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Recursion formulas for certain quadruple hypergeometric functions

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Abstract
A remarkably large number of hypergeometric (and generalized) functions and a variety of their extensions have been presented and investigated in the literature by many authors. In this paper, we introduce five new hypergeometric functions in four variables and then establish several recursion formulas for these new functions. Some interesting particular cases and consequences of the main results are also considered.

MSC: 15A15; 33C65

Keywords: Recursion formula; Quadruple hypergeometric functions

1 Introduction and preliminaries

The ordinary hypergeometric functions have been the subject of extensive researches by several prominent mathematicians. These functions play a crucial role in mathematical analysis, physics, engineering and applied sciences. Most of the special functions, which have various physical and technical applications and are closely connected with orthogonal polynomial and problems of mechanical quadrature, can be expressed in terms of generalized hypergeometric functions. Agarwal et al. [1, 2] established some properties for the generalized Gauss hypergeometric functions, which were introduced by Özergin et al. Rahman et al. [3] defined further extensions of hypergeometric and Appell's hypergeometric functions. Very recently, Saboor et al. [4] defined a new extension of Srivastava's triple hypergeometric functions, and the authors presented some of their properties such as integral representations, derivative formulas, and recurrence relations.

Many modern mathematics and theoretical physics problems lead to the study of the hypergeometric functions of several complex variables (see, e.g., [5–16]). These include, for example, problems in the representation theory, combinatorics, number theory, analytic continuation of integrals of the Mellin–Barnes type, and algebraic geometry. Moreover, hypergeometric-type functions are seen in several applications of physical and chemical problems [17–20].

In [21], Exton defined 21 complete hypergeometric functions in four variables denoted by the symbols $K_1, K_2, \dots, K_{21}$. In [22], Sharma and Parihar introduced 83 complete quadruple hypergeometric functions, denoted by $F_1^{(4)}, F_2^{(4)}, \dots, F_{83}^{(4)}$. Bin-Saad and Younis [23, 24] introduced 30 new quadruple hypergeometric functions written as

$X_1^{(4)}, X_2^{(4)}, \dots, X_{30}^{(4)}$. In [25] Younis et al. discovered the existence of six additional complete hypergeometric functions in four variables, $X_{85}^{(4)}, X_{86}^{(4)}, \dots, X_{90}^{(4)}$. Motivated by the above investigations, we introduce five new quadruple hypergeometric functions as follows:

$$\begin{aligned} &X_{95}^{(4)}(\rho_1,\rho_1,\rho_3,\rho_5,\rho_1,\rho_2,\rho_4,\rho_6;\omega_2,\omega_1,\omega_1,\omega_1;x,y,z,u) \\ &\quad= \sum_{m,n,p,q=0}^{\infty} \frac{(\rho_1)_{2m+n}(\rho_2)_n(\rho_3)_p(\rho_4)_p(\rho_5)_q(\rho_6)_q}{(\omega_1)_{n+p+q}(\omega_2)_m} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!} \\ &\qquad \left(|x| < \frac{1}{4}, |y| < 1, |z| < 1, |t| < 1\right), \end{aligned} \tag{1.1}$$

$$\begin{aligned} &X_{96}^{(4)}(\rho_1,\rho_1,\rho_2,\rho_3,\rho_1,\rho_2,\rho_2,\rho_4;\omega_1,\omega_2,\omega_1,\omega_2;x,y,z,u) \\ &\quad= \sum_{m,n,p,q=0}^{\infty} \frac{(\rho_1)_{2m+n}(\rho_2)_{2p+n}(\rho_3)_q(\rho_4)_q}{(\omega_1)_{m+p}(\omega_2)_{n+q}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!} \\ &\qquad \left(|x| < \frac{1}{4}, |y| < 1, |z| < \frac{1}{4}, |t| < 1\right), \end{aligned} \tag{1.2}$$

$$\begin{aligned} &X_{97}^{(4)}(\rho_1,\rho_1,\rho_2,\rho_3,\rho_1,\rho_2,\rho_2,\rho_4;\omega_1,\omega_1,\omega_2,\omega_1;x,y,z,u) \\ &\quad= \sum_{m,n,p,q=0}^{\infty} \frac{(\rho_1)_{2m+n}(\rho_2)_{2p+n}(\rho_3)_q(\rho_4)_q}{(\omega_1)_{m+n+q}(\omega_2)_{p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!} \\ &\qquad \left(|x| < \frac{1}{4}, |y| < 1, |z| < \frac{1}{4}, |t| < 1\right), \end{aligned} \tag{1.3}$$

$$\begin{aligned} &X_{98}^{(4)}(\rho_1,\rho_1,\rho_2,\rho_3,\rho_1,\rho_2,\rho_2,\rho_4;\omega_1,\omega_2,\omega_1,\omega_1;x,y,z,u) \\ &\quad= \sum_{m,n,p,q=0}^{\infty} \frac{(\rho_1)_{2m+n}(\rho_2)_{2p+n}(\rho_3)_q(\rho_4)_q}{(\omega_1)_{m+p+q}(\omega_2)_{n}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!} \\ &\qquad \left(|x| < \frac{1}{4}, |y| < 1, |z| < \frac{1}{4}, |t| < 1\right), \end{aligned} \tag{1.4}$$

$$\begin{aligned} &X_{99}^{(4)}(\rho_1,\rho_1,\rho_2,\rho_3,\rho_1,\rho_2,\rho_2,\rho_4;c,c,c,c;x,y,z,u) \\ &\quad= \sum_{m,n,p,q=0}^{\infty} \frac{(\rho_1)_{2m+n}(\rho_2)_{2p+n}(\rho_3)_q(\rho_4)_q}{(c)_{m+n+p+q}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!} \\ &\qquad \left(|x| < \frac{1}{4}, |y| < 1, |z| < \frac{1}{4}, |t| < 1\right), \end{aligned} \tag{1.5}$$

where $(a)_m$ is the Pochhammer symbol defined by

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} = a(a+1)\cdots(a+m-1),$$

for $m \geq 1$, $(a)_0 = 1$, and Γ being the well-known Gamma function.

Recently, many authors have obtained several recursion formulas involving classical hypergeometric functions. In [26], Opps et al. introduced the recursion formulas for Appell's function F_2 and gave its applications to radiation field problems. Wang [27] presented the recursion formulas for Appell functions F_1, F_2, F_3, and F_4. Sahai and Verma [28, 29] established the recursion formulas for Lauricella's triple functions, Srivastava hypergeometric functions in three variables, k-variable Lauricella functions, and the Srivastava–Daoust and related multivariable hypergeometric functions. In [25, 30], the authors gave the recursion formulas for Srivastava general triple hypergeometric function and Exton's triple hypergeometric functions. In this present paper, we establish several recursion formulas associated with the quadruple functions (1.1)–(1.5).

In the following, some abbreviated notations are used in this paper. We, for example, write $X_{95}^{(4)}$ for the series

$$X_{95}^{(4)}(\rho_1,\rho_1,\rho_3,\rho_5,\rho_1,\rho_2,\rho_4,\rho_6;\omega_2,\omega_1,\omega_1,\omega_1;x,y,z,u)$$

and $X_{95}^{(4)}(\rho_1+n)$ for

$$X_{95}^{(4)}(\rho_1+n,\rho_1+n,\rho_3,\rho_5,\rho_1+n,\rho_2,\rho_4,\rho_6;\omega_2,\omega_1,\omega_1,\omega_1;x,y,z,u).$$

The notation $X_{95}^{(4)}(\rho_1+n,\rho_2+n_1)$ stands for

$$X_{95}^{(4)}(\rho_1+n,\rho_1+n,\rho_3,\rho_5,\rho_1+n,\rho_2+n_1,\rho_4,\rho_6;\omega_2,\omega_1,\omega_1,\omega_1;x,y,z,u)$$

and $X_{95}^{(4)}(\rho_1+n,\rho_2+n_1,\omega_1+n_2)$ stands for

$$X_{95}^{(4)}(\rho_1+n,\rho_1+n,\rho_3,\rho_5,\rho_1+n,\rho_2+n_1,\rho_4,\rho_6;\omega_2,\omega_1+n_2,\omega_1+n_2,\omega_1+n_2;x,y,z,u),$$

etc.

2 Main results

Here, we present certain recursion formulas for the hypergeometric functions of four variables $X_{95}^{(4)},X_{96}^{(4)},\dots,X_{99}^{(4)}$. Throughout the paper, n denotes a nonnegative integer.

Theorem 2.1 *The following recursion formulas hold true for the numerator parameters* ρ_1, ρ_2, ρ_3, ρ_4, ρ_5, ρ_6 *of* $X_{95}^{(4)}$:

$$\begin{aligned} X_{95}^{(4)}(\rho_1+n) &= X_{95}^{(4)}+\frac{2x}{\omega_2}\sum_{n_1=1}^{n}(\rho_1+n_1)X_{95}^{(4)}(\rho_1+1+n_1,\omega_2+1)\\ &\quad+\frac{y\rho_2}{\omega_1}\sum_{n_1=1}^{n}X_{95}^{(4)}(\rho_1+n_1,\rho_2+1,\omega_1+1), \end{aligned} \tag{2.1}$$

$$\begin{aligned} X_{95}^{(4)}(\rho_1-n) &= X_{95}^{(4)}-\frac{2x}{\omega_2}\sum_{n_1=1}^{n}(\rho_1+1-n_1)X_{95}^{(4)}(\rho_1+2-n_1,\omega_2+1)\\ &\quad-\frac{y\rho_2}{\omega_1}\sum_{n_1=1}^{n}X_{95}^{(4)}(\rho_1+1-n_1,\rho_2+1,\omega_1+1), \end{aligned} \tag{2.2}$$

$$\begin{aligned} X_{95}^{(4)}(\rho_2+n) &= X_{95}^{(4)}+\frac{y\rho_1}{c}\sum_{n_1=1}^{n}X_{95}^{(4)}(\rho_1+1,\rho_2+n_1,c+1)\\ &\quad+\frac{z\rho_3}{c}\sum_{n_1=1}^{n}X_{95}^{(4)}(\rho_2+n_1,\rho_3+1,c+1), \end{aligned} \tag{2.3}$$

$$X_{95}^{(4)}(\rho_2-n)=X_{95}^{(4)}-\frac{y\rho_1}{c}\sum_{n_1=1}^{n}X_{95}^{(4)}(\rho_1+1,\rho_2+1-n_1,c+1)$$
$$-\frac{z\rho_3}{c}\sum_{n_1=1}^{n}X_{95}^{(4)}(\rho_2+1-n_1,\rho_3+1,c+1), \tag{2.4}$$

$$X_{95}^{(4)}(\rho_3+n)=X_{95}^{(4)}+\frac{z\rho_2}{c}\sum_{n_1=1}^{n}X_{95}^{(4)}(\rho_2+1,\rho_3+n_1,c+1), \tag{2.5}$$

$$X_{95}^{(4)}(\rho_3-n)=X_{95}^{(4)}-\frac{z\rho_2}{c}\sum_{n_1=1}^{n}X_{95}^{(4)}(\rho_2+1,\rho_3+1-n_1,c+1), \tag{2.6}$$

$$X_{95}^{(4)}(\rho_4+n)=X_{95}^{(4)}+\frac{u\rho_5}{c}\sum_{n_1=1}^{n}X_{95}^{(4)}(\rho_5+1,\rho_4+n_1,c+1), \tag{2.7}$$

$$X_{95}^{(4)}(\rho_4-n)=X_{95}^{(4)}-\frac{u\rho_5}{c}\sum_{n_1=1}^{n}X_{95}^{(4)}(\rho_5+1,\rho_4+1-n_1,c+1), \tag{2.8}$$

$$X_{95}^{(4)}(\rho_5+n)=X_{95}^{(4)}+\frac{u\rho_4}{c}\sum_{n_1=1}^{n}X_{95}^{(4)}(\rho_4+1,\rho_5+n_1,c+1), \tag{2.9}$$

$$X_{95}^{(4)}(\rho_5-n)=X_{95}^{(4)}-\frac{u\rho_4}{c}\sum_{n_1=1}^{n}X_{95}^{(4)}(\rho_4+1,\rho_5+1-n_1,c+1). \tag{2.10}$$

Proof From the definition of the hypergeometric function $X_{95}^{(4)}$ and the relation

$$(\rho_1+1)_{2m+n}=(\rho_1)_{2m+n}\left(1+\frac{2m}{\rho_1}+\frac{n}{\rho_1}\right), \tag{2.11}$$

we obtain the following contiguous relation:

$$X_{95}^{(4)}(\rho_1+1)=X_{95}^{(4)}+\frac{2x}{\omega_2}(\rho_1+1)X_{95}^{(4)}(\rho_1+2,\omega_2+1)$$
$$+\frac{y\rho_2}{\omega_1}X_{95}^{(4)}(\rho_1+1,\rho_2+1,\omega_1+1). \tag{2.12}$$

To find a contiguous relation for $X_{95}^{(4)}(\rho_1+2)$, we replace $\rho_1\to\rho_1+1$ in (2.12) and simplify. This leads to

$$X_{95}^{(4)}(\rho_1+2)=X_{95}^{(4)}+\frac{2x}{\omega_2}\sum_{n_1=1}^{2}(\rho_1+n_1)X_{95}^{(4)}(\rho_1+n_1+1,\omega_2+1)$$
$$+\frac{y\rho_2}{\omega_1}\sum_{n_1=1}^{2}X_{95}^{(4)}(\rho_1+n_1,\rho_2+1,\omega_1+1). \tag{2.13}$$

Iterating this process n times, we obtain (2.1). For the proof of (2.2), replace the parameter $\rho_1\to\rho_1-1$ in (2.12). This gives

$$X_{95}^{(4)}(\rho_1-1)=X_1-\frac{2x}{\omega_2}\rho_1X_{95}^{(4)}(\rho_1+1,\omega_2+1)-\frac{y\rho_2}{\omega_1}X_{95}^{(4)}(\rho_2+1,\omega_1+1). \tag{2.14}$$

Iteratively, we get (2.2).

The recursion formulas (2.3)–(2.10) can be proved in a similar manner. □

Theorem 2.2 *The following recursion formulas hold true for the numerator parameters* $\rho_2, \rho_3, \rho_4, \rho_5$ *of* $X_{95}^{(4)}$:

$$X_{95}^{(4)}(\rho_2+n)=\sum_{n_1\leq n}\binom{n}{n_1}\frac{(\rho_1)_{n_1}y^{n_1}}{(\omega_1)_{n_1}}X_{95}^{(4)}(\rho_1+n_1,\rho_2+n_1,\omega_1+n_1), \tag{2.15}$$

$$X_{95}^{(4)}(\rho_2-n)=\sum_{n_1\leq n}\binom{n}{n_1}\frac{(\rho_1)_{n_1}(-y)^{n_1}}{(\omega_1)_{n_1}}X_{95}^{(4)}(\rho_1+n_1,\omega_1+n_1), \tag{2.16}$$

$$X_{95}^{(4)}(\rho_3+n)=\sum_{n_1\leq n}\binom{n}{n_1}\frac{(\rho_4)_{n_1}z^{n_1}}{(\omega_1)_{n_1}}X_{95}^{(4)}(\rho_3+n_1,\rho_4+n_1,\omega_1+n_1), \tag{2.17}$$

$$X_{95}^{(4)}(\rho_3-n)=\sum_{n_1\leq n}\binom{n}{n_1}\frac{(\rho_4)_{n_1}(-z)^{n_1}}{(\omega_1)_{n_1}}X_{95}^{(4)}(\rho_4+n_1,\omega_1+n_1), \tag{2.18}$$

$$X_{95}^{(4)}(\rho_4+n)=\sum_{n_1\leq n}\binom{n}{n_1}\frac{(\rho_3)_{n_1}z^{n_1}}{(\omega_1)_{n_1}}X_{95}^{(4)}(\rho_3+n_1,\rho_4+n_1,\omega_1+n_1), \tag{2.19}$$

$$X_{95}^{(4)}(\rho_4-n)=\sum_{n_1\leq n}\binom{n}{n_1}\frac{(\rho_3)_{n_1}(-z)^{n_1}}{(\omega_1)_{n_1}}X_{95}^{(4)}(\rho_3+n_1,c+n_1), \tag{2.20}$$

$$X_{95}^{(4)}(\rho_5+n)=\sum_{n_1\leq n}\binom{n}{n_1}\frac{(\rho_6)_{n_1}u^{n_1}}{(\omega_1)_{n_1}}X_{95}^{(4)}(\rho_5+n_1,\rho_6+n_1,\omega_1+n_1), \tag{2.21}$$

$$X_{95}^{(4)}(\rho_5-n)=\sum_{n_1\leq n}\binom{n}{n_1}\frac{(\rho_6)_{n_1}(-u)^{n_1}}{(\omega_1)_{n_1}}X_{95}^{(4)}(\rho_6+n_1,\omega_1+n_1), \tag{2.22}$$

$$X_{95}^{(4)}(\rho_6+n)=\sum_{n_1\leq n}\binom{n}{n_1}\frac{(\rho_5)_{n_1}u^{n_1}}{(\omega_1)_{n_1}}X_{95}^{(4)}(\rho_5+n_1,\rho_6+n_1,\omega_1+n_1), \tag{2.23}$$

$$X_{95}^{(4)}(\rho_6-n)=\sum_{n_1\leq n}\binom{n}{n_1}\frac{(\rho_5)_{n_1}(-u)^{n_1}}{(\omega_1)_{n_1}}X_{95}^{(4)}(\rho_5+n_1,\omega_1+n_1). \tag{2.24}$$

Proof The proof of (2.15) is based upon the principle of mathematical induction on $n \in \mathbb{N}$. For $n = 1$, the result (2.15) is true obviously by (2.3). Suppose (2.15) is true for $n = m$, that is,

$$X_{95}^{(4)}(\rho_2+m)=\sum_{n_1\leq m}\binom{n}{n_1}\frac{(\rho_1)_{n_1}y^{n_1}}{(\omega_1)_{n_1}}X_{95}^{(4)}(\rho_1+n_1,\rho_2+n_1,\omega_1+n_1). \tag{2.25}$$

Replacing $\rho_2 \mapsto \rho_2 + 1$ in (2.25) and using the contiguous relation (2.3) for $n = 1$, we get

$$\begin{aligned}
&X_{95}^{(4)}(\rho_2+m+1)\\
&\quad=\sum_{n_1\leq m}\binom{n}{n_1}\frac{(\rho_1)_{n_1}y^{n_1}}{(\omega_1)_{n_1}}\biggl\{X_{95}^{(4)}(\rho_1+n_1,\rho_2+n_1,\omega_1+n_1)\\
&\qquad+\frac{(\rho_1+n_1)y}{(\omega_1+n_1)}X_{95}^{(4)}(\rho_1+n_1+1,\rho_2+n_1,\omega_1+n_1+1)\biggr\}.
\end{aligned} \tag{2.26}$$

Simplifying, (2.26) takes the form

$$\begin{aligned} &X_{95}^{(4)}(a_2+m+1) \\ &\quad = \sum_{n_1\leq m}\binom{n}{n_1}\frac{(\rho_1)_{n_1}y^{n_1}}{(\omega_1)_{n_1}}X_{95}^{(4)}(\rho_1+n_1,\rho_2+n_1,\omega_1+n_1) \\ &\qquad + \sum_{n_1\leq m+1}\binom{n}{n_1-1}\frac{(\rho_1)_{n_1}y^{n_1}}{(\omega_1)_{n_1}}X_{95}^{(4)}(\rho_1+n_1,\rho_2+n_1,\omega_1+n_1). \end{aligned} \tag{2.27}$$

Using Pascal's identity in (2.27), we have

$$X_{95}^{(4)}(\rho_2+m+1) = \sum_{n_1\leq m+1}\binom{n}{n_1}\frac{(\rho_1)_{n_1}y^{n_1}}{(\omega_1)_{n_1}}X_{95}^{(4)}(\rho_1+n_1,\rho_2+n_1,\omega_1+n_1). \tag{2.28}$$

This establishes (2.15) for $n = m + 1$. Hence, by induction, the result given in (2.15) is true for all values of n. The recursion formulas (2.16)–(2.24) can be proved in a similar manner. □

Theorem 2.3 *The following recursion formulas hold true for the denominator parameter c of* $X_{95}^{(4)}$:

$$\begin{aligned} &X_{95}^{(4)}(\omega_1-n) \\ &\quad = X_{95}^{(4)} + \rho_1\rho_2 y\sum_{n_1=1}^{n}\frac{1}{(\omega_1-n_1)(\omega_1+1-n_1)}X_{95}^{(4)}(\rho_1+1,\rho_2+1,\omega_1+2-n_1) \\ &\qquad + \rho_3\rho_4 z\sum_{n_1=1}^{n}\frac{1}{(\omega_1-n_1)(\omega_1+1-n_1)}X_{95}^{(4)}(\rho_3+1,\rho_4+1,\omega_1+2-n_1) \\ &\qquad + \rho_5\rho_6 u\sum_{n_1=1}^{n}\frac{1}{(\omega_1-n_1)(\omega_1+1-n_1)}X_{95}^{(4)}(\rho_5+1,\rho_6+1,\omega_1+2-n_1), \end{aligned} \tag{2.29}$$

$$\begin{aligned} &X_{95}^{(4)}(\omega_2-n) \\ &\quad = X_{95}^{(4)} + (\rho_1)_2 x\sum_{n_1=1}^{n}\frac{1}{(\omega_2-n_1)(\omega_2+1-n_1)}X_{95}^{(4)}(\rho_1+2,\omega_2+2-n_1). \end{aligned} \tag{2.30}$$

Proof Using the definition of hypergeometric function $X_{95}^{(4)}$ and the relation

$$\frac{1}{(\omega_1-1)_{n+p+q}} = \frac{1}{(\omega_1)_{n+p+q}}\left(1+\frac{n}{\omega_1-1}+\frac{p}{\omega_1-1}+\frac{q}{\omega_1-1}\right), \tag{2.31}$$

we have

$$\begin{aligned} X_{95}^{(4)}(\omega_1-1) = X_{95}^{(4)} &+ \frac{\rho_1\rho_2 y}{\omega_1(\omega_1-1)}X_{95}^{(4)}(\rho_1+1,\rho_2+1,\omega_1+1) \\ &+ \frac{\rho_3\rho_4 z}{\omega_1(\omega_1-1)}X_{95}^{(4)}(\rho_3+1,\rho_4+1,\omega_1+2-n_1) \\ &+ \frac{\rho_5\rho_6 u}{\omega_1(\omega_1-1)}X_{95}^{(4)}(\rho_5+1,\rho_6+1,\omega_1+2-n_1). \end{aligned} \tag{2.32}$$

Using this contiguous relation for $X_{95}^{(4)}$ with the parameter $\omega_1 - n$ for n times, we get (2.29). Recursion formula (2.30) can be proved in a similar manner. □

Theorem 2.4 *The following recursion formulas hold true for the numerator parameters* $\rho_1, \rho_2, \rho_3, \rho_4$ *of* $X_{96}^{(4)}$:

$$\begin{aligned} X_{96}^{(4)}(\rho_1+n) = X_{96}^{(4)} &+ \frac{2x}{\omega_1}\sum_{n_1=1}^{n}(\rho_1+n_1)X_{96}^{(4)}(\rho_1+1+n_1,\omega_1+1) \\ &+ \frac{y\rho_2}{\omega_2}\sum_{n_1=1}^{n}X_{96}^{(4)}(\rho_1+n_1,\rho_2+1,\omega_2+1), \end{aligned} \tag{2.33}$$

$$\begin{aligned} X_{96}^{(4)}(\rho_1-n) = X_{96}^{(4)} &- \frac{2x}{\omega_1}\sum_{n_1=1}^{n}(\rho_1+1-n_1)X_{96}^{(4)}(\rho_1+2-n_1,\omega_1+1) \\ &- \frac{y\rho_2}{\omega_2}\sum_{n_1=1}^{n}X_{96}^{(4)}(\rho_1+1-n_1,\rho_2+1,\omega_2+1), \end{aligned} \tag{2.34}$$

$$\begin{aligned} X_{96}^{(4)}(\rho_2+n) = X_{96}^{(4)} &+ \frac{2z}{\omega_1}\sum_{n_1=1}^{n}(\rho_2+n_1)X_{96}^{(4)}(\rho_2+1+n_1,\omega_1+1) \\ &+ \frac{y\rho_1}{\omega_2}\sum_{n_1=1}^{n}X_{96}^{(4)}(\rho_1+1,\rho_2+n_1,\omega_2+1), \end{aligned} \tag{2.35}$$

$$\begin{aligned} X_{96}^{(4)}(\rho_2-n) = X_{96}^{(4)} &- \frac{2z}{\omega_1}\sum_{n_1=1}^{n}(\rho_2+1-n_1)X_{96}^{(4)}(\rho_2+2-n_1,\omega_1+1) \\ &- \frac{y\rho_1}{\omega_2}\sum_{n_1=1}^{n}X_{96}^{(4)}(\rho_1+1,\rho_2+1-n_1,\omega_2+1), \end{aligned} \tag{2.36}$$

$$X_{96}^{(4)}(\rho_3+n) = X_{96}^{(4)} + \frac{z\rho_4}{\omega_2}\sum_{n_1=1}^{n}X_{96}^{(4)}(\rho_4+1,\rho_3+n_1,\omega_2+1), \tag{2.37}$$

$$X_{96}^{(4)}(\rho_3-n) = X_{96}^{(4)} - \frac{z\rho_4}{\omega_2}\sum_{n_1=1}^{n}X_{96}^{(4)}(\rho_4+1,\rho_3+1-n_1,\omega_2+1), \tag{2.38}$$

$$X_{96}^{(4)}(\rho_4+n) = X_{96}^{(4)} + \frac{u\rho_3}{\omega_2}\sum_{n_1=1}^{n}X_{96}^{(4)}(\rho_3+1,\rho_4+n_1,\omega_2+1), \tag{2.39}$$

$$X_{96}^{(4)}(\rho_4-n) = X_{96}^{(4)} - \frac{u\rho_3}{\omega_2}\sum_{n_1=1}^{n}X_{96}^{(4)}(\rho_3+1,\rho_4+1-n_1,\omega_2+1). \tag{2.40}$$

Theorem 2.5 *The following recursion formulas hold true for the numerator parameters* ρ_3, ρ_4 *of* $X_{96}^{(4)}$:

$$X_{96}^{(4)}(\rho_3+n) = \sum_{n_1\le n}\binom{n}{n_1}\frac{(\rho_4)_{n_1}z^{n_1}}{(\omega_2)_{n_1}}X_{96}^{(4)}(\rho_3+n_1,\rho_4+n_1,\omega_2+n_1), \tag{2.41}$$

$$X_{96}^{(4)}(\rho_3-n) = \sum_{n_1\le n}\binom{n}{n_1}\frac{(\rho_4)_{n_1}(-z)^{n_1}}{(\omega_2)_{n_1}}X_{96}^{(4)}(\rho_4+n_1,\omega_2+n_1), \tag{2.42}$$

$$X_{96}^{(4)}(\rho_4 + n) = \sum_{n_1 \leq n} \binom{n}{n_1} \frac{(\rho_3)_{n_1} u^{n_1}}{(\omega_2)_{n_1}} X_{96}^{(4)}(\rho_3 + n_1, \rho_4 + n_1, \omega_2 + n_1), \tag{2.43}$$

$$X_{96}^{(4)}(\rho_4 - n) = \sum_{n_1 \leq n} \binom{n}{n_1} \frac{(\rho_3)_{n_1} (-u)^{n_1}}{(\omega_2)_{n_1}} X_{96}^{(4)}(\rho_3 + n_1, \omega_2 + n_1). \tag{2.44}$$

Theorem 2.6 *The following recursion formulas hold true for the denominator parameters* ω_1, ω_2 *of* $X_{96}^{(4)}$:

$$\begin{aligned} &X_{96}^{(4)}(\omega_1 - n) \\ &\quad = X_{96}^{(4)} + (\rho_1)_2 x \sum_{n_1=1}^{n} \frac{1}{(\omega_1 - n_1)(\omega_1 + 1 - n_1)} X_{96}^{(4)}(\rho_1 + 2, \omega_1 + 2 - n_1) \\ &\quad\quad + (\rho_2)_2 z \sum_{n_1=1}^{n} \frac{1}{(\omega_1 - n_1)(\omega_1 + 1 - n_1)} X_{96}^{(4)}(\rho_2 + 2, \omega_1 + 2 - n_1), \end{aligned} \tag{2.45}$$

$$\begin{aligned} &X_{96}^{(4)}(\omega_2 - n) \\ &\quad = X_{96}^{(4)} + \rho_1 \rho_2 y \sum_{n_1=1}^{n} \frac{1}{(\omega_2 - n_1)(\omega_2 + 1 - n_1)} X_{96}^{(4)}(\rho_1 + 1, \rho_2 + 1, \omega_2 + 2 - n_1) \\ &\quad\quad + \rho_3 \rho_4 u \sum_{n_1=1}^{n} \frac{1}{(\omega_2 - n_1)(\omega_2 + 1 - n_1)} X_{96}^{(4)}(\rho_3 + 1, \rho_4 + 1, \omega_2 + 2 - n_1). \end{aligned} \tag{2.46}$$

Theorem 2.7 *The following recursion formulas hold true for the numerator parameters* ρ_1, ρ_2, ρ_3, ρ_4 *of* $X_{97}^{(4)}$:

$$\begin{aligned} X_{97}^{(4)}(\rho_1 + n) = {}& X_{97}^{(4)} + \frac{2x}{\omega_1} \sum_{n_1=1}^{n} (\rho_1 + n_1) X_{97}^{(4)}(\rho_1 + 1 + n_1, \omega_1 + 1) \\ & + \frac{y\rho_2}{\omega_1} \sum_{n_1=1}^{n} X_{97}^{(4)}(\rho_1 + n_1, \rho_2 + 1, \omega_1 + 1), \end{aligned} \tag{2.47}$$

$$\begin{aligned} X_{97}^{(4)}(\rho_1 - n) = {}& X_{97}^{(4)} - \frac{2x}{\omega_1} \sum_{n_1=1}^{n} (\rho_1 + 1 - n_1) X_{97}^{(4)}(\rho_1 + 2 - n_1, \omega_1 + 1) \\ & - \frac{y\rho_2}{\omega_1} \sum_{n_1=1}^{n} X_{97}^{(4)}(\rho_1 + 1 - n_1, \rho_2 + 1, \omega_1 + 1), \end{aligned} \tag{2.48}$$

$$\begin{aligned} X_{97}^{(4)}(\rho_2 + n) = {}& X_{97}^{(4)} + \frac{2z}{\omega_2} \sum_{n_1=1}^{n} (\rho_2 + n_1) X_{97}^{(4)}(\rho_2 + 1 + n_1, \omega_2 + 1) \\ & + \frac{y\rho_1}{\omega_1} \sum_{n_1=1}^{n} X_{97}^{(4)}(\rho_1 + 1, \rho_2 + n_1, \omega_1 + 1), \end{aligned} \tag{2.49}$$

$$\begin{aligned} X_{97}^{(4)}(\rho_2 - n) = {}& X_{97}^{(4)} - \frac{2z}{\omega_2} \sum_{n_1=1}^{n} (\rho_2 + 1 - n_1) X_{97}^{(4)}(\rho_2 + 2 - n_1, \omega_2 + 1) \\ & - \frac{y\rho_1}{\omega_1} \sum_{n_1=1}^{n} X_{97}^{(4)}(\rho_1 + 1, \rho_2 + 1 - n_1, \omega_1 + 1), \end{aligned} \tag{2.50}$$

$$X_{97}^{(4)}(\rho_3+n)=X_{97}^{(4)}+\frac{u\rho_4}{\omega_1}\sum_{n_1=1}^{n}X_{97}^{(4)}(\rho_4+1,\rho_3+n_1,\omega_1+1), \tag{2.51}$$

$$X_{97}^{(4)}(\rho_3-n)=X_{97}^{(4)}-\frac{u\rho_4}{\omega_1}\sum_{n_1=1}^{n}X_{97}^{(4)}(\rho_4+1,\rho_3+1-n_1,\omega_1+1), \tag{2.52}$$

$$X_{97}^{(4)}(\rho_4+n)=X_{97}^{(4)}+\frac{u\rho_3}{\omega_1}\sum_{n_1=1}^{n}X_{97}^{(4)}(\rho_3+1,\rho_4+n_1,\omega_1+1), \tag{2.53}$$

$$X_{97}^{(4)}(\rho_4-n)=X_{97}^{(4)}-\frac{u\rho_3}{\omega_1}\sum_{n_1=1}^{n}X_{97}^{(4)}(\rho_3+1,\rho_4+1-n_1,\omega_1+1). \tag{2.54}$$

Theorem 2.8 *The following recursion formulas hold true for the numerator parameters* ρ_3, ρ_4 *of the* $X_{97}^{(4)}$:

$$X_{97}^{(4)}(\rho_3+n)=\sum_{n_1\leq n}\binom{n}{n_1}\frac{(\rho_4)_{n_1}u^{n_1}}{(\omega_1)_{n_1}}X_{97}^{(4)}(\rho_3+n_1,\rho_4+n_1,\omega_1+n_1), \tag{2.55}$$

$$X_{97}^{(4)}(\rho_3-n)=\sum_{n_1\leq n}\binom{n}{n_1}\frac{(\rho_4)_{n_1}(-u)^{n_1}}{(\omega_1)_{n_1}}X_{97}^{(4)}(\rho_4+n_1,\omega_1+n_1), \tag{2.56}$$

$$X_{97}^{(4)}(\rho_4+n)=\sum_{n_1\leq n}\binom{n}{n_1}\frac{(\rho_3)_{n_1}u^{n_1}}{(\omega_1)_{n_1}}X_{97}^{(4)}(\rho_3+n_1,\rho_4+n_1,\omega_1+n_1), \tag{2.57}$$

$$X_{97}^{(4)}(\rho_4-n)=\sum_{n_1\leq n}\binom{n}{n_1}\frac{(\rho_3)_{n_1}(-u)^{n_1}}{(\omega_1)_{n_1}}X_{97}^{(4)}(\rho_3+n_1,\omega_1+n_1). \tag{2.58}$$

Theorem 2.9 *The following recursion formulas hold true for the denominator parameters* ω_1, ω_2 *of* $X_{97}^{(4)}$:

$$\begin{aligned}
&X_{97}^{(4)}(\omega_1-n)\\
&\quad=X_{97}^{(4)}+(\rho_1)_2x\sum_{n_1=1}^{n}\frac{1}{(\omega_1-n_1)(\omega_1+1-n_1)}X_{97}^{(4)}(\rho_1+2,\omega_1+2-n_1)\\
&\qquad+\rho_1\rho_2y\sum_{n_1=1}^{n}\frac{1}{(\omega_1-n_1)(\omega_1+1-n_1)}X_{97}^{(4)}(\rho_1+1,\rho_2+1,\omega_1+2-n_1)\\
&\qquad+\rho_3\rho_4u\sum_{n_1=1}^{n}\frac{1}{(\omega_1-n_1)(\omega_1+1-n_1)}X_{97}^{(4)}(\rho_3+1,\rho_4+1,\omega_1+2-n_1),
\end{aligned} \tag{2.59}$$

$$\begin{aligned}
&X_{97}^{(4)}(\omega_2-n)\\
&\quad=X_{97}^{(4)}+(\rho_2)_2z\sum_{n_1=1}^{n}\frac{1}{(\omega_2-n_1)(\omega_2+1-n_1)}X_{97}^{(4)}(\rho_2+2,\omega_2+2-n_1).
\end{aligned} \tag{2.60}$$

Theorem 2.10 *The following recursion formulas hold true for the numerator parameters* $\rho_1, \rho_2, \rho_3, \rho_4$ *of* $X_{98}^{(4)}$:

$$X_{98}^{(4)}(\rho_1+n) = X_{98}^{(4)} + \frac{2x}{\omega_1}\sum_{n_1=1}^{n}(\rho_1+n_1)X_{98}^{(4)}(\rho_1+1+n_1,\omega_1+1) + \frac{y\rho_2}{\omega_2}\sum_{n_1=1}^{n}X_{98}^{(4)}(\rho_1+n_1,\rho_2+1,\omega_2+1), \tag{2.61}$$

$$X_{98}^{(4)}(\rho_1-n) = X_{98}^{(4)} - \frac{2x}{\omega_1}\sum_{n_1=1}^{n}(\rho_1+1-n_1)X_{98}^{(4)}(\rho_1+2-n_1,\omega_1+1) - \frac{y\rho_2}{\omega_2}\sum_{n_1=1}^{n}X_{98}^{(4)}(\rho_1+1-n_1,\rho_2+1,\omega_2+1), \tag{2.62}$$

$$X_{98}^{(4)}(\rho_2+n) = X_{98}^{(4)} + \frac{2z}{\omega_1}\sum_{n_1=1}^{n}(\rho_2+n_1)X_{98}^{(4)}(\rho_2+1+n_1,\omega_1+1) + \frac{y\rho_1}{\omega_2}\sum_{n_1=1}^{n}X_{98}^{(4)}(\rho_1+1,\rho_2+n_1,\omega_2+1), \tag{2.63}$$

$$X_{98}^{(4)}(\rho_2-n) = X_{98}^{(4)} - \frac{2z}{\omega_1}\sum_{n_1=1}^{n}(\rho_2+1-n_1)X_{98}^{(4)}(\rho_2+2-n_1,\omega_1+1) - \frac{y\rho_1}{\omega_2}\sum_{n_1=1}^{n}X_{98}^{(4)}(\rho_1+1,\rho_2+1-n_1,\omega_2+1), \tag{2.64}$$

$$X_{98}^{(4)}(\rho_3+n) = X_{98}^{(4)} + \frac{u\rho_4}{\omega_1}\sum_{n_1=1}^{n}X_{98}^{(4)}(\rho_4+1,\rho_3+n_1,\omega_1+1), \tag{2.65}$$

$$X_{98}^{(4)}(\rho_3-n) = X_{98}^{(4)} - \frac{u\rho_4}{\omega_1}\sum_{n_1=1}^{n}X_{98}^{(4)}(\rho_4+1,\rho_3+1-n_1,\omega_1+1), \tag{2.66}$$

$$X_{98}^{(4)}(\rho_4+n) = X_{98}^{(4)} + \frac{u\rho_3}{\omega_1}\sum_{n_1=1}^{n}X_{98}^{(4)}(\rho_3+1,\rho_4+n_1,\omega_1+1), \tag{2.67}$$

$$X_{98}^{(4)}(\rho_4-n) = X_{98}^{(4)} - \frac{u\rho_3}{\omega_1}\sum_{n_1=1}^{n}X_{98}^{(4)}(\rho_3+1,\rho_4+1-n_1,\omega_1+1). \tag{2.68}$$

Theorem 2.11 *The following recursion formulas hold true for the numerator parameters* ρ_3, ρ_4 *of* $X_{98}^{(4)}$:

$$X_{98}^{(4)}(\rho_3+n) = \sum_{n_1\le n}\binom{n}{n_1}\frac{(\rho_4)_{n_1}u^{n_1}}{(\omega_1)_{n_1}}X_{98}^{(4)}(\rho_3+n_1,\rho_4+n_1,\omega_1+n_1), \tag{2.69}$$

$$X_{98}^{(4)}(\rho_3-n) = \sum_{n_1\le n}\binom{n}{n_1}\frac{(\rho_4)_{n_1}(-u)^{n_1}}{(\omega_1)_{n_1}}X_{98}^{(4)}(\rho_4+n_1,\omega_1+n_1), \tag{2.70}$$

$$X_{98}^{(4)}(\rho_4+n) = \sum_{n_1\le n}\binom{n}{n_1}\frac{(\rho_3)_{n_1}u^{n_1}}{(\omega_1)_{n_1}}X_{98}^{(4)}(\rho_3+n_1,\rho_4+n_1,\omega_1+n_1), \tag{2.71}$$

$$X_{98}^{(4)}(\rho_4 - n) = \sum_{n_1 \leq n} \binom{n}{n_1} \frac{(\rho_3)_{n_1}(-u)^{n_1}}{(\omega_1)_{n_1}} X_{98}^{(4)}(\rho_3 + n_1, \omega_1 + n_1). \tag{2.72}$$

Theorem 2.12 *The following recursion formulas hold true for the denominator parameters ω_1, ω_2 of $X_{98}^{(4)}$:*

$$\begin{aligned} X_{98}^{(4)}&(\omega_1 - n) \\ &= X_{98}^{(4)} + (\rho_1)_2 x \sum_{n_1=1}^{n} \frac{1}{(\omega_1 - n_1)(\omega_1 + 1 - n_1)} X_{98}^{(4)}(\rho_1 + 2, \omega_1 + 2 - n_1) \\ &\quad + (\rho_2)_2 z \sum_{n_1=1}^{n} \frac{1}{(\omega_1 - n_1)(\omega_1 + 1 - n_1)} X_{98}^{(4)}(\rho_2 + 2, \omega_1 + 2 - n_1) \\ &\quad + \rho_3\rho_4 u \sum_{n_1=1}^{n} \frac{1}{(\omega_1 - n_1)(\omega_1 + 1 - n_1)} X_{98}^{(4)}(\rho_3 + 1, \rho_4 + 1, \omega_1 + 2 - n_1), \end{aligned} \tag{2.73}$$

$$\begin{aligned} X_{98}^{(4)}&(\omega_2 - n) \\ &= X_{98}^{(4)} + \rho_1\rho_2 y \sum_{n_1=1}^{n} \frac{1}{(\omega_2 - n_1)(\omega_2 + 1 - n_1)} X_{98}^{(4)}(\rho_1 + 1, \rho_2 + 1, \omega_2 + 2 - n_1). \end{aligned} \tag{2.74}$$

Theorem 2.13 *The following recursion formulas hold true for the numerator parameters ρ_1, ρ_2, ρ_3, ρ_4 of $X_{99}^{(4)}$:*

$$\begin{aligned} X_{99}^{(4)}(\rho_1 + n) = X_{99}^{(4)} &+ \frac{2x}{c} \sum_{n_1=1}^{n} (\rho_1 + n_1) X_{99}^{(4)}(\rho_1 + 1 + n_1, c + 1) \\ &+ \frac{y\rho_2}{c} \sum_{n_1=1}^{n} X_{99}^{(4)}(\rho_1 + n_1, \rho_2 + 1, c + 1), \end{aligned} \tag{2.75}$$

$$\begin{aligned} X_{99}^{(4)}(\rho_1 - n) = X_{99}^{(4)} &- \frac{2x}{c} \sum_{n_1=1}^{n} (\rho_1 + 1 - n_1) X_{99}^{(4)}(\rho_1 + 2 - n_1, c + 1) \\ &- \frac{y\rho_2}{c} \sum_{n_1=1}^{n} X_{99}^{(4)}(\rho_1 + 1 - n_1, \rho_2 + 1, c + 1), \end{aligned} \tag{2.76}$$

$$\begin{aligned} X_{99}^{(4)}(\rho_2 + n) = X_{99}^{(4)} &+ \frac{2z}{c} \sum_{n_1=1}^{n} (\rho_2 + n_1) X_{99}^{(4)}(\rho_2 + 1 + n_1, c + 1) \\ &+ \frac{y\rho_1}{c} \sum_{n_1=1}^{n} X_{99}^{(4)}(\rho_1 + 1, \rho_2 + n_1, c + 1), \end{aligned} \tag{2.77}$$

$$\begin{aligned} X_{99}^{(4)}(\rho_2 - n) = X_{99}^{(4)} &- \frac{2z}{c} \sum_{n_1=1}^{n} (\rho_2 + 1 - n_1) X_{99}^{(4)}(\rho_2 + 2 - n_1, c + 1) \\ &- \frac{y\rho_1}{c} \sum_{n_1=1}^{n} X_{99}^{(4)}(\rho_1 + 1, \rho_2 + 1 - n_1, c + 1), \end{aligned} \tag{2.78}$$

$$X_{99}^{(4)}(\rho_3 + n) = X_{99}^{(4)} + \frac{u\rho_4}{c} \sum_{n_1=1}^{n} X_{99}^{(4)}(\rho_4 + 1, \rho_3 + n_1, c + 1), \tag{2.79}$$

$$X_{99}^{(4)}(\rho_3-n)=X_{99}^{(4)}-\frac{u\rho_4}{c}\sum_{n_1=1}^{n}X_{99}^{(4)}(\rho_4+1,\rho_3+1-n_1,c+1),\tag{2.80}$$

$$X_{99}^{(4)}(\rho_4+n)=X_{99}^{(4)}+\frac{u\rho_3}{c}\sum_{n_1=1}^{n}X_{99}^{(4)}(\rho_3+1,\rho_4+n_1,c+1),\tag{2.81}$$

$$X_{99}^{(4)}(\rho_4-n)=X_{99}^{(4)}-\frac{u\rho_3}{c}\sum_{n_1=1}^{n}X_{99}^{(4)}(\rho_3+1,\rho_4+1-n_1,c+1).\tag{2.82}$$

Theorem 2.14 *The following recursion formulas hold true for the numerator parameters ρ_3, ρ_4 of $X_{99}^{(4)}$:*

$$X_{99}^{(4)}(\rho_3+n)=\sum_{n_1\le n}\binom{n}{n_1}\frac{(\rho_4)_{n_1}u^{n_1}}{(c)_{n_1}}X_{99}^{(4)}(\rho_3+n_1,\rho_4+n_1,c+n_1),\tag{2.83}$$

$$X_{99}^{(4)}(\rho_3-n)=\sum_{n_1\le n}\binom{n}{n_1}\frac{(\rho_4)_{n_1}(-u)^{n_1}}{(c)_{n_1}}X_{99}^{(4)}(\rho_4+n_1,c+n_1),\tag{2.84}$$

$$X_{99}^{(4)}(\rho_4+n)=\sum_{n_1\le n}\binom{n}{n_1}\frac{(\rho_3)_{n_1}u^{n_1}}{(c)_{n_1}}X_{99}^{(4)}(\rho_3+n_1,\rho_4+n_1,c+n_1),\tag{2.85}$$

$$X_{99}^{(4)}(\rho_4-n)=\sum_{n_1\le n}\binom{n}{n_1}\frac{(\rho_3)_{n_1}(-u)^{n_1}}{(c)_{n_1}}X_{99}^{(4)}(\rho_3+n_1,c+n_1).\tag{2.86}$$

Theorem 2.15 *The following recursion formula holds true for the denominator parameter c of $X_{99}^{(4)}$:*

$$\begin{aligned}
&X_{99}^{(4)}(\omega_1-n)\\
&\quad=X_{99}^{(4)}+(\rho_1)_2x\sum_{n_1=1}^{n}\frac{1}{(c-n_1)(c+1-n_1)}X_{99}^{(4)}(\rho_1+2,c+2-n_1)\\
&\qquad+\rho_1\rho_2y\sum_{n_1=1}^{n}\frac{1}{(c-n_1)(c+1-n_1)}X_{99}^{(4)}(\rho_1+1,\rho_2+1,c+2-n_1)\\
&\qquad+(\rho_2)_2z\sum_{n_1=1}^{n}\frac{1}{(c-n_1)(c+1-n_1)}X_{99}^{(4)}(\rho_2+2,c+2-n_1)\\
&\qquad+\rho_3\rho_4u\sum_{n_1=1}^{n}\frac{1}{(c-n_1)(c+1-n_1)}X_{99}^{(4)}(\rho_3+1,\rho_4+1,c+2-n_1).
\end{aligned}\tag{2.87}$$

3 Conclusion

This paper presented recursion formulas for some hypergeometric functions of four variables. Also, some interesting particular cases and consequences of our results have been discussed. New structures of hypergeometric functions of four variables emerge with wide possibilities of application in physics and engineering within such a context. Therefore, the results of this work are various, significant, and it would be interesting and possible to develop this study in the future.

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