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# A fractional $q$-integral operator associated with a certain class of $q$-Bessel functions and $q$-generating series

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**Abstract**

This paper deals with Al-Salam fractional $q$-integral operator and its application to certain $q$-analogues of Bessel functions and power series. Al-Salam fractional $q$-integral operator has been applied to various types of $q$-Bessel functions and some power series of special type. It has been obtained for basic $q$-generating series, $q$-exponential and $q$-trigonometric functions as well. Various results and corollaries are provided as an application to this theory.

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## 1 Introduction

The theory of $q$-calculus is an old subject centered on the idea of deriving $q$-analogous results without using limits. Jackson was the first to develop the $q$-calculus theory in systematic way [1]. He defined the concept of the $q$-integral and the concept of the $q$-difference operator in a generic manner. In excellence, the theory of $q$-calculus allows to deal with sets of non-differentiable functions, different classes of orthogonal polynomials, integral operators, and various classes of special functions including $q$-hypergeometric functions, $q$-Bessel functions, $q$-gamma and $q$-beta functions, and many others, to mention but a few. It connects mathematics and physics and plays a significant role in various fields of physical sciences such as cosmic strings [2], conformal quantum mechanics [3], and nuclear physics of high energy [4]. It, further, applies to topics in number theory, combinatorics, orthogonal polynomials, basic hypergeometric functions, quantum theory, mechanics, and the theory of relativity.

The $q$-integrals from 0 to $\xi$ and from 0 to $\infty$ are, resp., defined by Jackson as [1]

$$\int_0^{\xi} f(t)\,d_q t = \xi(1-q)\sum_{j=0}^{\infty} q^j f\left(\xi q^j\right) \tag{1}$$

and

$$\int_0^{\infty/A} f(t)\,d_qt = (1-q)\sum_{j\in\mathbb{Z}} \frac{q^j}{A} f\left(\frac{q^j}{A}\right). \tag{2}$$

The $q$-analogue of the Bessel function

$$J_\mu(\xi) = \sum_{j=0}^{\infty} \frac{(-1)^j(\frac{\xi}{2})^{\mu+2j}}{j!\Gamma(\mu+j+1)} \tag{3}$$

of the first type, which was studied later by Hahn [5] and Ismail [6], is defined by [7] as

$$J_\mu^{(1)}(\xi;q) = \left(\frac{\xi}{2}\right)^\mu \sum_{j=0}^{\infty} \frac{(\frac{-\xi}{4}^2)^j}{(q,q)_{\mu+j}(q;q)_j}, \quad |\xi| < 2. \tag{4}$$

Jackson defines the $q$-analogue of the Bessel function of the second type as [7]

$$J_\mu^{(2)}(\xi;q) = \left(\frac{\xi}{2}\right)^\mu \sum_{j=0}^{\infty} \frac{q^{j(j+\mu)}(\frac{-\xi}{4}^2)^j}{(q;q)_{\mu+j}(q;q)_j}, \quad \xi \in \mathbb{C}. \tag{5}$$

Hahn [8] and Exton [9] introduced the third type $q$-Bessel function (called Hahn–Exton $q$-Bessel function) as

$$J_\mu^{(3)}(\xi;q) = \xi^\mu \sum_{j=0}^{\infty} \frac{(-1)^j q^{\frac{j(j-1)}{2}}(q\xi^2)^j}{(q;q)_{\mu+j}(q;q)_j}, \quad \xi \in \mathbb{C}. \tag{6}$$

The $q$-shifted factorials are defined, in literature, by fixing $\xi \in \mathbb{C}$ as

$$(\xi;q)_0 = 1; \qquad (\xi;q)_n = \prod_{j=0}^{n-1}\bigl(1-\xi q^k\bigr), \quad n = 1,2,\ldots; \qquad (\xi;q)_\infty = \lim_{n\to\infty}(\xi;q)_n. \tag{7}$$

This indeed gives

$$(\xi;q)_x = \frac{(\xi;q)_\infty}{(\xi q^x;q)_\infty}, \quad x \in \mathbb{R}. \tag{8}$$

For $\xi \in \mathbb{C}$, we mean

$$[\xi]_q = \frac{1-q^\xi}{1-q}.$$

Hence, for $n \in \mathbb{N}$, we obtain

$$\bigl([n]_q\bigr)! = \frac{(q;q)_n}{(1-q)^n}.$$

Due to [10, (1.5), (1.6)], we, resp., write

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} \tag{9}$$

and

$$\begin{bmatrix} \alpha \\ k \end{bmatrix}_q = \frac{(q^{-\alpha};q)_k}{(q;q)_k}(-1)^k q^{\alpha k-\binom{k}{2}} = \frac{\Gamma_q(\alpha+1)}{\Gamma_q(k+1)\Gamma_q(\alpha-k)}. \tag{10}$$

The $q$-analogue of the exponential function of the second type is given by

$$e_q(\xi) = \sum_{j=0}^{\infty} \frac{\xi^j}{(q;q)_j} = \frac{1}{(\xi;q)_\infty}, \quad |\xi| < 1, \tag{11}$$

whereas the $q$-analogue of the exponential function of the first type is given by

$$E_q(\xi) = \sum_{j=0}^{\infty} \frac{(-1)^j q^{j\frac{(j-1)}{2}} \xi^j}{(q;q)_j} = (\xi;q)_\infty, \quad \xi \in \mathbb{C}.$$

Consequently, the following formula holds:

$$\left(q^{\xi+m};q\right)_\infty = \frac{(q^\xi;q)_\infty}{(q^\xi;q)_m}, \quad m \in \mathbb{N}. \tag{12}$$

For real arguments $t$, the $q$-analogues of the gamma function are given by [11]

$$\Gamma_q(t) = \int_0^{\frac{1}{1-q}} x^{t-1} E_q(-qx)\, d_q x \quad \text{and} \quad \hat{\Gamma}_q(t) = \int_0^{\infty} x^{t-1} e_q(-x)\, d_q x. \tag{13}$$

Henceforth, for $t \in \mathbb{R}$ and $n \in \mathbb{N}$, the following auxiliary results hold:

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t), \qquad \Gamma_q(n+1) = [n]_q! \quad \text{and} \quad \Gamma_q(t+1) = \frac{1-q^t}{1-q}\Gamma_q(t). \tag{14}$$

The theory of fractional calculus was born in early 1695 due to a very deep question raised in a letter of L'Hospital to Leibniz [12–16]. During a long period of time (300 years), the fractional calculus has kept the attention of top level mathematicians. It has become a very useful tool for tackling dynamics of complex systems from various branches of science and engineering. The fractional $q$-calculus is the $q$-extension of the ordinary fractional calculus. Integral operators have attained their popularity due to their wide range of applications in various fields of science and engineering [17–22] and [23–34]. In [35, 36] Al-Salam and Agarwal studied certain $q$-fractional integrals and derivatives. Recently, perhaps due to explosion in research within the fractional calculus setting, new developments in the theory of fractional $q$-difference calculus, specifically, the $q$-analogues of the integral and the differential fractional operator properties were made, see, e.g., [37–39]. In [36, p. 966], Al-Salam defines a fractional $q$-integral operator in the form of the basic integral

$$K_q^\eta f(x) = \frac{q^{-\eta} x^\eta}{\Gamma_q(\alpha)} \int_x^\infty (y-x)_{\alpha-1} y^{-\eta-\alpha} f\left(yq^{1-\alpha}\right) d(y;q), \tag{15}$$

provided $\alpha \neq 0,-1,-2,\ldots$. With the aid of series definition (1), the above equation can be expressed as

$$K_q^{\eta} f(x) = (1-q)^{\alpha} \sum_{k=0}^{\alpha} (-1)^k q^{k(\eta+\alpha)+\frac{1}{2}k(k+1)} \begin{bmatrix} -\alpha \\ k \end{bmatrix} f\left(xq^{-\alpha-k}\right). \tag{16}$$

Consequently, by applying (9), (2) can be expressed as

$$K_q^{\eta,\alpha} f(x) = (1-q)^{\alpha} \sum_{k=0}^{\infty} (-1)^k q^{k(\eta+\alpha)+\frac{1}{2}k(k+1)} \left( \frac{(q;q)_{-\alpha}}{(q;q)_k (q;q)_{-\alpha-k}} \right) f\left(xq^{-\alpha-k}\right).$$

Therefore, it follows that

$$K_q^{\eta,\alpha} f(x) = \frac{(q;q)_{-\alpha}}{(1-q)^{-\alpha}} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(\eta+\alpha)+\frac{1}{2}k(k+1)}}{(q;q)_k (q;q)_{-\alpha-k}} f\left(xq^{-\alpha-k}\right). \tag{17}$$

In what follows, we discuss the Al-Salam fractional $q$-integral (15) on some special functions. We apply it to various types of $q$-Bessel functions and some power series of special type. In Sect. 1, we already recalled some definitions and notations from the fractional $q$-calculus theory. In Sect. 2, we apply the Al-Salam fractional $q$-integral to a finite product of $q$-Bessel functions. In Sect. 3, we apply the Al-Salam fractional $q$-integral to a power series. We also include some new applications. In Sect. 4, we apply the Al-Salam $q$-integral operator to some $q$-generating series.

## 2 Main results

**Theorem 1** *Let* $\{J_{2\mu_1}^{(1)}(2\sqrt{\delta_1 t};q),\ldots,J_{2\mu_n}^{(1)}(2\sqrt{\delta_n t};q)\}$ *be a set of first kind q-Bessel functions and*

$$f(t) = t^{\Delta-1} \prod_{j=1}^{n} J_{2\mu_j}^{(1)}(2\sqrt{\delta_j t};q). \tag{18}$$

*Then, for some* $B = q^{-\alpha(\Delta-1)} \frac{(q;q)_{-\alpha}}{(1-q)^{-\alpha}} x^{\Delta-1}$, *we have*

$$\begin{aligned} K_q^{\eta,\alpha} f(x) &= B \prod_{j=1}^{n} \left(\delta_j x q^{-\alpha}\right)^{\mu_j} \sum_{m=0}^{\infty} \delta_j x^m q^{-\alpha m} \frac{(q^{2\mu_j+m+1};q)_{\infty}}{\Gamma_q(m+1)} \\ &\quad \times \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(-m+\eta+\alpha+\frac{1}{2}k+\frac{3}{2}-\Delta)}}{\Gamma_q(k+1)\Gamma_q(1-\alpha-k)}. \end{aligned}$$

*Proof* By employing (18), the fractional $q$-integral (17) reveals

$$\begin{aligned} K_q^{\eta,\alpha} f(x) &= \frac{(q;q)_{-\alpha}}{(1-q)^{-\alpha}} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(\eta+\alpha)+\frac{1}{2}k(k+1)}}{(q;q)_k (q;q)_{-\alpha-k}} f\left(xq^{-\alpha-k}\right) \\ &= \frac{(q;q)_{-\alpha}}{(1-q)^{-\alpha}} \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(\eta+\alpha)+\frac{1}{2}k(k+1)}}{(q;q)_k (q;q)_{-\alpha-k}} \left(xq^{-\alpha-k}\right)^{\Delta-1} \\ &\quad \times \prod_{j=1}^{n} J_{2\mu_j}^{(1)}\left(2\sqrt{\delta_j x q^{-\alpha-k}};q\right) \end{aligned}$$

$$= \frac{(q;q)_{-\alpha}}{(1-q)^{-\alpha}} x^{\Delta-1} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(\eta+\alpha)+\frac{1}{2}k(k+1)-(\alpha+k)(\Delta-1)}}{(q;q)_k (q;q)_{-\alpha-k}} \times \prod_{j=1}^{n} J_{2\mu_j}^{(1)}\bigl(2\sqrt{\delta_j x q^{-\alpha-k}};q\bigr).$$

By taking into account the definition of the Bessel function $J_\nu^{(1)}$ given in (4), jointly with simple computations, the above equation reduces to yield

$$\begin{aligned} K_q^{\eta,\alpha} f(x) &= \frac{(q;q)_{-\alpha} x^{\Delta-1}}{(1-q)^{-\alpha}} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(\eta+\alpha)+\frac{1}{2}k(k+1)-(\alpha+k)(\Delta-1)}}{(q;q)_k (q;q)_{-\alpha-k}} \prod_{j=1}^{n} \bigl(a_j x q^{-\alpha-k}\bigr)^{\mu_j} \\ &\quad \times \sum_{m=0}^{\infty} \frac{(\delta_j x q^{-\alpha-k})^m}{(q,q)_{2\mu_j+m}(q,q)_m} \\ &= q^{-\alpha(\Delta-1)} \frac{(q;q)_{-\alpha} x^{\Delta-1}}{(1-q)^{-\alpha}} \prod_{j=1}^{n} \bigl(\delta_j x q^{-\alpha}\bigr)^{\mu_j} \sum_{m=0}^{\infty} \frac{(\delta_j x q^{-\alpha})^m q^{-km}}{(q;q)_{2\mu_j+m}(q;q)_m} \\ &\quad \times \sum_{k=0}^{\infty} (-1)^k \frac{q^{-km+k(\eta+\alpha)+\frac{1}{2}k(k+1)-k(\Delta-1)}}{(q;q)_k (q;q)_{-\alpha-k}}. \end{aligned}$$

Hence, by the fact [40, Equ. (8)]

$$(\zeta;q)_x = \frac{(\zeta;q)_\infty}{(\zeta q^x;q)_\infty}, \tag{19}$$

we obtain

$$\begin{aligned} K_q^{\eta,\alpha} f(x) &= q^{-\alpha(\Delta-1)} \frac{(q;q)_{-\alpha}}{(1-q)^{-\alpha}} x^{\Delta-1} \prod_{j=1}^{n} \delta_j^{\mu_j} \mu_j x q^{\alpha\mu_j-\alpha} q^{\mu_j} \\ &\quad \times \sum_{m=0}^{\infty} \frac{\delta_j^m x^m q^{-\alpha m} (q^{2\mu_j+m+1};q)_\infty}{(q;q)_m} \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(-m+\eta+\alpha+\frac{1}{2}k+\frac{3}{2}-\Delta)}}{\Gamma_q(k+1)\Gamma_q(1-\alpha-k)}. \end{aligned} \tag{20}$$

This completes the proof of the theorem. □

Now the identity

$$(q;q)_\alpha = \Gamma_q(\alpha+1) \tag{21}$$

leads to the following useful remark.

*Remark* 2 Let $\{J_{2\mu_1}^{(1)}(2\sqrt{\delta_1 t};q),\dots,J_{2\mu_n}^{(1)}(2\sqrt{\delta_n t};q)\}$ be a set of first kind $q$-Bessel functions and $f(t) = t^{\Delta-1}\prod_{j=1}^{n} J_{2\mu_j}^{(1)}(2\sqrt{\delta_j t};q)$. Then, for some $B = q^{-\alpha(\Delta-1)}\frac{(q;q)_{-\alpha}}{(1-q)^{-\alpha}} x^{\Delta-1}$, we have

$$\begin{aligned} K_q^{\eta,\alpha} f(x) &= B \prod_{j=1}^{n} \bigl(\delta_j x q^{-\alpha}\bigr)^{\mu_j} \sum_{m=0}^{\infty} \delta_j x^m q^{-\alpha m} \frac{(q^{2\mu_j+m+1};q)_\infty}{\Gamma_q(m+1)} \\ &\quad \times \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(-m+\eta+\alpha+\frac{1}{2}k+\frac{3}{2}-\Delta)}}{\Gamma_q(k+1)\Gamma_q(1-\alpha-k)}. \end{aligned}$$

*Proof* Indeed, from Theorem 1 and (21), we have

$$
\begin{aligned}
K_q^{\eta,\alpha} f(x) &= \frac{q^{-\alpha(\Delta-1)} x^{\Delta-1}}{(q;q)_\infty (1-q)^{-\alpha}} \Gamma_q(1-\alpha) \prod_{j=1}^{n} \left(\delta_j x q^{-\alpha\mu_j-\alpha}\right)^{\mu_j} \\
&\quad \times \sum_{m=0}^{\infty} \frac{(\delta_j x q^{-\alpha})^m (q^{\mu_j+m+1};q)_\infty}{\Gamma_q(m+1)} \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(-m+\eta+\alpha+\frac{1}{2}k+\frac{3}{2}-\Delta)}}{\Gamma_q(k+1)\Gamma_q(1-\alpha-k)} \\
&= B \prod_{j=1}^{n} \left(\delta_j x q^{-\alpha}\right)^{\mu_j} \sum_{m=0}^{\infty} \delta_j x^m q^{-\alpha m} \frac{(q^{2\mu_j+m+1};q)_\infty}{\Gamma_q(m+1)} \\
&\quad \times \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(-m+\eta+\alpha+\frac{1}{2}k+\frac{3}{2}-\Delta)}}{\Gamma_q(k+1)\Gamma_q(1-\alpha-k)}.
\end{aligned}
$$

This completes the proof of the remark. □

**Theorem 3** *Let* $J^{(2)}_{2\mu_1}(2\sqrt{\delta_1 t};q),\dots,J^{(2)}_{2\mu_n}(2\sqrt{\delta_n t};q)$ *and* $f(t)=t^{\Delta-1}\prod_{j=1}^{n}J^{(2)}_{2\mu_j}(2\sqrt{\delta_j t};q)$. *Then, for some* $A=\frac{(q;q)_{-\alpha}}{(1-q)^{-\alpha}}x^{\Delta-1}q^{-\alpha(\Delta-1)}$, *we have*

$$
\begin{aligned}
K_q^{\eta,\alpha} f(x) &= A \prod_{j=1}^{n} \delta_j^{\mu_j} x^{\mu_j} q^{-\alpha\mu_j} \sum_{m=0}^{\infty} q^{m(m+\mu_j)} \frac{(-\delta_j x q^{-\alpha-k})^m (q^{\mu_i+m+1};q_\infty)}{(q;q)_\infty \Gamma_q(m+k)} \\
&\quad \times \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(\eta+\alpha+k+\mu_j+\frac{3}{2}-\Delta)}}{\Gamma_q(1+k)\Gamma_q(1-\alpha-k)}.
\end{aligned}
$$

*Proof* Let the hypothesis of the theorem be satisfied. Then we have

$$
K_q^{\eta,\alpha} f(x) = (1-q)^{\alpha} \sum_{k=0}^{\infty} (-1)^k q^{k(\eta+\alpha)+\frac{1}{2}k(k+1)} \begin{bmatrix} -\alpha \\ k \end{bmatrix} f\left(xq^{-\alpha-k}\right).
$$

Therefore, in view of (18) and (3), we write

$$
\begin{aligned}
K_q^{\eta,\alpha} f(x) &= (1-q)^{\alpha} \sum_{k=0}^{\infty} (-1)^k q^{k(\eta+\alpha)+\frac{1}{2}k(k+1)} \frac{(q;q)_{-\alpha}}{(q;q)_k (q;q)_{-\alpha-k}} f\left(xq^{-\alpha-k}\right) \\
&= \frac{(q;q)_{-\alpha}}{(1-q)^{-\alpha}} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(\eta+\alpha)+\frac{1}{2}k(k+1)}}{(q;q)_k (q;q)_{-\alpha-k}} \left(xq^{-\alpha-k}\right)^{\Delta-1} \\
&\quad \times \prod_{j=1}^{n} J^{(2)}_{2\mu_j}\left(\sqrt[2]{\delta_j x q^{-\alpha-k}};q\right) \\
&= \frac{(q;q)_{-\alpha}}{(1-q)^{-\alpha}} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(\eta+\alpha)+\frac{1}{2}k(k+1)}}{(q;q)_k (q;q)_{-\alpha-k}} \left(xq^{-\alpha-k}\right)^{\Delta-1} \\
&\quad \times \prod_{j=1}^{n} \left(\delta_j x q^{-\alpha-k}\right)^{\mu_j} \sum_{m=0}^{\infty} \frac{q^{m(m+\mu_j)}(-\delta_j x q^{-\alpha-k})^m}{(q;q)_{\mu_j+m}(q;q)_m}
\end{aligned}
$$

$$= \frac{(q;q)_{-\alpha}}{(1-q)^{-\alpha}} x^{\Delta-1} q^{-\alpha(\Delta-1)} \sum_{k=0}^{\infty} (-1)^{q} \frac{q^{k(\eta+\alpha)+\frac{1}{2}k(k+1)-k(\Delta-1)}}{(q;q)_k (q;q)_{-\alpha-k}}$$
$$\times \prod_{j=1}^{n} \delta_j^{\mu_j} x^{\mu_j} q^{-(\alpha+k)\mu_j} \sum_{m=0}^{\infty} \frac{q^{m(m+\mu_j)}(-\delta_j x q^{-\alpha-k})^m}{(q;q)_{\mu_j+m}(q,q)_m}.$$

Hence, it yields

$$K_q^{\eta,\alpha} f(x) = \frac{(q;q)_{-\alpha}}{(1-q)^{-\alpha}} x^{\Delta-1} q^{-\alpha(\Delta-1)} \prod_{j=1}^{n} a_j^{\mu_j} x^{\mu_j} q^{-(\alpha+k)\mu_j}$$
$$\times \sum_{m=0}^{\infty} \frac{q^{m(m+\mu_j)}(-a_j x q^{-\alpha-k})^m}{(q;q)_{\mu_j+m}(q;q)_m} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(\eta+\alpha)+\frac{1}{2}k(k+1)-k(\Delta-1)+\mu_j k}}{(q;q)_k (q;q)_{-\alpha-k}}. \tag{22}$$

By the fact $(q;q)_k = \Gamma_q(1+k)$ and the identity

$$(\zeta;q)_x = \frac{(q;q)_\infty}{(\zeta q^x;q)_\infty}, \tag{23}$$

we write

$$K_q^{\eta,\alpha} f(x) = A \prod_{j=1}^{n} \delta_j^{\mu_j} x^{\mu_j} q^{-\alpha\mu_j} \sum_{m=0}^{\infty} \frac{q^{m(m+\mu_j)}(\delta_j x q^{-\alpha-k})^m}{(q;q)_\infty (q;q)_m} \left(q^{\mu_i+m+1};q\right)_\infty$$
$$\times \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(\eta+\alpha)+\frac{1}{2}k(k+1)-k(\Delta-1)+\mu_j k}}{(q;q)_k (q;q)_{-\alpha-k}}$$
$$= A \prod_{j=1}^{n} \delta_j^{\mu_j} x^{\mu_j} q^{-\alpha\mu_j} \sum_{m=0}^{\infty} q^{m(m+\mu_j)} \frac{(-\delta_j x q^{-\alpha-k})^m (q^{\mu_i+m+1};q)_\infty}{(q;q)_\infty \Gamma_q(m+k)}$$
$$\times \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(\eta+\alpha+k+\mu_j+\frac{3}{2}-\Delta)}}{\Gamma_q(1+k)\Gamma_q(1-\alpha-k)}.$$

This completes the proof of the theorem. □

**Theorem 4** *Let $J_{2\mu_1}^{(3)}(2\sqrt{q^{-1}\delta_1 t};q), \dots, J_{2\mu_n}^{(3)}(2\sqrt{q^{-1}\delta_n t};q)$ be n q-Bessel functions and*

$$f(t) = t^{\Delta-1} \prod_{j=1}^{n} \delta^{\mu_j} J_{2\mu_j}^{(3)}\left(2\sqrt{q^{-1}\delta_n t};q\right).$$

*Then we have*

$$K_q^{\eta,\alpha} f(x) = \frac{x^{\Delta-1}\Gamma_q(1-\alpha)(1-q)^\alpha}{(q;q)_\infty} \prod_{j=1}^{n} \delta_j^{\mu_j} x^{\mu_j} q^{(-\alpha-k)\mu_j}$$
$$\times \sum_{m=0}^{\infty} (-1)^m \frac{q^{m\frac{(m-1)}{2}+m(-\alpha)} x^m \delta_j^m (q^{2\mu_j+m+1};q)_\infty}{(q;q)_m}$$
$$\times \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(\eta+\alpha)+\frac{1}{2}k(k+1)+(\Delta-1)(-\alpha-k)-mk}}{\Gamma_q(k+1)\Gamma_q(-\alpha-k)}.$$

*Proof* By (2) and (6), we obtain

$$\begin{aligned}
K_q^{\eta,\alpha} f(x) &= (1-q)^\alpha \sum_{k=0}^{\infty} (-1)^k q^{k(\eta+\alpha)+\frac{1}{2}k(k+1)} \begin{bmatrix} -\alpha \\ k \end{bmatrix} f\left(xq^{-\alpha-k}\right) \\
&= (1-q)^\alpha \sum_{k=0}^{\infty} (-1)^k q^{k(\eta+\alpha)+\frac{1}{2}k(k+1)} \begin{bmatrix} -\alpha \\ k \end{bmatrix} \left(xq^{-\alpha-k}\right)^{\Delta-1} \\
&\quad \times \prod_{j=1}^{n} q^{\mu j} J_{2\mu_j}^{(3)} \left(\sqrt{q^{-1}\delta_j x q^{-\alpha-k}}; q\right) \\
&= (1-q)^\alpha \sum_{k=0}^{\infty} (-1)^k q^{k(\eta+\alpha)+\frac{1}{2}k(k+1)} \begin{bmatrix} -\alpha \\ k \end{bmatrix} \left(xq^{-\alpha-k}\right)^{\Delta-1} \\
&\quad \times \prod_{j=1}^{n} q^{\mu j} \left(q^{-1}\delta_j x q^{-\alpha-k}\right)^{\mu j} \sum_{m=0}^{\infty} (-1)^m \frac{q^{m\frac{(m-1)}{2}} (qq^{-1}\delta_j q^{-\alpha-k})^m}{(q;q)_{2\mu_j+m} (q;q)_m}.
\end{aligned}$$

Equations (10), (21), and simple simplifications reveal

$$\begin{aligned}
K_q^{\eta,\alpha} f(x) &= x^{\Delta-1} (1-q)^\alpha \sum_{k=0}^{\infty} (-1)^k q^{k(\eta+\alpha)+\frac{1}{2}k(k+1)+(\Delta-1)(-\alpha-k)} \frac{\Gamma_q(1-\alpha)}{\Gamma_q(k+1)\Gamma_q(-\alpha-k)} \\
&\quad \times \prod_{j=1}^{n} \delta_j^{\mu_j} x^{\mu_j} q^{(-\alpha-k)\mu_j} \sum_{m=0}^{\infty} (-1)^m \frac{q^{m\frac{(m-1)}{2}+m(-\alpha-k)} x^m \delta_j^m (q^{2\mu_j+m+1};q)_\infty}{(q;q)_\infty (q;q)_m} \\
&= \frac{x^{\Delta-1}\Gamma_q(1-\alpha)(1-q)^\alpha}{(q;q)_\infty} \prod_{j=1}^{n} \delta_j^{\mu_j} x^{\mu_j} q^{(-\alpha-k)\mu_j} \\
&\quad \times \sum_{m=0}^{\infty} (-1)^m \frac{q^{m\frac{(m-1)}{2}+m(-\alpha)} x^m \delta_j^m (q^{2\mu_j+m+1};q)_\infty}{(q;q)_m} \\
&\quad \times \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(\eta+\alpha)+\frac{1}{2}k(k+1)+(\Delta-1)(-\alpha-k)-mk}}{\Gamma_q(k+1)\Gamma_q(-\alpha-k)}.
\end{aligned}$$

This completes the proof of the theorem. □

## 3 The fractional $q$-integral of the power series

This section is briefly devoted to the application of the fractional $q$-integral to functions of a power series form. Some corollaries associated with polynomials and unit functions are also deduced.

**Theorem 5** *Let $g(x) = \sum_{i=0}^{\infty} r_i x^i$ be a power series and $\beta$ be a positive real number. If $f(x) = (x^{\beta-1}g)(x)$, then we have*

$$K_q^{\eta,\alpha} f(x) = \frac{q^{-\alpha\beta+\alpha} x^{\beta-1} (q;q)_{-\alpha}}{(1-q)^{-\alpha}} \sum_{i=0}^{\infty} r_i q^{-\alpha i} x^i \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(\eta+\alpha)+\frac{1}{2}k+\frac{1}{2}-i}}{\Gamma_q(k)\Gamma_q(-\alpha-k)}.$$

*Proof* Let $g(x)=\sum_{i=0}^{\infty} r_i x^i$ be a power series and $\beta$ be a positive real number. From (26) it follows

$$K_q^{\eta,\alpha} f(x) = \frac{(q;q)_{-\alpha}}{(1-q)^{-\alpha}} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(\eta+\alpha)+\frac{1}{2}k(k+1)}}{(q;q)_k (q;q)_{-\alpha-k}} f\left(xq^{-\alpha-k}\right)$$
$$= \frac{(q;q)_{-\alpha}}{(1-q)^{-\alpha}} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(\eta+\alpha)+\frac{1}{2}k(k+1)}}{(q;q)_k (q;q)_{-\alpha-k}} \left(xq^{-\alpha-k}\right)^{\beta-1} \sum_{i=0}^{\infty} r_i \left(xq^{-\alpha-k}\right)^i. \tag{24}$$

Interchanging the order of summation in (24) leads to

$$K_q^{\eta,\alpha} f(x) = \frac{q^{-\alpha\beta+\alpha} x^{\beta-1} (q;q)_{-\alpha}}{(1-q)^{-\alpha}} r_i q^{-\alpha i} x^i \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(\eta+\alpha)+\frac{1}{2}k(k+1)-k_i}}{(q;q)_k (q;q)_{-\alpha-k}}.$$

Employing (21) indeed gives

$$K_q^{\eta,\alpha} f(x) = \frac{q^{-\alpha\beta+\alpha} x^{\beta-1} (q;q)_{-\alpha}}{(1-q)^{-\alpha}} \sum_{i=0}^{\infty} r_i q^{-\alpha i} x^i \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(\eta+\alpha)+\frac{1}{2}k+\frac{1}{2}-i}}{\Gamma_q(k)\Gamma_q(-\alpha-k)}.$$

Hence, the proof of the theorem is completed. □

**Corollary 6** *Let $\beta > 0$ be a real number. Then we have*

$$K_q^{\eta,\alpha}\left(x^{\beta-1}\right) = \frac{q^{-\alpha\beta+\alpha} x^{\beta-1}}{(1-q)^{-\alpha}} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k((\eta+\alpha)+\frac{1}{2}k+\frac{1}{2})}}{\Gamma_q(k)\Gamma_q(-\alpha-k)}.$$

*This result follows from setting $r_0 = 1$ and $r_i = 0$ for $i = 1, 2, 3, \ldots$.*

**Corollary 7** *We have*

$$K_q^{\eta,\alpha}(1) = \frac{(q;q)_{-\alpha}}{(1-q)^{-\alpha}} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(\eta+\alpha+\frac{1}{2}k+\frac{1}{2})}}{\Gamma_q(k)\Gamma_q(-\alpha-k)}. \tag{25}$$

## 4 $K_q^{\eta,\alpha}$ of $q$-generating Heines series

The basic $q$-generating series of the first type is defined by [41] as

$${}_r\phi_s(\delta_1,\ldots,\delta_r; b_1,\ldots,b_s, q, \zeta) = \sum_{i\geq 0} \frac{(\delta_1;q)_1,\ldots,(\delta_r;q)_i}{(q;q)_i,(b_1;q)_i,\ldots,(b_s;q)_i} \left((-1)^i q^{(2^i)}\right)^{1+s-r} \zeta^i,$$

where

$$\left(2^i\right) = \frac{i(i-1)}{2}, \qquad r > s+1, \qquad q > 0. \tag{26}$$

The basic $q$-generating series of the second type is given as

$${}_r\psi_s(\delta_1,\ldots,\delta_r; \grave{\delta}_1,\ldots,\grave{\delta}_s, q, \zeta) = \sum_{i\geq 0}^{\infty} \frac{(\delta_1;q)_1,\ldots,(\delta_r;q)_i}{(q;q)_i,(\grave{\delta}_1;q)_i,\ldots,(\grave{\delta}_s;q)_i} \left((-1)^i q^{(2^i)}\right)^{s-r} \zeta^i. \tag{27}$$

The parameters $b_1,\ldots,b_s$ are given so that the denominator factors in terms of the series are never zero, and the basic series terminates when one of its numerator parameters is of type $q^{-n}$, $n=0,1,2,\ldots$ .

**Theorem 8** *Let $\beta$ and $\gamma$ be real numbers. Then, provided $\beta>0$, we have*

$$\begin{aligned}
&K_q^{\eta,\alpha}\left(x^{\beta-1}\right)_r\phi_s(\delta_1,\ldots,\delta_r;\grave{\delta}_1,\ldots,\grave{\delta}_s;q,\gamma x)\\
&\quad=\frac{q^{-\alpha\beta+\alpha}x^{\beta-1}(q;q)_{-\alpha}}{(1-q)^{-\alpha}}\\
&\qquad\times\sum_{i=0}^{\infty}r_iq^{-\alpha i}x^i\sum_{k=0}^{\infty}(-1)^k\frac{q^{k(\eta+\alpha+\frac{1}{2}k+\frac{1}{2}-i)}}{\Gamma_q(k)\Gamma_q(_{-\alpha-k})}.
\end{aligned}$$

*Proof* Let $\beta$ and $\gamma$ be real numbers. Then, by (17), we write

$$\begin{aligned}
&K_q^{\eta,\alpha}\left(x^{\beta-1}\right)_r\phi_s(\delta_1,\ldots,\delta_r;\grave{\delta}_1,\ldots,\grave{\delta}_s;q,\gamma x)\\
&\quad=\frac{(q;q)_{-\alpha}}{(1-q)^{-\alpha}}\sum_{k=0}^{\infty}(-1)^k\frac{q^{k(\eta+\alpha)+\frac{1}{2}k(k+1)}}{(q;q)_k(q;q)_{-\alpha-k}}\\
&\qquad\times f\left(xq^{-\alpha-k}\right)\left(xq^{-\alpha-k}\right)^{\beta-1}{}_r\phi_s\left(\delta_1,\ldots,\delta_r;\grave{\delta}_1,\ldots,\grave{\delta}_s;q,\gamma xq^{-\alpha-k}\right).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&{}_r\phi_s\left(\delta_1,\ldots,\delta_r;\grave{\delta}_1,\ldots,\grave{\delta}_s;q,\gamma xq^{-\alpha-k}\right)\sum_{i\geq 0}^{\infty}\frac{(\delta_1;q)_i,\ldots,(\delta_r;q)_i}{(q;q)_i,(\grave{\delta}_1;q)_i,\ldots,(\grave{\delta}_s;q)_i}\left((-1)^iq^{(2^i)}\right)^{s-r}\\
&\quad\times\left(\gamma xq^{-\alpha-k}\right)^i=\sum_{i\geq 0}^{\infty}r_ix^i,
\end{aligned}$$

where

$$r_i=\frac{(\delta_1;q)_i,\ldots,(\delta_r;q)_i}{(q;q)_i,(\grave{\delta}_1;q)_i,\ldots,(\grave{\delta}_s;q)_i}\left((-1)^iq^{(2^i)}\right)^{s-r}\gamma^iq^{(-\alpha-k)i}. \tag{28}$$

Therefore, by Theorem 5 we get

$$\begin{aligned}
K_q^{\eta,\alpha}\left(x^{\beta-1}\right)_r\phi_s(\delta_1,\ldots,\delta_r;\grave{\delta}_1,\ldots,\grave{\delta}_s;q,\gamma x)&=\frac{q^{-\alpha\beta+\alpha}x^{\beta-1}(q;q)_{-\alpha}}{(1-q)^{-\alpha}}\sum_{i=0}^{\infty}r_iq^{-\alpha i}x^i\\
&\quad\times\sum_{k=0}^{\infty}(-1)^k\frac{q^{k(\eta+\alpha+\frac{1}{2}k+\frac{1}{2}-i)}}{\Gamma_q(k)\Gamma_q(_{-\alpha-k})}.
\end{aligned}\tag{29}$$

This completes the proof of the theorem. □

**Theorem 9** *Let $\beta > 0$ and $r$ be real numbers. Then we have*

$$K_q^{\eta,\alpha}\left(x_r^{\beta-1}\psi_s(\delta_1,\ldots,\delta_r;\grave{\delta}_1,\ldots,\grave{\delta}_s;q,\gamma x)\right) = \frac{q^{-\alpha\beta+\alpha}x^{\beta-1}(q;q)_{-\alpha}}{(1-q)^{-\alpha}}\sum_{i=0}^{\infty} r_i q^{-\alpha i}x^i \times \sum_{k=0}^{\infty}(-1)^k \frac{q^{k(\eta+\alpha+\frac{1}{2}k+\frac{1}{2}-i)}}{\Gamma_q(k)\Gamma_q(-\alpha-k)}.$$

*Proof* By taking into account (20), we write

$$K_q^{\eta,\alpha}\left(x^{\beta-1}{}_r\psi_s\right) = \frac{(q;q)_{-\alpha}}{(1-q)^{-\alpha}}\sum_{k=0}^{\infty}(-1)^k \frac{q^{k(\eta+\alpha)+\frac{1}{2}k+\frac{1}{2}}}{(q;q)_k(q;q)_{-\alpha,k}} f\left(xq^{-\alpha-k}\right)^{\beta-1} \times f\left(xq^{-\alpha-k}\right)^{\beta-1}{}_r\psi_s\left(\delta_1,\ldots,\delta_r;\grave{\delta}_1,\ldots,\grave{\delta}_s;q^{-\alpha-k}\right). \tag{30}$$

However,

$${}_r\psi_s\left(\delta_1,\ldots,\delta_r;\grave{\delta}_1,\ldots,\grave{\delta}_s;q^{-\alpha-k}\right) = \sum_{i\geq 0}^{\infty}\frac{(\delta_1;q)_i,\ldots,(\delta_r;q)_i}{(q;q)_i,(\grave{\delta}_1;q)_i,\ldots,(\grave{\delta}_s;q)_i}\left((-1)^i q^{(2^i)}\right)^{s-r} \times \left(\gamma x q^{-\alpha-k}\right)^i = \sum_{i\geq 0}^{\infty} r_i x^i,$$

where

$$r_i = \frac{(\delta_1;q)_i,\ldots,(\delta_r;q)_i}{(q;q)_i,(\grave{\delta}_1;q)_i,\ldots,(\grave{\delta}_s;q)_i}\left((-1)^i q^{(2^i)}\right)^{s-r}\gamma^i q^{(-\alpha-k)i}. \tag{31}$$

Hence, by Theorem 5 it follows

$$K_q^{\eta,\alpha}\left(x^{\beta-1}{}_r\psi_s\right) = \frac{q^{-\alpha\beta+\alpha}x^{\beta-1}(q;q)_{-\alpha}}{(1-q)^{-\alpha}}\sum_{i=0}^{\infty} r_i q^{-\alpha i}x^i\sum_{k=0}^{\infty}(-1)^k \frac{q^{k(\eta+\alpha+\frac{1}{2}k+\frac{1}{2}-i)}}{\Gamma_q(k)\Gamma_q(-\alpha-k)}.$$

This completes the proof of the theorem. □

**Corollary 10** *Let $\gamma$ be a real number. Then we have*

$$K_q^{\eta,\alpha}\left(E_q(\gamma x)\right) = \frac{(q;q)_{-\alpha}}{(1-q)^{-\alpha}}\sum_{i=0}^{\infty} r_i q^{-\alpha i}x^i\sum_{k=0}^{\infty}(-1)^k \frac{q^{k(\eta+\alpha+\frac{1}{2}k-i)}}{\Gamma_q(k)\Gamma_q(-\alpha-k)}.$$

*Proof* By setting $\beta = 0$, $r = 0$, and $s = 0$, the result easily follows from Theorem 8. The proof is completed. □

**Corollary 11** *Let $\gamma$ be a real number. Then we have*

$$\left(K_q^{\eta,\alpha} e_q(\gamma x)\right) = \frac{(q;q)_{-\alpha}}{(1-q)^{-\alpha}}\sum_{i=0}^{\infty} r_i q^{-\alpha i}x^i\sum_{k=0}^{\infty}(-1)^k \frac{q^{k(\eta+\alpha+\frac{1}{2}k+\frac{1}{2}-i)}}{\Gamma_q(k)\Gamma_q(-\alpha-k)}.$$

*Proof* By setting $\beta = 1$, $r = 0$, and $s = 0$, Theorem 8 completes the proof of the corollary. □

The proof of the following corollary is straightforward. Details are therefore deleted.

**Corollary 12** *Let $\gamma$ be a real number. Then we have*

$$(i) K_q^{\eta,\alpha}\big(\sinh_q(\gamma x)\big) = K_q^{\eta,\alpha}\left(\frac{E_q(\gamma x) - E_q(-\gamma x)}{2}\right) = \frac{(q;q)_{-\alpha}}{(1-q)^{-\alpha}} \sum_{i=0}^{\infty} r_i q^{-\alpha i} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(\eta+\alpha+\frac{1}{2}k-i)}}{\Gamma_q(k)\Gamma_q(_{-\alpha-k})} x^i \frac{(1+(-1)^{i+1})}{2}.$$

$$(ii) K_q^{\eta,\alpha}\big(\cosh_q(\gamma x)\big) = K_q^{\eta,\alpha}\left(\frac{E_q(\gamma x) - E_q(-\gamma x)}{2}\right) = \frac{(q;q)_{-\alpha}}{(1-q)^{-\alpha}} \sum_{i=0}^{\infty} r_i q^{-\alpha i} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(\eta+\alpha+\frac{1}{2}-i)}}{\Gamma_q(k)\Gamma_q(_{-\alpha-k})} x^i \frac{(1+(-1)^{i})}{2}.$$

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