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Computation of Fourier transform representations involving the generalized Bessel matrix polynomials

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Abstract

Motivated by the recent studies and developments of the integral transforms with various special matrix functions, including the matrix orthogonal polynomials as kernels, in this article we derive the formulas for Fourier cosine and sine transforms of matrix functions involving generalized Bessel matrix polynomials. With the help of these transforms several results are obtained, which are extensions of the corresponding results in the standard cases. The results given here are of general character and can yield a number of (known and new) results in modern integral transforms.

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1 Introduction

In the past few decades, the orthogonal matrix polynomials have attracted a lot of research interest due to their close relations and various applications in many areas of mathematics, engineering, probability theory, graph theory, and physics; for example, see [1–9]. In [4], extensions to the matrix framework of the classical families of Legendre, Laguerre, Jacobi, Chebyshev, Gegenbauer, and Hermite polynomials have been introduced. Meanwhile, one particular orthogonal polynomial family which frequently appears in the recent studies and applications [10–12] is that of generalized Bessel polynomials, which in its matrix form is also defined in [4, 13]. Later on, distinct works on the generalized Bessel matrix polynomials have been discussed (see [14–17]).

Nowadays, many integral transforms (see, e.g., Fourier, Laplace, Beta, Hankel, Mellin, Whittaker transforms, etc.), with various special functions (also with the new generalized special matrix functions) as kernels, have begun to play an important role in modeling of various physical, engineering, automatization, and biological phenomena, as well as in several other branches of science (see, for instance, [8, 18–30]).

Fourier transform (FT) is an integral transform that is used in solving different problems in mathematical physics, applied statistics, and engineering (see, [31, 32]). The idea of Fourier transform is a natural extension of the idea of Fourier series. In particular, Fourier transform can accommodate aperiodic functions, which Fourier series cannot do. Recently, many results on Fourier transform and its applications have been contributed by Nicola and Trapasso [33], Urieles et al. [34], Ghodadra and Fülöp [35], Bergold and Lasser [36], and Al-Lail and Qadir [37].

On the contrary, matrix Fourier expansions and Fourier series in orthonormal matrix polynomials have been introduced by B. Osihnker in [38, 39]. Defez and Jóbdar [40, 41] introduced basic properties of matrix Fourier series and Fourier approximation for functions of matrix argument. Recently, Groenevelt and Koelink [42] discussed the generalized Fourier transform with hypergeometric function and matrix-valued orthogonal polynomials as kernels. Also, applications of matrix summability to Fourier transforms were established by Ş. Yildiz [43].

Motivated by some of these aforementioned investigations of the Fourier transforms of matrix-valued orthogonal polynomials, in our investigation here we study the Fourier-type transforms of the generalized Bessel matrix polynomials $\mathcal{Y}_n(\xi;F,L)$, $\xi\in\mathbb{C}$, for (square) matrix parameters F and L. In particular, we obtain several Fourier cosine and sine transforms of functions involving generalized Bessel matrix polynomials with powers of the matrix, as well as matrix exponential, trigonometric, binomial, and Bessel functions. Moreover, pertinent integral transforms of the different results given here, including simpler and earlier ones, are also investigated.

2 Auxiliary toolbox

In this section, we recall some definitions, lemmas, and terminology which will be used to prove the main results.

Let $\mathbb{C}$ and $\mathbb{N}$ denote the sets of complex numbers and positive integers, respectively, and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. Let $\mathbb{C}^n$ denote the n-dimensional complex vector space and $\mathbb{C}^{n\times n}$ denote the space of all square matrices with n rows and n columns whose entries are complex numbers.

Definition 2.1 ([4]) For a matrix F in $\mathbb{C}^{n\times n}$, the spectrum $\sigma(F)$ is the set of all eigenvalues of F for which we denote

$$\alpha(F)=\max\bigl\{\operatorname{Re}(\xi):\xi\in\sigma(F)\bigr\}\quad\text{and}\quad\widetilde{\alpha}(F)=\min\bigl\{\operatorname{Re}(\xi):\xi\in\sigma(F)\bigr\},\tag{1}$$

where $\alpha(F)$ refers to the spectral abscissa of F and for which $\widetilde{\alpha}(F)=-\alpha(-F)$. A matrix F is said to be positive stable if and only if $\widetilde{\alpha}(F)>0$.

Definition 2.2 ([44]) If F and L are commuting matrices in $\mathbb{C}^{n\times n}$ and $w\in\mathbb{C}$, then

$$\begin{aligned}\cos\bigl[(F\pm L)w\bigr]&=\cos(Fw)\cos(Lw)\mp\sin(Fw)\sin(Lw),\\ \sin\bigl[(F\pm L)w\bigr]&=\sin(Fw)\cos(Lw)\pm\cos(Fw)\sin(Lw).\end{aligned}\tag{2}$$

Remark 2.1 If $F,L\in\mathbb{C}^{1\times 1}=\mathbb{C}$, then the identities in Definition 2.2 reduce to those in the scalar setting.

Definition 2.3 ([4, 45]) Let F be a positive stable matrix in $\mathbb{C}^{n\times n}$. The gamma matrix function $\Gamma(F)$ is defined as

$$\Gamma(F)=\int_0^\infty e^{-w}w^{F-I}\,dw;\quad w^{F-I}=\exp\bigl((F-I)\ln w\bigr),\tag{3}$$

where I is the identity matrix in $\mathbb{C}^{n\times n}$.

Definition 2.4 ([4, 45]) The reciprocal gamma function denoted by $\Gamma^{-1}(w)=\frac{1}{\Gamma(w)}$ is an entire function of the complex variable ξ. Then the image of $\Gamma^{-1}(w)$ acting on $F\in\mathbb{C}^{n\times n}$ denoted by $\Gamma^{-1}(F)$ is a well-defined matrix and invertible, as well as

$$F+nI\quad\text{is invertible for all integers } n\in\mathbb{N}_0.\tag{4}$$

By applying the matrix functional calculus to F, which is a positive stable matrix in $\mathbb{C}^{n\times n}$, the Pochhammer symbol of a matrix argument defined by

$$(F)_n=\begin{cases}F(F+I)\cdots(F+(n-1)I)=\Gamma^{-1}(F)\Gamma(F+nI), & n\geq 1,\\ I, & n=0.\end{cases}\tag{5}$$

Note that, if $F=-sI$, where s is a positive integer, then $(F)_n=0$, whenever $n>s$.

Now, from properties of the gamma matrix function, we give some lemmas which will be needed in the proof of some theorems.

Lemma 2.1 *Let S be a matrix in $\mathbb{C}^{n\times n}$ such that $\widetilde{\alpha}(S)>0$ and $w\in\mathbb{C}$ with* $\operatorname{Re}(w)>0$. *The following integral formulas hold*:

$$\int_0^\infty \xi^{S-I}e^{-w\xi}\,d\xi=w^{-S}\Gamma(S)\tag{6}$$

and

$$\int_0^\infty \xi^{S-I}e^{-wi\xi}\,d\xi=e^{-\frac{1}{2}i\pi S}w^{-S}\Gamma(S);\quad i=\sqrt{-1}.\tag{7}$$

We thus observe that

$$\int_0^\infty \xi^{S-I}\cos(w\xi)\,d\xi=\cos\left(\frac{1}{2}\pi S\right)w^{-S}\Gamma(S)\tag{8}$$

and

$$\int_0^\infty \xi^{S-I}\sin(w\xi)\,d\xi=\sin\left(\frac{1}{2}\pi S\right)w^{-S}\Gamma(S).\tag{9}$$

Putting $S=I-R\in\mathbb{C}^{n\times n}$ in (8) *and* (9), *we get*

$$\int_0^\infty \xi^{-R}\cos(w\xi)\,d\xi=\frac{\pi w^{R-I}}{2}\sec\left(\frac{1}{2}\pi R\right)\Gamma^{-1}(R),\quad \widetilde{\alpha}(R)>0,\tag{10}$$

and

$$\int_0^\infty \xi^{-R}\sin(w\xi)\,d\xi = \frac{\pi w^{R-I}}{2}\csc\left(\frac{1}{2}\pi R\right)\Gamma^{-1}(R), \quad \widetilde{\alpha}(R) > 0. \tag{11}$$

Similarly, we can present the following lemma.

Lemma 2.2 *Let S be a matrix in $\mathbb{C}^{n\times n}$ such that $\widetilde{\alpha}(S) > 0$, $\lambda, w \in \mathbb{C}$ with* $\mathrm{Re}(\lambda) > 0$ *and* $\mathrm{Re}(w) > 0$. *The following integral formulas hold*:

$$\int_0^\infty \xi^{S-I}e^{-\lambda\xi}\cos(w\xi)\,d\xi = \cos\left(\arctan\left(\frac{w}{\lambda}\right)S\right)\left(\lambda^2+w^2\right)^{-\frac{1}{2}S}\Gamma(S) \tag{12}$$

and

$$\int_0^\infty \xi^{S-I}e^{-\lambda\xi}\sin(w\xi)\,d\xi = \sin\left(\arctan\left(\frac{w}{\lambda}\right)S\right)\left(\lambda^2+w^2\right)^{-\frac{1}{2}S}\Gamma(S). \tag{13}$$

Definition 2.5 ([4, 46]) Let k and r be finite positive integers. The generalized hypergeometric matrix function is defined by the matrix power series

$${}_k\mathbf{H}_r[\mathbf{F};\mathbf{L};w] = \sum_{m=0}^{\infty}\prod_{i=1}^{k}(F_i)_m\prod_{j=1}^{r}\left[(L_j)_m\right]^{-1}\frac{w^m}{m!}, \tag{14}$$

where $\mathbf{F} = F_i$, $1 \le i \le k$, and $\mathbf{L} = L_j$, $1 \le j \le r$, are commutative matrices in $\mathbb{C}^{n\times n}$ with $L_j + mI$ being invertible for all integers $m \in \mathbb{N}_0$.

Note that for $k = 1$, $r = 0$, we have the binomial-type matrix function ${}_1\mathbf{H}_0(F_1;-;w)$, $|w| < 1$, as follows:

$${}_1\mathbf{H}_0(F;-;w) = (1-w)^{-F_1} = I + F_1 w + \frac{F_1(F_1+I)w^2}{2!} + \cdots + \frac{(F_1)_n w^n}{n!} + \cdots.$$

Also, note that for $k = 2$, $r = 1$, we get the Gauss hypergeometric matrix function ${}_2\mathbf{H}_1$ in the form

$${}_2\mathbf{H}_1(F_1,F_2;L_1;w) = \sum_{s=0}^{\infty}(F_1)_s(F_2)_s\left[(L_1)_s\right]^{-1}\frac{w^s}{s!}.$$

Several special matrix functions, including the matrix orthogonal polynomials, are also presented in terms of the generalized hypergeometric matrix function in [4, 46].

Definition 2.6 ([4, 13, 16]) Let F and L be commuting matrices in $\mathbb{C}^{n\times n}$ such that L is an invertible matrix. For any natural number $n \in \mathbb{N}_0$ and $\xi \in \mathbb{C}$, the nth generalized Bessel matrix polynomial $\mathcal{Y}_n(\xi;F,L)$ is defined as

$$\begin{aligned}\mathcal{Y}_n(\xi;F,L) &= \sum_{s=0}^{n} \frac{(-1)^s}{s!}(-nI)_s\bigl(F+(n-1)I\bigr)_s\bigl(\xi L^{-1}\bigr)^s \\ &= \sum_{s=0}^{n} \frac{n}{s!(n-s)!}\bigl(F+(n-1)I\bigr)_s\bigl(\xi L^{-1}\bigr)^s \\ &= {}_2\mathbf{H}_0\left[\begin{matrix}-nI, F+(n-1)I \\ -\end{matrix};-\xi L^{-1}\right].\end{aligned} \tag{15}$$

Remark 2.2 If the matrices $F, L \in \mathbb{C}^{1\times 1} = \mathbb{C}$, then the generalized Bessel matrix polynomial in (15) reduces to generalized Bessel polynomials in [10–12].

Definition 2.7 ([47, 48]) Let a matrix $F \in \mathbb{C}^{n\times n}$ satisfy the condition:

$$\beta \text{ is not a negative integer for every } \beta \in \sigma(F), \tag{16}$$

then Bessel matrix function $\mathbf{J}_F(w)$ of the first kind associated to F is given by

$$\mathbf{J}_F(w) = \sum_{s=0}^{\infty} \frac{(-1)^s}{(s)!}\Gamma^{-1}\bigl(F+(s+1)I\bigr)\left(\frac{w}{2}\right)^{F+2sI}, \quad w \in \mathbb{C}, \tag{17}$$

and the modified Bessel matrix functions $\mathbf{I}_F(w)$ and $\mathbf{K}_F(w)$ are respectively defined as

$$\mathbf{I}_F(w) = \sum_{s=0}^{\infty} \frac{1}{(s)!}\Gamma^{-1}\bigl(F+(s+1)I\bigr)\left(\frac{w}{2}\right)^{F+2sI} \tag{18}$$

and

$$\mathbf{K}_F(w) = \frac{\pi}{2}\bigl[\sin(\pi F)\bigr]^{-1}\bigl\{\mathbf{I}_{-F}(w) - \mathbf{I}_F(w)\bigr\}. \tag{19}$$

Definition 2.8 ([31, 32]) Let $f(\xi)$ be a function of ξ specified for $\xi > 0$. Then the complex Fourier transform of $f(\xi)$ associated with complex frequency w is defined by

$$\mathcal{F}(w) = \mathcal{F}\bigl\{f(\xi)\bigr\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)e^{-i\xi w}\,d\xi, \quad w \in \mathbb{C}, \tag{20}$$

together with the requirement of $|\mathcal{F}(w)| < \infty$.

Similarly, the inverse Fourier transform, denoted by $\mathcal{F}^{-1}\{\mathcal{F}(w)\} = f(\xi)$, is defined by

$$f(\xi) = \mathcal{F}^{-1}\bigl\{\mathcal{F}(w)\bigr\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathcal{F}(w)e^{i\xi w}\,dw. \tag{21}$$

The cosine and sine transformations, respectively, are defined similarly as follows:

$$\mathcal{F}^c(w) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(\xi)\cos(\xi w)\,d\xi, \tag{22}$$

$$f(\xi)=\sqrt{\frac{2}{\pi}}\int_0^\infty \mathcal{F}^c(w)\cos(\xi w)\,dw,$$
$$\mathcal{F}^s(w)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(\xi)\sin(\xi w)\,d\xi, \tag{23}$$

and

$$f(\xi)=\sqrt{\frac{2}{\pi}}\int_0^\infty \mathcal{F}^s(w)\sin(\xi w)\,dw.$$

Note that if $f(\xi)$ is an even function, then $\mathcal{F}(w)=\mathcal{F}^c(w)$, and if $f(\xi)$ is an odd function, then $\mathcal{F}(w)=i\mathcal{F}^s(w)$.

The following lemma will be required in the proof of our theorems.

Lemma 2.3 ([18]) *From the basic formulae of the Fourier cosine transform, if $f(\xi)$ is replaced by $\xi^{2n}f(\xi)$, then*

$$\mathcal{F}^c\{\xi^{2n}f(\xi)\}(w)=(-1)^n\frac{d^{2n}}{dw^{2n}}\bigl(\mathcal{F}^c\{f\}(w)\bigr).$$

Also, if

$$f(\xi)=\bigl(\lambda^2+\xi^2\bigr)^{-(S+\frac{1}{2})};\quad \widetilde{\alpha}(S)>-\frac{1}{2},$$

then

$$\mathcal{F}^c(w)=\sqrt{2}(w/2\lambda)^S\Gamma^{-1}\left(S+\frac{1}{2}I\right)\boldsymbol{K}_S(\lambda w),$$

where S is a positive stable matrix in $\mathbb{C}^{n\times n}$, $w,\lambda\in\mathbb{C}$ with $\mathrm{Re}(w)>0$, $\mathrm{Re}(\lambda)>0$, and $\boldsymbol{K}_S(w)$ is the modified Bessel matrix function in (19).

Remark 2.3 Physically, the Fourier transform $\mathcal{F}(w)$ can be interpreted as an integral superposition of an infinite number of sinusoidal oscillations with different wavenumbers w (or different wavelengths $\tau=\frac{2\pi}{w}$). Thus, the definition of the Fourier transform is restricted to absolutely integrable functions. This restriction is too strong for many physical applications (see [31, 32]).

3 Statement and proof of main theorems

In this section, we investigate several new interesting Fourier cosine and sine transforms of functions involving generalized Bessel matrix polynomials asserted in the following theorems:

Theorem 3.1 *Let S, F and L be commuting matrices in $\mathbb{C}^{n\times n}$, and let $\mathcal{Y}_n(\lambda\xi;F,L)$ be given in* (15). *For the function*

$$f(\xi)=\xi^S\mathcal{Y}_n(\lambda\xi;F,L), \tag{24}$$

we have

$$\mathcal{F}^c(w) = -\sqrt{\frac{2}{\pi}} w^{-(S+I)}\Gamma(S+I)\sum_{r=0}^{n}(-nI)_r\big(F+(n-1)I\big)_r(S+I)_r \times \frac{(-\lambda(Lw)^{-1})^r \sin[(S+rI)\pi/2]}{r!}, \tag{25}$$

$$\mathcal{F}^s(w) = \sqrt{\frac{2}{\pi}} w^{-(S+I)}\Gamma(S+I)\sum_{r=0}^{n}(-nI)_r\big(F+(n-1)I\big)_r(S+I)_r \times \frac{(-\lambda(Lw)^{-1})^r \cos[(S+rI)\pi/2]}{r!}, \tag{26}$$

where $w, \lambda \in \mathbb{C}$ *are such that* $\mathrm{Re}(w) > 0$, $\mathrm{Re}(\lambda) > 0$, *and* $\widetilde{\alpha}(S) > -1$.

Proof To prove (25) from Definition 2.6 and (22), we observe that

$$\begin{aligned}\mathcal{F}^c(w) &= \sqrt{\frac{2}{\pi}}\int_0^\infty \xi^S \mathcal{Y}_n(\lambda\xi;F,L)\cos(\xi w)\,d\xi \\ &= \sqrt{\frac{2}{\pi}}\sum_{r=0}^{n}(-nI)_r\big(F+(n-1)I\big)_r\frac{(-\lambda L^{-1})^r}{r!}\int_0^\infty \xi^{S+rI}\cos(w\xi)\,d\xi.\end{aligned}$$

From the integral (8), we obtain

$$\begin{aligned}\mathcal{F}^c(w) &= \sqrt{\frac{2}{\pi}}\sum_{r=0}^{n}(-nI)_r\big(F+(n-1)I\big)_r\frac{(-\lambda L^{-1})^r}{r!} \\ &\quad\times w^{-(S+(r+1)I)}\Gamma\big(S+(r+1)I\big)\cos\big[\big(S+(r+1)I\big)\pi/2\big] \\ &= -\sqrt{\frac{2}{\pi}} w^{-(S+I)}\Gamma(S+I)\sum_{r=0}^{n}(-nI)_r\big(F+(n-1)I\big)_r \\ &\quad\times (S+I)_r\frac{(-\lambda(wL)^{-1})^r}{r!}\sin\big[(S+rI)\pi/2\big],\end{aligned}$$

which is the claimed result in (25).

Now, we prove (26), from (24) in (23), we have

$$\begin{aligned}\mathcal{F}^s(w) &= \sqrt{\frac{2}{\pi}}\int_0^\infty \xi^S \mathcal{Y}_n(\lambda\xi;F,L)\sin(\xi w)\,d\xi \\ &= \sqrt{\frac{2}{\pi}}\sum_{r=0}^{n}(-nI)_r\big(F+(n-1)I\big)_r\frac{(-\lambda L^{-1})^r}{r!}\int_0^\infty \xi^{S+rI}\sin(w\xi)\,d\xi.\end{aligned}$$

By invoking relation (9), we obtain

$$\begin{aligned}\mathcal{F}^s(w) &= \sqrt{\frac{2}{\pi}}\sum_{r=0}^{n}(-nI)_r\big(F+(n-1)I\big)_r\frac{(-\lambda L^{-1})^r}{r!} \\ &\quad\times w^{-(S+(r+1)I)}\Gamma\big(S+(r+1)I\big)\sin\big[\big(S+(r+1)I\big)\pi/2\big]\end{aligned}$$

$$= -\sqrt{\frac{2}{\pi}} w^{-(S+I)} \Gamma(S+I) \sum_{r=0}^{n} (-nI)_r \big(F + (n-1)I\big)_r \times (S+I)_r \frac{(-\lambda (wL)^{-1})^r}{r!} \cos\big[(S+rI)\pi/2\big],$$

which is the desired result in (26). □

Now by taking advantage of the previous results, we obtain the following corollaries:

Corollary 3.1.1 *If r is odd, then* (25) *reduces to*

$$\mathcal{F}^c(w) = \sqrt{\frac{2}{\pi}} w^{-(S+I)} \Gamma(S+I) \cos(S\pi/2) \times \sum_{r=1}^{n} (-nI)_r \big(F + (n-1)I\big)_r (S+I)_r \frac{(-1)^{\frac{r+1}{2}} (-\lambda (wL)^{-1})^r}{r!},$$

and if r is even, then (25) *reduces to*

$$\mathcal{F}^c(w) = -\sqrt{\frac{2}{\pi}} w^{-(S+I)} \Gamma(S+I) \sin(S\pi/2) \times \sum_{r=0}^{n} (-nI)_r \big(F + (n-1)I\big)_r (S+I)_r \frac{(-1)^{\frac{r}{2}} (-\lambda (wL)^{-1})^r}{r!}.$$

Corollary 3.1.2 *If r is odd, then* (26) *reduces to*

$$\mathcal{F}^s(w) = \sqrt{\frac{2}{\pi}} w^{-(S+I)} \Gamma(S+I) \sin(S\pi/2) \times \sum_{r=1}^{n} (-nI)_r \big(F + (n-1)I\big)_r (S+I)_r \frac{(-1)^{\frac{r+1}{2}} (-\lambda (wL)^{-1})^r}{r!},$$

and if r is even, then (26) *reduces to*

$$\mathcal{F}^s(w) = \sqrt{\frac{2}{\pi}} w^{-(S+I)} \Gamma(S+I) \cos(S\pi/2) \times \sum_{r=0}^{n} (-nI)_r \big(F + (n-1)I\big)_r (S+I)_r \frac{(-1)^{\frac{r}{2}} (-\lambda (wL)^{-1})^r}{r!}.$$

Corollary 3.1.3 *Replacing the Bessel matrix polynomials* $\mathcal{Y}_n(\lambda\xi; F, L)$ *by* $\mathcal{Y}_n(\lambda\xi^2; F, L)$ *and choosing* $S = 0$ *in* (25) *and* (26), *that is, if*

$$f(\xi) = \mathcal{Y}_n\big(\lambda\xi^2; F, L\big),$$

then, we obtain the following results:

$$\mathcal{F}^c(w) = 0,$$

and

$$\mathcal{F}^s(w) = \sqrt{\frac{2}{\pi}} w_4^{-1} \mathcal{H}_0 \left[\begin{matrix} -nI, F + (n-1)I, \frac{1}{2}I, I \\ - \end{matrix} ; 4\lambda \left(Lw^2\right)^{-1} \right].$$

Also, a consequence of Theorem 3.1 is the following theorem:

Theorem 3.2 *Let S, F, and L be commuting matrices in $\mathbb{C}^{n\times n}$, and let $\mathcal{Y}_n(\lambda\xi; F, L)$ be given in (15). For the function*

$$f(\xi) = \xi^S \cos(\mu\xi) \mathcal{Y}_n(\lambda\xi; F, L), \tag{27}$$

we have

$$\begin{aligned} \mathcal{F}^c(w) = {}& \frac{-1}{2} \sqrt{\frac{2}{\pi}} \Gamma(I+S) \\ & \times \sum_{r=0}^{n} (-nI)_r \bigl(F + (n-1)I\bigr)(S+I)_r \frac{(\lambda L^{-1})^r \sin[(S+rI)\pi/2]}{r!} \\ & \times \bigl\{ (w+\mu)^{-(S+(r+1)I)} + \bigl|(w-\mu)\bigr|^{-(S+(r+1)I)} \bigr\}, \end{aligned} \tag{28}$$

and

$$\begin{aligned} \mathcal{F}^s(w) = {}& \frac{1}{2} \sqrt{\frac{2}{\pi}} \Gamma(I+S) \\ & \times \sum_{r=0}^{n} (-nI)_r \bigl(F + (n-1)I\bigr)(S+I)_r \frac{(\lambda L^{-1})^r \cos[(S+rI)\pi/2]}{r!} \\ & \times \bigl\{ (w+\mu)^{-(S+(r+1)I)} + |w-\mu|^{-(S+(r+1)I)} \bigr\} \end{aligned} \tag{29}$$

where $w, \mu, \lambda \in \mathbb{C}$ are such that $\mathrm{Re}(w) > 0$, $\mathrm{Re}(\mu) > 0$, $\mathrm{Re}(\lambda) > 0$, $\mathrm{Re}(w) > \mathrm{Re}(\mu)$, *and* $\widetilde{\alpha}(S+I) > 0$.

Proof To describe the relation in (28), the proof is easy, using the well-known identities in (2). In a similar way, we can get the result in (29). □

Theorem 3.3 *Let S, F, and L be commuting matrices in $\mathbb{C}^{n\times n}$. If*

$$f(\xi) = \xi^{-S} \mathcal{Y}_n(\lambda; F, L\xi), \tag{30}$$

then,

$$\begin{aligned} \mathcal{F}^c(w) = {}& \sqrt{\frac{2}{\pi}} w^{(S-I)} \Gamma(I-S) \\ & \times \sum_{r=0}^{n} (-nI)_r \bigl(F + (n-1)I\bigr) \bigl[(S)_r\bigr]^{-1} \frac{(\lambda w L^{-1})^r \sin[(S+rI)\pi/2]}{r!}, \end{aligned} \tag{31}$$

and

$$\mathcal{F}^s(w) = \sqrt{\frac{2}{\pi}} w^{(S-I)} \Gamma(I-S) \times \sum_{r=0}^{n} (-nI)_r \big(F+(n-1)I\big)\big[(S)_r\big]^{-1} \frac{(\lambda w(L^{-1})^r \cos[(S+rI)\pi/2]}{r!}, \tag{32}$$

where $\mathrm{Re}(w) > 0$, $\mathrm{Re}(\lambda) > 0$, *and* $\widetilde{\alpha}(I-S) > 0$.

Proof The proofs of the two results in (31) and (32) can be obtained by the use of the two formulas in (10) and (11) with Definition 2.6. □

Theorem 3.4 *Let S, F, and L be commuting matrices in $\mathbb{C}^{n\times n}$, and let $\mathcal{Y}_n(\lambda\xi;F,L)$ be given in* (15). *For the function*

$$f(\xi) = \xi^{S-I} e^{-\mu\xi} \mathcal{Y}_n(\lambda\xi;F,L), \tag{33}$$

we have

$$\mathcal{F}^c(w) = \sqrt{\frac{2}{\pi}} \big(\mu^2+w^2\big)^{\frac{-S}{2}} \Gamma(S) \times \sum_{r=0}^{n} (-nI)_r \big(F+(n-1)I\big)_r (S)_r \frac{(-\lambda L^{-1})^r \cos[(S+rI)\arctan(w/\mu)]}{r!}, \tag{34}$$

and

$$\mathcal{F}^s(w) = \sqrt{\frac{2}{\pi}} \big(\mu^2+w^2\big)^{\frac{-S}{2}} \Gamma(S) \times \sum_{r=0}^{n} (-nI)_r \big(F+(n-1)I\big)_2 (S)_r \frac{(-\lambda L^{-1})^r \sin[(S+rI)\arctan(w/\mu)]}{r!}, \tag{35}$$

where $w,\mu,\lambda \in \mathbb{C}$ are such that $\mathrm{Re}(w) > 0$, $\mathrm{Re}(\mu) > 0$, $\mathrm{Re}(\lambda) > 0$, *and* $\widetilde{\alpha}(S) > 0$.

Proof Using (15) and applying formula (22) on the right-hand side of (33) reveals that

$$\begin{aligned}\mathcal{F}^c(w) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \xi^{S-I} e^{-\mu\xi} \mathcal{Y}_n(\lambda\xi;F,L)\cos(\xi w)\,d\xi \\ &= \sqrt{\frac{2}{\pi}} \sum_{r=0}^{n} (-nI)_r \big(F+(n-1)I\big)_r \frac{(-\lambda L^{-1})^r}{r!} \\ &\quad \times \int_0^\infty \xi^{S-(1-r)I} e^{-\mu\xi} \cos(\xi w)\,d\xi.\end{aligned}$$

Using (12), we get

$$\begin{aligned}\mathcal{F}^c(w) &= \sqrt{\frac{2}{\pi}} \sum_{r=0}^{n} (-nI)_r \big(F+(n-1)I\big)_r \frac{(-\lambda L^{-1})^r}{r!} \\ &\quad \times \big(\mu^2+w^2\big)^{\frac{-1}{2}S} \Gamma(S+rI)\cos\big[(S+rI)\arctan(w/\mu)\big],\end{aligned}$$

which implies formula (34).

Likewise, we can get the result in (35) by using (13). □

Theorem 3.5 *For the function*

$$f(\xi) = \xi^{2n} e^{-\mu\xi^2} \mathcal{Y}_n\big(\lambda\xi^2; F, L\big), \tag{36}$$

where $\mathcal{Y}_n(-; F, L)$ is as in (15), we have

$$\begin{aligned}
\mathcal{F}^c(w) &= \sqrt{\frac{1}{2\pi}} \Gamma(n+1/2) \mu^{-(\frac{1}{2}+2n)} \big(-w^2/4\big)^n e^{(-w^2/4\mu)} \\
&\quad \times \sum_{r=0}^{n} (-nI)_r \big(F + (n-1)I\big)_r \frac{(\lambda w^2 (4\mu^2 L)^{-1})^r}{r!} \\
&\quad \times \mathcal{Y}_{n+r}\big(4\mu; 3/2 - 2(n+r), w^2\big),
\end{aligned} \tag{37}$$

where $w, \mu, \lambda \in \mathbb{C}$ are such that $\operatorname{Re}(w) > 0$, $\operatorname{Re}(\mu) > 0$, *and* $\operatorname{Re}(\lambda) > 0$.

Proof To establish our result in (37), using (36) in (22), we arrive at

$$\begin{aligned}
\mathcal{F}^c(w) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \xi^{2n} e^{-\mu\xi^2} \mathcal{Y}_n\big(\lambda\xi^2; F, L\big) \cos(\xi w)\, d\xi \\
&= \sqrt{\frac{2}{\pi}} \sum_{r=0}^{n} (-nI)_r \big(F + (n-1)I\big)_r \frac{(-\lambda L^{-1})^r}{r!} \\
&\quad \times \int_0^\infty \xi^{2n+2r} e^{-\mu\xi^2} \sum_{s=0}^{\infty} \frac{(-w^2\xi^2)^s}{(2s)!}\, d\xi \\
&= \sqrt{\frac{1}{2\pi}} \sum_{r=0}^{n} \sum_{s=0}^{\infty} (-nI)_r \big(F + (n-1)I\big)_r \frac{(-w^2)^s (-\lambda L^{-1})^r}{\Gamma(2s+1) r!} \\
&\quad \times \int_0^\infty \xi^{n+r+s-\frac{1}{2}} e^{-\mu\xi}\, d\xi \\
&= \sqrt{\frac{1}{2\pi}} \sum_{r=0}^{n} \sum_{s=0}^{\infty} (-nI)_r \big(F + (n-1)I\big)_r \frac{(-\lambda L^{-1})^r}{r!} \\
&\quad \times \frac{\sqrt{\pi}\,\Gamma(n+r+s+\frac{1}{2})}{\Gamma(\frac{1}{2})(\frac{1}{2})_s s! 2^{2s} \mu^{(n+r+s+\frac{1}{2})}} \big(-w^2\big)^s.
\end{aligned} \tag{38}$$

After changing the order of summation and simplifying, we obtain

$$\begin{aligned}
\mathcal{F}^c(w) &= \sum_{s=0}^{\infty} \frac{(n+r+\frac{1}{2})_s (-w^2/4\mu)^s}{s!(\frac{1}{2})_s} \\
&= \sqrt{\frac{1}{2\pi}} \mu^{-(n+\frac{1}{2})} \Gamma\left(n+\frac{1}{2}\right) e^{(-w^2/4\mu)} \\
&\quad \times \sum_{r=0}^{n} (-nI)_r \big(F + (n-1)I\big)_r \left(n+\frac{1}{2}\right)_r \frac{(-\lambda(\mu L)^{-1})^r}{r!} \\
&\quad \times \frac{(-w^2/4\mu)^{n+r}}{(\frac{1}{2})_{n+r}} \sum_{s=0}^{n+r} (-n-r)_s \left(\frac{1}{2}-n-r\right)_s \frac{(-4\mu/w^2)^s}{s!},
\end{aligned} \tag{39}$$

which implies formula (37). □

Theorem 3.6 *Let $\mathcal{Y}_n(\lambda\xi;F,L)$ be given in* (15). *For the function*

$$f(\xi) = \xi^{2n+1} e^{-\mu\xi^2} \mathcal{Y}_n\big(\lambda\xi^2;F,L\big), \tag{40}$$

we have

$$\begin{aligned} \mathcal{F}^s(w) &= \frac{1}{2\sqrt{2}} \mu^{-(2n+3/2)} \big(-w^2/4\big)^n e^{(-w^2/4\mu)} \\ &\quad \times \sum_{r=0}^{n} (-nI)_r \big(F + (n-1)I\big)_r \frac{(\lambda w(4\mu^2 L)^{-1})^r}{r!} \mathcal{Y}_{n+r}\left(4\mu; \frac{1}{2} - 2(n+r), w^2\right), \end{aligned} \tag{41}$$

where $w,\mu,\lambda \in \mathbb{C}$ are such that $\operatorname{Re}(w) > 0$, $\operatorname{Re}(\mu) > 0$, *and* $\operatorname{Re}(\lambda) > 0$.

Proof To describe the relation (41), from (40) in (23), we see that

$$\begin{aligned} \mathcal{F}^s(w) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \xi^{2n+1} e^{-\mu\xi^2} \mathcal{Y}_n\big(\lambda\xi^2;F,L\big) \sin(\xi w)\, d\xi \\ &= \sqrt{\frac{2}{\pi}} \sum_{r=0}^{n} (-nI)_r \big(F + (n-1)I\big)_r \frac{(-\lambda L^{-1})^r}{r!} \\ &\quad \times \int_0^\infty \xi^{2n+2r+1} e^{-\mu\xi^2} \sin(\xi w)\, d\xi \\ &= \sqrt{\frac{2}{\pi}} \sum_{r=0}^{n} \sum_{s=0}^{\infty} (-nI)_r \big(F + (n-1)I\big)_r \\ &\quad \times \frac{(-1)^s (w)^{2s+1} (-\lambda L^{-1})^r \Gamma(n+r+s+3/2)}{2\mu^{(n+r+s+3/2)} (2s+1)! r!} \\ &= \sqrt{\frac{1}{2\pi}} w \mu^{-(n+3/2)} \Gamma(n+3/2) \\ &\quad \times \sum_{r=0}^{n} (-nI)_r \big(F + (n-1)I\big)_r (n+3/2)_r \frac{(-\lambda(\mu L)^{-1})^r}{r!} \\ &\quad \times e^{(-w^2/4\mu)} \sum_{s=0}^{n+r} \left\{ \frac{(-n-r)_s}{s!(3/2)_s} \right\} \big(w^2/4\mu\big)^s. \end{aligned}$$

After simplification, we obtain the desired result

$$\begin{aligned} \mathcal{F}^s(w) &= \frac{1}{2\sqrt{2}} \mu^{-(2n+3/2)} \left(\frac{-w^2}{4}\right)^n e^{(-w^2/4\mu)} \\ &\quad \times \sum_{r=0}^{n} (-nI)_r \big(F + (n-1)I\big)_r \frac{(\lambda w(4\mu^2 L)^{-1})^r}{r!} \\ &\quad \times {}_2\mathcal{H}_0 \left[\begin{matrix} -n-r, -\frac{1}{2}-n-r \\ - \end{matrix} ; -4\mu/w^2 \right]. \end{aligned}$$

This completes the proof of Theorem 3.6. □

Theorem 3.7 *Let S and F be positive stable and commuting matrices in $\mathbb{C}^{n\times n}$. For the function*

$$f(\xi) = \xi^{S+\frac{1}{2}I} e^{-\mu\xi^2} \mathcal{Y}_n\bigl(1; F, \mu\xi^2\bigr), \tag{42}$$

we have

$$\begin{aligned} \mathcal{F}^c(w) &= (-1)^n \sqrt{\frac{\pi}{2}} \mu^{-(S/4+1/4)} (S/2 + F + 3/4I)_n \\ &\quad \times \Gamma^{-1}\bigl((n-3/4)I - S/2\bigr) \csc\bigl[\pi(S/2 + 1/4I)\bigr] \\ &\quad \times {}_2\mathcal{H}_2 \left[\begin{matrix} (7/4-n)I + S/2, (3/4-n)I + S/2 - F \\ \frac{1}{2}, 3/4I + S/2 + F \end{matrix} ; -w^2/4\mu \right], \end{aligned} \tag{43}$$

where $w, \mu \in \mathbb{C}$ are such that $\mathrm{Re}(w) > 0$, $\mathrm{Re}(\mu) > 0$, *and* $\widetilde{\alpha}(S + 3/2I) < 2n + \widetilde{\alpha}(2F)$, *as well as*

$$\begin{aligned} \mathcal{F}^s(w) &= w(-1)^n \sqrt{\frac{\pi}{2}} \mu^{-(S/2+3/4)} (S/2 + F - 1/4I)_n \\ &\quad \times \Gamma^{-1}\bigl((n+1/4)I - S/2\bigr) \csc\bigl[\pi(S/2 + 3/4I)\bigr] \\ &\quad \times {}_2\mathcal{H}_2 \left[\begin{matrix} (3/4-n)I + S/2, (n-1/4)I + S/2 + F \\ \frac{3}{2}I, -1/4I + S/2 + F \end{matrix} ; -w^2/4\mu \right], \end{aligned} \tag{44}$$

where $w, \mu \in \mathbb{C}$ are such that $\mathrm{Re}(w) > 0$, $\mathrm{Re}(\mu) > 0$, *and* $\widetilde{\alpha}(S + 1/2I) > 0$, $\widetilde{\alpha}(S) < 2n + 1/2$.

Proof To demonstrate the truth of these results, making use of (22) with (42), we observe that

$$\begin{aligned} \mathcal{F}^c(w) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \xi^{S+\frac{1}{2}I} e^{-\mu\xi^2} \mathcal{Y}_n\bigl(1; F, \mu\xi^2\bigr) \cos(\xi w)\, d\xi \\ &= \sqrt{\frac{1}{2\pi}} \sum_{r=0}^{n} \sum_{s=0}^{\infty} (-nI)_r \bigl(F + (n-1)I\bigr)_r \frac{(-1/\mu)^r (-w^2)^s}{r!(2s)!} \\ &\quad \times \int_0^\infty \xi^{S/2+(1/4-r+s-1)I} e^{-\mu\xi}\, d\xi \\ &= \sqrt{\frac{1}{2\pi}} \sum_{r=0}^{n} \sum_{s=0}^{\infty} (-nI)_r \bigl(F + (n-1)I\bigr)_r \frac{(-1/\mu)^r (-w^2)^s}{r!(2s)!} \\ &\quad \times \mu^{(-S/2+(r-s-1/4)I)} \Gamma\bigl(S/2 + (s-r+1/4)I\bigr). \end{aligned}$$

Thus, after a simplification, we find that

$$\begin{aligned} \mathcal{F}^c(w) &= \sqrt{\frac{1}{2\pi}} \mu^{-(S/2+1/4I)} \Gamma(S/2 + 1/4I) \\ &\quad \times \sum_{s=0}^{\infty} (S/2 + 1/4I)_s \Gamma\bigl(-S/2 - F - (3/4+s-1)I\bigr) \Gamma\bigl(-S/2 - (3/4+s)I\bigr) \\ &\quad \times \Gamma^{-1}\bigl(-S/2 - F - (3/4+s+n-1)I\bigr) \end{aligned}$$

$$\times \Gamma^{-1}\big(-S/2-(3/4+s-n)I\big)\frac{(-w^2/4\mu)^s}{s!(\frac{1}{2})_s}$$

$$= \sqrt{\frac{1}{2\pi}}\mu^{-(S/2+1/4I)}\Gamma(S/2+1/4I)\Gamma(-S/2-3/4I)\Gamma(-S/2-F+1/4I)$$

$$\times \Gamma^{-1}\big(-S/2-F+(1/4-n)I\big)\Gamma^{-1}\big(-S/2-(3/4-n)I\big)$$

$$\times \sum_{s=0}^{\infty}(S/2+1/4I)_s\big(S/2+(7/2-n)I\big)_s\big(S/2+F+(\mu/4-n)I\big)_s$$

$$\times \big[(S/2+F+3/4I)_s\big]^{-1}\big[(S/2+1/4I)_s\big]^{-1}\frac{(-w^2/4\mu)^s}{s!(\frac{1}{2})_s}.$$

The above equation gives the proof of (43).

In a similar way and by using (23) with (42), we can get the result in (44).

Hence, the demonstration of Theorem 3.7 is finished. □

Theorem 3.8 *Let S, F, and L be commuting matrices in $\mathbb{C}^{n\times n}$. If*

$$f(\xi) = \xi^S\frac{1}{(\mu^2+\xi^2)}J_S(\lambda\xi)\mathcal{Y}_n\big(\xi^2;F,L\big), \tag{45}$$

then, we have

$$\mathcal{F}^s(w) = \sqrt{\frac{2}{\pi}}\mu^{S-I}\sinh(\mu w)K_S(\lambda\mu)\mathcal{Y}_n\big(\mu^2;F,-L\big), \tag{46}$$

where $w,\mu,\lambda \in \mathbb{C}$ are such that $\operatorname{Re}(w) > 0$, $\operatorname{Re}(\mu) > 0$, $\operatorname{Re}(\lambda) > 0$, and S is a positive stable matrix in $\mathbb{C}^{n\times n}$ such that $-1 < \widetilde{\alpha}(S) < 0$, $J_S(x)$ is the Bessel matrix function defined in (17) and $K_S(x)$ is the modified Bessel matrix function defined in (19).

Proof The proof of this result indeed follows from applying (23) on (45), we have

$$\mathcal{F}^s(w) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\xi^S\frac{1}{(\mu^2+\xi^2)}J_S(\lambda\xi)\mathcal{Y}_n\big(\xi^2;F,L\big)\sin(\xi w)\,d\xi$$

$$= \sqrt{\frac{2}{\pi}}\sum_{r=0}^{n}(-nI)_r\big(F+(n-1)I\big)_r\frac{(-L^{-1})^r}{r!}$$

$$\times \int_0^{\infty}\xi^{(2r+1)I-S}\frac{1}{(\mu^2+\xi^2)}J_S(\lambda\xi)\sin(\xi w)\,d\xi.$$

Using the Fourier sine transform (see [18, p. 426]), we obtain

$$\mathcal{F}^s(w) = \sqrt{\frac{2}{\pi}}\sum_{r=0}^{n}(-nI)_r\big(F+(n-1)I\big)_r$$

$$\times \frac{L^{-r}}{r!}\mu^{(S+(2r-1)I)}\sinh(\mu w)K_S(\mu\lambda)$$

$$= \sqrt{\frac{2}{\pi}}\mu^{S-I}\sinh(\mu w)K_S(\mu\lambda)\sum_{r=0}^{n}(-nI)_r\big(F+(n-1)I\big)_r\frac{(\mu^2L^{-1})^r}{r!}.$$

This completes the proof of equation (46) asserted in Theorem 3.8. □

Similarly, we can arrive at the following result.

Theorem 3.9 *Let $\mathcal{Y}_n(\xi^2;F,L)$ be given in (15). For the function*

$$f(\xi)=\xi^{I-S}\frac{1}{(\mu^2+\xi^2)}J_S(\lambda\xi)\mathcal{Y}_n\big(\xi^2;F,L\big), \tag{47}$$

we have

$$\mathcal{F}^s(w)=\sqrt{\frac{\pi}{2}}\mu^{-S}e^{(-\mu w)}J_S(\lambda w)\mathcal{Y}_n\big(-\mu^2;F,-L\big), \tag{48}$$

where $w,\mu,\lambda\in\mathbb{C}$ are such that $\operatorname{Re}(w)>0,\operatorname{Re}(\mu)>0,\operatorname{Re}(\lambda)>0,\operatorname{Re}(w)>\operatorname{Re}(\lambda)$, S is a positive stable matrix in $\mathbb{C}^{n\times n}$ such that $\widetilde{\alpha}(I-S)>0$, and S, F, and L are commuting matrices in $\mathbb{C}^{n\times n}$.

Theorem 3.10 *Let $\mathcal{Y}_n(\xi^2;F,L)$ be given in (15). For the function*

$$f(\xi)=\big(\lambda^2+\xi^2\big)^{-(S+\frac{1}{2}I)}\mathcal{Y}_n\big(\xi^2;F,L\big), \tag{49}$$

we have

$$\begin{aligned}\mathcal{F}^c(w)={}&\frac{\sqrt{8}}{\pi}(2F)^{-S}\Gamma^{-1}\left(S+\frac{1}{2}I\right)\\&\times\sum_{r=0}^{n}(-nI)_r\big(F+(n-1)I\big)_r\frac{L^{-r}}{r!}\frac{d^{2r}(w^SK_S(\lambda w))}{dw^{2r}},\end{aligned} \tag{50}$$

where $w,\lambda\in\mathbb{C}$ are such that $\operatorname{Re}(w)>0,\operatorname{Re}(\lambda)>0$, S is a positive stable matrix in $\mathbb{C}^{n\times n}$ such that $\widetilde{\alpha}(S)>-\frac{1}{2}$, $\boldsymbol{K}_S(x)$ is modified Bessel matrix function defined in (19), and S, F, and L are commuting matrices in $\mathbb{C}^{n\times n}$.

Proof In order to establish the result (50), with the help of the Lemma 2.3, we get

$$\begin{aligned}\mathcal{F}^c(w)&=\sqrt{\frac{2}{\pi}}\int_0^\infty\big(\lambda^2+\xi^2\big)^{-(S+\frac{1}{2}I)}\mathcal{Y}_n\big(\xi^2;F,L\big)\cos(w\xi)\,d\xi\\&=\sqrt{\frac{2}{\pi}}\sum_{r=0}^{n}(-nI)_r\big(F+(n-1)I\big)_r\frac{(-L)^{-r}}{r!}\\&\quad\times\int_0^\infty\xi^{2r}\big(\lambda^2+\xi^2\big)^{-(S+\frac{1}{2}I)}\cos(w\xi)\,d\xi\\&=\sqrt{\frac{2}{\pi}}\sum_{r=0}^{n}(-nI)_r\big(F+(n-1)I\big)_r\frac{(-L)^{-r}}{r!}\\&\quad\times\sqrt{\frac{2}{\pi}}(-1)^r\sqrt{2}\Gamma^{-1}\left(S+\frac{1}{2}I\right)(2F)^{-S}\frac{d^{2r}(w^S\mathbf{K}_S(\lambda w))}{dw^{2r}}.\end{aligned}$$

This completes the proof. □

Similarly, we can arrive at the following result.

Theorem 3.11 *Let $\mathcal{Y}_n(\xi^2;F,L)$ be given in* (15). *For the function*

$$f(\xi) = \xi\left(\lambda^2+\xi^2\right)^{-(n+\frac{1}{2})}\mathcal{Y}_n\left(\xi^2;F,L\right), \tag{51}$$

we have

$$\mathcal{F}^c(w) = \frac{-\sqrt{8}}{\pi\,\Gamma(n+\frac{1}{2})}(2F)^{-n}\frac{d^{2r+1}(w^n K_n(\lambda w))}{dw^{2r+1}}, \tag{52}$$

where $w,\lambda\in\mathbb{C}$ are such that $\operatorname{Re}(w)>0$, $\operatorname{Re}(\lambda)>0$, and $K_n(w)$ is the modified Bessel function defined in [18].

Theorem 3.12 *Let $\mathcal{Y}_n(\xi;F,L)$ be given in* (15). *For the function*

$$f(\xi) = \xi^{-1}e^{-\lambda/\xi}\mathcal{Y}_n(\xi;F,L), \tag{53}$$

we have

$$\begin{aligned}\mathcal{F}^c(w) = {}&\sqrt{\frac{2}{\pi}}\sum_{r=0}^{n}(-nI)_r\left(F+(n-1)I\right)_r\frac{(-L)^{-r}(w/\lambda)^{\frac{r}{2}}}{r!}\\&\times\left\{e^{i\frac{\pi r}{4}}K_r(2\sqrt{i\lambda w})+e^{-i\frac{\pi r}{4}}K_r(2\sqrt{-i\lambda w})\right\},\end{aligned} \tag{54}$$

where $w,\lambda\in\mathbb{C}$ are such that $\operatorname{Re}(w)>0$ and $\operatorname{Re}(\lambda)>0$, $\xi\in\mathbb{C}\setminus\{0\}$, and $K_r(w)$ is the modified Bessel function defined in [18].

Proof Using (15) and (53) in (22), we observe that

$$\begin{aligned}\mathcal{F}^c(w) &= \sqrt{\frac{2}{\pi}}\int_0^\infty \xi^{-1}e^{-\lambda/\xi}\mathcal{Y}_n(\xi;F,L)\cos(w\xi)\,d\xi\\&= \sqrt{\frac{2}{\pi}}\sum_{r=0}^{n}(-nI)_r\left(F+(n-1)I\right)_r\frac{(-L)^{-r}}{r!}\\&\quad\times\int_0^\infty \xi^{r-1}e^{-\lambda/\xi}\cos(w\xi)\,d\xi.\end{aligned} \tag{55}$$

Applying the formula in [18, p. 403], we see that

$$\begin{aligned}\mathcal{F}^c(w) = {}&\sqrt{\frac{2}{\pi}}\sum_{r=0}^{n}(-nI)_r\left(F+(n-1)I\right)_r\frac{(-L)^{-r}(w/\lambda)^{\frac{r}{2}}}{r!}\\&\times\left\{e^{i\frac{\pi r}{4}}K_r(2\sqrt{i\lambda w})+e^{-i\frac{\pi r}{4}}K_r(2\sqrt{-i\lambda w})\right\}.\end{aligned} \tag{56}$$

This completes the proof of Theorem 3.12. □

4 Conclusion

In this manuscript, motivated by the recent studies and developments of the integral transforms with various special matrix functions, including the matrix orthogonal polynomials as kernels and their applications [14, 17, 49, 50], we introduce some Fourier cosine and sine transforms of generalized Bessel matrix polynomials, together with certain elementary matrix functions, as well as exponential and Bessel functions. It is obvious that the results presented here, which involve certain matrices in $\mathbb{C}^{n\times n}$, reduce to the corresponding scalar ones when $n = 1$.

In fact, a remarkably large number of Fourier cosine and sine formulas involving a variety of functions have been published (see, e.g., [51, pp. 7–108]). In this connection, we conclude this manuscript by posing the following problem for further investigation.

Open problem. Try to give matrix versions of the results for Fourier integral transforms (or formulas) involving a variety of special functions (see, e.g., [51, pp. 7–108]).

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