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On generalized Bessel–Maitland function

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Abstract

An approach to the generalized Bessel–Maitland function is proposed in the present paper. It is denoted by $\mathcal{J}^{\mu}_{\nu,\lambda}$, where $\mu > 0$ and $\lambda, \nu \in \mathbb{C}$ get increasing interest from both theoretical mathematicians and applied scientists. The main objective is to establish the integral representation of $\mathcal{J}^{\mu}_{\nu,\lambda}$ by applying Gauss's multiplication theorem and the representation for the beta function as well as Mellin–Barnes representation using the residue theorem. Moreover, the *m*th derivative of $\mathcal{J}^{\mu}_{\nu,\lambda}$ is considered, and it turns out that it is expressed as the Fox–Wright function. In addition, the recurrence formulae and other identities involving the derivatives are derived. Finally, the monotonicity of the ratio between two modified Bessel–Maitland functions $\mathcal{I}^{\mu}_{\nu,\lambda}$ defined by $\mathcal{I}^{\mu}_{\nu,\lambda}(z) = i^{-2\lambda-\nu}\mathcal{J}^{\mu}_{\nu,\lambda}(iz)$ of a different order, the ratio between modified Bessel–Maitland and hyperbolic functions, and some monotonicity results for $\mathcal{I}^{\mu}_{\nu,\lambda}(z)$ are obtained where the main idea of the proofs comes from the monotonicity of the quotient of two Maclaurin series. As an application, some inequalities (like Turán-type inequalities and their reverse) are proved. Further investigations on this function are underway and will be reported in a forthcoming paper.

MSC: 30C45; 30C50

Keywords: Generalized Bessel–Maitland and Struve functions; Integral representation; Mellin–Barnes integral representation; Differential properties; Monotonicity

1 Introduction

Special functions (abbreviated as SFs) are potentially useful in diverse fields of mathematical physics and engineering. They represent a crucial tool to provide solutions to differential equations and systems, used as mathematical models. This fact follows from the point of view of the applied scientists and engineers dealing with the practical application of differential equations. In this connection, substantial efforts have been carried out on the special functions and their properties to attract particular attention. For a thorough treatment of such theory and its more recent achievements, we refer the reader, e.g., to [4–9, 11, 17–19, 26, 29, 30], while more general aspects of the theory are given in [3, 23, 34].

The modified Bessel–Maitland and modified Struve functions are related to the modified Bessel function. Their properties can be helpful in a variety of areas in mathematical physics. A list of applications of modified Bessel function can be found in various problems that arise in wave mechanics, fluid mechanics, electrical engineering, quantum billiards,

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biophysics, mathematical physics, finite elasticity, probability and statistics, special relativity, and so on. Regarding treatises on the subject, we refer, e.g., to [12, 20–22, 28, 33] and the references therein.

The famous Turán's inequality for the Legendre polynomials $\mathrm{P}_n(x)$, that is, $(\mathrm{P}_n(x))^2 - \mathrm{P}_{n+1}(x)\mathrm{P}_{n-1}(x) \geq 0$, for $-1 \leq x \leq 1$, $n \in \mathbb{N}$, proved by Szegö [31] and Turán [32], still attracts the attention of mathematicians and has been extended to several orthogonal polynomials and special functions. Some of the results have been applied in problems that arise in information theory and credit risk modeling can be found in [24]. In [16], Carey and Gordy offered a time-independent model prediction in determining the value of assets at which firms declare bankruptcy. Using the models of Metron [25] and Black and Cox [15], Carey et al. assumed that the firm's asset follows a geometric Brownian motion. It is shown that the bank's optimal foreclosure solves a first-order condition involving a ratio of contiguous Kummer functions for which a Turán-type inequality appeared in the study of the model. A proof of this essential Turán-type inequality is established in [13]. For a general background on the applications of the Kummer function in economic theory and econometrics, see [1].

The organization of this paper is as follows. Section 2 is devoted to obtaining the integral representation as well as Mellin–Barnes integral representation for the generalized Bessel–Maitland function. Moreover, the mth derivative of $\mathcal{J}^{\mu}_{\nu,\lambda}$ is considered, and it turns out that it is expressed as the Fox–Wright function. In addition, the recurrence relations and other identities involving the derivatives are derived generalizing some of the works of [9] and [18]. At the end of the paper, the monotonicity of the ratio between two modified Bessel–Maitland functions $\mathcal{I}^{\mu}_{\nu,\lambda}$ defined by $\mathcal{I}^{\mu}_{\nu,\lambda}(z) = i^{-2\lambda-\nu}\mathcal{J}^{\mu}_{\nu,\lambda}(iz)$ of a different order, the ratio between modified Bessel–Maitland and hyperbolic functions, and some monotonicity results for $\mathcal{I}^{\mu}_{\nu,\lambda}(z)$ are obtained where the main idea of the proofs comes from the monotonicity of the quotient of two Maclaurin series. As an application, some inequalities (like Turán-type inequalities and their reverse) are proved.

We proceed to recall the generalized Bessel–Maitland function $\mathcal{J}^{\mu}_{\nu,\lambda}(z)$ defined by Pathak in [27] as

$$\begin{aligned}\mathcal{J}^{\mu}_{\nu,\lambda}(z) &= \left(\frac{z}{2}\right)^{2\lambda+\nu} \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+\lambda+1)\Gamma(n\mu+\lambda+\nu+1)} \left(\frac{z}{2}\right)^{2n} \\ &= \left(\frac{z}{2}\right)^{2\lambda+\nu} {}_1\Psi_2\left[\begin{matrix}(1,1)\\(\lambda+1,1),(\lambda+\nu+1,\mu)\end{matrix}\middle| -\frac{z^2}{4}\right],\end{aligned} \tag{1.1}$$

for $\mu > 0$, $\lambda, \nu \in \mathbb{C}$, and $z \in \mathbb{C}\backslash(-\infty, 0]$, where ${}_p\Psi_q$ denotes the Fox–Wright generalization of the hypergeometric function. It is defined by

$${}_p\Psi_q\left[\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_q,B_q)\end{matrix}\middle| z\right] =: {}_p\Psi_q\left[\begin{matrix}(\mathbf{a}_p,\mathbf{A}_p)\\(\mathbf{b}_q,\mathbf{B}_q)\end{matrix}\middle| z\right] = \sum_{n=0}^{\infty} \psi_n \frac{z^n}{n!}, \tag{1.2}$$

with

$$\psi_n = \frac{\Gamma(a_1+A_1n)\ldots\Gamma(a_p+A_pn)}{\Gamma(b_1+B_1n)\ldots\Gamma(b_q+B_qn)},$$

for $a_i, b_j \in \mathbb{C}$ and $A_i, B_j \in \mathbb{R}$ ($i = 1, \ldots, p$, $j = 1, \ldots, q$). It is worth noting that the above series converges absolutely in the whole complex z-plane when $\Delta := \sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i > -1$, while if $\Delta = -1$, series (1.1) converges absolutely for $|z| < \rho$ and $|z| = \rho$ under the condition $\Re\{\sigma\} > 1/2$, where

$$\rho = \left(\prod_{i=1}^{p} A_i^{-A_i}\right)\left(\prod_{j=1}^{q} B_j^{-B_j}\right), \qquad \sigma = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2}.$$

On the other hand, the modified Bessel-Maitland function can be defined as $\mathcal{I}_{\nu,\lambda}^{\mu}(z) = i^{-2\lambda-\nu}\mathcal{J}_{\nu,\lambda}^{\mu}(iz)$ which has the power series expansion

$$\mathcal{I}_{\nu,\lambda}^{\mu}(z) = \left(\frac{z}{2}\right)^{2\lambda+\nu} \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+\lambda+1)\Gamma(n\mu+\lambda+\nu+1)}\left(\frac{z}{2}\right)^{2n}. \tag{1.3}$$

Remark 1.1 We note the following special cases:

$${}_{\mu}\mathrm{S}_{\nu,1}(z) := \mathcal{J}_{\nu,\frac{1}{2}}^{\mu}(z) = \left(\frac{z}{2}\right)^{\nu+1} \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+3/2)\Gamma(n\mu+\nu+3/2)}\left(\frac{z}{2}\right)^{2n},$$

for $\mu > 0$ and $z, \nu \in \mathbb{C}$, where ${}_{\mu}\mathrm{S}_{\nu,1}(z)$ has been introduced by Ali et al. [9];

$${}_{\mu}\mathrm{J}_{\nu}(z) := \mathcal{J}_{\nu,0}^{\mu}(z) = \left(\frac{z}{2}\right)^{\nu} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n\mu+\nu+1)}\left(\frac{z}{2}\right)^{2n},$$

for $\mu > 0$, $\nu \in \mathbb{C}$ and $|z| < \infty$ with $|\arg z| < \pi$, where ${}_{\mu}\mathrm{J}_{\nu}(z)$ has been introduced by Galué [18];

$$\mathrm{H}_{\nu}(z) := \mathcal{J}_{\nu,1/2}^{1}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+3/2)\Gamma(n+\nu+3/2)}\left(\frac{z}{2}\right)^{2n+\nu+1}, \quad z, \nu \in \mathbb{C},$$

where $\mathrm{H}_{\nu}(z)$ is the well-known *Struve function of order* ν;

$$\mathrm{J}_{\nu}(z) := \mathcal{J}_{\nu,0}^{1}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)}\left(\frac{z}{2}\right)^{\nu+2n},$$

for $z, \nu \in \mathbb{C}$, $z \neq 0$ and $\Re\{\nu\} > -1$, where $\mathrm{J}_{\nu}(z)$ is the *Bessel function of order* ν.

We shall base our discussion upon the following definitions. A function $f : X \to \mathbb{R}$ is convex if

$$f\big(\alpha x + (1-\alpha)y\big) \leq \alpha f(x) + (1-\alpha)f(y),$$

for all $x, y \in X$ and $\alpha \in [0,1]$. If the above inequality is ($<$), then f is strictly convex. Moreover, f is (strictly) concave if $-f$ is (strictly) convex. In addition, if f is differentiable, then f is convex (concave) if and only if f is increasing (decreasing) and if f is twice differentiable, then f is convex (concave) if and only if f'' is nonnegative (nonpositive). A function g is log-convex or superconvex on $[a,b]$ if $g>0$ and $\log g$ is convex on $[a,b]$, that is,

$$\log g\big(\alpha x+(1-\alpha)y\big)\le \alpha\log g(x)+(1-\alpha)\log g(y),$$

or equivalently,

$$g\big(\alpha x+(1-\alpha)y\big)\le \big(g(x)\big)^{\alpha}\big(g(y)\big)^{1-\alpha},$$

for all $x,y\in[a,b]$ and $\alpha\in[0,1]$. It is worthwhile mentioning that g is log-concave if the above inequality is reversed. A real-valued function $f(x)$ is called absolutely monotonic on $(0,\infty)$ if it has derivatives of all orders and satisfies $f^{(n)}(x)\ge 0$ for all $x\in(0,\infty)$ and $n\ge 0$.

After this preparation, we can pass on the main results of the present paper.

2 Some properties of the generalized Bessel–Maitland function

In this section, we shall recall Gauss's multiplication theorem [2] which states that

$$\Gamma(mz)=(2\pi)^{(1-m)/2}m^{mz-1/2}\prod_{i=1}^{m}\Gamma\left(z+\frac{i-1}{m}\right),\quad z\neq 0,-\frac{1}{m},-\frac{2}{m},\dots,m\in\mathbb{N},\tag{2.1}$$

where $\mathbb{N}=\{1,2,3,\dots\}$. Therefore

$$\Gamma(n\mu+\ell)=\Gamma\left(\mu\left(n+\frac{\ell}{\mu}\right)\right)=(2\pi)^{(1-\mu)/2}\mu^{\mu n+\ell-1/2}\prod_{i=1}^{\mu}\Gamma\left(n+\frac{i+\ell-1}{\mu}\right),$$

for $\ell\neq-\mu n,-\mu n-1,\dots$ and $\mu\in\mathbb{N}$. Moreover,

$$\Gamma(\ell)=(2\pi)^{(1-\mu)/2}\mu^{\ell-1/2}\prod_{i=1}^{\mu}\Gamma\left(\frac{i+\ell-1}{\mu}\right),$$

that is,

$$\prod_{i=1}^{\mu}\Gamma\left(\frac{i+\ell-1}{\mu}\right)=\frac{\Gamma(\ell)}{(2\pi)^{(1-\mu)/2}\mu^{\ell-1/2}}.\tag{2.2}$$

From (2.1) and (2.2), it further follows that

$$\Gamma(n\mu+\ell)=\mu^{\mu n}\Gamma(\ell)\prod_{i=1}^{\mu}\left(\frac{i+\ell-1}{\mu}\right)_n,\tag{2.3}$$

where $(a)_n$ represents the *Pochhammer symbol* defined by

$$(a)_n:=\begin{cases}1, & \text{if } n=0,\\ a(a+1)(a+2)\dots(a+n-1), & \text{if } n\in\mathbb{N}.\end{cases}$$

Putting $\ell=\lambda+\nu+1$, we find

$$\Gamma(n\mu+\lambda+\nu+1)=\mu^{\mu n}\Gamma(\lambda+\nu+1)\prod_{i=1}^{\mu}\left(\frac{i+\lambda+\nu}{\mu}\right)_n.$$

Using the series expansion (1.1) of $\mathcal{J}_{\nu,\lambda}^{\mu}(z)$, we obtain

$$\begin{aligned}\mathcal{J}_{\nu,\lambda}^{\mu}(z) &= \frac{z^{2\lambda+\nu}}{2^{2\lambda+\nu}\Gamma(\lambda+1)\Gamma(\lambda+\nu+1)} \\ &\quad\times \sum_{n=0}^{\infty} \frac{1}{(\lambda+1)_n(\frac{\lambda+\nu+1}{\mu})_n\cdots(\frac{\lambda+\nu+\mu}{\mu})_n}\left(-\frac{z^2}{4\mu^{\mu}}\right)^n \\ &= \frac{z^{2\lambda+\nu}}{2^{2\lambda+\nu}\Gamma(\lambda+1)\Gamma(\lambda+\nu+1)} \\ &\quad\times {}_1\mathrm{F}_{\mu+1}\left(1;\lambda+1,\frac{\lambda+\nu+1}{\mu},\ldots,\frac{\lambda+\nu+\mu}{\mu};-\frac{z^2}{4\mu^{\mu}}\right),\end{aligned}$$

where ${}_p\mathrm{F}_q$ stands for the generalized hypergeometric function defined by

$${}_p\mathrm{F}_q(a_1,\ldots,a_p,b_1,\ldots,b_q) = \sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!}.$$

As a special case, letting $\mu=1$, $\lambda=0$, $\nu=1/2$, and $\nu=-1/2$, respectively, we obtain

$$\begin{aligned}\mathcal{J}_{\frac{1}{2},0}^{1}(z) &= \sqrt{\frac{2z}{\pi}}\,{}_0\mathrm{F}_1\left(-;\frac{3}{2};-\frac{z^2}{4}\right) = \sqrt{\frac{2}{\pi z}}\sin z, \\ \mathcal{J}_{-\frac{1}{2},0}^{1}(z) &= \sqrt{\frac{2}{\pi z}}\,{}_0\mathrm{F}_1\left(-;\frac{1}{2};-\frac{z^2}{4}\right) = \sqrt{\frac{2}{\pi z}}\cos z.\end{aligned}$$

We then proceed to establish the integral representation for $\mathcal{J}_{\nu,\lambda}^{\mu}(z)$. Let us recall that

$$\frac{1}{\Gamma(x+y)} = \frac{k}{\Gamma(x)\Gamma(y)}\int_0^1 t^{kx-1}\left(1-t^k\right)^{y-1}dt,$$

for $\Re\{x\}>0$, $\Re\{y\}>0$. Letting $k=2$, $x=n+\lambda+1/2$, and $y=(\lambda+\nu+i)/\mu-\lambda-1/2$ in the aforementioned integration, we get

$$\begin{aligned}\frac{1}{\Gamma(n+(\lambda+\nu+i)/\mu)} &= \frac{2}{\Gamma(n+\lambda+1/2)\Gamma((\lambda+\nu+i)/\mu-\lambda-1/2)} \\ &\quad\times\int_0^1 t^{2n+2\lambda}\left(1-t^2\right)^{(\lambda+\nu+i)/\mu-\lambda-3/2}dt,\end{aligned}$$

and so

$$\begin{aligned}\mathcal{J}_{\nu,\lambda}^{\mu}(z) &= \sum_{n=0}^{\infty}\frac{(-1)^n(z/2)^{2n+2\lambda+\nu}}{\Gamma(n+\lambda+1)}(2\pi)^{(\mu-1)/2}\mu^{-(\mu n+\lambda+\nu+1/2)} \\ &\quad\times\prod_{i=1}^{\mu}\frac{2}{\Gamma(n+\lambda+1/2)\Gamma((\lambda+\nu+i)/\mu-\lambda-1/2)} \\ &\quad\times\int_0^1 t^{2n+2\lambda}\left(1-t^2\right)^{(\lambda+\nu+i)/\mu-\lambda-3/2}dt,\end{aligned}$$

where $\Re\{(\lambda+\nu+1)/\mu-\lambda-1/2\}>0$. Interchanging the order of the integral and summation, we obtain

$$\mathcal{J}^{\mu}_{\nu,\lambda}(z)=2(2\pi)^{(\mu-1)/2}\mu^{-(\lambda+\nu+1/2)}\left(\frac{z}{2}\right)^{2\lambda+\nu}\prod_{i=1}^{\mu}\frac{1}{\Gamma((\lambda+\nu+i)/\mu-\lambda-1/2)}$$
$$\times\int_0^1\left(1-t^2\right)^{(\lambda+\nu+i)/\mu-\lambda-3/2}\sum_{n=0}^{\infty}\frac{(-1)^n(z/2)^{2n}\mu^{-\mu n}t^{2n+2\lambda}}{\Gamma(n+\lambda+1)\Gamma(n+\lambda+1/2)}\,dt. \tag{2.4}$$

Using *Legendre's formula*

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2z-1}}\Gamma(2z),$$

for $z=n+\lambda+\frac{1}{2}$, we get

$$\Gamma\left(n+\lambda+\frac{1}{2}\right)\Gamma(n+\lambda+1)=\frac{\sqrt{\pi}}{2^{2n+2\lambda}}\Gamma(2n+2\lambda+1). \tag{2.5}$$

Substituting in (2.4), we find

$$\mathcal{J}^{\mu}_{\nu,\lambda}(z)=\frac{2^{1-\nu}}{\sqrt{\pi}}(2\pi)^{(\mu-1)/2}\mu^{-(\lambda+\nu+1/2)}z^{2\lambda+\nu}\prod_{i=1}^{\mu}\frac{1}{\Gamma((\lambda+\nu+i)/\mu-\lambda-1/2)}$$
$$\times\int_0^1\left(1-t^2\right)^{(\lambda+\nu+i)/\mu-\lambda-3/2}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}\mu^{-\mu n}t^{2n+2\lambda}}{\Gamma(2n+2\lambda+1)}\,dt$$
$$=\frac{2^{1-\nu}(2\pi)^{(\mu-1)/2}\mu^{-(\lambda+\nu+1/2)}z^{2\lambda+\nu}}{\sqrt{\pi}\,\Gamma(2\lambda+1)}\prod_{i=1}^{\mu}\frac{1}{\Gamma((\lambda+\nu+i)/\mu-\lambda-1/2)}$$
$$\times\int_0^1\left(1-t^2\right)^{(\lambda+\nu+i)/\mu-\lambda-3/2}t^{2\lambda}{}_1\mathrm{F}_2\left(1;\frac{1}{2}+\lambda,1+\lambda;-\frac{t^2z^2}{4\mu^{\mu}}\right)dt. \tag{2.6}$$

Suppose that

$$\mathrm{T}(z):={}_1\mathrm{F}_2\left(1;\frac{1}{2}+\lambda,1+\lambda;-\frac{t^2z^2}{4\mu^{\mu}}\right).$$

By applying the following integral representation for ${}_1\mathrm{F}_2(a;b,c;z)$:

$${}_1\mathrm{F}_2(a;b,c;z):=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1(1-u)^{c-a-1}u^{a-1}{}_0\mathrm{F}_1(-;b;uz)\,du,$$

with $\Re\{c\}>\Re\{a\}>0$, then $\mathrm{T}(z)$ takes the form

$$\mathrm{T}(z)=\lambda\int_0^1(1-u)^{\lambda-1}_0\mathrm{F}_1\left(-;\frac{1}{2}+\lambda;-\frac{ut^2z^2}{4\mu^{\mu}}\right)du, \tag{2.7}$$

with $\lambda>0$. Now, using the integral representation for ${}_0\mathrm{F}_1(-;b;t)$,

$${}_0\mathrm{F}_1(-;b;z)=\frac{2\Gamma(b)}{\sqrt{\pi}\,\Gamma(b-1/2)}\int_0^1\left(1-\nu^2\right)^{b-3/2}\cosh(2\nu\sqrt{z})\,d\nu,$$

with $\Re\{b\} > 1/2$, we obtain

$$\begin{aligned}
\mathrm{T}(z) &= \frac{2\lambda\Gamma(\frac{1}{2}+\lambda)}{\sqrt{\pi}\Gamma(\lambda)}\int_0^1\int_0^1 (1-u)^{\lambda-1}\big(1-v^2\big)^{\lambda-1}\cosh\big(ivt\sqrt{uz^2\mu^{-\mu}}\big)\,du\,dv \\
&= \frac{2\lambda\Gamma(\frac{1}{2}+\lambda)}{\sqrt{\pi}\Gamma(\lambda)}\int_0^1\int_0^1 (1-u)^{\lambda-1}\big(1-v^2\big)^{\lambda-1}\cos\big(vt\sqrt{uz^2\mu^{-\mu}}\big)\,du\,dv,
\end{aligned}$$

with $\lambda > 0$, and so (2.6) becomes

$$\begin{aligned}
\mathcal{J}_{\nu,\lambda}^{\mu}(z) &= \frac{4\lambda\Gamma(\frac{1}{2}+\lambda)}{\pi\Gamma(\lambda)\Gamma(2\lambda+1)}(2\pi)^{(\mu-1)/2}\mu^{-(\lambda+\nu+1/2)}z^{2\lambda+\nu}2^{-\nu}\prod_{i=1}^{\mu}\frac{1}{\Gamma((\lambda+\nu+i)/\mu-\lambda-1/2)} \\
&\quad\times\int_0^1\int_0^1\int_0^1\big(1-t^2\big)^{(\lambda+\nu+i)/\mu-\lambda-3/2}t^{2\lambda}(1-u)^{\lambda-1} \\
&\quad\times\big(1-v^2\big)^{\lambda-1}\cos\big(vt\sqrt{uz^2\mu^{-\mu}}\big)\,dt\,du\,dv.
\end{aligned}$$

Remark 2.1 Setting $\lambda = 1/2$ and 0 in (2.6), we obtain the corresponding results of [9] and [18], respectively.

We are now presenting Mellin–Barnes integral representation as well as some differential results related to the generalized Bessel–Maitland function defined by *(1.1).*

Theorem 2.1 *Let $\lambda > -1$, $\mu > 0$, $\nu \geq 0$, and $z \in \mathbb{C}\setminus(-\infty, 0]$, then $\mathcal{J}_{\nu,\lambda}^{\mu}(z)$ can be represented by the Mellin–Barnes integral as*

$$\mathcal{J}_{\nu,\lambda}^{\mu}(z) = \frac{1}{2\pi i}\int_{\mathcal{L}}\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\lambda+1-s)\Gamma(\lambda+\nu+1-\mu s)}\left(\frac{z}{2}\right)^{-2s+\nu+2\lambda}ds, \tag{2.8}$$

where the contour of integration $\mathcal{L}$ beginning at $c - i\infty$ and ending at $c + i\infty$ for any $c > 0$ and separates all poles at $s = -n$ ($n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$) to the left and at $s = n + 1$ to the right.

Proof The poles of the integrand in (2.8) are at the points $s = 0, -1, -2, \ldots$. Consider the straight line contour $c - i\infty$ to $c + i\infty$ for any $c > 0$, then all the poles lie to the left of the contour. Thus, any infinite semi-circle can enclose all these poles and the residue theorem applies to find

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\mathcal{L}}\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\lambda+1-s)\Gamma(\lambda+\nu+1-\mu s)}\left(\frac{z}{2}\right)^{-2s+\nu+2\lambda}ds \\
&\quad= \sum_{k=0}^{\infty}\operatorname*{Res}_{s=-n}\left[\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\lambda+1-s)\Gamma(\lambda+\nu+1-\mu s)}\left(\frac{z}{2}\right)^{-2s+\nu+2\lambda}\right] \\
&\quad= \sum_{n=0}^{\infty}\lim_{s\to-n}\left[\frac{(s+n)\Gamma(s)\Gamma(1-s)}{\Gamma(\lambda+1-s)\Gamma(\lambda+\nu+1-\mu s)}\left(\frac{z}{2}\right)^{-2s+\nu+2\lambda}\right] \\
&\quad= \sum_{n=0}^{\infty}\lim_{s\to-n}\frac{\Gamma(s+n+1)\Gamma(1-s)}{(s+n-1)\cdots s\Gamma(\lambda+1-s)\Gamma(\lambda+\nu+1-\mu s)}\left(\frac{z}{2}\right)^{-2s+\nu+2\lambda} \\
&\quad= \sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(\lambda+1+n)\Gamma(\lambda+\nu+1+\mu n)}\left(\frac{z}{2}\right)^{2n+\nu+2\lambda} = \mathcal{J}_{\nu,\lambda}^{\mu}(z).
\end{aligned}$$

□

Theorem 2.2 *For $\mu > 0$, $\lambda, \nu \in \mathbb{C}$, and $z \in \mathbb{C} \setminus (-\infty, 0]$, we have*

$$\frac{d^m}{dz^m}\big(\mathcal{J}^{\mu}_{\nu,\lambda}(z)\big) = (-1)^m \left(\frac{z}{2}\right)^{2\lambda+\nu+m} 2^{-m} \times {}_2\Psi_3\left[\begin{array}{c} (2m+2\lambda+\nu+1,2),(1,1) \\ (m+\lambda+1,1),(\lambda+\nu+1+\mu m,\mu),(m+2\lambda+\nu+1,2) \end{array}\middle| -\frac{z^2}{4}\right].$$

Proof By making use of the series representation (1.1), we find

$$\begin{aligned}
&\frac{d^m}{dz^m}\big(\mathcal{J}^{\mu}_{\nu,\lambda}(z)\big) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+2\lambda+\nu}\Gamma(n+\lambda+1)\Gamma(n\mu+\lambda+\nu+1)} \frac{d^m}{dz^m}\big(z^{2n+2\lambda+\nu}\big) \\
&= \sum_{n=m}^{\infty} \frac{(-1)^n(2n+2\lambda+\nu)\cdots(2n+2\lambda+\nu-m-1)}{2^{2n+2\lambda+\nu}\Gamma(n+\lambda+1)\Gamma(n\mu+\lambda+\nu+1)} \frac{(2n+2\lambda+\nu)!}{(2n+2\lambda+\nu)!} z^{2n+2\lambda+\nu-m} \\
&= \sum_{n=m}^{\infty} \frac{(-1)^n}{2^{2n+2\lambda+\nu}\Gamma(n+\lambda+1)\Gamma(n\mu+\lambda+\nu+1)} \frac{(2n+2\lambda+\nu)! z^{2n+2\lambda+\nu-m}}{(2n+2\lambda+\nu-m)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n+m}}{2^{2n+2m+2\lambda+\nu}\Gamma(n+m+\lambda+1)\Gamma(n\mu+m\mu+\lambda+\nu+1)} \\
&\quad \times \frac{(2n+2m+2\lambda+\nu)! z^{2n+2\lambda+\nu+m}}{(2n+m+2\lambda+\nu)!} \\
&= (-1)^m \left(\frac{z}{2}\right)^{2\lambda+\nu+m} 2^{-m} \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+m+\lambda+1)} \\
&\quad \times \frac{\Gamma(2n+2m+2\lambda+\nu+1)}{\Gamma(n\mu+m\mu+\nu+\lambda+1)\Gamma(2n+m+2\lambda+\nu+1)} \left(-\frac{z^2}{4}\right)^n,
\end{aligned}$$

and the proof is complete. □

Theorem 2.3 *If $\lambda \geq 0$, $\nu, \mu > 0$, and $z \in \mathbb{C} \setminus (-\infty, 0]$, the following identities hold:*

$$\frac{d}{dz}\big(z^{-\nu}\mathcal{J}^{\mu}_{\nu,\lambda}(z)\big) = -2^{\mu-1} z^{1-\nu-\mu} \mathcal{J}^{\mu}_{\nu+\mu,\lambda}(z), \tag{2.9}$$

$$\frac{d}{dz}\Big(z^{-\nu-2\lambda+\frac{2(\lambda+\nu)}{\mu}} \mathcal{J}^{\mu}_{\nu,\lambda}(z)\Big) = \frac{1}{\mu} z^{-2\lambda-\nu+\frac{2(\lambda+\nu)}{\mu}} \mathcal{J}^{\mu}_{\nu-1,\lambda}(z), \quad \textit{for } 2(\lambda+\nu) > \mu. \tag{2.10}$$

Proof By using the series expansion (1.1), the left-hand side of (2.9) becomes

$$\begin{aligned}
&\frac{d}{dz}\big(z^{-\nu}\mathcal{J}^{\mu}_{\nu,\lambda}(z)\big) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n z^{2\lambda+2n-1}}{2^{2n+2\lambda+\nu-1}\Gamma(n+\lambda)\Gamma(n\mu+\lambda+\nu+1)} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} z^{2\lambda+2n+1}}{2^{2n+2\lambda+\nu+1}\Gamma(n+\lambda+1)\Gamma(n\mu+\mu+\lambda+\nu+1)}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{z^{1-\nu-\mu}}{2^{1-\mu}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2\lambda+2n+\nu+\mu}}{2^{2n+2\lambda+\nu+\mu}\Gamma(n+\lambda+1)\Gamma(n\mu+\mu+\lambda+\nu+1)} \\
&= -2^{\mu-1} z^{1-\nu-\mu} \mathcal{J}^{\mu}_{\nu+\mu,\lambda}(z),
\end{aligned}
$$

which ends the proof of (2.9). A similar argument is used to prove (2.10) as follows:

$$
\begin{aligned}
&\frac{d}{dz}\left(z^{-\nu-2\lambda+\frac{2(\lambda+\nu)}{\mu}} \mathcal{J}^{\mu}_{\nu,\lambda}(z)\right) \\
&= \frac{2}{\mu} \sum_{n=0}^{\infty} \frac{(-1)^n (n\mu+\lambda+\nu) z^{2n-1+\frac{2(\lambda+\nu)}{\mu}}}{2^{2n+2\lambda+\nu}\Gamma(n+\lambda+1)\Gamma(n\mu+\lambda+\nu+1)} \\
&= \frac{2}{\mu} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n-1+\frac{2(\lambda+\nu)}{\mu}}}{2^{2n+2\lambda+\nu}\Gamma(n+\lambda+1)\Gamma(n\mu+\lambda+\nu)} \\
&= \frac{z^{-2\lambda-\nu+\frac{2(\lambda+\nu)}{\mu}}}{\mu} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+2\lambda+\nu-1}}{2^{2n+2\lambda+\nu-1}\Gamma(n+\lambda+1)\Gamma(n\mu+\lambda+\nu)} \\
&= \frac{1}{\mu} z^{-2\lambda-\nu+\frac{2(\lambda+\nu)}{\mu}} \mathcal{J}^{\mu}_{\nu-1,\lambda}(z).
\end{aligned}
$$

Hence, the proof of the present theorem is complete. □

The theorem above generalizes the results given in [18] for $\lambda = 0$. Now, it is worth mentioning that (2.9) is equivalent to

$$
-\nu z^{-\nu-1} \mathcal{J}^{\mu}_{\nu,\lambda}(z) + z^{-\nu}\left(\mathcal{J}^{\mu}_{\nu,\lambda}(z)\right)' = -2^{\mu-1} z^{1-\nu-\mu} \mathcal{J}^{\mu}_{\nu+\mu,\lambda}(z),
$$

that is,

$$
-\nu \mathcal{J}^{\mu}_{\nu,\lambda}(z) + z\left(\mathcal{J}^{\mu}_{\nu,\lambda}(z)\right)' = -2^{\mu-1} z^{2-\mu} \mathcal{J}^{\mu}_{\nu+\mu,\lambda}(z). \tag{2.11}
$$

Furthermore, from (2.10), we have

$$
\begin{aligned}
&z^{-\nu-2\lambda+\frac{2(\lambda+\nu)}{\mu}} \left(\mathcal{J}^{\mu}_{\nu,\lambda}(z)\right)' + \left(-\nu - 2\lambda + \frac{2(\lambda+\nu)}{\mu}\right) z^{-\nu-2\lambda+\frac{2(\lambda+\nu)}{\mu}-1} \left(\mathcal{J}^{\mu}_{\nu,\lambda}(z)\right) \\
&= \frac{1}{\mu} z^{-2\lambda-\nu+2(\lambda+\nu)/\mu} \mathcal{J}^{\mu}_{\nu-1,\lambda}(z),
\end{aligned}
$$

which leads to

$$
z\left(\mathcal{J}^{\mu}_{\nu,\lambda}(z)\right)' + \left(-\nu - 2\lambda + \frac{2(\lambda+\nu)}{\mu}\right)\left(\mathcal{J}^{\mu}_{\nu,\lambda}(z)\right) = \frac{1}{\mu} z\left(\mathcal{J}^{\mu}_{\nu-1,\lambda}(z)\right). \tag{2.12}
$$

From (2.11) and (2.12), it is easy to observe that

$$
\left(\frac{2(\lambda+\nu)}{\mu} - 2\lambda\right)\left(\mathcal{J}^{\mu}_{\nu,\lambda}(z)\right) = 2^{\mu-1} z^{2-\mu} \mathcal{J}^{\mu}_{\nu+\mu,\lambda}(z) + \frac{1}{\mu} z\left(\mathcal{J}^{\mu}_{\nu-1,\lambda}(z)\right). \tag{2.13}
$$

Theorem 2.4 *Let* $\lambda \geq 0$, $\nu, \mu > 0$, *and* $z \in \mathbb{C} \setminus (-\infty, 0]$. *The following identities hold*:

$$\left(z^{1-\frac{2}{\mu}} \frac{d}{dz}\right)^m \left(z^{-\nu-2\lambda+\frac{2(\lambda+\nu)}{\mu}} \mathcal{J}_{\nu,\lambda}^{\mu}(z)\right) = \frac{1}{\mu^m} z^{\frac{2}{\mu}(\lambda+\nu-m)-2\lambda-\nu+m} \mathcal{J}_{\nu-m,\lambda}^{\mu}(z), \tag{2.14}$$

with $2(\lambda + \nu - m + 1) > \mu$ *and*

$$\left(\frac{1}{z} \frac{d}{dz}\right)^m \left(z^{-\nu} \mathcal{J}_{\nu,\lambda}^{\mu}(z)\right) = (-1)^m 2^{(\mu-1)m} z^{-\nu-\mu m} \mathcal{J}_{\nu+\mu m,\lambda}^{\mu}(z). \tag{2.15}$$

Proof We proceed by induction on m. When $m = 1$, the identity holds. Assume that it holds when $m = k$ for some integer $k \geq 1$. We have to show that it still holds when $m = k + 1$ as follows:

$$\begin{aligned}
&\left(z^{1-\frac{2}{\mu}} \frac{d}{dz}\right)^{k+1} \left(z^{-\nu-2\lambda+\frac{2(\lambda+\nu)}{\mu}} \mathcal{J}_{\nu,\lambda}^{\mu}(z)\right) \\
&= \left(\frac{1}{z^{\frac{2}{\mu}-1}} \frac{d}{dz}\right)\left(z^{1-\frac{2}{\mu}} \frac{d}{dz}\right)^{k} \left(z^{-\nu-2\lambda+\frac{2(\lambda+\nu)}{\mu}} \mathcal{J}_{\nu,\lambda}^{\mu}(z)\right) \\
&= \left(\frac{1}{z^{\frac{2}{\mu}-1}} \frac{d}{dz}\right)\left[z^{\frac{2}{\mu}(\lambda+\nu-1)-2\lambda-\nu+k} \mathcal{J}_{\nu-k,\lambda}^{\mu}(z)\right] \\
&= \frac{1}{z^{\frac{2}{\mu}-1}} \frac{d}{dz}\left[\frac{z^{\frac{2}{\mu}(\lambda+\nu-1)-2\lambda-\nu+k}}{\mu^k} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+2\lambda+\nu-k}}{2^{2n+2\lambda+\nu-k}\Gamma(n+\lambda+1)\Gamma(\lambda+\nu+n\mu-k)}\right] \\
&= \frac{1}{z^{\frac{2}{\mu}-1}} \frac{d}{dz}\left[\frac{1}{\mu^k} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+2\frac{(\lambda+\nu-k)}{\mu}}}{2^{2n+2\lambda+\nu+k\mu}\Gamma(n+\lambda+1)\Gamma(\lambda+\nu+n\mu-k)}\right] \\
&= \frac{2}{z^{\frac{2}{\mu}-1}} \frac{1}{\mu^{k+1}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+2\frac{(\lambda+\nu-k)}{\mu}-1}}{2^{2n+2\lambda+\nu-k}\Gamma(n+\lambda+1)\Gamma(\lambda+\nu+n\mu-k-1)} \\
&= \frac{1}{\mu^{k+1}} z^{\frac{2}{\mu}(\lambda+\nu-k-1)-2\lambda-\nu+k+1} \mathcal{J}_{\nu-k-1,\lambda}^{\mu}(z).
\end{aligned}$$

Therefore, the identity also holds when $m = k + 1$; and consequently it holds for every integer $m \geq 1$. Similarly, relation (2.15) may be proved. □

The above theorem generalizes the result given in [18] for $\lambda = 0$.

3 Monotonicity properties of the modified Bessel–Maitland function

We proceed to state the following lemma which will be used in proving the theoretical results of this section.

Lemma 3.1 ([14]) *Let* $a_n \in \mathbb{R}$ *and* $b_n > 0$ *for* $n \in \mathbb{N}_0$. *If* $A(z) = \sum_{n\geq 0} a_n z^n$ *and* $B(z) = \sum_{n\geq 0} b_n z^n$ *are a convergent power series in* $|z| < R$ *and the sequence* $\{a_n/b_n\}_{n\geq 0}$ *is increasing* (*decreasing*), *then the quotient* $A(z)/B(z)$ *is increasing* (*decreasing*) *on* $(0, R)$.

Bear in mind that the above lemma can be applied if the power series is of the form

$$A(z) = \sum_{n\geq 0} a_n z^{2n} \quad \text{and} \quad B(z) = \sum_{n\geq 0} b_n z^{2n},$$

or

$$A(z) = \sum_{n\geq 0} a_n z^{2n+1} \quad \text{and} \quad B(z) = \sum_{n\geq 0} b_n z^{2n+1}.$$

Furthermore, let us define the normalized form of the modified Bessel–Maitland function by

$$\begin{aligned}\mathfrak{J}^{\mu}_{\nu,\lambda}(z) &= \Gamma(\lambda+1)\Gamma(\lambda+\nu+1)2^{2\lambda+\nu}z^{1-2\lambda-\nu}\mathcal{I}^{\mu}_{\nu,\lambda}(z) \\ &= \sum_{n=0}^{\infty}\frac{z^{2n+1}}{4^n(\lambda+1)_n(\lambda+\nu+1)_{n\mu}}.\end{aligned} \tag{3.1}$$

Now, we are ready to pass on the main results of this section.

Theorem 3.1 *Let k be a nonnegative integer. Then the following assertions hold*:

(i) *If* $\nu,\nu_1\geq 0$, $\mu>0$, $\lambda>-1$, *and* $\nu>\nu_1$, *then* $z\mapsto \mathfrak{J}^{\mu}_{\nu,\lambda}(z)/\mathfrak{J}^{\mu}_{\nu_1,\lambda}(z)$ *is decreasing on* $(0,\infty)$;

(ii) *If* $\mu\in\mathbb{N}$, $\nu>1/2$, *and* $\lambda\geq 0$, *then* $z\mapsto(\mathfrak{J}^{\mu}_{\nu,\lambda}(z))^{(2k+1)}/\cosh z$ *is strictly decreasing on* $(0,\infty)$;

(iii) *If* $\mu\in\mathbb{N}$, $\nu>1/2$, *and* $\lambda\geq 0$, *then* $z\mapsto(\mathfrak{J}^{\mu}_{\nu,\lambda}(z))^{(2k+2)}/\sinh z$ *is strictly decreasing on* $(0,\infty)$;

(iv) *If* $\mu\in\mathbb{N}$, $\nu>1/2$, *and* $\lambda\geq 0$, *then* $z\mapsto(\mathfrak{J}^{\mu}_{\nu,\lambda}(z))^{(2k+1)}/(\cosh z+z\sinh z)$ *is strictly decreasing on* $(0,\infty)$.

Proof (i) Using the power series expansion of $\mathfrak{J}^{\mu}_{\nu,\lambda}(z)$, we have

$$\frac{\mathfrak{J}^{\mu}_{\nu,\lambda}(z)}{\mathfrak{J}^{\mu}_{\nu_1,\lambda}(z)} = \frac{\sum_{n=0}^{\infty}\mathrm{b}_n(\nu)z^{2n+1}}{\sum_{n=0}^{\infty}\mathrm{b}_n(\nu_1)z^{2n+1}}.$$

In view of Lemma 3.1, it is enough to establish the monotonicity of

$$\mathrm{g}_n(z) = \frac{\mathrm{b}_n(\nu)}{\mathrm{b}_n(\nu_1)} = \frac{(\lambda+\nu_1+1)_{n\mu}}{(\lambda+\nu+1)_{n\mu}}.$$

So,

$$\frac{\mathrm{g}_{n+1}(z)}{\mathrm{g}_n(z)} = \frac{\Gamma(n\mu+\mu+\lambda+\nu_1+1)\Gamma(n\mu+\lambda+\nu+1)}{\Gamma(n\mu+\lambda+\nu_1+1)\Gamma(n\mu+\mu+\lambda+\nu+1)}.$$

Suppose that

$$\beta(\nu) = \frac{\Gamma(n\mu+\lambda+\nu+1)}{\Gamma(n\mu+\mu+\lambda+\nu+1)}. \tag{3.2}$$

Differentiating (3.2) logarithmically with respect to ν, we find

$$\frac{\beta'(\nu)}{\beta(\nu)} = \Psi(n\mu+\lambda+\nu+1)-\Psi(n\mu+\mu+\lambda+\nu+1).$$

Here, $\Psi(z)$ denotes for digamma function defined by $\Psi(z) = \Gamma'(z)/\Gamma(z)$. Using the well-known formula

$$\Psi(z) = -\gamma + \int_0^1 \frac{t^{z-1} - 1}{t-1}\,dt, \tag{3.3}$$

where γ is the Euler–Mascheroni constant given by

$$\gamma = \lim_{n\to\infty}\left(\sum_{m=1}^{n} \frac{1}{m} - \log n\right) = 0.5772156649,$$

we have

$$\frac{\beta'(\nu)}{\beta(\nu)} = \int_0^1 \frac{t^{n\mu+\lambda+\nu}(1-t^{\mu})}{t-1}\,dt \le 0,$$

which implies that β is a decreasing function with respect to ν, that is, if $\nu > \nu_1$, then $\mathrm{g}_{n+1}(z) \le \mathrm{g}_n(z)$, and the result follows.

(ii) Straightforward computations using (3.1) show that

$$\left(\mathfrak{J}^{\mu}_{\nu,\lambda}(z)\right)^{(2k+1)} = \sum_{n=0}^{\infty} \frac{(2n+2k+1)!}{4^{n+k}(\lambda+1)_{n+k}(\lambda+\nu+1)_{(n+k)\mu}(2n)!} z^{2n},$$

and

$$\left(\mathfrak{J}^{\mu}_{\nu,\lambda}(z)\right)^{(2k+2)} = \sum_{n=0}^{\infty} \frac{(2n+2k+3)!}{4^{n+k+1}(\lambda+1)_{n+k+1}(\lambda+\nu+1)_{(n+k+1)\mu}(2n+1)!} z^{2n+1}.$$

On the other hand,

$$\mathfrak{J}^{1}_{\frac{1}{2},0}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\frac{3}{2})}{4^n n!\Gamma(n+\frac{3}{2})} z^{2n+1} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1} = \sinh z,$$
$$\mathfrak{J}^{1}_{-\frac{1}{2},0}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{2})}{4^n n!\Gamma(n+\frac{1}{2})} z^{2n+1} = z\sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n} = z\cosh z,$$

and

$$\begin{aligned}\left(\mathfrak{J}^{1}_{-\frac{1}{2},0}(z)\right)' &= \sum_{n=0}^{\infty} \frac{(2n+1)\Gamma(\frac{1}{2})}{4^n n!\Gamma(n+\frac{1}{2})} z^{2n} = \sum_{n=1}^{\infty} \frac{z^{2n}}{\Gamma(2n)} + \sum_{n=0}^{\infty} \frac{z^{2n}}{\Gamma(2n+1)} \\ &= \sum_{n=0}^{\infty} \frac{z^{2n+2}}{\Gamma(2n+2)} + \sum_{n=0}^{\infty} \frac{z^{2n}}{\Gamma(2n+1)} \\ &= z\sinh z + \cosh z.\end{aligned}$$

Thanks to Lemma 3.1, it suffices to study the monotonicity of the sequence $\{\mathrm{c}_n\}_{n\ge 0}$ where

$$\mathrm{c}_n = \frac{(2n+2k+1)!}{4^{n+k}(\lambda+1)_{n+k}(\lambda+\nu+1)_{(n+k)\mu}}.$$

Using the fact that

$$\Gamma\bigl(\lambda+\nu+1+(n+1+k)\mu\bigr) \geq \Gamma\bigl(\lambda+\nu+(n+k)\mu+2\bigr) \\ = \bigl(\lambda+\nu+1+(n+k)\mu\bigr)\Gamma\bigl(\lambda+\nu+1+(n+k)\mu\bigr),$$

for $\mu \in \mathbb{N}$, we obtain

$$\frac{\mathrm{c}_{n+1}}{\mathrm{c}_n} \leq \frac{(2n+2k+3)(2n+2k+2)}{4(\lambda+n+k+1)(\lambda+\nu+1+n\mu+k\mu)} < 1,$$

whenever $\mu \in \mathbb{N}$, $\nu > 1/2$, and $\lambda \geq 0$.

(iii) According to Lemma 3.1, it is enough to show the monotonicity of

$$\mathrm{d}_n = \frac{(2n+2k+3)!}{4^{n+k+1}(\lambda+1)_{n+k+1}(\lambda+\nu+1)_{(n+k+1)\mu}}.$$

Hence,

$$\frac{\mathrm{d}_{n+1}}{\mathrm{d}_n} \leq \frac{(2n+2k+5)(2n+2k+4)}{4(\lambda+n+k+2)(\lambda+\nu+1+n\mu+k\mu+\mu)} < 1,$$

if $\mu \in \mathbb{N}$, $\nu > 1/2$, and $\lambda \geq 0$.

(iv) Consider the sequence $\{\mathrm{h}_n\}_{n\geq 0}$, where

$$\mathrm{h}_n = \frac{(2n+2k+1)!}{4^{n+k}(2n+1)(\lambda+1)_{n+k}(\lambda+\nu+1)_{(n+k)\mu}}.$$

Further computations show that

$$\frac{\mathrm{h}_{n+1}}{\mathrm{h}_n} \leq \frac{(2n+2k+3)(2n+2k+2)(2n+1)}{4(\lambda+n+k+1)(\lambda+\nu+1+n\mu+k\mu)(2n+3)} < 1,$$

for $\mu \in \mathbb{N}$, $\nu > 1/2$, and $\lambda \geq 0$, which ends the proof. □

Theorem 3.2 *The following assertions hold*:

(i) *If* $\lambda > -1$ *and* $\nu, \mu > 0$, *then* $z \mapsto \mathfrak{J}^{\mu}_{\nu,\lambda}(z)$ *is an absolutely monotonic function on* $(0,\infty)$;

(ii) *If* $\lambda > -1$, $\nu, \mu > 0$, *and* $z > 0$, *then* $\lambda \mapsto \mathfrak{J}^{\mu}_{\nu,\lambda}(z)$ *is decreasing on* $(-1,\infty)$;

(iii) *If* $\lambda > -1$, $\nu, \mu > 0$, *and* $z > 0$, *then* $\lambda \mapsto \mathfrak{J}^{\mu}_{\nu,\lambda}(z)$ *is log-convex on* $(-1,\infty)$. *Furthermore, the following reverse Turán-type inequality holds*:

$$\bigl(\mathfrak{J}^{\mu}_{\nu,\lambda}(z)\bigr)^2 \leq \mathfrak{J}^{\mu}_{\nu,\lambda-1}(z)\mathfrak{J}^{\mu}_{\nu,\lambda-1}(z);$$

(iv) *If* $\lambda > -1$, $\nu, \mu > 0$, *and* $z > 0$, *then* $\nu \mapsto \mathfrak{J}^{\mu}_{\nu,\lambda}(z)$ *is decreasing on* $(0,\infty)$;

(v) *If* $\lambda > -1$, $\nu, \mu > 0$, *and* $z > 0$, *then* $\nu \mapsto \mathfrak{J}^{\mu}_{\nu,\lambda}(z)$ *is log-convex on* $(0,\infty)$. *Moreover, the following reverse Turán-type inequality is valid*:

$$\bigl(\mathfrak{J}^{\mu}_{\nu,\lambda}(z)\bigr)^2 \leq \mathfrak{J}^{\mu}_{\nu-1,\lambda}(z)\mathfrak{J}^{\mu}_{\nu+1,\lambda}(z).$$

Proof (i) The proof follows using the fact that the power series of $\mathfrak{J}^{\mu}_{\nu,\lambda}(z)$ has a nonnegative coefficients for $\lambda > -1$, $\nu, \mu > 0$, and $z > 0$ (see [10]).

To complete the proof of (ii)–(v), let us assume that

$$\mathrm{U}^{\mu}_{\nu,\lambda}(z) := \frac{\Gamma(\lambda+1)\Gamma(\lambda+\nu+1)}{4^n\Gamma(n+\lambda+1)\Gamma(n\mu+\lambda+\nu+1)}. \tag{3.4}$$

(ii) The proof follows by taking the logarithmic derivative of (3.4) for $\mathrm{U}^{\mu}_{\nu,\lambda}(z)$ as follows:

$$\frac{\partial}{\partial\lambda}\log \mathrm{U}^{\mu}_{\nu,\lambda}(z) = \Psi(\lambda+1) + \Psi(\lambda+\nu+1) - \Psi(n+\lambda+1) - \Psi(n\mu+\lambda+\nu+1).$$

Using representation (3.3), we get

$$\frac{\partial}{\partial\lambda}\log \mathrm{U}^{\mu}_{\nu,\lambda}(z) = \int_0^1 \frac{t^{\lambda}(1-t^n) + t^{\lambda+\nu}(1-t^{n\mu})}{t-1}\,dt \le 0,$$

for $\lambda > -1$ and $\nu, \mu > 0$, it follows that $\lambda \mapsto \mathrm{U}^{\mu}_{\nu,\lambda}(z)$ is decreasing on $(-1,\infty)$. Since the infinite sum of decreasing functions is also decreasing, this leads to $\lambda \mapsto \mathfrak{J}^{\mu}_{\nu,\lambda}(z)$ is decreasing on $(-1,\infty)$ for $\lambda > -1$, $\nu, \mu > 0$, and $z > 0$.

(iii) Since

$$\frac{\partial^2}{\partial\lambda^2}\log \mathrm{U}^{\mu}_{\nu,\lambda}(z) = \Psi'(\lambda+1) + \Psi'(\lambda+\nu+1) - \Psi'(n+\lambda+1) - \Psi'(n\mu+\lambda+\nu+1),$$

and by using the well-known formula

$$\Psi'(t) = \sum_{k=0}^{\infty}\frac{1}{(t+k)^2}, \quad t \in \mathbb{R}\backslash\{0,-1,-2,\ldots\},$$

we find

$$\Psi'(\lambda+1) - \Psi'(n+\lambda+1) = \sum_{k=0}^{\infty}\frac{n^2 + 2n(\lambda+1+k)}{(\lambda+1+k)^2(n+\lambda+1+k)^2} \ge 0,$$

and

$$\Psi'(\lambda+\nu+1) - \Psi'(n\mu+\lambda+\nu+1) = \sum_{k=0}^{\infty}\frac{n^2\mu^2 + 2n\mu(\lambda+\nu+1+k)}{(\lambda+\nu+1+k)^2(n\mu+\lambda+\nu+1+k)^2} \ge 0,$$

which implies

$$\frac{\partial^2}{\partial\lambda^2}\log \mathrm{U}^{\mu}_{\nu,\lambda}(z) \ge 0,$$

for $\lambda > -1$, $\nu, \mu > 0$, and $z > 0$, that is, the function $\lambda \mapsto \mathrm{U}^{\mu}_{\nu,\lambda}(z)$ is log-convex on $(0,\infty)$. Since the infinite sum of log-convex functions is log-convex too, this leads to $\lambda \mapsto \mathfrak{J}^{\mu}_{\nu,\lambda}(z)$ is log-convex on $(0,\infty)$ for $\lambda > -1$, $\nu, \mu > 0$, and $z > 0$. On the other hand,

$$\log \mathfrak{J}^{\mu}_{\nu,\lambda}(z) \le \frac{1}{2}\big(\log \mathfrak{J}^{\mu}_{\nu,\lambda-1}(z) + \log \mathfrak{J}^{\mu}_{\nu,\lambda+1}(z)\big),$$

or equivalently $(\mathfrak{J}^{\mu}_{\nu,\lambda}(z))^2 \le \mathfrak{J}^{\mu}_{\nu,\lambda-1}(z)\mathfrak{J}^{\mu}_{\nu,\lambda-1}(z)$.

(iv) From (3.4), we obtain

$$\frac{\partial}{\partial\nu}\log \mathrm{U}^{\mu}_{\nu,\lambda}(z) = \Psi(\lambda+\nu+1) - \Psi(n\mu+\lambda+\nu+1) \le 0, \tag{3.5}$$

which implies that $\nu \mapsto \mathfrak{J}^{\mu}_{\nu,\lambda}(z)$ is decreasing on $(0,\infty)$.

(v) From (3.5), we have

$$\frac{\partial^2}{\partial\nu^2}\log \mathrm{U}^{\mu}_{\nu,\lambda}(z) = \Psi'(\lambda+\nu+1) - \Psi'(n\mu+\lambda+\nu+1) \ge 0,$$

this leads to $\nu \mapsto \mathfrak{J}^{\mu}_{\nu,\lambda}(z)$ is log-convex on $(0,\infty)$ and the proof is complete. □

4 Conclusions

In our present investigation, the integral representation of $\mathcal{J}^{\mu}_{\nu,\lambda}$ as well as Mellin–Barnes representation with the help of Gauss's multiplication theorem, the well-known representation for the beta function, and the residue theorem have been established. Further, the mth derivative of $\mathcal{J}^{\mu}_{\nu,\lambda}$ has been considered, and it can be expressed as the Fox–Wright function. Additionally, the recurrence relations and other identities involving the derivatives have been discussed. We end up showing the monotonicity of the ratio between two modified Bessel–Maitland functions $\mathcal{I}^{\mu}_{\nu,\lambda}$ of a different order, the ratio between modified Bessel–Maitland and hyperbolic functions, and some monotonicity results for $\mathcal{I}^{\mu}_{\nu,\lambda}(z)$ where the main idea of the proofs comes from the monotonicity of the quotient of two Maclaurin series. This makes it possible to construct some inequalities (like Turán-type inequalities and their reverse) as an application.

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References

1. Abadir, K.M.: An introduction to hypergeometric functions for economists. Econom. Rev. **18**, 287–330 (1999)
2. Abramowitz, M., Stegun, I.A.: A Handbook of Mathematical Functions, Courier Co., New York (1965)
3. Agarwal, P., Agarwal, R.P., Ruzhansky, M.: Special Functions and Analysis of Differential Equations, 1st edn. CRC Press, Boca Raton (2020)
4. Agarwal, P., Chand, M., Choi, J., Singh, G.: Certain fractional integrals and image formulas of generalized *k*-Bessel function. Commun. Korean Math. Soc. **33**(2), 423–436 (2018)
5. Agarwal, P., Chand, M., Singh, G.: Certain fractional kinetic equations involving the product of generalized *k*-Bessel function. Alex. Eng. J. **55**(4), 3053–3059 (2016)
6. Agarwal, P., Jain, S., Chand, M., Dwivedi, S.K., Singh, G., Kumar, S.: Bessel functions associated with Saigo-Maeda fractional derivative operators. J. Fract. Calc. Appl. **5**(2), 96–106 (2014)

7. Agarwal, P., Ntouyas, S.K., Jain, S., Chand, M., Singh, G.: Fractional kinetic equations involving generalized *k*-Bessel function via Sumudu transform. Alex. Eng. J. **57**, 1937–1942 (2018)
8. Agarwal, P., Qi, F., Chand, M., Singh, G.: Some fractional differential equations involving generalized hypergeometric functions. J. Appl. Anal. **25**(1), 37–44 (2019)
9. Ali, R.M., Mondal, S.R., Nisar, K.S.: Monotonicity properties of the generalized Struve functions. J. Korean Math. Soc. **54**(2), 575–598 (2017)
10. Askey, R., Pollard, H.: Some absolutely monotonic and completely monotonic functions. SIAM J. Math. Anal. **5**, 58–63 (1974)
11. Baricz, Á.: Geometric properties of generalized Bessel functions. Publ. Math. (Debr.) **73**, 155–178 (2008)
12. Baricz, Á.: Bounds for modified Bessel functions of the first and second kinds. Proc. Edinb. Math. Soc. **53**, 575–599 (2010)
13. Barnard, R.W., Gordy, M.B., Richards, K.C.: A note on Turán type and mean inequalities for the Kummer function. J. Math. Anal. Appl. **349**(1), 259–263 (2009)
14. Biernacki, M., Krzyż, J.: On the monotonicity of certain functionals in the theory of analytic function. Ann. Univ. Mariae Curie-Skłodowska, Sect. A **9**, 135–147 (1957)
15. Black, F., Cox, J.: Valuing corporate securities: some effects of bond indenture provisions. J. Finance **31**(2), 351–367 (1976)
16. Carey, M., Gordy, M.B.: The bank as grim reaper: debt composition and recoveries on defaulted debt. Preprint (2007)
17. Chand, M., Agarwal, P., Hammouch, Z.: Certain sequences involving product of *k*-Bessel function. Int. J. Appl. Comput. Math. **4**(4), 101 (2018)
18. Galué, L.: A generalized Bessel function. Integral Transforms Spec. Funct. **14**(5), 395–401 (2003)
19. Gupta, I.S., Debnath, L.: Some properties of the Mittag-Leffler functions. Integral Transforms Spec. Funct. **18**, 329–336 (2007)
20. Gurtin, M.E.: Topics Infinite Elasticity. CBMS-NSF Regional Conference Series in Applied Mathematics. SIAM, Philadelphia (1981)
21. Hinch, E.J., Schubert, G.: Strong streaming induced by a moving thermal wave. J. Fluid Mech. **47**(2), 291–304 (1971)
22. Hoppe, W., Lohmann, W., Markl, H., Zeigler, H. (eds.): Biophysics Springer, Berlin (1983)
23. Lebedev, N.N.: Special Functions and Their Applications. Dover, New York (1972)
24. Mceliece, R.J., Reznick, B., Shearer, J.B.: A Turán inequality arising in information theory. SIAM J. Math. Anal. **12**(6), 931–934 (1981)
25. Merton, R.C.: On the pricing of corporate debt: the risk structure of interest rates. J. Finance **29**(2), 449–470 (1974)
26. Orhan, H., Yağmur, N.: Geometric properties of generalized Struve functions. An. Ştiinţ. Univ. 'Al.I. Cuza' Iaşi, Mat. **63**(2), 229–244 (2017)
27. Pathak, R.S.: Certain convergence theorems and asymptotic properties of a generalization of Lommel and Maitland transformations. Proc. Natl. Acad. Sci. **A–36**(1), 81–86 (1966)
28. Rizzoni, G.: Fundamentals of Electrical Engineering. McGraw-Hill, New York (2009)
29. Shukla, A.K., Prajapati, J.C.: On a generalization of Mittag-Leffler function and its properties. J. Math. Anal. Appl. **336**(2), 797–811 (2007)
30. Singh, G., Agarwal, P., Chand, M., Jain, S.: Certain fractional kinetic equations involving generalized *k*-Bessel function. Trans. A. Razmadze Math. Inst. **172**, 559–570 (2018)
31. Szegö, G.: On an inequality of P. Turán concerning Legendre polynomials. Bull. Am. Math. Soc. **54**, 401–405 (1948)
32. Turán, P.: On the zeros of the polynomials of Legendre. Căsopis Pest. Mat. Fys. **75**, 113–122 (1950)
33. Waalkens, H., Wiersig, J., Dullin, H.R.: Elliptic quantum billiard. Ann. Phys. **260**, 50–90 (1997)
34. Watson, G.N.: A Treatise on the Theory of Bessel Functions. Cambridge University Press, Cambridge (1966)

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