{\Delta}_2-d_2\Omega^*)}\biggl[\int_0^1 e^{-d_2(1-\rho)}\int_0^\rho\frac{(\rho-r)^{\alpha-2}}{\Gamma(\alpha-1)}u_i'(r)\,\mathrm{d}r\,\mathrm{d}\rho\\&\quad+\int_0^p\frac{(p-\rho)^{\chi-1}}{\Gamma(\xi)}\int_0^\rho e^{-d_2(\rho-\iota)}\int_0^\iota\frac{(\iota-r)^{\alpha-2}}{\Gamma(\alpha-1)}u_i'(r)\,\mathrm{d}r\,\mathrm{d}\iota\,\mathrm{d}\rho\biggr]\end{aligned}$$

for all $s\in[0,1]$ and $i=1,\dots,k$. Since

$$\begin{aligned}\bigl|h_i(s)-h_{t_i}(s)\bigr|&=\frac{1}{d_1}\int_0^s e^{-d_2(s-\rho)}\int_0^\rho\frac{(\rho-r)^{\alpha-2}}{\Gamma(\alpha-1)}\bigl|u_i(r)-u_{t_i}(r)\bigr|\,\mathrm{d}r\,\mathrm{d}\rho\\&\quad+\frac{1-e^{-d_2s}+(d_2^2-d_2)s}{d_1(\tilde{\Delta}_2-d_2\Omega^*)}\\&\quad\times\biggl[\int_0^1 e^{-d_2(1-\rho)}\int_0^\rho\frac{(\rho-r)^{\alpha-2}}{\Gamma(\alpha-1)}\bigl|u_i(r)-u_{t_i}(r)\bigr|\,\mathrm{d}r\,\mathrm{d}\rho\\&\quad+\int_0^p\frac{(p-\rho)^{\chi-1}}{\Gamma(\xi)}\int_0^\rho e^{-d_2(\rho-\iota)}\int_0^\iota\frac{(\iota-r)^{\alpha-2}}{\Gamma(\alpha-1)}\bigl|u_i(r)-u_{t_i}(r)\bigr|\,\mathrm{d}r\,\mathrm{d}\iota\,\mathrm{d}\rho\biggr]\\&\le M_{i_1}\|\theta_i\|_{\mathcal{L}^1}\bigl\|(y_1-z_1,\dots,y_k-z_k)\bigr\|,\\\bigl|h_i'(s_2)-h_i'(s_1)\bigr|&\le\biggl|\frac{1}{d_1}\int_0^\rho\frac{(\rho-r)^{\alpha_i-2}}{\Gamma(\alpha_i-1)}\hat{u}_i'(r)\,\mathrm{d}r\,\mathrm{d}\rho\\&\quad-\frac{1}{d_1}\int_0^\rho\frac{(\rho-r)^{\alpha_i-2}}{\Gamma(\alpha_i-1)}\hat{u}_i'(r)\,\mathrm{d}r\,\mathrm{d}\rho\biggr|\\&\quad+\frac{(-d_2e^{-d_2s_1}-d_2e^{-d_2s_2})+|d_2^2-d_2|}{d_1|\tilde{\Delta}_{2_i}-d_2\Omega_i^*|}\\&\quad\times\biggl[\int_0^1 e^{-d_2(1-\rho)}\int_0^\rho\frac{(\rho-r)^{\alpha_i-2}}{\Gamma(\alpha_i-1)}\bigl|\hat{u}'_i(r)\bigr|\,\mathrm{d}r\,\mathrm{d}\rho\\&\quad+\int_0^p\frac{(p-\rho)^{\chi_i-1}}{\Gamma(\chi)_i}\int_0^\rho e^{-d_2(\rho-\iota)}\int_0^\iota\frac{(\iota-r)^{\alpha-2}}{\Gamma(\alpha_i-1)}\bigl|\hat{u}'_i(r)\bigr|\,\mathrm{d}r\,\mathrm{d}\iota\,\mathrm{d}\rho\biggr]\\&\le M_{i_2}\|\theta_i\|_{\mathcal{L}^1}\bigl\|(y_1-z_1,\dots,y_k-z_k)\bigr\|,\end{aligned}$$

and

$$\begin{aligned}\bigl|h_i''(s_2)-h_i''(s_1)\bigr|&\le\biggl|\frac{1}{d_1}\int_0^\rho\frac{(\rho-r)^{\alpha_i-2}}{\Gamma(\alpha_i-1)}\hat{u}_i''(r)\,\mathrm{d}r\,\mathrm{d}\rho\\&\quad-\frac{1}{d_1}\int_0^\rho\frac{(\rho-r)^{\alpha_i-2}}{\Gamma(\alpha_i-1)}\hat{u}_i''(r)\,\mathrm{d}r\,\mathrm{d}\rho\biggr|\end{aligned}$$

$$
\begin{aligned}
&+ \frac{(d_2^2 e^{-d_2 s_1} + d_2^2 e^{-d_2 s_2})}{d_1|\tilde{\Delta}_{2_i} - d_2\Omega_i^*|} \times \left[\int_0^1 e^{-d_2(1-\rho)} \int_0^\rho \frac{(\rho - r)^{\alpha_i - 2}}{\Gamma(\alpha_i - 1)} \left|\hat{u''}_i(r)\right| \mathrm{d}r\, \mathrm{d}\rho \right. \\
&\left. + \int_0^p \frac{(p-\rho)^{\chi_i - 1}}{\Gamma(\chi)_i} \int_0^\rho e^{-d_2(\rho - \iota)} \int_0^\iota \frac{(\iota - r)^{\alpha - 2}}{\Gamma(\alpha_i - 1)} \left|\hat{u''}_i(r)\right| \mathrm{d}r\, \mathrm{d}\iota\, \mathrm{d}\rho \right] \\
&\leq M_{i_3} \|\theta_i\|_{\mathcal{L}^1} \left\|(y_1 - z_1, \dots, y_k - z_k)\right\|,
\end{aligned}
$$

we get $\|h_i - h_{t_i}\| \leq (M_{i_1} + M_{i_2} + M_{i_3})\|\theta_i\|_{\mathcal{L}^1} \|(y_1 - z_1, \dots, y_k - z_k)\|$ for all $i = 1, \dots, k$. Hence,

$$
\begin{aligned}
&\left\|(h_1, \dots, h_k) - (h_{t_1}, \dots, h_{t_k})\right\| \\
&= \sum_{i=1}^k \|h_i - h_{t_i}\|_i \leq \sum_{i=1}^k M_{i_3} \|\theta_i\|_{\mathcal{L}^1} \left\|(y_1 - z_1, \dots, y_k - z_k)\right\| \\
&\leq A \left\|(y_1, \dots, y_k) - (z_1, \dots, z_k)\right\|.
\end{aligned}
$$

This implies that

$$
\mathrm{PH}_{d_Q}\big(H(y_1, \dots, y_k), H(z_1, \dots, z_k)\big) \leq A \left\|(y_1, \dots, y_k) - (z_1, \dots, z_k)\right\|,
$$

and so H is a closed-valued contractive multifunction. Now, by using Lemma 2.4 and Theorem 2.3, we deduce that H has a fixed point which is a solution for the non-hybrid inclusion system. □

We now present two examples to illustrate our main results.

Example 3.4 Consider the fractional two-dimensional fractional sequential differential inclusion system

$$
\begin{cases}
0.07({}^cD^{2.64} + 0.21{}^cD^{1.64})\left(\frac{v(s)}{0.0006 + \frac{s}{1000}(\arcsin v(s) + \sin({}^RI^{0.03} v(s)))}\right) \\
\quad \in [0, (s + \frac{1}{3}) \sin v(s) + \frac{1}{5}, \sin v(s) + (s + \frac{1}{5}) v'(s) \cos v(s)], \\
0.069({}^cD^{2.64} + 0.20{}^cD^{1.64})\left(\frac{v(s)}{0.0005 + \frac{s}{1000}(\arcsin v(s) + \sin({}^RI^{0.03} v(s)))}\right) \\
\quad \in [0, (s + \frac{1}{2}) \sin v(s) + \frac{1}{4}, \sin v(s) + (s + \frac{1}{4}) v'(s) \cos v(s)]
\end{cases}
\tag{21}
$$

with hybrid integro-derivative boundary conditions

$$
\begin{cases}
\left(\frac{v(s)}{0.0006 + \frac{s}{1000}(\arcsin v(s) + \sin({}^RI^{0.03} v(s)))}\right)|_{s=0} = 0, \\
{}^cD^1\left(\frac{v(s)}{0.0006 + \frac{s}{1000}(\arcsin v(s) + \sin({}^RI^{0.03} v(s)))}\right)|_{s=0} \\
\quad + {}^cD^2\left(\frac{v(s)}{0.0006 + \frac{s}{1000}(\arcsin v(s) + \sin({}^RI^{0.03} v(s)))}\right)|_{s=0} = 0, \\
\left(\frac{v(s)}{0.0006 + \frac{s}{1000}(\arcsin v(s) + \sin({}^RI^{0.03} v(s)))}\right)|_{s=1} \\
\quad + {}^RI^{0.32}\left(\frac{v(s)}{0.0006 + \frac{s}{1000}(\arcsin v(s) + \sin({}^RI^{0.03} v(s)))}\right)|_{s=0.4} = 0,
\end{cases}
\tag{22}
$$

where $s \in [0, 1]$, $\alpha = 2.64$, $d_{1_1} = 0.07$, $d_{1_2} = 0.069$, $d_{2_1} = 0.21$, $d_{2_2} = 0.20$, $\rho = 0.03$, and $\xi = 0.32$. Then we have $\tilde{\Delta}_1 \simeq 0.1576$, $\tilde{\Delta}_2 \simeq 0.008$, and $\Omega^* \simeq 0.1323$. Define the continuous map $\alpha : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ by $\alpha(s, v_1(s), v_2(s)) = 0.0006 + \frac{s}{1000}(\arcsin v_1(s) + \sin({}^RI^{0.03} v_2(s)))$

with $\alpha^* = \sup_{s\in[0,1]} |\alpha(s,0,0)| = 0.0007$. Let $\nu, \nu' \in \mathbb{R}$. Then we have

$$\begin{aligned}&\bigl|\alpha\bigl(s,\nu(s),\nu'(s)^{R}I^{\gamma}\nu(s)\bigr) - \alpha\bigl(s,\nu'(s),{}^{R}I^{\gamma}\nu'(s)\bigr)\bigr|\\ &\quad\le \nu(s)\biggl[1+\frac{s^{\gamma}}{\Gamma(\gamma+1)}\biggr]\bigl|\nu(s)-\nu'(s)\bigr| = \frac{s}{1000}\biggl[1+\frac{s^{0.04}}{\Gamma(1.04)}\biggr]\bigl|\nu(s)-\nu'(s)\bigr|,\end{aligned}$$

where $\nu(s) = \frac{s}{1000}$ and $\nu^* = \sup_{s\in[0,1]} |\nu(s)| = \frac{1}{1000}$. Note that the Lipschitz constant of the function α is $\nu^*[1+\frac{1}{\Gamma(\gamma+1)}] = \frac{1}{1000}[1+\frac{1}{\Gamma(1.03)}] \simeq 0.012021 > 0$. Consider the set-valued map $\mathcal{S} : [0,1]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ defined by

$$\begin{aligned}\mathcal{S}\bigl(s,\nu(s),\nu'(s)\bigr) = \biggl[0, &\biggl(s+\frac{1}{3}\biggr)\sin\nu(s)+\frac{1}{5}, \sin\nu(s)+\biggl(s+\frac{1}{5}\biggr)\nu'(s)\cos\nu(s), 0,\\ &\biggl(s+\frac{1}{2}\biggr)\sin\nu(s)+\frac{1}{4}, \sin\nu(s)+\biggl(s+\frac{1}{4}\biggr)\nu'(s)\cos\nu(s)\biggr].\end{aligned}$$

Since

$$|\nu| \le \max\biggl[0, \biggl(s+\frac{1}{4}\biggr)\sin\nu(s)+\frac{1}{2}\sin\nu(s)+\biggl(s+\frac{1}{5}\biggr)\nu'(s)\cos\nu(s)\biggr] \le s+0.35$$

for all $\nu \in \mathcal{S}(s,\nu(s))$,

$$\bigl\|\mathcal{S}\bigl(s,\nu(s),\nu'(s)\bigr)\bigr\| = \sup\bigl\{|\hat{\vartheta}| : \hat{\vartheta} \in \mathcal{S}\bigl(s,\nu(s),\nu'(s)\bigr)\bigr\} \le s+0.35.$$

Here, put $\theta(s) = s+0.35$ for all $s\in[0,1]$. Then

$$\|\theta\|_{\mathcal{L}^1} = \int_0^1 \bigl|\theta(r)\bigr|\,\mathrm{d}r = \int_0^1 (r+0.35)\,\mathrm{d}r = 1.15$$

and $M \simeq 117.7012$. Choose $q > 0.2474259$. Then

$$\nu^*\biggl[1+\frac{1}{\Gamma(\gamma+1)}\biggr]M\|q\|_{\mathcal{L}^1} \simeq (0.002022)(117.6114)(1.15) \simeq 0.343974.$$

Now, by using Theorem 3.1, hybrid system (21)–(22) has a solution.

Example 3.5 Consider the fraction two-dimensional hybrid differential inclusion system

$$\begin{cases}0.07({}^{c}D^{2.35}+0.21{}^{c}D^{1.35})q(s)\\ \quad\in [0, \frac{2e^s}{8}\cos q(s), \frac{-2e^s}{8}q'(s)\sin q(s), \frac{-2e^s}{8}q''(s)\sin q(s)+\frac{-2e^s}{8}q'(s)\cos q(s)],\\ 0.06({}^{c}D^{2.35}+0.20{}^{c}D^{1.35})q(s)\\ \quad\in [0, \frac{3e^s}{8}\cos q(s), \frac{-3e^s}{8}q'(s)\sin q(s), \frac{-3e^s}{8}q''(s)\sin q(s)+\frac{-3e^s}{8}q'(s)\cos q(s)]\end{cases} \tag{23}$$

with three-point integro-derivative boundary conditions

$$q(0) = 0, \qquad q'(0)+q''(0) = 0, \qquad q(1)+{}^{R}I^{0.32}q(0.4) = 0, \tag{24}$$

for all $s\in[0,1]$, where ${}^{c}D^{j}$ is the Caputo derivative of order $j \in \{2.35, 1.35\}$ and ${}^{R}I^{0.32}$ is the Riemann–Liouville integral of order 0.32. Put $\alpha = 2.35$, $d_{1_1} = 0.07$, $d_{1_2} = 0.06$, $d_{2,1} = 0.21$,

$d_{2,2} = 0.20$, and $\xi = 0.32$. One can find that $\tilde{\Delta}_1 \simeq 0.1246$, $\tilde{\Delta}_2 \simeq 0.007$, $\Omega^* \simeq 0.1656$, and $M \simeq 151.6013$. Define the set-valued map $\mathcal{S} : [0,1] \times \mathcal{Q} \to \mathcal{P}(\mathcal{Q})$ by

$$\begin{aligned}&\mathcal{S}\bigl(s, q(s), q'(s), q''(s)\bigr)\\&\quad= \left[0, \frac{2e^s}{8}\cos q(s), \frac{-2e^s}{8}q'(s)\sin q(s), \frac{-2e^s}{8}q''(s)\sin q(s) + \frac{-2e^s}{8}q'(s)\cos q(s)\right]\end{aligned}$$

for all $s \in [0,1]$. Consider the function $\delta \in \mathcal{C}_{\mathbb{R}\geq 0}([0,1])$ defined by $\delta(s) = \frac{2e^s}{8}$ for all s with $\|\delta\| = \frac{2e}{8} \simeq 1.8361$. Define the nondecreasing nonnegative function $\psi : [0,\infty) \to [0,\infty)$ by $\psi(s) = \frac{s}{2}$ for all $s > 0$. Note that ψ has the upper semi-continuity property

$$\lim \inf_{s\to\infty} \bigl(s - \psi(s)\bigr) > 0$$

and $\psi(s) < s$ for all $s > 0$. For every $q, q_i \in \mathcal{Q}$, we have

$$\begin{aligned}&\mathrm{PH}_{d_{\mathcal{Q}}}\bigl(\mathcal{S}\bigl(s, q(s), q'(s), q''(s)\bigr), \mathcal{S}\bigl(s, q_i'(s), q_i''\bigr)\bigr)\\&\quad\leq \frac{2e^s}{8}\frac{1}{2}\bigl(|q - q_i|\bigr) = \frac{2e^s}{8}\psi\bigl(|q - q_i|\bigr) \leq \delta(s)\psi\bigl(|q - q_i|\bigr)\frac{1}{M\|\delta\|},\end{aligned}$$

where $\frac{1}{M\|\delta\|} \simeq 0.002007$. Consider the operator $\mathcal{K} : \mathcal{Q} \to \mathcal{P}(\mathcal{Q})$ defined by

$$\mathcal{K}(q) = \bigl\{z \in \mathcal{Q} : \text{there is } \hat{v} \in (\mathcal{SEL})_{\mathcal{S},q} \text{ such that } z(s) = h(s) \text{ for any } s \in [0,1]\bigr\},$$

where

$$\begin{aligned}h(s) &= \frac{1}{0.07}\int_0^s e^{-0.21(s-\rho)}\int_0^\rho \frac{(\rho - r)^{2.35-2}}{\Gamma(2.35-1)}\hat{\vartheta}(r)\,\mathrm{d}r\,\mathrm{d}\rho\\&\quad+ \frac{1 - e^{-0.21s} + ((0.21)^2 - 0.21)s}{0.07(0.007 - (0.21)(0.1565))}\biggl[\int_0^1 e^{-0.21(1-\rho)}\int_0^\rho \frac{(\rho - r)^{2.53-2}}{\Gamma(2.35-1)}\hat{\vartheta}(r)\,\mathrm{d}r\,\mathrm{d}\rho\\&\quad+ \int_0^{0.4}\frac{(0.4-\rho)^{0.32-1}}{\Gamma(0.32)}\int_0^\rho e^{-0.21(\rho-\iota)}\int_0^\iota \frac{(\iota - r)^{2.35-2}}{\Gamma(2.35-1)}\hat{\vartheta}(r)\,\mathrm{d}r\,\mathrm{d}\iota\,\mathrm{d}\rho\biggr].\end{aligned}$$

Now, by using Theorem 3.3, the non-hybrid two-dimensional inclusion system (23)–(24) has a solution.

4 Conclusion

Today, most researchers try to review complicated versions of systems of differential equations to increase the ability to better model different versions of events in the world. One of the appropriate methods in this way is an investigation of hybrid and non-hybrid differential inclusion systems. We can use fractional sequential operators and inclusion systems for better modeling of some natural phenomena, but we first need to increase our abilities in the study of such systems. In this work, we examined two-hybrid and non-hybrid differential inclusion systems with integral boundary conditions. Finally, we provided two examples to illustrate our main results. The novelty of this work was mixing different ideas and techniques and also using a modern nonlinear technique for concluding the results.

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The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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