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Study on asymptotic behavior of stochastic Lotka–Volterra system in a polluted environment

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Abstract

A three-species non-autonomous stochastic Lotka–Volterra food web system in a polluted environment is proposed, and the existence of positive periodic solutions of this system is established by constructing a proper Lyapunov function. Then the extinction property and its threshold between persistence and extinction are discussed by using Itô's formula and the strong law of large numbers of martingale, and the sufficient condition of a.s. exponential stability of equilibrium point is obtained. Finally, the conclusions are tested by several numerical simulations.

Keywords: Lotka–Volterra food web system; Positive periodic solution; Lyapunov function; Exponential stability

1 Introduction

Lotka–Volterra predator-prey model has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology owing to its universal importance, which can well explain the dynamic relationship between predators and their preys [1, 2]. Among those predator-prey models, a three-species omnivorous food web system plays an important role. Its form is as follows [3]:

$$\begin{cases} \dot{x}(t) = x(t)[D_1(t) - a_{11}(t)x(t) - a_{12}(t)y(t) - a_{13}(t)z(t)], \\ \dot{y}(t) = y(t)[-D_2(t) + a_{21}(t)x(t) - a_{22}(t)z(t)], \\ \dot{z}(t) = z(t)[-D_3(t) + a_{31}(t)x(t) + a_{32}(t)y(t)], \end{cases} \tag{1}$$

where $x(t)$, $y(t)$, $z(t)$ denote prey, middle-predator (also prey), and omnivorous top-predator. $D_1(t)$ is the intrinsic growth rate of $x(t)$, $D_2(t)$ and $D_3(t)$ are the death rate of $y(t)$ and $z(t)$ respectively. $a_{11}(t)$ denotes the coefficient of intraspecific competition in the resource, $a_{21}(t)$, $a_{31}(t)$, $a_{32}(t)$ measure the contributions of the victim to the growth of consumer, $a_{12}(t)$, $a_{13}(t)$, $a_{22}(t)$ are the rate of consumption [3, 4].

Nowadays, a lot of scholars have been studying the deterministic food web system [3–5]. Hsu et al. proposed the sufficient conditions of extinction, persistence, uniform persistence, and the existence condition of periodic solutions of the system [3]. Namba ana-

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lyzed bifurcation and chaos of the system [4]. Krikorian proposed the conditions of global asymptotic stability and global boundedness of the system solution [5]. However, all the research works are focused on the deterministic system. On account of the influence of white noise in an environment, it is hard to simulate reality efficiently and to protect the future of population precisely by the deterministic system. Thus, it is necessary to put stochastic perturbation into consideration to describe the influence of white noise in a food web system. Considering the mutual influence with the functional responses only depends on prey density, Liu [6] proposed half-saturation constant, established sufficient conditions for the existence of an ergodic stationary distribution to the model. Inspired by [6], our attention has been paid to the behavior of solutions of the food web system when there are stochastic noises. Also, as we know, environmental pollution is another important factor for population survival, which enabled more and more studies on the influence of pollution on population [7–10]. Thus, we will consider both stochastic perturbation and environmental pollution into the system to establish how the environmental noise and pollution affect the behavior of solutions on the food web system with omnivory. Environmental noise always has an influence on the intrinsic growth rate of population, that is,

$$D_1(t) \to D_1(t) + \alpha(t)\dot{B}_1(t),$$
$$D_2(t) \to D_2(t) + \beta(t)\dot{B}_2(t),$$
$$D_3(t) \to D_3(t) + \gamma(t)\dot{B}_3(t),$$

where $B_i(t)$ ($i = 1, 2, 3$) are dependent standard Brownian motions, $\alpha(t)$, $\beta(t)$, $\gamma(t)$ are disturbing intensities of three intrinsic growth rates.

Then, putting environmental pollution into system (1), we can obtain the important model in this work:

$$\begin{cases} dx(t) = x(t)[D_1(t) - a_{11}(t)x(t) - a_{12}(t)y(t) - a_{13}(t)z(t) - \delta_1(t)S(t)]\,dt \\ \qquad\quad + \alpha(t)x(t)\,dB_1(t), \\ dy(t) = y(t)[-D_2(t) + a_{21}(t)x(t) - a_{22}(t)z(t) - \delta_2(t)S(t)]\,dt + \beta(t)y(t)\,dB_2(t), \\ dz(t) = z(t)[-D_3(t) + a_{31}(t)x(t) + a_{32}(t)y(t) - \delta_3(t)S(t)]\,dt + \gamma(t)y(t)\,dB_3(t), \\ dS(t) = [k(t)T(t) - g(t)S(t) - m(t)S(t)]\,dt, \\ dT(t) = [-h(t)T(t) + f(t)]\,dt, \end{cases} \qquad (2)$$

where $S(t)$, $T(t)$ denote the toxin concentrations in organism and environment separately at time t [9]. Thus, $0 \le S(t)$, $T(t) \le 1$ for every $t \ge 0$. The input rate of the exotic toxin $f(t)$ is a control function, and this kind of toxin-population model was proposed by Hallam et al. [7].

On the other hand, the periodic solution of a predator-prey stochastic system has been studied in [11–14], where the persistence and the global stability of periodic solution of a three-species omnivorous food web system were studied by Zhou et al. [11]. Ma et al. discussed the persistence of periodic solution and uniformly asymptotic stability of a discrete competitive system [12]. However, almost all the references focused on the discrete system, only a few scholars discussed the continuous system, among which there are those about a non-omnivorous system [15, 16].

Inspired by the existing research results, we study the periodic solution, extinction, and exponential stability of a stochastic omnivorous food web system in a polluted environment. The differences of conclusion between this paper and others (e.g. Hsu et al. [3], Namba et al. [4], Liu [6], Zhou et al. [11], Zu et al. [15]) are as follows: (a) introduction of white noise in this system; (b) consideration of the effect of environmental pollution, which generalizes the results in [3] and [6].

The rest of the paper is organized as follows. In Sect. 2, some assumptions, definitions, and lemmas are given. Then the existence of positive periodic solutions is proved by constructing a proper Lyapunov function in Sect. 3. In Sect. 4, by using Itô's formula and the strong law of large numbers of martingale, the conditions of extinction of solution are discussed. In Sect. 5, the exponential stability of the equilibrium is considered. Then, numerical simulations are provided in Sect. 6. Finally, conclusions and discussion are given in Sect. 7.

2 Preliminaries

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right-continuous and $\mathscr{F}_0$ contains all p-null sets). For convenience's sake, let

$$g^u := \max_{t\in[0,\theta]} g(t), \qquad g^l := \min_{t\in[0,\theta]} g(t),$$

$$\langle g \rangle := \frac{1}{t}\int_0^t g(s)\,ds,$$

$$g^* := \limsup_{t\to +\infty} g(t), \qquad g_* := \liminf_{t\to +\infty} g(t),$$

where $g(t)$ is a continuous and bounded function, θ is a positive constant. In addition, several assumptions are given as follows.

Assumption 2.1 All the parameters $D_i(t)$, $a_{ij}(t)$ ($i = 1, 2, 3; j = 1, 2, 3$) of system (2) are continuous functions with period θ and positive upper and lower bound.

Assumption 2.2 The parameters $\delta_i(t)$ ($i = 1, 2, 3$) of system (2) are bounded continuous functions.

Assumption 2.3 $D_i(t)$, $a_{ij}(t)$, $\delta_i(t)$, $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $k(t)$, $g(t)$, $m(t)$, $h(t)$, $f(t)$ ($i = 1, 2, 3; j = 1, 2, 3$) are all constants which are recorded briefly as D_i, a_{ij}, δ_i, α, β, γ, k, g, m, h, f ($i = 1, 2, 3; j = 1, 2, 3$).

Resources in an ecological environment are always limited in reality, so increase in population density will not be unlimited. In this regard, we propose the following assumptions.

Assumption 2.4 The population density x, y, z are bounded.

Let

$$\Omega_1 := \big\{w \in \Omega \,|\, \big(x(t, w(t)), y(t, w(t)), z(t, w(t)), S(t, w(t)), T(t, w(t))\big) \in \Delta, t \geq 0\big\},$$

where Δ denotes the subset of $\mathbb{R}^l$. In this paper, we assume that sample paths always stay in Ω_1.

Definition 2.5 ([17]) For the stochastic process $\zeta(t) = \zeta(t,\omega)(-\infty < t < +\infty)$, if $\zeta(t_1 + h),\ldots,\zeta(t_n + h)$ are dependent on h ($h = k\theta$ ($k = 1,2,\ldots$)) for arbitrary finite-time series $t_1,\ldots,t_n$, then $\zeta(t)$ is called a period stochastic process with period θ.

In reference [17], Khasminskii proposed that Markov process $r(t)$ has θ as its period if and only if its transition probability function also has the same period, and function $F_0(t,A) = F\{X(t) \in A\}$ satisfies

$$F_0(s,A) = \int_{\mathbb{R}^l} F_0(s,dr)F(s,r,s+\theta,A) := F_0(s+\theta,A),$$

where $A \in \mathscr{B}$, $\mathscr{B}$ denotes σ-algebra.

For the $l-$ dimensional stochastic differential equation

$$dw(t) = f\big(w(t),t\big)\,dt + g\big(w(t),t\big)\,dB(t), \quad w(0) = w_0, \tag{3}$$

where vectors $f(w,t)$, $g(w,t)$, ($t \in [t_0, T]$), $w(t) \in \mathbb{R}^l$ are both continuous function vectors and satisfy the following conditions:

$$\begin{aligned}
\big|f(t,w) - f(t,\bar{w})\big| + \big|g(t,w) - g(t,\bar{w})\big| &\leq B|w - \bar{w}|, \\
\big|f(t,w)\big| + \big|g(t,w)\big| &\leq B\big(1 + |w|\big),
\end{aligned} \tag{4}$$

where B is a constant. The definition of a.s. exponential stability is given as follows.

Definition 2.6 ([18]) If

$$\limsup_{t\to\infty} \frac{1}{t} \ln\big|w(t,w_0)\big| < 0,$$

then the equilibrium $w = 0$ of system (3) is a.s. exponentially stable.

Let E denote a given open set and $G = H \times \mathbb{R}^l$. Let $\mathbb{C}^2$ denote a set defined in G and consist of continuous differentiate function on t and two times continuous differentiate function to w_i ($i = 1,2,\ldots,l$).

Lemma 2.7 ([17]) *If Assumption* 1.1 *holds, and the parameters of Eq.* (3) *satisfy condition* (4) *on* $H \times E$, *also there exists a function* $V(t,w) \in \mathbb{C}^2$ *with period* θ *satisfying the following conditions*:

(I) $\inf_{|z|>R} V(t,w) \to \infty$ *when* $R \to \infty$;

(II) $\mathscr{L}V(t,w) \leq -1$ *holds outside some compact set,*

then a solution of Eq. (3) *exists, which is a Markovian process with* θ *as its period.*

Lemma 2.8 ([17]) *It assumes that* $Z(t) = (Z_1(t), Z_2(t),\ldots,Z_l(t))$ ($l \in \mathbb{N}$) *denotes a bounded function defined in* $\mathbb{R}^l$, $\{t_{0,n}\}$ *denotes an arbitrary unbounded positive sequence of real numbers. Then, for every* $k \in \{1,2,\ldots,l\}$, *there exists a set of sequence* $\{t_{k,n}\}$ *such that* $\{t_{k,n}\}$ *is the subsequence of* $\{t_{k-1,n}\}$ *and converges to the largest limit of sequence* $\{Z_k(t_{k,n})\}$.

Lemma 2.9 *Let Assumption* 1.2 *hold, then for every given initial value* $(x(0), y(0), z(0))$, *there exists a unique solution* $(x(t), y(t), z(t))$, *which will stay in* $\mathbb{R}_+^3$ *with probability* 1.

Proof The proof of Lemma 2.9 is similar to Theorem 2.1 in [19], so we omit it here.　□

3　Existence of positive periodic solutions

We consider that Assumption 1.1 always holds. Taking a biological meaning of the model into account, we discuss the solution of system (2) with the initial condition $(x(0), y(0), z(0)) \in \mathbb{R}_+^3$ in $\mathbb{R}_+^3 = \{(x, y, z) \in \mathbb{R}^3 | x > 0, y > 0, z > 0\}$. On the basis of Lemma 2.7, together with constructing a proper Lyapunov function, the sufficient condition of a positive period solution of (2) will be obtained. For convenience, we let

$$
\lambda = \frac{1}{\theta} \int_0^\theta \left(-2a_{21}^u a_{31}^u \left[D_1(t) - \frac{\alpha^2(t)}{2} \right] + a_{11}^l a_{31}^l \left[D_2(t) + \frac{\beta^2(t)}{2} \right] \right.
$$
$$
\left. + a_{11}^l a_{21}^l \left[D_3(t) + \frac{\gamma^2(t)}{2} \right] \right) dt.
$$

Theorem 3.1 *If* $\lambda > 0$ *and*

$$
a_{12}^l \geq a_{21}^u, a_{13}^l \geq a_{31}^u, a_{22}^l \geq a_{32}^u, \tag{5}
$$
$$
2a_{31}^u a_{12}^l \geq a_{11}^l a_{32}^u, \tag{6}
$$

then for system (2), *there exists at least one periodic solution with* θ *as its period*.

Proof (4) holds in that parameters of system (2) are all continuous and bounded periodic functions. Now we will prove that conditions (I) and (II) of Lemma 2.7 hold. The Lyapunov function $V : [0, +\infty) \times \mathbb{R}_+^3 \rightarrow \mathbb{R}$ on $\mathbb{C}^2$ is defined as

$$
V(t, x, y, z) = M\left(2a_{21}^u a_{31}^u \ln x + a_{11}^l a_{31}^l \ln y + a_{11}^l a_{21}^l \ln z\right) + \frac{(x + y + z)^2}{2} + M\varpi
$$
$$
= V_1(x, y, z) + V_2(x, y, z) + V_3(t),
$$

where $M = (2/\lambda) \max\{1, \sup_{(x,y,z) \in \mathbb{R}_+^3} Q(x, y, z)\}$,

$$
Q(x, y, z) = -\frac{1}{2}a_{11}^l x^3 + \left(D_1^u + \frac{(\alpha^2)^u}{2} \right)x^2 + \frac{(\beta^2)^u}{2}y^2 + \frac{(\gamma^2)^u}{2}z^2 - \left(D_1^l + D_2^l\right)xy
$$
$$
+ \left(-D_3^l + D_1^u\right)xz + \left(-D_3^l - D_2^l\right)yz,
$$

$V_2(x, y, z) = \frac{(x+y+z)^2}{2}$, $V_3(t) = M\varpi$, obviously $M\lambda \geq 2$. Let

$$
\begin{aligned}
\dot{\varpi} = & -\frac{1}{\theta} \int_0^\theta \left(-2a_{21}^u a_{31}^u \left[D_1(t) - \frac{\alpha^2(t)}{2} \right] \right. \\
& + a_{11}^l a_{31}^l \left[D_2(t) + \frac{\beta^2(t)}{2} \right] + a_{11}^l a_{21}^l \left[D_3(t) + \frac{\gamma^2(t)}{2} \right] \right) dt \\
& - 2a_{21}^u a_{31}^u \left[D_1(t) - \frac{\alpha^2(t)}{2} \right] + a_{11}^l a_{31}^l \left[D_2(t) + \frac{\beta^2(t)}{2} \right] \\
& + a_{11}^u a_{21}^u \left[D_3(t) + \frac{\gamma^2(t)}{2} \right] \\
= & -\lambda + \left(-2a_{21}^u a_{31}^u \left[D_1(t) - \frac{\alpha^2(t)}{2} \right] \right. \\
& + a_{11}^l a_{31}^l \left[D_2(t) + \frac{\beta^2(t)}{2} \right] + a_{11}^l a_{21}^l \left[D_3(t) + \frac{\gamma^2(t)}{2} \right] \right).
\end{aligned}
\tag{7}
$$

It is obvious that $\varpi(t)$ is a periodic function with period θ. As a matter of fact, integrating (7) from t to $t + \theta$, we get

$$
\begin{aligned}
\varpi(t + \theta) - \varpi(\theta) = & \int_t^{t+\theta} \dot{\varpi}(s) \, ds \\
= & -\int_0^\theta \left(-2a_{21}^u a_{31}^u \left[D_1(s) - \frac{\alpha^2(s)}{2} \right] + a_{11}^l a_{31}^l \left[D_2(s) + \frac{\beta^2(s)}{2} \right] \right. \\
& \left. + a_{11}^l a_{21}^l \left[D_3(s) + \frac{\gamma^2(s)}{2} \right] \right) ds + \int_0^\theta \left(-2a_{21}^u a_{31}^u \left[D_1(s) - \frac{\alpha^2(s)}{2} \right] \right. \\
& \left. + a_{11}^l a_{31}^l \left[D_2(s) + \frac{\beta^2(s)}{2} \right] + a_{11}^l a_{21}^l \left[D_3(s) + \frac{\gamma^2(s)}{2} \right] \right) ds = 0.
\end{aligned}
$$

Now we will prove that condition (I) of Lemma 2.7 holds. Since quadratic terms of $V(t, x, y, z)$ are all positive, then

$$
\inf_{(x,y,z) \in \mathbb{R}_+^3 \setminus E_\kappa} V(t, x, y, z) \to \infty, \quad \text{when } \kappa \to \infty,
$$

where $E_\kappa = (\frac{1}{\kappa}, \kappa) \times (\frac{1}{\kappa}, \kappa) \times (\frac{1}{\kappa}, \kappa)$. Now we will prove that condition (II) of Lemma 2.7 holds. By using Itô's formula and condition (6), we have

$$
\begin{aligned}
\mathscr{L} V_1(x, y, z) = & M \left(2a_{21}^u a_{31}^u \left[D_1(t) - a_{11}(t)x - a_{12}(t)y - a_{13}(t)z - \delta_1(t)S(t) \right] \right. \\
& + a_{11}^l a_{31}^l \left[-D_2(t) + a_{21}(t)x - a_{22}(t)z - \delta_2(t)S(t) \right] \\
& + a_{11}^l a_{21}^l \left[-D_3(t) + a_{31}(t)x + a_{32}(t)y - \delta_2(t)S(t) \right] \\
& \left. - a_{21}^l a_{31}^l \frac{\alpha^2(t)}{2} - a_{11}^l a_{31}^l \frac{\beta^2(t)}{2} + a_{11}^u a_{21}^u \frac{\gamma^2(t)}{2} \right) \\
\leq & M \left(2a_{21}^u a_{31}^u \left[D_1(t) - \frac{\alpha^2(t)}{2} \right] - a_{11}^l a_{31}^l \left[D_2(t) + \frac{\beta^2(t)}{2} \right] \right. \\
& \left. - a_{11}^l a_{21}^l \left[D_3(t) + \frac{\gamma^2(t)}{2} \right] \right).
\end{aligned}
\tag{8}
$$

Considering (7) and (8) together, we get

$$\mathscr{L}(V_1 + V_3) = -M\lambda. \tag{9}$$

Similarly, we get the following conclusion by using (5):

$$
\begin{aligned}
\mathscr{L}V_2(x,y,z) &= (x + y + z)\Big(D_1(t)x - a_{11}(t)x^2 - a_{12}(t)xy - a_{13}(t)xz - \delta_1(t)S - D_2(t)y \\
&\quad + a_{21}(t)xy - a_{22}(t)yz - \delta_2(t)S - D_3(t)z + a_{31}(t)xz + a_{32}(t)yz - \delta_3(t)S \\
&\quad + \frac{\alpha^2(t)}{2}x^2 + \frac{\beta^2(t)}{2}y^2 + \frac{\gamma^2(t)}{2}z^2 \Big) \\
&\leq -a_{11}^l x^3 + \Big[D_1^u + \frac{(\alpha^2)^u}{2} \Big]x^2 + \Big[-D_2^l + \frac{(\beta^2)^u}{2} \Big]y^2 + \Big[-D_3^l + \frac{(\beta^2)^u}{2} \Big]z^2 \\
&\quad + \big(-D_2^l + D_2^u\big)xy + \big(-D_3^l + D_1^u\big)xz - \big(D_3^l + D_2^l\big)yz. \tag{10}
\end{aligned}
$$

Considering (9) and (10), we get

$$
\begin{aligned}
\mathscr{L}V(t,x,y,z) &\leq -M\lambda - a_{11}^l x^3 + \Big[D_1^u + \frac{(\alpha^2)^u}{2} \Big]x^2 + \Big[-D_2^l + \frac{(\beta^2)^u}{2} \Big]y^2 \\
&\quad + \Big[-D_3^l + \frac{(\beta^2)^u}{2} \Big]z^2 + \big(-D_2^l + D_2^u\big)xy \\
&\quad + \big(-D_3^l + D_1^u\big)xz - \big(D_3^l + D_2^l\big)yz \\
&= -M\lambda - \frac{1}{2}a_{11}^l x^3 - D_2^l y^2 - D_3^l z^2 + Q(x,y,z). \tag{11}
\end{aligned}
$$

Now we define a bounded close set

$$\mathscr{D} = \Big\{ (x,y,z) \in \mathbb{R}_+^3 : \epsilon \leq x \leq \frac{1}{\epsilon}, \epsilon \leq y \leq \frac{1}{\epsilon}, \epsilon \leq z \leq \frac{1}{\epsilon} \Big\},$$

where $0 < \epsilon < 1$. We choose ϵ small enough such that

$$-M\lambda - \frac{a_{11}^l}{2\epsilon^3} + Q_{\sup} \leq -1, \tag{12}$$

$$-M\lambda - \frac{D_2^l}{\epsilon^2} + Q_{\sup} \leq -1, \tag{13}$$

$$-M\lambda - \frac{D_3^l}{\epsilon^2} + Q_{\sup} \leq -1, \tag{14}$$

where $Q_{\sup} = \sup_{(x,y,z) \in \mathbb{R}_+^3} Q(x,y,z)$. Let

$$\mathscr{D}_\epsilon^1 = \big\{ (x,y,z) \in \mathbb{R}_+^3 : 0 < x < \epsilon \big\}, \qquad \mathscr{D}_\epsilon^2 = \big\{ (x,y,z) \in \mathbb{R}_+^3 : 0 < y < \epsilon \big\},$$

$$\mathscr{D}_\epsilon^3 = \big\{ (x,y,z) \in \mathbb{R}_+^3 : 0 < z < \epsilon \big\}, \qquad \mathscr{D}_\epsilon^4 = \Big\{ (x,y,z) \in \mathbb{R}_+^3 : x > \frac{1}{\epsilon} \Big\},$$

$$\mathscr{D}_\epsilon^5 = \Big\{ (x,y,z) \in \mathbb{R}_+^3 : y > \frac{1}{\epsilon} \Big\}, \qquad \mathscr{D}_\epsilon^6 = \Big\{ (x,y,z) \in \mathbb{R}_+^3 : z > \frac{1}{\epsilon} \Big\}.$$

A complementary set of $\mathscr{D}$ can be denoted as $\mathscr{D}^C = \mathscr{D}_\epsilon^1 \cup \mathscr{D}_\epsilon^2 \cup \mathscr{D}_\epsilon^3 \cup \mathscr{D}_\epsilon^4 \cup \mathscr{D}_\epsilon^5 \cup \mathscr{D}_\epsilon^6$. Now we prove that $\mathscr{L}V(t,x,y,z) \leq -1$ is valid on $[0,+\infty) \times \mathscr{D}^C$.

Case 1. When $(t,x,y,z) \in [0,+\infty) \times (\mathscr{D}_\epsilon^1 \cup \mathscr{D}_\epsilon^2 \cup \mathscr{D}_\epsilon^3)$, we have

$$\mathscr{L}V(t,x,y,z) \leq -M\lambda + Q(x,y,z) \leq -M\lambda + Q_{\sup}$$
$$\leq -M\lambda + \frac{M\lambda}{2} = -\frac{M\lambda}{2} \leq -1. \tag{15}$$

Case 2. When $(t,x,y,z) \in [0,+\infty) \times \mathscr{D}_\epsilon^4$, on the basis of (12), we have

$$\mathscr{L}V(t,x,y,z) \leq -M\lambda - \frac{a_{11}^l}{2}x^3 + Q(x,y,z) \leq -M\lambda - \frac{a_{11}^l}{2\epsilon^3} + Q_{\sup} \leq -1. \tag{16}$$

Case 3. When $(t,x,y,z) \in [0,+\infty) \times \mathscr{D}_\epsilon^5$, from (13), we get

$$\mathscr{L}V(t,x,y,z) \leq -M\lambda - D_2^l y^2 + Q(x,y,z) \leq -M\lambda - \frac{D_2^l}{\epsilon^2} + Q_{\sup} \leq -1. \tag{17}$$

Case 4. When $(t,x,y,z) \in [0,+\infty) \times \mathscr{D}_\epsilon^6$, on the basis of (14), we get

$$\mathscr{L}V(t,x,y,z) \leq -M\lambda - D_3^l y^2 + Q(x,y,z) \leq -M\lambda - \frac{D_3^l}{\epsilon^2} + Q_{\sup} \leq -1. \tag{18}$$

Thus, from (15)–(18), we get

$$\mathscr{L}V(t,x,y,z) \leq -1, \quad \forall (t,x,y,z) \in [0,+\infty) \times \mathscr{D}^C. \tag{19}$$

So Lemma 2.7 (II) is true, and there exists a periodic solution of system (2) with period θ. Besides, from Lemma 2.9, there exists a unique positive solution of system (2). Thus there exists at least one periodic solution of system (2) with period θ. □

Thanks to Part 3 in [3], we obtain the condition of a periodic solution of system (2), which is an expansion of Theorem 1 in [6].

4 Extinction of solution

We assume that Assumptions (2.1) and (2.2) always hold.

Theorem 4.1 *If* $\langle r_1(t)\rangle^* = \limsup_{t\to+\infty} \frac{1}{t}\int_0^t (D_1(s) - \frac{1}{2}\alpha^2(s))\,ds < 0$, *then system* (2) *will go to extinction with probability* 1.

Proof By using Itô's formula to (2), we get

$$d\ln x = \left(D_1(t) - a_{11}(t) - a_{12}(t) - a_{13}(t)z - \delta_1(t)S - \frac{1}{2}\alpha^2\right)dt + \alpha(t)\,dB_1(t). \tag{20}$$

Integrating both sides on the above formula, we get

$$\frac{\ln x(t) - \ln x_0}{t} = \langle r_1(t)\rangle - \langle a_{11}(t)x\rangle - \langle a_{12}(t)y\rangle - \langle a_{13}(t)z\rangle - \langle\delta_1(t)S\rangle$$
$$+ \frac{\int_0^t \alpha_t(s)\,dB_1(s)}{t}, \tag{21}$$

where $r_1(t) = D_1(t) - \frac{1}{2}\alpha^2(t)$. Similarly, we have

$$\frac{\ln y(t) - \ln y_0}{t} = \langle r_2(t) \rangle + \langle a_{21}(t)x \rangle - \langle a_{22}(t)z \rangle - \langle \delta_2(t)S \rangle + \frac{\int_0^t \beta_t(s)\, dB_1(s)}{t}, \tag{22}$$

$$\frac{\ln z(t) - \ln z_0}{t} = \langle r_3(t) \rangle + \langle a_{31}(t)x \rangle + \langle a_{32}(t)y \rangle - \langle \delta_3(t)S \rangle + \frac{\int_0^t \gamma_t(s)\, dB_3(s)}{t}, \tag{23}$$

where $r_2(t) = -D_2(t) - \frac{1}{2}\beta^2(t) < 0$, $r_3(t) = -D_3(t) - \frac{1}{2}\gamma^2(t) < 0$. Let $M_1(t) = \int_0^t \alpha(t)\, dB_1(t)$, $M_2(t) = \int_0^t \beta(t)\, dB_2(t)$, $M_3(t) = \int_0^t \gamma(t)\, dB_3(t)$, we find that $M_i(t), i = 1, 2, 3$, are local martingales. So from the strong law of large numbers, we get

$$\lim_{t \to +\infty} M_i(t)/t = 0 \quad \text{a.s.} \tag{24}$$

Taking the upper limit on both sides of (21), together with (24), we get

$$\left(\frac{1}{t} \ln x(t) \right)^* \leq \langle r_1(t) \rangle^* - \langle a_{11}(t)x \rangle_* - \langle a_{12}(t)y \rangle_* - \langle a_{13}(t)z \rangle_* - \langle \delta_1(t)S \rangle_*$$

$$\leq \langle r_1(t) \rangle^* < 0. \tag{25}$$

Thus, we have $\lim_{t \to +\infty} x(t) = 0$. Besides, if $\langle r_1(t) \rangle^* < 0$, then $\langle x(t) \rangle^* = 0$. Similarly, taking the upper limit on both sides of (22) and taking (24) into consideration, we can get

$$\left(\frac{1}{t} \ln y(t) \right)^* \leq \langle r_2(t) \rangle^* + \langle a_{21}(t)x(t) \rangle^* - \langle a_{22}(t)z \rangle_* - \langle \delta_2(t)S \rangle_*$$

$$\leq \langle r_2(t) \rangle^* < 0. \tag{26}$$

So $\lim_{t \to +\infty} y(t) = 0$. By using the same method, we can get $\langle x(t) \rangle^* = 0$, $\langle y(t) \rangle^* = 0$, then

$$\left(\frac{1}{t} \ln z(t) \right)^* \leq \langle r_3(t) \rangle^* + \langle a_{31}(t)x(t) \rangle^* + \langle a_{32}(t)y \rangle^* - \langle \delta_3(t)S \rangle_* \leq \langle r_2(t) \rangle^* < 0,$$

so $\lim_{t \to +\infty} z(t) = 0$. $\qquad \square$

Corollary 4.2 *Let $(x(t), y(t), z(t))$ be a solution of system* (2) *on the initial condition* $(x(0), y(0), z(0))$, *then we have the following conclusions about population x of system* (2):

  (a) *If $\langle r_1(t) \rangle^* = 0$, then x will not be a.s. persistent in mean;*
  (b) *If $\langle r_1(t) \rangle^* > \langle \delta_1(t) \rangle_*$ and $(a_{11}(t))_* \langle r_2(t) \rangle^* + \langle r_1(t) \rangle^* a_{21}^u < 0$, then x will be a.s. weakly persistent;*
  (c) *For all $s > 1$, there exists $L(s)$ such that $E[x^s(t)] \leq L(s)$,*
where $r_2(t) = -D_2(t) - \frac{1}{2}\beta^2(t)$.

Remark 4.3 [20] $x(t)$ is called weakly persistent if $x^* = \limsup_{t \to \infty} x(t) > 0$.

Proof The proof is similar to [20], so we omit it here. $\qquad \square$

5 Exponential stability of the equilibrium

Now we consider that Assumptions (2.3) and (2.4) always hold. For all given $\sigma_1 > 0$, $\sigma_2 > 0$, $\ldots, \sigma_5 > 0$, the stochastic process is defined as follows:

$$p\big(X(t)\big) = \sigma_1 x(t) + \sigma_2 y(t) + \sigma_3 z(t) + \sigma_4\big(1 - S(t)\big) + \sigma_5\big(1 - T(t)\big) \tag{27}$$

and

$$U_x = p^{-1}x, \qquad U_y = p^{-1}y, \qquad U_z = p^{-1}z, \qquad U_S = p^{-1}(1 - S), \qquad U_T = p^{-1}(1 - T).$$

Then, for every $t > 0$, we get

$$0 < U_x(t) \le \frac{1}{\sigma_1}, \qquad 0 < U_y(t) \le \frac{1}{\sigma_2}, \qquad 0 < U_z(t) \le \frac{1}{\sigma_3},$$

$$0 < U_S(t) \le \frac{1}{\sigma_4}, \qquad 0 < U_T(t) \le \frac{1}{\sigma_5}.$$

In other words, the above stochastic process has an upper bound $\max\{\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \frac{1}{\sigma_3}, \frac{1}{\sigma_4}, \frac{1}{\sigma_5}\}$. So

$$\sigma_1 U_x(t) + \sigma_2 U_y(t) + \sigma_3 U_z(t) + \sigma_4 U_S(t) + \sigma_5 U_T(t) = 1. \tag{28}$$

According to the assumptions, the stochastic process $p(X(t))$ is also bounded, that is,

$$p\big(X(t)\big) < M, \quad t > 0, \tag{29}$$

where M is a constant.

Theorem 5.1 *We suppose that $E^*(0, 0, 0, S^*, T^*)$ denotes the equilibrium of system* (2). *If*

$$\sigma_5 f \ge D_1 M, \tag{30}$$

$$D_1 > (g + m) \wedge h + \eta^2, \tag{31}$$

$$\sigma_1 a_{12} \ge \sigma_2 a_{21}, \qquad \sigma_1 a_{13} \ge \sigma_3 a_{31}, \qquad \sigma_2 a_{22} \ge \sigma_3 a_{32}, \tag{32}$$

where $\eta = \alpha \wedge \beta \wedge \gamma$, then the equilibrium point $E^*(0, 0, 0, S^*, T^*)$ is a.s. exponentially stable.

Proof For convenience's sake, we let $X(t) := (x(t), y(t), z(t), S(t), T(t))$. Defining a stochastic process $p(X(t))$ similar to (27), we find that $p(X(t)) > 0$ for all $t > 0$ (since a sample path will come into Ω_1). Then we define

$$V\big(X(t)\big) = \ln p\big(X(t)\big).$$

In order to prove Theorem 5.1, we only have to prove that $p(X(t))$ will converge to zero a.s. By using Itô's formula, we rewrite the stochastic process $V(X(t))$ as

$$V\big(X(t)\big) = V\big(X(0)\big) + \int_0^t \mathscr{L}V\big(X(\xi)\big)\,d\xi + M(t), \tag{33}$$

where $M(t) = \sum_{i=1}^{3} M_i(t)$ is a local martingale, here

$$M_1(t) = \int_0^t \frac{\alpha x(u)}{p(X(u))}\, dB_1(u),$$

$$M_2(t) = \int_0^t \frac{\beta y(u)}{p(X(u))}\, dB_2(u),$$

$$M_3(t) = \int_0^t \frac{\gamma z(u)}{p(X(u))}\, dB_3(u).$$

Applying the strong law of large numbers of martingale, we get

$$\lim_{t\to\infty} \frac{1}{t} \sum_{i=1}^{3} M_i(t) = 0, \quad \text{a.s.} \tag{34}$$

Taking limits on both sides of (33) and using (34), we get

$$\limsup_{t\to\infty} \frac{1}{t} V(X(t)) = \limsup_{t\to\infty} \frac{1}{t} \int_0^t \mathscr{L}V(X(u))\, du, \quad \text{a.s.} \tag{35}$$

In order to prove that $p(X(t))$ will converge to zero a.s., we only have to prove that

$$\limsup_{t\to\infty} \mathscr{L}V(X(t)) < 0, \quad \text{a.s.} \tag{36}$$

From (32), together with Itô's formula, we get

$$\mathscr{L}V(X) = \frac{1}{p}\big[\sigma_1 x(D_1 - a_{11}x - a_{12}y - a_{13}z - \delta_1 S) + \sigma_2 y(-D_2 + a_{21}x - a_{22}z - \delta_2 S)$$

$$+ \sigma_3 z(-D_3 + a_{31}x + a_{32}y - \delta_3 S) - \sigma_4(kT - gS - mS) - \sigma_5(-hT + f)\big]$$

$$- \frac{3}{p^2}\big[(\sigma_1\alpha x)^2 + (\sigma_2\beta y)^2 + (\sigma_3\gamma z)^2\big]$$

$$\le \frac{1}{p}[\sigma_1 D_1 x - \sigma_1\delta_1 Sx - \sigma_2 D_2 y - \sigma_2\delta_2 Sy - \sigma_3 D_3 z - \sigma_3\delta_3 Sz - \sigma_4 kT + \sigma_4 gS$$

$$+ \sigma_4 mS + \sigma_5 hT - \sigma_5 f] - \frac{3}{p^2}\big[(\sigma_1\alpha x)^2 + (\sigma_2\beta y)^2 + (\sigma_3\gamma z)^2\big]. \tag{37}$$

For every sample path of three-dimensional Brownian motion $w(t)$, there exists an unbounded increasing sequence $\{\tau_n^w\}$ such that

$$\lim_{n\to\infty} \mathscr{L}V(X(\tau_n, w(\tau_n))) = \limsup_{n\to\infty} \mathscr{L}V(X(t, w(t))).$$

Now we fix a sequence. From Lemma 2.8, there exists a subsequence $\{t_n^w\}$ such that the following limit exists:

$$\lim_{n\to\infty} \big(U_x(X(t_n^w, w(t_n^w))), U_y(X(t_n^w, w(t_n^w))),$$

$$U_z(X(t_n^w, w(t_n^w))), S(t_n^w, w(t_n^w)), T(t_n^w, w(t_n^w))),$$

which can guarantee the definition of the following limits:

$$\bar{x} = \lim_{n\to\infty} U_x\big(X(t_n)\big), \qquad \bar{y} = \lim_{n\to\infty} U_y\big(X(t_n)\big), \qquad \bar{z} = \lim_{n\to\infty} U_z\big(X(t_n)\big),$$

$$\bar{S} = \lim_{n\to\infty} U_S\big(X(t_n)\big), \qquad \bar{T} = \lim_{n\to\infty} U_T\big(X(t_n)\big), \tag{38}$$

$$\tilde{S} = \lim_{n\to\infty} S(t_n), \qquad \tilde{T} = \lim_{n\to\infty} T(t_n).$$

Putting (38) into (28), we get

$$\sigma_1\bar{x} + \sigma_2\bar{y} + \sigma_3\bar{z} + \sigma_4\bar{S} + \sigma_5\bar{T} = 1. \tag{39}$$

Let

$$\Psi = \lim_{n\to\infty} \mathscr{L}\big(\ln p(t_n)\big),$$

then, applying (29) and (37), (38) can be rewritten as

$$\begin{aligned}
\Psi \leq{}& \sigma_1 D_1\bar{x} - \sigma_1\delta_1\tilde{S}\bar{x} - \sigma_2 D_2\bar{y} - \sigma_2\delta_2\tilde{S}\bar{y} - \sigma_3 D_3\bar{z} \\
& - \sigma_3\delta_3\tilde{S}\bar{z} - \sigma_4 k\bar{T} + \sigma_4 g\bar{S} + \sigma_4 m\bar{S} + \sigma_5 h\bar{T} \\
& - \frac{\sigma_5}{M}f - 3\big[(\sigma_1\alpha\bar{x})^2 + (\sigma_2\beta\bar{y})^2 + (\sigma_3\gamma\bar{z})^2\big].
\end{aligned} \tag{40}$$

From equation (39), we get

$$\sigma_1\bar{x} + \sigma_2\bar{y} + \sigma_3\bar{z} = 1 - \sigma_4\bar{S} - \sigma_5\bar{T}, \qquad \sigma_1\bar{x} + \sigma_2\bar{y} + \sigma_3\bar{z} \leq 1.$$

In terms of the above two equalities, we obtain the following estimation:

$$\begin{aligned}
& -(\sigma_1\alpha\bar{x})^2 - (\sigma_2\beta\bar{y})^2 - (\sigma_3\gamma\bar{z})^2 \\
& \quad = -\eta^2\big[(\sigma_1\bar{x})^2 + (\sigma_2\bar{y})^2 + (\sigma_3\bar{z})^2\big] \\
& \quad \leq -\frac{1}{3}\eta^2[\sigma_1\bar{x} + \sigma_2\bar{y} + \sigma_3\bar{z}]^2 \\
& \quad = -\frac{1}{3}\eta^2[\sigma_1\bar{x} + \sigma_2\bar{y} + \sigma_3\bar{z}](1 - \sigma_4\bar{S} - \sigma_5\bar{T}) \\
& \quad \leq -\frac{1}{3}\eta^2(\sigma_1\bar{x} + \sigma_2\bar{y} + \sigma_3\bar{z}) + \frac{1}{3}\eta^2(\sigma_1\bar{x} + \sigma_2\bar{y} + \sigma_3\bar{z})\sigma_4\bar{S} \\
& \qquad + \frac{1}{3}\eta^2(\sigma_1\bar{x} + \sigma_2\bar{y} + \sigma_3\bar{z})\sigma_5\bar{T} \\
& \quad \leq -\frac{1}{3}\eta^2(\sigma_1\bar{x} + \sigma_2\bar{y} + \sigma_3\bar{z}) + \frac{1}{3}\eta^2\sigma_4\bar{S} + \frac{1}{3}\eta^2\sigma_5\bar{T},
\end{aligned}$$

where $\eta = \min\{\alpha, \beta, \gamma\}$. Putting the above inequality into (40), we get

$$\begin{aligned}
\Psi \leq{}& \sigma_1 D_1\bar{x} - \sigma_1\delta_1\tilde{S}\bar{x} - \sigma_2 D_2\bar{y} - \sigma_2\delta_2\tilde{S}\bar{y} - \sigma_3 D_3\bar{z} \\
& - \sigma_3\delta_3\tilde{S}\bar{z} - \sigma_4 k\bar{T} + \sigma_4 g\bar{S} + \sigma_4 m\bar{S} + \sigma_5 h\bar{T} \\
& - \frac{\sigma_5}{M}f - \eta^2\sigma_1\bar{x} - \eta^2\sigma_2\bar{y} - \eta^2\sigma_3\bar{z} + \eta^2\sigma_4\bar{S} + \eta^2\sigma_5\bar{T}.
\end{aligned} \tag{41}$$

Owing to $\sigma_1 \bar{x} = 1 - \sigma_2 \bar{y} - \sigma_3 \bar{z} - \sigma_4 \bar{S} - \sigma_5 \bar{T}$, we get

$$
\begin{aligned}
\Psi \leq{} & D_1 - D_1 \sigma_2 \bar{y} - D_1 \sigma_3 \bar{z} - D_1 \sigma_4 \bar{S} - D_1 \sigma_5 \bar{T} - \sigma_2 D_2 \bar{y} - \sigma_3 D_3 \bar{z} - \sigma_4 k \bar{T} + \sigma_4 g \bar{S} \\
& + \sigma_4 m \bar{S} + \sigma_5 h \bar{T} - \frac{\sigma_5}{M} f - \eta^2 \sigma_1 \bar{x} - \eta^2 \sigma_2 \bar{y} - \eta^2 \sigma_3 \bar{z} + \eta^2 \sigma_4 \bar{S} + \eta^2 \sigma_5 \bar{T} \\
\leq{} & -\eta^2 \sigma_1 \bar{x} - \left(D_1 \sigma_2 + D_2 \sigma_2 + \eta^2 \sigma_2\right) \bar{y} - \left(D_1 \sigma_3 + D_3 \sigma_3 + \eta^2 \sigma_3\right) \bar{z} \\
& - \left(D_1 \sigma_4 - \sigma_4 g - \sigma_4 m - \eta^2 \sigma_4\right) \bar{S} - \left(D_1 \sigma_5 - \sigma_5 h - \eta^2 \sigma_5\right) \bar{T} \\
:={} & A_1 \bar{x} + A_2 \bar{y} + A_3 \bar{z} + A_4 \bar{S} + A_5 \bar{T}.
\end{aligned}
$$

From (31), we know that $A_4, A_5 < 0$, so parameters A_1, A_2, A_3, A_4, A_5 are all negative. Besides, from (39), we know that $\bar{x}, \bar{y}, \bar{z}, \bar{S}, \bar{T}$ are not completely zeroes, thus $\Psi < 0$. □

In system (2), if $\alpha = \beta = \gamma = 0$, then stochastic system (2) will be a deterministic system with environmental pollution. For the deterministic system, the following conclusion holds.

Theorem 5.2 *If* (30)–(32) *hold, then the deterministic system is exponentially stable a.s.*

Proof The proof of this theorem is similar to that of Theorem 5.1. In the process of the proof, (40) will be simplified as follows:

$$
\begin{aligned}
\Psi \leq{} & \sigma_1 D_1 \bar{x} - \sigma_1 \delta_1 \tilde{S} \bar{x} - \sigma_2 D_2 \bar{y} - \sigma_2 \delta_2 \tilde{S} \bar{y} - \sigma_3 D_3 \bar{z} \\
& - \sigma_3 \delta_3 \tilde{S} \bar{z} - \sigma_4 k \bar{T} + \sigma_4 g \bar{S} + \sigma_4 m \bar{S} + \sigma_5 h \bar{T} - \frac{\sigma_5}{M} f.
\end{aligned}
$$

The following proof is similar to Theorem 5.1, thus can be omitted here. □
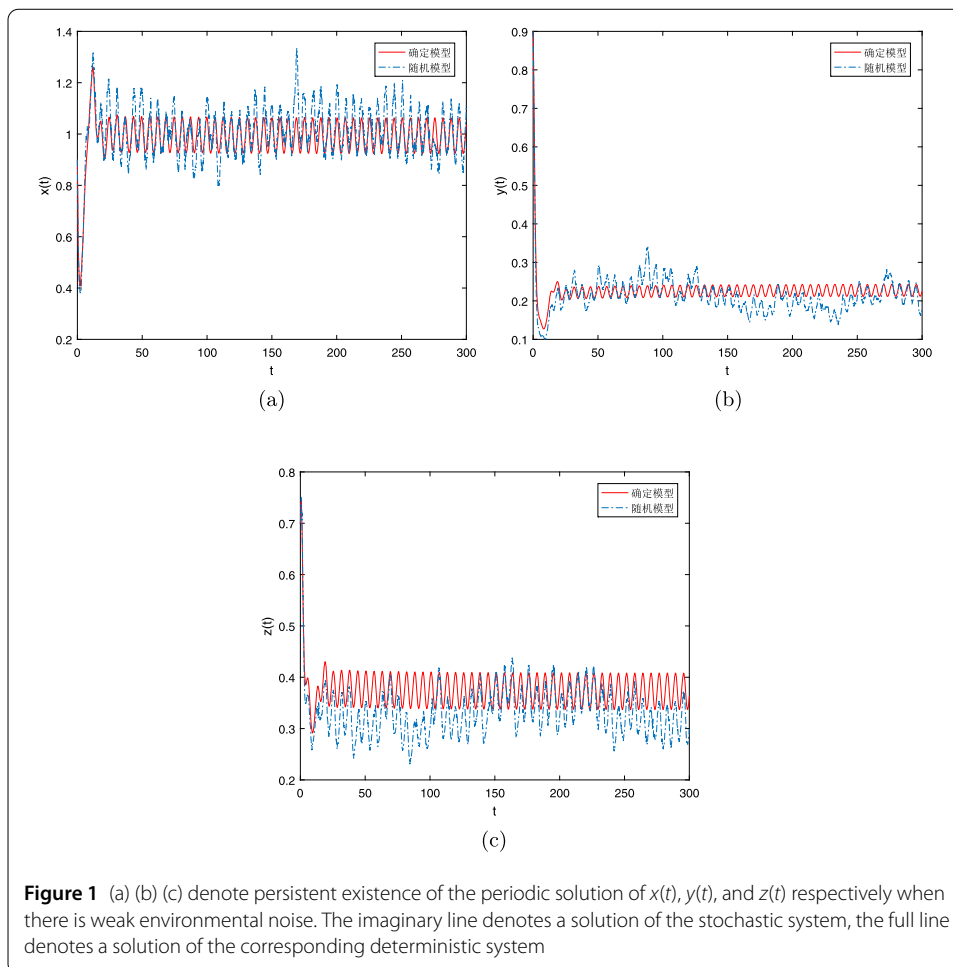
6 Numerical simulations

We now verify the rationality of the above theorems' conclusion by several examples. Selecting parameters of system (2), we get the following equation set:

$$
\begin{cases}
dx(t) = x(t)[(0.8 + 0.1 \sin t) - (0.4 + 0.1 \sin t)x(t) - (0.8 + 0.1 \sin t)y(t) \\
\qquad - (0.6 + 0.1 \sin t)z(t) - 0.01 S(t)]\, dt + \alpha(t)x(t)\, dB_1(t), \\
dy(t) = y(t)[-(0.3 + 0.2 \sin t) + (0.5 + 0.2 \sin t)x(t) - (0.5 + 0.1 \sin t)z(t) \\
\qquad - 0.1 S(t)]\, dt + \beta(t)y(t)\, dB_2(t), \\
dz(t) = z(t)[-(0.36 + 0.3 \sin t) + (0.3 + 0.2 \sin t)x(t) + (0.3 + 0.1 \sin t)y(t) \\
\qquad - 0.01 S(t)]\, dt + \gamma(t)z(t)\, dB_3(t), \\
dS(t) = [0.1 T(t) - 0.05 S(t)]\, dt, \\
dT(t) = [-0.2 T(t) + f(t)]\, dt,
\end{cases} \tag{42}
$$

with the initial condition

$$
x(0) = y(0) = 0.9, \qquad z(0) = 0.7, \qquad S(0) = 0.5, \qquad T(0) = 0.5.
$$

$\alpha(t), \beta(t)$, and $\gamma(t)$ will be offered separately in Examples 6.1 and 6.2. Besides, we suppose that the control function $f(t)$ subjects to uniform distribution of $[0, 1]$.

Figure 1 (a) (b) (c) denote persistent existence of the periodic solution of $x(t)$, $y(t)$, and $z(t)$ respectively when there is weak environmental noise. The imaginary line denotes a solution of the stochastic system, the full line denotes a solution of the corresponding deterministic system

Example 6.1　Let $\alpha(t) = \beta(t) = 0.6 + 0.1\sin t$, $\gamma(t) = 0.5 + 0.2\sin t$.

　Taking notice of $\lambda = 0.102 > 0$, we find that the conditions of Theorem 3.1 are obviously satisfied, so there exists at least one periodic solution of system (2). Meanwhile, if the tendency of pollution can be controlled within a certain range, then the population of system (2) will be persistent in terms of survival. From Fig. 1, we know that the weaker stochastic disturbance, the weaker disturbance to population x, while the stronger disturbance to population y and z. In addition, for an arbitrary initial value, a solution of the deterministic model will come into the periodic orbit, while a stochastic one will fluctuate around the periodic orbit when stochastic noise is smaller.

Example 6.2　Let $0.5\alpha^2(t) = 0.9 + 0.2\sin t$, $\beta(t) = \gamma(t) = 0.1 + 0.1\sin t$.

　For $\langle r_1(t)\rangle^* = -0.1 < 0$, the condition of Theorem 4.1 will be satisfied. Therefore, for an arbitrary initial value, system (2) will tend to extinction in the sense of probability. From Fig. 2, we know that $x(t)$, $y(t)$, $z(t)$ will tend to zero in a time, which means that population will go to extinction.
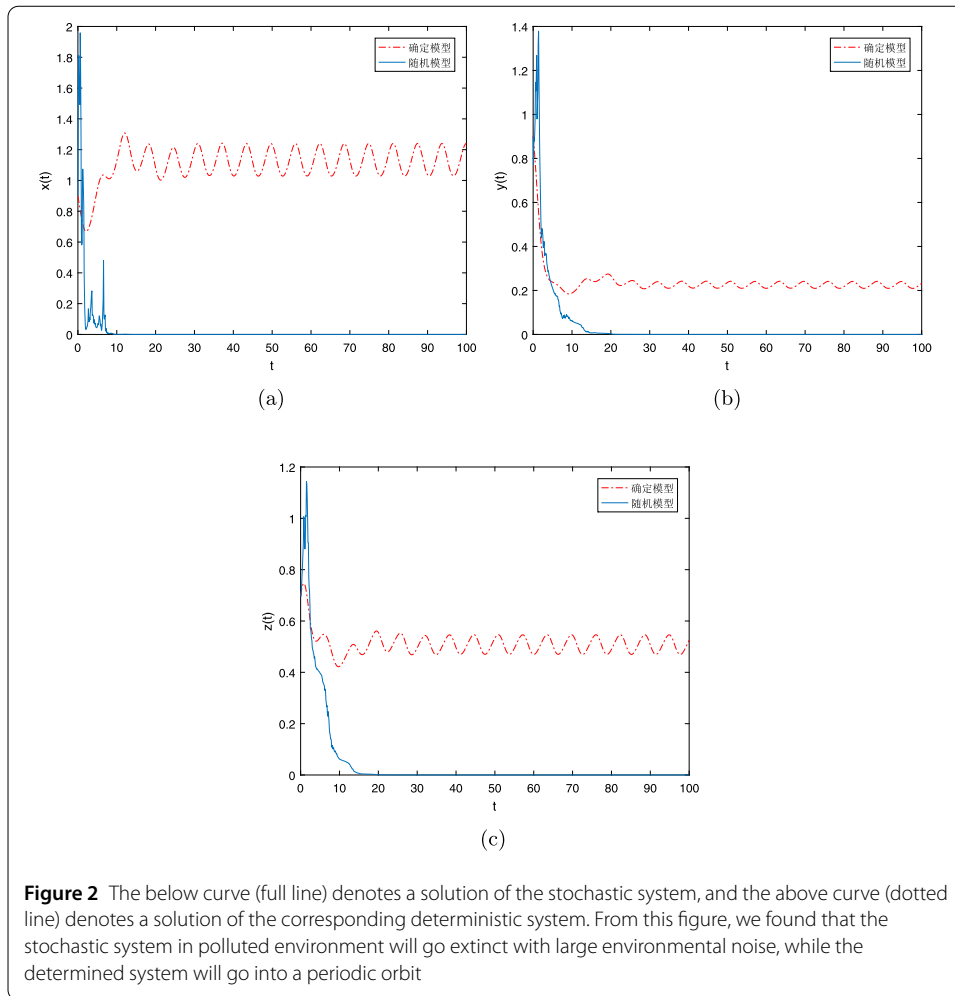
Figure 2 The below curve (full line) denotes a solution of the stochastic system, and the above curve (dotted line) denotes a solution of the corresponding deterministic system. From this figure, we found that the stochastic system in polluted environment will go extinct with large environmental noise, while the determined system will go into a periodic orbit
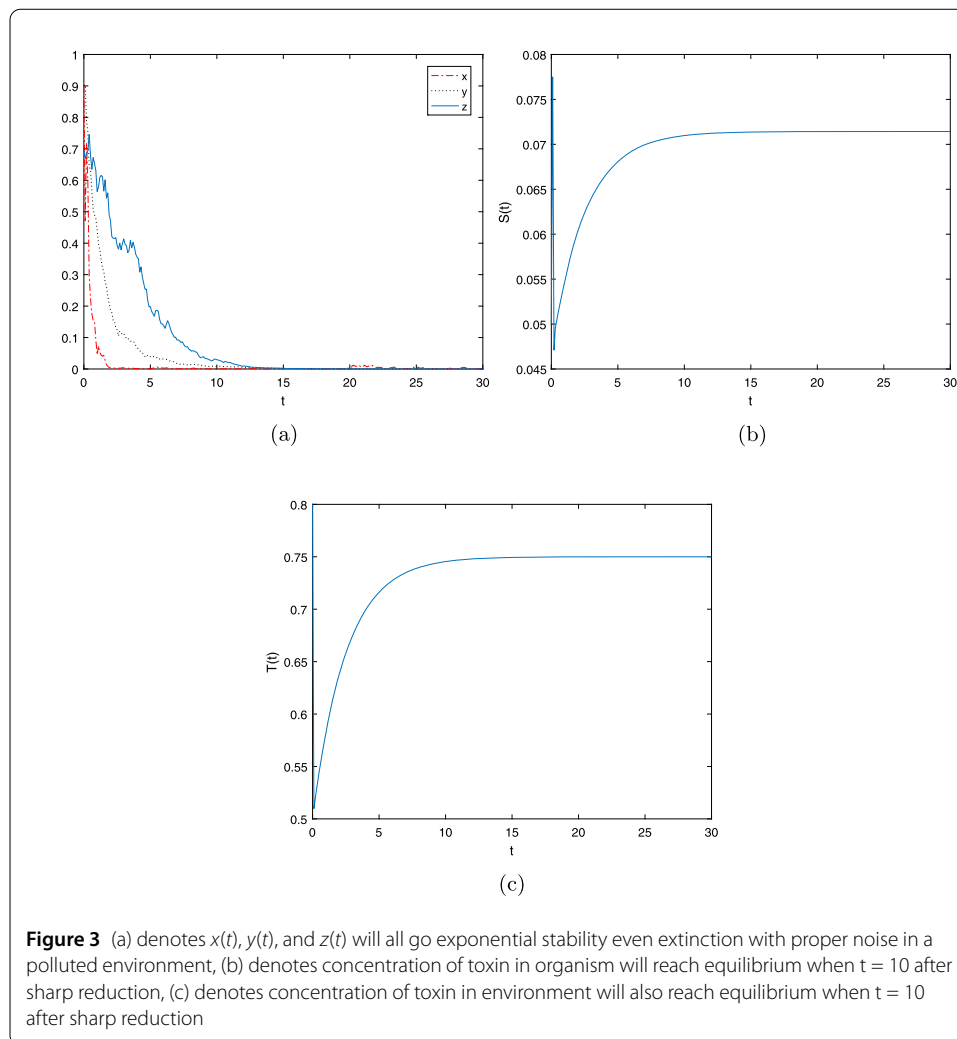
Example 6.3 Choose the parameters of system (2) as follows:

$$
\begin{cases}
dx(t) = x(t)(0.8 - 0.4x(t) - 0.2y(t) - 0.6z(t) - 0.01S(t))\, dt + 0.6x(t)\, dB_1(t), \\
dy(t) = y(t)(-0.3 + 0.5x(t) - 0.5z(t) - 0.1S(t))\, dt + 0.2y(t)\, dB_2(t), \\
dz(t) = z(t)(-0.4 + 0.3x(t) + 0.3y(t) - 0.01S(t))\, dt + 0.2z(t)\, dB_3(t), \\
dS(t) = [0.1T(t) - 0.05S(t)]\, dt, \qquad dT(t) = [-0.4T(t) + 0.3]\, dt,
\end{cases}
\tag{43}
$$

with the initial value

$$x(0) = y(0) = 0.9, \qquad z(0) = 0.7, \qquad S(0) = 0.05, \qquad T(0) = 0.8.$$

Obviously, the condition of Theorem 5.1 holds, so equilibrium $E^*(0, 0, 0, S^*, T^*)$ of system (43) is exponentially stable a.s. From Fig. 3, we find that exponential stability of equilibrium E^* holds. From Fig. 3(a) we can see that densities of $x, y,$ and z will decrease sharply with white noise and pollution. When t equals 2, 5, 8 respectively, the decreasing rate of $x, y,$ and z will slow down, but when t equals 3, 15, 15 respectively, $x, y,$ and z will go to extinction. From Fig. 3(b) and Fig. 3(c) we know that toxin concentration $S(t)$ and $T(t)$ will

Figure 3 (a) denotes $x(t)$, $y(t)$, and $z(t)$ will all go exponential stability even extinction with proper noise in a polluted environment, (b) denotes concentration of toxin in organism will reach equilibrium when t = 10 after sharp reduction, (c) denotes concentration of toxin in environment will also reach equilibrium when t = 10 after sharp reduction

reach equilibrium when t equals 10, which indicates that population will tend to extinction under certain white noise and pollution.

7 Conclusions and discussion

This paper is concerned with a stochastic three-species food web system with omnivory and environmental pollution. First, by using stochastic analysis theory, we establish a sufficient condition for the existence of a positive periodic solution of system (2). Next, we investigate the condition of extinction and a.s. exponential stability of equilibrium E^* under some assumptions. Finally, some numerical simulations are introduced to support the main results. In Sect. 4, we prove that if we control the environmental pollution within a certain range, then system (2) can keep persistent existence, otherwise not. While small noises have small influence on prey $x(t)$ under the impact of the intrinsic growth rate, they have large influence on middle-predator $y(t)$ and omnivorous top-predator $z(t)$. Furthermore, under a certain range of white noise and environmental pollution, species will go to extinction, otherwise species will go into a periodic orbit under some certain pollution when there is no influence of stochastic noise.

Although there are many research works focusing on the condition of extinction and exponential stability of a food web system, there are few works on the influence of both stochastic noise and environment pollution, which do harm to the stability and persistent existence. In general, inspired by [3] and [6], we finally obtain the conditions of existence of periodic solutions, extinction, and exponential stability under some assumptions, which is different from the existing conclusions.

To the best of our knowledge, there are many related recent literature works on fractional integral [21–23] and derivative in the field of mathematical modeling and applied sciences, such as [24–26]. In the near future, we will focus on the fractional order food web model with stochastic noise and environmental pollution to find out the behavior of system solution, which will extend the results of this work. Furthermore, there are also other literature works on oscillations of periodic solutions in some timescale model, such as [27, 28] and so on, which give us an open view of continuing to explore how timescale impacts the behavior of system solution.

Availability of data and materials
All data, models, and code generated or used during the study appear in the submitted article.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors made equal contributions. All authors read and approved the final manuscript.

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