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A generalized neutral-type inclusion problem in the frame of the generalized Caputo fractional derivatives

Adel Lachouri[1], Mohammed S. Abdo[2], Abdelouaheb Ardjouni[3], Sina Etemad[4] and Shahram Rezapour[4,5*]

*Correspondence:
sh.rezapour@azaruniv.ac.ir;
sh.rezapour@mail.cmuh.org.tw;
rezapourshahram@yahoo.ca
[4]Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran
[5]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan
Full list of author information is available at the end of the article

Abstract
In this paper, we study the existence of solutions for a generalized sequential Caputo-type fractional neutral differential inclusion with generalized integral conditions. The used fractional operator has the generalized kernel in the format of $(\vartheta(t) - \vartheta(s))$ along with differential operator $\frac{1}{\vartheta'(t)}\frac{\mathrm{d}}{\mathrm{d}t}$. We obtain existence results for two cases of convex-valued and nonconvex-valued multifunctions in two separated sections. We derive our findings by means of the fixed point principles in the context of the set-valued analysis. We give two suitable examples to validate the theoretical results.

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1 Introduction

Fractional differential inclusions as a generalization of fractional differential equations are established to be of considerable interest and value in optimizations and stochastic processes [1]. Fractional differential inclusions additionally help us study dynamical systems in which speeds are not remarkably specific by the condition of the system, regardless of relying upon it. In recent periods the theory of fractional differential equations has gained a lot of interest in all areas of mathematics; see [2–4]. Also, fractional differential equations and fractional differential inclusions appear naturally in a variety of scientific fields and have a wide range of applications; see [5–8]. Almeida [9] introduced a new operator called the ψ-Caputo fractional derivative combining a fractional operator with other different types of fractional derivatives and thus opened a new window to modern and complicated applications.

Throughout the years, many researchers have been interested in discussing the existence of solutions for fractional differential equations and fractional differential inclusions involving various types of fractional derivatives; see [10–34].

In 2018, Asawasamrit et al. [35] studied the following class of fractional differential equations involving the Hilfer fractional derivatives:

$$\begin{cases} {}^{H}D^{b_1,b_2}\varphi(\mathfrak{t}) = h(\mathfrak{t},\varphi(\mathfrak{t})), \quad \mathfrak{t}\in[\mathfrak{a},b], \\ \varphi(\mathfrak{a}) = 0, \qquad \varphi(b) = \sum_{i=1}^{m}\theta_i I_{\mathfrak{a}+}^{\lambda_i}\varphi(\sigma_i), \quad \sigma_i\in[\mathfrak{a},b], \end{cases} \tag{1}$$

where $b_1 \in (1,2)$, $b_2 \in [0,1]$, $\theta_i \in \mathbb{R}$, $\lambda_i > 0$, ${}^{H}D^{b_1,b_2}$ is the b_1-Hilfer fractional derivative of type b_2, $I_{\mathfrak{a}+}^{\lambda_i}$ is the λ_i-Riemann–Liouville fractional integral, and $h \in C([\mathfrak{a},b]\times\mathbb{R},\mathbb{R})$.

Mali and Kucche [36] discussed the existence and stability of ψ-Hilfer-type implicit BVP for given fractional differential equations (1), and then Wongcharoen et al. [37] studied the set-valued case of (1) in the same year. Adjimi et al. [38] used fixed point theorems to prove the uniqueness and existence of possible solutions to the generalized Caputo-type problem

$$\begin{cases} {}^{C}D_{0+}^{b_1,\vartheta}({}^{C}D_{0+}^{b_2,\vartheta}\varphi(\mathfrak{t}) - \mathcal{K}(\mathfrak{t},\varphi(\mathfrak{t}))) = \psi(\mathfrak{t},\varphi(\mathfrak{t})), \\ \varphi(\mathfrak{a}) = 0, \qquad I_{0+}^{b_3,\vartheta}\varphi(T) = 0, \quad \mathfrak{a}\in(0,T), \end{cases} \tag{2}$$

where $\mathfrak{t}\in[0,T)$, ${}^{C}D_{0+}^{\theta,\vartheta}$ is the ϑ-Caputo fractional derivative of order $\theta \in \{b_1, b_2 \in (0,1]\}$. $I_{0+}^{b_3,\vartheta}$ is the ϑ-Riemann–Liouville fractional integral of order $b_3 > 0$, and $\mathcal{K}, \psi \in C([0,T]\times\mathbb{R},\mathbb{R})$.

Motivated by the aforementioned works and inspired by [9], we prove the existence of solutions to the following nonlinear neutral fractional differential inclusion involving ϑ-Caputo fractional derivative with ϑ-Riemann–Liouville fractional integral boundary conditions:

$$\begin{cases} {}^{C}D_{0+}^{b_1,\vartheta}({}^{C}D_{0+}^{b_2;\vartheta}\varphi(\mathfrak{t}) - \mathcal{K}(\mathfrak{t},\varphi(\mathfrak{t}))) \in \mathcal{H}(\mathfrak{t},\varphi(\mathfrak{t})), \\ \varphi(\mathfrak{a}) = 0, \qquad I_{0+}^{b_3;\vartheta}\varphi(T) = 0, \quad \mathfrak{a}\in(0,T), \end{cases} \tag{3}$$

where $\mathfrak{t}\in[0,T)$, and $\mathcal{H} : [0,T]\times\mathbb{R} \to \mathfrak{P}(\mathbb{R})$ is a set-valued map from $[0,T]\times\mathbb{R}$ to the collection $\mathfrak{P}(\mathbb{R}) \subset \mathbb{R}$.

We obtain the desired results for the suggested ϑ-Caputo inclusion FBVP (3) involving convex and nonconvex set-valued maps using some well-known fixed point theorems. We also construct two examples to validate our results. Reported findings are new in the frame of the generalized sequential Caputo fractional derivatives implemented on a novel neutral-type generalized fractional differential inclusion.

Observe that our problem (3) involves a general structure and is reduced to an Erdelyi–Kober-type (and Hadamard-type) inclusion problem when we take $\vartheta(\mathfrak{t}) = \mathfrak{t}^{\eta}$ (and $\vartheta(\mathfrak{t}) = \log(\mathfrak{t})$, respectively). Moreover, problem (3) is more general than problem (2).

This paper is organized as follows. Some fundamentals ideas of fractional calculus and theory of multifunctions are presented in Sect. 2. The main results on the existence of solutions to the ϑ-Caputo inclusion problem (3) using some fixed point theorems are obtained in Sect. 3. Two examples are provided in Sect. 4. In the final section, we give conclusive remarks.

2 Preliminary notions

2.1 Fractional calculus

In this section, we present some basic concepts on fractional calculus and necessary lemmas.

Let $\mathfrak{J}_T = [0, T]$. By $\mathfrak{C} = C(\mathfrak{J}_T, \mathbb{R})$ we denote the Banach space of all continuous functions $z : \mathfrak{J}_T \to \mathbb{R}$ with the norm

$$\|z\| = \sup\{|z(\mathrm{t})| : \mathrm{t} \in \mathfrak{J}_T\},$$

and by $L^1(\mathfrak{J}_T, \mathbb{R})$ we denote the Banach space of Lebesgue-integrable functions $z : \mathfrak{J}_T \to \mathbb{R}$ with the norm

$$\|z\|_{L^1} = \int_{\mathfrak{J}_T} |z(\mathrm{t})| \, d\mathrm{t}.$$

Let $z : \mathfrak{J}_T \to \mathbb{R}$ be integrable, and let $\vartheta \in C^n(\mathfrak{J}_T, \mathbb{R})$ be increasing such that $\vartheta'(\mathrm{t}) \neq 0$ for all $\mathrm{t} \in \mathfrak{J}_T$.

Definition 2.1 ([39]) The b_1-ϑ-Riemann–Liouville integral of a function z is given by

$$I_{0+}^{b_1;\vartheta} z(\mathrm{t}) = \frac{1}{\Gamma(b_1)} \int_0^{\mathrm{t}} \vartheta'(\varsigma)\big(\vartheta(\mathrm{t}) - \vartheta(\varsigma)\big)^{b_1-1} z(\varsigma)\, d\varsigma, \quad b_1 > 0.$$

Definition 2.2 ([39]) The b_1-ϑ-Riemann–Liouville fractional derivative of a function z is defined by

$$D_{0+}^{b_1;\vartheta} z(\mathrm{t}) = \left(\frac{1}{\vartheta'(\mathrm{t})}\frac{d}{d\mathrm{t}}\right)^n I_{0+}^{(n-b_1);\vartheta} z(\mathrm{t}),$$

where $n = [b_1] + 1$.

Definition 2.3 ([9, 39, 40]) The ϑ-Caputo fractional derivative of a function $z \in AC^n(\mathfrak{J}_T, \mathbb{R})$ of order b_1 is defined by

$${}^C D_{0+}^{b_1;\vartheta} z(\mathrm{t}) = I_{0+}^{(n-b_1);\vartheta} z^{[n]}(\mathrm{t}),$$

where $z^{[n]}(\mathrm{t}) = (\frac{1}{\vartheta'(\mathrm{t})}\frac{d}{d\mathrm{t}})^n z(\mathrm{t})$, and $n = [b_1] + 1$, $n \in \mathbb{N}$.

Lemma 2.4 ([39, 40]) *Let $b_1, b_2, \mu > 0$. Then*
1) $I_{0+}^{b_1;\vartheta}(\vartheta(\varsigma) - \vartheta(0))^{b_2-1}(\mathrm{t}) = \frac{\Gamma(b_2)}{\Gamma(b_1+b_2)}(\vartheta(\mathrm{t}) - \vartheta(0))^{b_1+b_2-1}$,
2) ${}^C D_{0+}^{b_1;\vartheta}(\vartheta(\varsigma) - \vartheta(0))^{b_2-1}(\mathrm{t}) = \frac{\Gamma(b_2)}{\Gamma(b_2-b_1)}(\vartheta(\mathrm{t}) - \vartheta(0))^{b_2-b_1-1}$.

Lemma 2.5 ([39]) *If $z \in AC^n(\mathfrak{J}_T, \mathbb{R})$ and $b_1 \in (n-1, n)$, then*

$$I_{0+}^{b_1;\vartheta\,C} D_{0+}^{b_1;\vartheta}\, z(\mathrm{t}) = z(\mathrm{t}) - \sum_{k=0}^{n-1} \frac{z^{[n]}(0^+)}{k!}\big(\vartheta(\mathrm{t}) - \vartheta(0)\big)^k.$$

In particular, for $b_1 \in (0, 1)$, we have

$$I_{0+}^{b_1;\vartheta\,C} D_{0+}^{b_1;\vartheta}\, z(\mathrm{t}) = z(\mathrm{t}) - z(0).$$

Regarding problem (3), we indicate the following essential lemma, which was proven in [38].

Lemma 2.6 ([38]) *Let*

$$\begin{aligned} &\phi_T = \vartheta(T) - \vartheta(0), \qquad \phi_{\mathfrak{a}} = \vartheta(\mathfrak{a}) - \vartheta(0), \\ &\Lambda = \frac{\phi_T^{b_2}\phi_{\mathfrak{a}}^{b_3}}{\Gamma(b_2+1)\Gamma(b_3+1)} - \frac{\phi_T^{b_2+b_3}}{\Gamma(b_2+b_3+1)} \neq 0, \end{aligned} \tag{4}$$

and $\mathcal{K}, q \in \mathfrak{C}$. *Then the solution of linear-type problem*

$$\begin{cases} {}^C D_{0+}^{b_1,\vartheta}({}^C D_{0+}^{b_2;\vartheta}\varphi(\mathfrak{t}) - \mathcal{K}(\mathfrak{t})) = q(\mathfrak{t}), \mathfrak{t} \in [0,T), \\ \varphi(\mathfrak{a}) = 0, \qquad I_{0+}^{b_3;\vartheta}\varphi(T) = 0, \quad \mathfrak{a} \in (0,T), \end{cases} \tag{5}$$

is given by

$$\begin{aligned} \varphi(\mathfrak{t}) &= I_{0+}^{b_2;\vartheta}\mathcal{K}(\mathfrak{t}) + I_{0+}^{b_1+b_2;\vartheta}q(\mathfrak{t}) \\ &\quad + \frac{(\vartheta(\mathfrak{t}) - \vartheta(0))^{b_2}}{\Lambda\Gamma(b_2+1)}\Bigg[I_{0+}^{b_2+b_3;\vartheta}\mathcal{K}(T) + I_{0+}^{b_1+b_2+b_3;\vartheta}q(T) \\ &\quad - \frac{\phi_T^{b_3}}{\Gamma(b_3+1)}\big(I_{0+}^{b_2;\vartheta}\mathcal{K}(\mathfrak{a}) + I_{0+}^{b_1+b_2;\vartheta}q(\mathfrak{a})\big)\Bigg] \\ &\quad + \frac{1}{\Lambda}\Bigg[\frac{\phi_T^{b_2+b_3}}{\Gamma(b_2+b_3+1)}\big(I_{0+}^{b_2;\vartheta}\mathcal{K}(\mathfrak{a}) + I_{0+}^{b_1+b_2;\vartheta}q(\mathfrak{a})\big) \\ &\quad - \frac{\phi_{\mathfrak{a}}^{b_2}}{\Gamma(b_2+1)}\big(I_{0+}^{b_2+b_3;\vartheta}\mathcal{K}(T) + I_{0+}^{b_1+b_2+b_3;\vartheta}q(T)\big)\Bigg]. \end{aligned} \tag{6}$$

2.2 Multifunction theory

We present some concepts regarding the multifunctions (set-valued maps) [41]. For this aim, consider the Banach space $(\mathfrak{C}, \|\cdot\|)$ and $\mathfrak{S} : \mathfrak{C} \to \mathfrak{P}(\mathfrak{C})$ as a multifunction that:

(I) is closed-(convex-)valued if $\mathfrak{S}(\varphi)$ is a closed (convex) set for each $\varphi \in \mathfrak{C}$;

(II) is bounded if $\mathfrak{S}(\mathcal{B}) = \cup_{\varphi\in\mathcal{B}}\mathfrak{S}(\varphi)$ is bounded with respect to φ for any bounded set $\mathcal{B} \subset \mathfrak{C}$, that is,

$$\sup_{\varphi\in\mathcal{B}}\big\{\{\sup|f| : f \in \mathfrak{S}(\varphi)\}\big\} < \infty;$$

(III) is measurable whenever for each $\eta \in \mathbb{R}$, the function

$$\mathfrak{t} \mapsto d(\eta, \mathfrak{S}(\mathfrak{t})) = \inf\{|\eta - \lambda| : \lambda \in \mathfrak{S}(\mathfrak{t})\}$$

is measurable.

For other notions such as the complete continuity or upper semicontinuity (u.s.c.), see [41]. Furthermore, the set of selections of $\mathcal{H}$ is given by

$$\mathcal{R}_{\mathcal{H},\eta} = \big\{\omega \in L^1(\mathfrak{J}_T, \mathbb{R}) | \omega(\mathfrak{t}) \in \mathcal{H}(\mathfrak{t}, \eta) \quad \forall (\text{a.e.})\,\mathfrak{t} \in \mathfrak{J}_T\big\}.$$

Next, we define

$$\mathfrak{P}_{\delta}(\mathfrak{C}) = \{\mathcal{W} \in \mathfrak{P}(\mathfrak{C}) : \mathcal{W} \neq \emptyset \text{ and has property } \delta\},$$

where $\mathfrak{P}_{cl}$, $\mathfrak{P}_{c}$, $\mathfrak{P}_{b}$, and $\mathfrak{P}_{cp}$ denote the classes of all closed, convex, bounded, and compact sets in $\mathfrak{C}$.

Definition 2.7 ([42]) A multifunction $\mathcal{H} : \mathfrak{J}_T \times \mathbb{R} \to \mathfrak{P}(\mathbb{R})$ is Carathéodory if $\mathrm{t} \mapsto \mathcal{H}(\mathrm{t}, \varphi)$ is measurable for each $\varphi \in \mathbb{R}$, and $\varphi \to \mathcal{H}(\mathrm{t}, \varphi)$ is u.s.c. for almost all $\mathrm{t} \in \mathfrak{J}_T$.

Furthermore, $\mathcal{H}$ is called L^1-Carathéodory if for each $l > 0$, there is $k^* \in L^1(\mathfrak{J}_T, \mathbb{R}^+)$ such that

$$\|\mathcal{H}(\mathrm{t}, \varphi)\| = \sup\{|\omega| : \omega \in \mathcal{H}(\omega, \varphi)\} \leq k^*(\mathrm{t})$$

for every $\|k^*\| \leq l$ and for almost all $\mathrm{t} \in \mathfrak{J}_T$.

The forthcoming lemmas are required to attain the desired outcomes in the current research study.

Lemma 2.8 ([42]) *Let $\mathfrak{C}$ and S be two Banach spaces, and let*

$$Gb(\mathfrak{S}) = \{(\varphi, \phi) \in \mathfrak{C} \times S, \phi \in \mathfrak{S}(\varphi)\}$$

be the graph of $\mathfrak{S}$. If $\mathfrak{S} : \mathfrak{C} \to \mathfrak{P}_{cl}(S)$ is u.s.c. Then $Gb(\mathfrak{S})$ is closed in $\mathfrak{C} \times S$. Moreover, if $\mathfrak{S}$ is completely continuous and has a closed graph, then $\mathfrak{S}$ is u.s.c.

Lemma 2.9 ([43]) *Let $\mathfrak{C}$ be a separable Banach space, let $\mathcal{H} : \mathfrak{J}_T \times \mathfrak{C} \to \mathfrak{P}_{cp,c}(\mathfrak{C})$ be L^1-Carathéodory, and let $\mathcal{Z} : L^1(\mathfrak{J}_T, \mathfrak{C}) \to C(\mathfrak{J}_T, \mathfrak{C})$ be linear and continuous. Then*

$$\mathcal{Z} \circ \mathcal{R}_{\mathcal{H}} : C(\mathfrak{J}_T, \mathfrak{C}) \to \mathfrak{P}_{cp,c}(C(\mathfrak{J}_T, \mathfrak{C})), \quad \varphi \mapsto (\mathcal{Z} \circ \mathcal{R}_{\mathcal{H}})(\varphi) = \mathcal{Z}(\mathcal{R}_{\mathcal{H},\varphi}),$$

is a map with closed graph in $C(\mathfrak{J}_T, \mathfrak{C}) \times C(\mathfrak{J}_T, \mathfrak{C})$.

Theorem 2.10 (Nonlinear alternative for contractive maps [42]) *Let $\mathfrak{C}$ be a Banach space, and let $\mathcal{D}$ be a bounded neighborhood of $0 \in \mathfrak{C}$. Let $\Phi_1 : \mathfrak{C} \to \mathfrak{P}_{cp,c}(\mathfrak{C})$ and $\Phi_2 : \overline{\mathcal{D}} \to \mathfrak{P}_{cp,c}(\mathfrak{C})$ be two set-valued operators satisfying*:

(i) Φ_1 is a contraction, and

(ii) Φ_2 is u.s.c. and compact.

If $\tilde{S} = \Phi_1 + \Phi_2$, then either

(a) $\tilde{S}$ has a fixed-point in $\overline{\mathcal{D}}$, or

(b) there exist $\varphi \in \partial\mathcal{D}$ and $\mu \in (0,1)$ such that $\varphi \in \mu\tilde{S}(\varphi)$.

Theorem 2.11 (Nadler–Covitz fixed point theorem [44]) *Let $\mathfrak{C}$ be a complete metric space. If $\mathcal{H} : \mathfrak{C} \to \mathfrak{P}_{cl}(\mathfrak{C})$ is a contraction, then $\mathcal{H}$ has a fixed point.*

3 Existence results for set-valued problems

In this section, we establish the main existence theorems.

Definition 3.1 The function $\varphi \in C^1(\mathfrak{J}_T,\mathbb{R})$ is a solution of (3) if there is $\omega \in L^1(\mathfrak{J}_T,\mathbb{R})$ such that $\omega(\mathfrak{t}) \in \mathcal{H}(\mathfrak{t},\varphi)$ for every $\mathfrak{t} \in \mathfrak{J}_T$ satisfying the generalized integral boundary conditions

$$\varphi(\mathfrak{a}) = 0, \qquad I_{0+}^{b_3;\vartheta}\varphi(T) = 0, \quad \mathfrak{a} \in (0,T),$$

and

$$\begin{aligned}
\varphi(\mathfrak{t}) &= I_{0+}^{b_2;\vartheta}\mathcal{K}\big(\mathfrak{t},\varphi(\mathfrak{t})\big) + I_{0+}^{b_1+b_2;\vartheta}\omega(\mathfrak{t}) + \frac{(\vartheta(\mathfrak{t})-\vartheta(0))^{b_2}}{\Lambda\Gamma(b_2+1)} \\
&\quad \times \bigg[I_{0+}^{b_2+b_3;\vartheta}\mathcal{K}\big(T,\varphi(T)\big) + I_{0+}^{b_1+b_2+b_3;\vartheta}\omega(T) \\
&\quad - \frac{\phi_T^{b_3}}{\Gamma(b_3+1)}\big(I_{0+}^{b_2;\vartheta}\mathcal{K}\big(\mathfrak{a},\varphi(\mathfrak{a})\big) + I_{0+}^{b_1+b_2;\vartheta}\omega(\mathfrak{a})\big)\bigg] \\
&\quad + \frac{1}{\Lambda}\bigg[\frac{\phi_T^{b_2+b_3}}{\Gamma(b_2+b_3+1)}\big(I_{0+}^{b_2;\vartheta}\mathcal{K}\big(\mathfrak{a},\varphi(\mathfrak{a})\big) + I_{0+}^{b_1+b_2;\vartheta}\omega(\mathfrak{a})\big) \\
&\quad - \frac{\phi_{\mathfrak{a}}^{b_2}}{\Gamma(b_2+1)}\big(I_{0+}^{b_2+b_3;\vartheta}\mathcal{K}\big(T,\varphi(T)\big) + I_{0+}^{b_1+b_2+b_3;\vartheta}\omega(T)\big)\bigg].
\end{aligned}$$

3.1 Case 1: convex-valued multifunctions

The first theorem deals with convex-valued multifunction $\mathcal{H}$ using the nonlinear alternative for contractive maps (Theorem 2.10). For convenience, we put

$$\begin{aligned}
\zeta_1 &= \frac{\phi_T^{b_1+b_2}}{\Gamma(b_1+b_2+1)} \\
&\quad + \frac{\phi_T^{b_2}}{|\Lambda|\Gamma(b_2+1)}\bigg[\frac{\phi_T^{b_1+b_2+b_3}}{\Gamma(b_1+b_2+b_3+1)} + \frac{\phi_T^{b_1+b_2+b_3}}{\Gamma(b_3+1)\Gamma(b_1+b_2+1)}\bigg] \\
&\quad + \frac{1}{|\Lambda|}\bigg[\frac{\phi_T^{b_1+2b_2+b_3}}{\Gamma(b_2+b_3+1)\Gamma(b_1+b_2+1)} + \frac{\phi_T^{b_1+2b_2+b_3}}{\Gamma(b_2+1)\Gamma(b_1+b_2+b_3+1)}\bigg], \\
\zeta_2 &= \frac{\phi_T^{b_2}}{\Gamma(b_2+1)} + \frac{\phi_T^{b_2}}{|\Lambda|\Gamma(b_2+1)}\bigg[\frac{\phi_T^{b_2+b_3}}{\Gamma(b_2+b_3+1)} + \frac{\phi_T^{b_2+b_3}}{\Gamma(b_3+1)\Gamma(b_2+1)}\bigg] \\
&\quad + \frac{1}{|\Lambda|}\bigg[\frac{\phi_T^{2b_2+b_3}}{\Gamma(b_2+b_3+1)\Gamma(b_2+1)} + \frac{\phi_T^{2b_2+b_3}}{\Gamma(b_2+1)\Gamma(b_2+b_3+1)}\bigg].
\end{aligned} \tag{7}$$

Theorem 3.2 *Suppose that*:

($Hyp1$) *The set-valued map* $\mathcal{H}:\mathfrak{J}_T\times\mathbb{R}\to\mathfrak{P}_{cp,c}(\mathbb{R})$ *is* L^1-*Carathéodory*;

($Hyp2$) *There exist* $\widetilde{R}_1 \in C(\mathfrak{J}_T,\mathbb{R}^+)$ *and a nondecreasing function* $\widetilde{R}_2 \in C((0,+\infty),(0,+\infty))$ *such that*

$$\big\|\mathcal{H}(\mathfrak{t},\varphi)\big\|_{\mathfrak{P}} = \sup\big\{|\eta| : \eta \in \mathcal{H}(\mathfrak{t},\varphi)\big\} \le \widetilde{R}_1(\mathfrak{t})\widetilde{R}_2\big(\|\varphi\|\big), \quad \forall(\mathfrak{t},\varphi) \in \mathfrak{J}_T\times\mathbb{R};$$

($Hyp3$) *There is a constant* $n_{\mathcal{K}} < \zeta_2^{-1}$ *such that*

$$\big|\mathcal{K}(\mathfrak{t},\varphi) - \mathcal{K}(\mathfrak{t},\overline{\varphi})\big| \le n_{\mathcal{K}}|\varphi - \overline{\varphi}|;$$

$(Hyp4)$ *There is* $\psi_{\mathcal{K}} \in C(\mathfrak{J}_T, \mathbb{R}^+)$ *such that*

$$\left|\mathcal{K}(\mathfrak{t},\varphi)\right| \leq \psi_{\mathcal{K}}(\mathfrak{t}), \quad \forall(\mathfrak{t},\varphi) \in \mathfrak{J}_T \times \mathbb{R};$$

$(Hyp5)$ *There is* $\mathcal{L} > 0$ *such that*

$$\frac{\mathcal{L}}{\zeta_1\|\widetilde{R}_1\|\widetilde{R}_2(\mathcal{L}) + \zeta_2\|\psi_{\mathcal{K}}\|} > 1. \tag{8}$$

Then (3) *has a solution on* $\mathfrak{J}_T$.

Proof First, to switch the neutral-type fractional differential inclusion (3) into a fixed-point problem, we define $\tilde{\mathcal{S}} : \mathfrak{C} \to \mathfrak{P}(\mathfrak{C})$ as

$$\tilde{\mathcal{S}}(\varphi) = \left\{ \begin{array}{l} \widetilde{\phi} \in \mathfrak{C} : \\ \widetilde{\phi}(\mathfrak{t}) = \begin{cases} I_{0+}^{b_2;\vartheta}\mathcal{K}(\mathfrak{t},\varphi(\mathfrak{t})) + I_{0+}^{b_1+b_2;\vartheta}\omega(\mathfrak{t}) \\ \quad + \frac{(\vartheta(\mathfrak{t})-\vartheta(0))^{b_2}}{\Lambda\Gamma(b_2+1)}[I_{0+}^{b_2+b_3;\vartheta}\mathcal{K}(T,\varphi(T)) + I_{0+}^{b_1+b_2+b_3;\vartheta}\omega(T) \\ \quad - \frac{\phi_T^{b_3}}{\Gamma(b_3+1)}(I_{0+}^{b_2;\vartheta}\mathcal{K}(\mathfrak{a},\varphi(\mathfrak{a})) + I_{0+}^{b_1+b_2;\vartheta}\omega(\mathfrak{a}))] \\ \quad + \frac{1}{\Lambda}[\frac{\phi_T^{b_2+b_3}}{\Gamma(b_2+b_3+1)}(I_{0+}^{b_2;\vartheta}\mathcal{K}(\mathfrak{a},\varphi(\mathfrak{a})) + I_{0+}^{b_1+b_2;\vartheta}\omega(\mathfrak{a})) \\ \quad - \frac{\phi_{\mathfrak{a}}^{b_2}}{\Gamma(b_2+1)}(I_{0+}^{b_2+b_3;\vartheta}\mathcal{K}(T,\varphi(T)) + I_{0+}^{b_1+b_2+b_3;\vartheta}\omega(T))]. \end{cases} \end{array} \right\} \tag{9}$$

for $\omega \in \mathcal{R}_{\mathcal{H},\varphi}$. Consider two operators $\Phi_1 : \mathfrak{C} \to \mathfrak{C}$ and $\Phi_2 : \mathfrak{C} \to \mathfrak{P}(\mathfrak{C})$ defined as

$$\begin{aligned} \Phi_1\varphi(\mathfrak{t}) = {} & I_{0+}^{b_2;\vartheta}\mathcal{K}\big(\mathfrak{t},\varphi(\mathfrak{t})\big) + \frac{(\vartheta(\mathfrak{t})-\vartheta(0))^{b_2}}{\Lambda\Gamma(b_2+1)} \\ & \times \left[I_{0+}^{b_2+b_3;\vartheta}\mathcal{K}\big(T,\varphi(T)\big) - \frac{\phi_T^{b_3}}{\Gamma(b_3+1)}I_{0+}^{b_2;\vartheta}\mathcal{K}\big(\mathfrak{a},\varphi(\mathfrak{a})\big)\right] \\ & + \frac{1}{\Lambda}\left[\frac{\phi_T^{b_2+b_3}}{\Gamma(b_2+b_3+1)}I_{0+}^{b_2;\vartheta}\mathcal{K}\big(\mathfrak{a},\varphi(\mathfrak{a})\big) - \frac{\phi_{\mathfrak{a}}^{b_2}}{\Gamma(b_2+1)}I_{0+}^{b_2+b_3;\vartheta}\mathcal{K}\big(T,\varphi(T)\big)\right], \end{aligned}$$

and

$$\Phi_2(\varphi) = \left\{ \begin{array}{l} \widetilde{\phi} \in \mathfrak{C} : \\ \widetilde{\phi}(\mathfrak{t}) = \begin{cases} I_{0+}^{b_1+b_2;\vartheta}\omega(\mathfrak{t}) \\ \quad + \frac{(\vartheta(\mathfrak{t})-\vartheta(0))^{b_2}}{\Lambda\Gamma(b_2+1)}[I_{0+}^{b_1+b_2+b_3;\vartheta}\omega(T) - \frac{\phi_T^{b_3}}{\Gamma(b_3+1)}I_{0+}^{b_1+b_2;\vartheta}\omega(\mathfrak{a})] \\ \quad + \frac{1}{\Lambda}[\frac{\phi_T^{b_2+b_3}}{\Gamma(b_2+b_3+1)}I_{0+}^{b_1+b_2;\vartheta}\omega(\mathfrak{a}) - \frac{\phi_{\mathfrak{a}}^{b_2}}{\Gamma(b_2+1)}I_{0+}^{b_1+b_2+b_3;\vartheta}\omega(T)] \end{cases} \end{array} \right\}.$$

Obviously, $\tilde{\mathcal{S}} = \Phi_1 + \Phi_2$. In what follows, we will show that the operators satisfy the hypotheses of the nonlinear alternative for contractive maps (Theorem 2.10). First, we define the bounded set

$$B_c = \{\varphi \in \mathfrak{C} : \|\varphi\| \leq c\}, \quad c > 0, \tag{10}$$

and show that Φ_1 and Φ_2 define the set-valued operators $\Phi_1, \Phi_2 : B_c \to \mathfrak{P}_{cp,c}(\mathfrak{C})$. To do this, we show that Φ_1 and Φ_2 are compact and convex-valued. We consider two steps.

Step 1. Φ_2 is bounded on bounded sets of $\mathfrak{C}$.
Let B_c be bounded in $\mathfrak{C}$. For $\widetilde{\phi} \in \Phi_2(\varphi)$ and $\varphi \in B_c$, there exists $\omega \in \mathcal{R}_{\mathcal{H},\varphi}$ such that

$$\begin{aligned}\widetilde{\phi}(\mathfrak{t}) &= I_{0+}^{b_1+b_2;\vartheta}\omega(\mathfrak{t}) + \frac{(\vartheta(\mathfrak{t})-\vartheta(0))^{b_2}}{\Lambda\Gamma(b_2+1)}\left[I_{0+}^{b_1+b_2+b_3;\vartheta}\omega(T) - \frac{\phi_T^{b_3}}{\Gamma(b_3+1)}I_{0+}^{b_1+b_2;\vartheta}\omega(\mathfrak{a})\right]\\ &\quad + \frac{1}{\Lambda}\left[\frac{\phi_T^{b_2+b_3}}{\Gamma(b_2+b_3+1)}I_{0+}^{b_1+b_2;\vartheta}\omega(\mathfrak{a}) - \frac{\phi_{\mathfrak{a}}^{b_2}}{\Gamma(b_2+1)}I_{0+}^{b_1+b_2+b_3;\vartheta}\omega(T)\right].\end{aligned}$$

Under assumption (Hyp2), for any $\mathfrak{t} \in \mathfrak{J}_T$, we have

$$\begin{aligned}\left|\widetilde{\phi}(\mathfrak{t})\right| &\le I_{0+}^{b_1+b_2;\vartheta}\left|\omega(\mathfrak{t})\right|\\ &\quad + \frac{(\vartheta(\mathfrak{t})-\vartheta(0))^{b_2}}{|\Lambda|\Gamma(b_2+1)}\left[I_{0+}^{b_1+b_2+b_3;\vartheta}\left|\omega(T)\right| + \frac{\phi_T^{b_3}}{\Gamma(b_3+1)}I_{0+}^{b_1+b_2;\vartheta}\left|\omega(\mathfrak{a})\right|\right]\\ &\quad + \frac{1}{|\Lambda|}\left[\frac{\phi_T^{b_2+b_3}}{\Gamma(b_2+b_3+1)}I_{0+}^{b_1+b_2;\vartheta}\left|\omega(\mathfrak{a})\right| + \frac{\phi_{\mathfrak{a}}^{b_2}}{\Gamma(b_2+1)}I_{0+}^{b_1+b_2+b_3;\vartheta}\left|\omega(T)\right|\right]\\ &\le \|\tilde{R}_1\|\tilde{R}_2(c)(T)\\ &\quad \times\Bigg(\frac{\phi_T^{b_1+b_2}}{\Gamma(b_1+b_2+1)}\\ &\quad + \frac{\phi_T^{b_2}}{|\Lambda|\Gamma(b_2+1)}\left[\frac{\phi_T^{b_1+b_2+b_3}}{\Gamma(b_1+b_2+b_3+1)} + \frac{\phi_T^{b_1+b_2+b_3}}{\Gamma(b_3+1)\Gamma(b_1+b_2+1)}\right]\\ &\quad + \frac{1}{|\Lambda|}\left[\frac{\phi_T^{b_1+2b_2+b_3}}{\Gamma(b_2+b_3+1)\Gamma(b_1+b_2+1)} + \frac{\phi_T^{b_1+2b_2+b_3}}{\Gamma(b_2+1)\Gamma(b_1+b_2+b_3+1)}\right]\Bigg).\end{aligned}$$

Thus

$$\|\widetilde{\phi}\| \le \zeta_1\|\widetilde{R}_1\|\widetilde{R}_2(c).$$

Step 2. Φ_2 maps bounded sets of $\mathfrak{C}$ into equicontinuous sets.
Let $\varphi \in B_c$ and $\widetilde{\phi} \in \Phi_2(\varphi)$. Then there is a function $\omega \in \mathcal{R}_{\mathcal{H},\varphi}$ such that

$$\begin{aligned}\widetilde{\phi}(\mathfrak{t}) &= I_{0+}^{b_1+b_2;\vartheta}\omega(\mathfrak{t}) + \frac{(\vartheta(\mathfrak{t})-\vartheta(0))^{b_2}}{\Lambda\Gamma(b_2+1)}\left[I_{0+}^{b_1+b_2+b_3;\vartheta}\omega(T) - \frac{\phi_T^{b_3}}{\Gamma(b_3+1)}I_{0+}^{b_1+b_2;\vartheta}\omega(\mathfrak{a})\right]\\ &\quad + \frac{1}{\Lambda}\left[\frac{\phi_T^{b_2+b_3}}{\Gamma(b_2+b_3+1)}I_{0+}^{b_1+b_2;\vartheta}\omega(\mathfrak{a}) - \frac{\phi_{\mathfrak{a}}^{b_2}}{\Gamma(b_2+1)}I_{0+}^{b_1+b_2+b_3;\vartheta}\omega(T)\right], \quad \mathfrak{t} \in \mathfrak{J}_T.\end{aligned}$$

Let $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathfrak{J}_T$ with $\mathfrak{t}_1 < \mathfrak{t}_2$. Then

$$\begin{aligned}&\left|\widetilde{\phi}(\mathfrak{t}_2) - \widetilde{\phi}(\mathfrak{t}_1)\right|\\ &\quad \le \frac{\|\tilde{R}_1\|\tilde{R}_2(c)}{\Gamma(b_1+b_2+1)}\left(\left(\vartheta(\mathfrak{t}_2)-\vartheta(0)\right)^{b_1+b_2} - \left(\vartheta(\mathfrak{t}_1)-\vartheta(0)\right)^{b_1+b_2}\right)\\ &\quad\quad \times \frac{(\vartheta(\mathfrak{t}_2)-\vartheta(0))^{b_2} - (\vartheta(\mathfrak{t}_1)-\vartheta(0))^{b_2}}{|\Lambda|\Gamma(b_2+1)}\left[I_{0+}^{b_1+b_2+b_3;\vartheta}\omega(T) + \frac{\phi_T^{b_3}}{\Gamma(b_3+1)}I_{0+}^{b_1+b_2;\vartheta}\omega(\mathfrak{a})\right].\end{aligned}$$

As $\mathrm{t}_1 \to \mathrm{t}_2$, we obtain

$$\left|\widetilde{\phi}(\mathrm{t}_2) - \widetilde{\phi}(\mathrm{t}_1)\right| \to 0.$$

Hence $\Phi_2(B_c)$ is equicontinuous. From steps 1–2, by the Arzelà–Ascoli theorem, Φ_2 is completely continuous.

Step 3. $\Phi_2(\varphi)$ is convex for every $\varphi \in \mathfrak{C}$.

Let $\widetilde{\phi}_1, \widetilde{\phi}_2 \in \Phi_2(\varphi)$. Then there exist $\omega_1, \omega_2 \in \mathcal{R}_{\mathcal{H},\varphi}$ such that for each $\mathrm{t} \in \mathfrak{J}_T$,

$$\begin{aligned}\widetilde{\phi}_j(\mathrm{t}) &= I_{0+}^{b_1+b_2;\vartheta}\omega_j(\mathrm{t}) + \frac{(\vartheta(\mathrm{t}) - \vartheta(0))^{b_2}}{\Lambda\Gamma(b_2+1)}\left[I_{0+}^{b_1+b_2+b_3;\vartheta}\omega_j(T) - \frac{\phi_T^{b_3}}{\Gamma(b_3+1)}I_{0+}^{b_1+b_2;\vartheta}\omega_j(\mathfrak{a})\right] \\ &\quad + \frac{1}{\Lambda}\left[\frac{\phi_T^{b_2+b_3}}{\Gamma(b_2+b_3+1)}I_{0+}^{b_1+b_2;\vartheta}\omega_j(\mathfrak{a}) - \frac{\phi_{\mathfrak{a}}^{b_2}}{\Gamma(b_2+1)}I_{0+}^{b_1+b_2+b_3;\vartheta}\omega_j(\mathfrak{T})\right], \quad j = 1, 2.\end{aligned}$$

Let $\sigma \in [0,1]$. Then, for each $\mathrm{t} \in \mathfrak{J}_T$, we write

$$\begin{aligned}\big(\sigma\widetilde{\phi}_1(\mathrm{t}) + (1-\sigma)\widetilde{\phi}_2(\mathrm{t})\big) &= I_{0+}^{b_1+b_2;\vartheta}\big(\sigma\omega_1(\mathrm{t}) + (1-\sigma)\omega_2(\mathrm{t})\big) \\ &\quad + \frac{(\vartheta(\mathrm{t}) - \vartheta(0))^{b_2}}{\Lambda\Gamma(b_2+1)}\bigg[I_{0+}^{b_1+b_2+b_3;\vartheta}\big(\sigma\omega_1(T) + (1-\sigma)\omega_2(T)\big) \\ &\quad - \frac{\phi_T^{b_3}}{\Gamma(b_3+1)}I_{0+}^{b_1+b_2;\vartheta}\big(\sigma\omega_1(\mathfrak{a}) + (1-\sigma)\omega_2(\mathfrak{a})\big)\bigg] \\ &\quad + \frac{1}{\Lambda}\bigg[\frac{\phi_T^{b_2+b_3}}{\Gamma(b_2+b_3+1)}I_{0+}^{b_1+b_2;\vartheta}\big(\sigma\omega_1(\mathfrak{a}) + (1-\sigma)\omega_2(\mathfrak{a})\big) \\ &\quad - \frac{\phi_{\mathfrak{a}}^{b_2}}{\Gamma(b_2+1)}I_{0+}^{b_1+b_2+b_3;\vartheta}\big(\sigma\omega_1(T) + (1-\sigma)\omega_2(T)\big)\bigg].\end{aligned}$$

Since $\mathcal{H}$ has convex values, $\mathcal{R}_{\mathcal{H},\varphi}$ is convex, and $[\sigma\omega_1(\mathrm{t}) + (1-\sigma)\omega_2(\mathrm{t})] \in \mathcal{R}_{\mathcal{H},\varphi}$. Thus $\sigma\widetilde{\phi}_1 + (1-\sigma)\widetilde{\phi}_2 \in \Phi_2(\varphi)$. In consequence, Φ_2 is convex-valued. Additionally, Φ_1 is compact and convex-valued.

Step 4. We check that the graph of Φ_2 is closed.

Let $\varphi_n \to \varphi_*$, $\widetilde{\phi}_n \in \Phi_2(\varphi_n)$, and $\widetilde{\phi}_n \to \widetilde{\phi}_*$. We prove that $\widetilde{\phi}_* \in \Phi_2(\varphi_*)$. Since $\widetilde{\phi}_n \in \Phi_2(\varphi_n)$, there exists $\omega_n \in \mathcal{R}_{\mathcal{H},\varphi_n}$ such that

$$\begin{aligned}\widetilde{\phi}_n(\mathrm{t}) &= I_{0+}^{b_1+b_2;\vartheta}\omega_n(\mathrm{t}) + \frac{(\vartheta(\mathrm{t}) - \vartheta(0))^{b_2}}{\Lambda\Gamma(b_2+1)}\left[I_{0+}^{b_1+b_2+b_3;\vartheta}\omega_n(T) - \frac{\phi_T^{b_3}}{\Gamma(b_3+1)}I_{0+}^{b_1+b_2;\vartheta}\omega_n(\mathfrak{a})\right] \\ &\quad + \frac{1}{\Lambda}\left[\frac{\phi_T^{b_2+b_3}}{\Gamma(b_2+b_3+1)}I_{0+}^{b_1+b_2;\vartheta}\omega_n(\mathfrak{a}) - \frac{\phi_{\mathfrak{a}}^{b_2}}{\Gamma(b_2+1)}I_{0+}^{b_1+b_2+b_3;\vartheta}\omega_n(\mathfrak{T})\right], \quad \mathrm{t} \in \mathfrak{J}_T.\end{aligned}$$

Therefore we have to show that there is $\omega_* \in \mathcal{R}_{\mathcal{H},\varphi_*}$ such that for each $\mathrm{t} \in \mathfrak{J}_T$,

$$\begin{aligned}\widetilde{\phi}_*(\mathrm{t}) &= I_{0+}^{b_1+b_2;\vartheta}\omega_*(\mathrm{t}) + \frac{(\vartheta(\mathrm{t}) - \vartheta(0))^{b_2}}{\Lambda\Gamma(b_2+1)}\left[I_{0+}^{b_1+b_2+b_3;\vartheta}\omega_*(T) - \frac{\phi_T^{b_3}}{\Gamma(b_3+1)}I_{0+}^{b_1+b_2;\vartheta}\omega_*(\mathfrak{a})\right] \\ &\quad + \frac{1}{\Lambda}\left[\frac{\phi_T^{b_2+b_3}}{\Gamma(b_2+b_3+1)}I_{0+}^{b_1+b_2;\vartheta}\omega_*(\mathfrak{a}) - \frac{\phi_{\mathfrak{a}}^{b_2}}{\Gamma(b_2+1)}I_{0+}^{b_1+b_2+b_3;\vartheta}\omega_*(T)\right], \quad \mathrm{t} \in \mathfrak{J}_T.\end{aligned}$$

Define the continuous linear operator $\mathcal{Z} : L^1(\mathfrak{J}_T, \mathbb{R}) \to C(\mathfrak{J}_T, \mathbb{R})$ by

$$\begin{aligned}
\omega \to \mathcal{Z}(\omega)(\mathfrak{t}) \\
&= I_{0+}^{b_1+b_2;\vartheta}\omega(\mathfrak{t}) \\
&\quad + \frac{(\vartheta(\mathfrak{t})-\vartheta(0))^{b_2}}{\Lambda\Gamma(b_2+1)}\left[I_{0+}^{b_1+b_2+b_3;\vartheta}\omega(T) - \frac{\phi_T^{b_3}}{\Gamma(b_3+1)}I_{0+}^{b_1+b_2;\vartheta}\omega(\mathfrak{a})\right] \\
&\quad + \frac{1}{\Lambda}\left[\frac{\phi_T^{b_2+b_3}}{\Gamma(b_2+b_3+1)}I_{0+}^{b_1+b_2;\vartheta}\omega(\mathfrak{a}) - \frac{\phi_{\mathfrak{a}}^{b_2}}{\Gamma(b_2+1)}I_{0+}^{b_1+b_2+b_3;\vartheta}\omega(T)\right], \quad \mathfrak{t} \in \mathfrak{J}_T.
\end{aligned}$$

Note that

$$\begin{aligned}
\|\widetilde{\phi}_n - \widetilde{\phi}_*\| = \Bigg\| & I_{0+}^{b_1+b_2;\vartheta}\big(\omega_n(\mathfrak{t}) - \omega_*(\mathfrak{t})\big) \\
&+ \frac{(\vartheta(\mathfrak{t})-\vartheta(0))^{b_2}}{\Lambda\Gamma(b_2+1)}\Bigg[I_{0+}^{b_1+b_2+b_3;\vartheta}\big(\omega_n(T)-\omega_*(T)\big) \\
&- \frac{\phi_T^{b_3}}{\Gamma(b_3+1)}I_{0+}^{b_1+b_2;\vartheta}\big(\omega_n(\mathfrak{a})-\omega_*(\mathfrak{a})\big)\Bigg] \\
&+ \frac{1}{\Lambda}\Bigg[\frac{\phi_T^{b_2+b_3}}{\Gamma(b_2+b_3+1)}I_{0+}^{b_1+b_2;\vartheta}\big(\omega_n(\mathfrak{a})-\omega_*(\mathfrak{a})\big) \\
&- \frac{\phi_{\mathfrak{a}}^{b_2}}{\Gamma(b_2+1)}I_{0+}^{b_1+b_2+b_3;\vartheta}\big(\omega_n(T)-\omega_*(T)\big)\Bigg] \to 0\Bigg\|
\end{aligned}$$

as $n \to \infty$. By Lemma 2.9, $\mathcal{Z} \circ \mathcal{R}_{\mathcal{H},\varphi}$ is a closed graph map. On the other hand,

$$\widetilde{\phi}_n \in \mathcal{Z}(\mathcal{R}_{\mathcal{H},\varphi_n}).$$

Since $\varphi_n \to \varphi_*$, Lemma 2.9 gives

$$\begin{aligned}
\widetilde{\phi}_*(\mathfrak{t}) = I_{0+}^{b_1+b_2;\vartheta}\omega_*(\mathfrak{t}) &+ \frac{(\vartheta(\mathfrak{t})-\vartheta(0))^{b_2}}{\Lambda\Gamma(b_2+1)}\left[I_{0+}^{b_1+b_2+b_3;\vartheta}\omega_*(T) - \frac{\phi_T^{b_3}}{\Gamma(b_3+1)}I_{0+}^{b_1+b_2;\vartheta}\omega_*(\mathfrak{a})\right] \\
&+ \frac{1}{\Lambda}\left[\frac{\phi_T^{b_2+b_3}}{\Gamma(b_2+b_3+1)}I_{0+}^{b_1+b_2;\vartheta}\omega_*(\mathfrak{a}) - \frac{\phi_{\mathfrak{a}}^{b_2}}{\Gamma(b_2+1)}I_{0+}^{b_1+b_2+b_3;\vartheta}\omega_*(T)\right]
\end{aligned}$$

for some $\omega_* \in \mathcal{R}_{\mathcal{H},\varphi_*}$. Thus Φ_2 has a closed graph, In consequence, Φ_2 is compact and u.s.c.

Step 5. Φ_1 is a contraction in $\mathfrak{C}$.

Let $\varphi, \overline{\varphi} \in \mathfrak{C}$. By the assumption (Hyp3) we get

$$\begin{aligned}
&\big|\Phi_1\varphi(\mathfrak{t}) - \Phi_1\overline{\varphi}(\mathfrak{t})\big| \\
&\quad \le \Bigg(\frac{\phi_T^{b_2}}{\Gamma(b_2+1)} + \frac{\phi_T^{b_2}}{|\Lambda|\Gamma(b_2+1)}\left[\frac{\phi_T^{b_2+b_3}}{\Gamma(b_2+b_3+1)} + \frac{\phi_T^{b_2+b_3}}{\Gamma(b_3+1)\Gamma(b_2+1)}\right] \\
&\qquad + \frac{1}{|\Lambda|}\left[\frac{\phi_T^{2b_2+b_3}}{\Gamma(b_2+b_3+1)\Gamma(b_2+1)} + \frac{\phi_T^{2b_2+b_3}}{\Gamma(b_2+1)\Gamma(b_2+b_3+1)}\right]\Bigg)n_{\mathcal{K}}\|\varphi - \overline{\varphi}\|.
\end{aligned}$$

Thus

$$\|\Phi_1\varphi - \Phi_1\overline{\varphi}\| \leq n_{\mathcal{K}}\zeta_2\|\varphi - \overline{\varphi}\|.$$

As $n_{\mathcal{K}}\zeta_2 < 1$, we infer that Φ_1 is a contraction.

Thus the operators Φ_1 and Φ_2 satisfy assumptions of Theorem 2.10. So, it yields that either condition (a) $\tilde{\mathcal{S}}$ has a fixed-point in $\overline{\mathcal{D}}$ or (b) there exist $\varphi \in \partial\mathcal{D}$ and $\mu \in (0,1)$ with $\varphi \in \mu\tilde{\mathcal{S}}(\varphi)$. We show that conclusion (b) is not possible. If $\varphi \in \mu\Phi_1(\varphi) + \mu\Phi_2(\varphi)$ for $\mu \in (0,1)$, then there is $\omega \in \mathcal{R}_{\mathcal{H},\varphi}$ such that

$$\begin{aligned}
\bigl|\varphi(\mathfrak{t})\bigr| &= \biggl|\mu I_{0+}^{b_2;\vartheta}\mathcal{K}\bigl(\mathfrak{t},\varphi(\mathfrak{t})\bigr) + \mu I_{0+}^{b_1+b_2;\vartheta}\omega(\mathfrak{t}) \\
&\quad + \frac{\mu(\vartheta(\mathfrak{t}) - \vartheta(0))^{b_2}}{\Lambda\Gamma(b_2+1)}\biggl[I_{0+}^{b_2+b_3;\vartheta}\mathcal{K}\bigl(T,\varphi(T)\bigr) + I_{0+}^{b_1+b_2+b_3;\vartheta}\omega(T) \\
&\quad - \frac{\phi_T^{b_3}}{\Gamma(b_3+1)}\bigl(I_{0+}^{b_2;\vartheta}\mathcal{K}\bigl(\mathfrak{a},\varphi(\mathfrak{a})\bigr) + I_{0+}^{b_1+b_2;\vartheta}\omega(\mathfrak{a})\bigr)\biggr] \\
&\quad + \frac{\mu}{\Lambda}\biggl[\frac{\phi_T^{b_2+b_3}}{\Gamma(b_2+b_3+1)}\bigl(I_{0+}^{b_2;\vartheta}\mathcal{K}\bigl(\mathfrak{a},\varphi(\mathfrak{a})\bigr) + I_{0+}^{b_1+b_2;\vartheta}\omega(\mathfrak{a})\bigr) \\
&\quad - \frac{\phi_{\mathfrak{a}}^{b_2}}{\Gamma(b_2+1)}\bigl(I_{0+}^{b_2+b_3;\vartheta}\mathcal{K}\bigl(T,\varphi(T)\bigr) + I_{0+}^{b_1+b_2+b_3;\vartheta}\omega(T)\bigr)\biggr]\biggr| \\
&\leq \zeta_1\|\widetilde{R}_1\|\widetilde{R}_2(\varphi) + \zeta_2\|\psi_{\mathcal{K}}\|.
\end{aligned}$$

Thus

$$\bigl|\varphi(\mathfrak{t})\bigr| \leq \zeta_1\|\widetilde{R}_1\|\widetilde{R}_2(\varphi) + \zeta_2\|\psi_{\mathcal{K}}\|, \quad \forall\mathfrak{t} \in \mathfrak{J}_T. \tag{11}$$

If condition (b) of Theorem 2.10 is true, then there are $\mu \in (0,1)$ and $\varphi \in \partial\mathcal{D}$ with $\varphi = \mu\tilde{\mathcal{S}}(\varphi)$. Then φ is a solution of (3) with $\|\varphi\| = \mathcal{L}$. Now by (11) we get

$$\frac{\mathcal{L}}{\zeta_1\|\widetilde{R}_1\|\widetilde{R}_2(\mathcal{L}) + \zeta_2\|\psi_{\mathcal{K}}\|} \leq 1,$$

contradicting to (8). Thus it follows from Theorem 2.10 that $\tilde{\mathcal{S}}$ has a fixed-point, which is a solution of (3), and the proof is completed. □

3.2 Case 2: nonconvex-valued multifunctions

In this section, we obtain another existence criterion for ϑ-Caputo fractional differential inclusion (3) under new assumptions. We will show our desired existence with a nonconvex-valued multifunction by using a theorem of Nadler and Covitz (Theorem 2.11).

Consider $(\mathfrak{C}, d)$ as a metric space. Consider $H^d : \mathfrak{P}(\mathfrak{C}) \times \mathfrak{P}(\mathfrak{C}) \to \mathbb{R}^+ \cup \{\infty\}$ defined by

$$H^d(\tilde{B}, \tilde{C}) = \max\Bigl\{\sup_{\tilde{b}\in\tilde{B}} d(\tilde{b}, \tilde{C}), \sup_{\tilde{c}\in\tilde{C}} d(\tilde{B}, \tilde{c})\Bigr\},$$

where $d(\tilde{B}, \tilde{c}) = \inf_{\tilde{b}\in\tilde{B}} d(\tilde{b}, \tilde{c})$ and $d(\tilde{b}, \tilde{C}) = \inf_{\tilde{c}\in\tilde{C}} d(\tilde{b}, \tilde{c})$. Then $(\mathfrak{P}_{b,cl}(\mathfrak{C}), \mathcal{H}^d)$ is a metric space (see [45]).

Definition 3.3 The multi-function $\tilde{\mathcal{S}} : \mathfrak{C} \to \mathfrak{P}_{cl}(\mathfrak{C})$ is a λ-Lipschitz if and only if there is $\lambda > 0$ such that

$$H^d\big(\tilde{\mathcal{S}}(\varphi), \tilde{\mathcal{S}}(\eta)\big) \le \lambda d(\varphi, \eta) \quad \text{for any } \varphi, \eta \in \mathfrak{C}.$$

In another case, if $\lambda < 1$, then $\tilde{\mathcal{S}}$ is a contraction.

Theorem 3.4 *Consider* (*Hyp*3) *and assume that*:

(*Hyp*6) $\mathcal{H} : \mathfrak{J}_T \times \mathbb{R} \to \mathfrak{P}_{cp}(\mathbb{R})$ *is such that* $\mathcal{H}(\cdot, \varphi) : \mathfrak{J}_T \to \mathfrak{P}_{cp}(\mathbb{R})$ *is measurable* $\forall \varphi \in \mathbb{R}$,

(*Hyp*7) $H^d(\mathcal{H}(\mathfrak{t}, \varphi), \mathcal{H}(\mathfrak{t}, \overline{\varphi})) \le \tilde{\mathfrak{m}}(\mathfrak{t})|\varphi - \overline{\varphi}|$ *for almost all* $\mathfrak{t} \in \mathfrak{J}_T$ *and* $\varphi, \overline{\varphi} \in \mathbb{R}$ *with* $\tilde{m} \in C(\mathfrak{J}_T, \mathbb{R}^+)$ *and* $d(0, \mathcal{H}(\mathfrak{t}, 0)) \le \tilde{m}(\mathfrak{t})$ *for almost all* $\mathfrak{t} \in \mathfrak{J}_T$.

Then the neutral-type fractional differential inclusion (3) *has one solution on* $\mathfrak{J}_T$ *if*

$$\|\tilde{\mathfrak{m}}\|\zeta_1 + n_{\mathcal{K}}\zeta_2 < 1,$$

where ζ_1, ζ_2 *are given in* (7).

Proof By virtue of assumption (Hyp6) and Theorem III.6 in [46], $\mathcal{H}$ has a measurable selection and thus, $\mathcal{R}_{\mathcal{H},\varphi} \neq \varnothing$. In the sequel, we prove that the operator $\tilde{\mathcal{S}} : \mathfrak{C} \to \mathfrak{P}(\mathfrak{C})$ defined in (9) satisfies the assumptions of Nadler and Covitz fixed-point theorem (Theorem 2.11). To Prove the closedness of $\tilde{\mathcal{S}}(\varphi)$ for all $\varphi \in \mathfrak{C}$, let $\{u_n\}_{n\ge 0} \in \tilde{\mathcal{S}}(\varphi)$ be such that $u_n \to u$ $(n \to \infty)$ in $\mathfrak{C}$. In such a case, $u \in \mathfrak{C}$ and there is $\omega_n \in \mathcal{R}_{\mathcal{H},\varphi_n}$ such that

$$\begin{aligned} u_n(\mathfrak{t}) &= I_{0+}^{b_2;\vartheta}\mathcal{K}\big(\mathfrak{t}, \varphi(\mathfrak{t})\big) + I_{0+}^{b_1+b_2;\vartheta}\omega_n(\mathfrak{t}) \\ &+ \frac{(\vartheta(\mathfrak{t}) - \vartheta(0))^{b_2}}{\Lambda\Gamma(b_2+1)}\Bigg[I_{0+}^{b_2+b_3;\vartheta}\mathcal{K}\big(T, \varphi(T)\big) + I_{0+}^{b_1+b_2+b_3;\vartheta}\omega_n(T) \\ &- \frac{\phi_T^{b_3}}{\Gamma(b_3+1)}\big(I_{0+}^{b_2;\vartheta}\mathcal{K}\big(\mathfrak{a}, \varphi(\mathfrak{a})\big) + I_{0+}^{b_1+b_2;\vartheta}\omega_n(\mathfrak{a})\big)\Bigg] \\ &+ \frac{1}{\Lambda}\Bigg[\frac{\phi_T^{b_2+b_3}}{\Gamma(b_2+b_3+1)}\big(I_{0+}^{b_2;\vartheta}\mathcal{K}\big(\mathfrak{a}, \varphi(\mathfrak{a})\big) + I_{0+}^{b_1+b_2;\vartheta}\omega_n(\mathfrak{a})\big) \\ &- \frac{\phi_{\mathfrak{a}}^{b_2}}{\Gamma(b_2+1)}\big(I_{0+}^{b_2+b_3;\vartheta}\mathcal{K}\big(T, \varphi(T)\big) + I_{0+}^{b_1+b_2+b_3;\vartheta}\omega_n(T)\big)\Bigg]. \end{aligned}$$

Accordingly, there exists a subsequence ω_n which converges to ω in $L^1(\mathfrak{J}_T, \mathbb{R})$, because $\mathcal{H}$ has compact values. As a result, $\omega \in \mathcal{R}_{\mathcal{H},\varphi}$ and we get

$$\begin{aligned} u_n(\mathfrak{t}) \to u(\mathfrak{t}) &= I_{0+}^{b_2;\vartheta}\mathcal{K}\big(\mathfrak{t}, \varphi(\mathfrak{t})\big) + I_{0+}^{b_1+b_2;\vartheta}\omega(\mathfrak{t}) \\ &+ \frac{(\vartheta(\mathfrak{t}) - \vartheta(0))^{b_2}}{\Lambda\Gamma(b_2+1)}\Bigg[I_{0+}^{b_2+b_3;\vartheta}\mathcal{K}\big(T, \varphi(T)\big) + I_{0+}^{b_1+b_2+b_3;\vartheta}\omega(T) \\ &- \frac{\phi_T^{b_3}}{\Gamma(b_3+1)}\big(I_{0+}^{b_2;\vartheta}\mathcal{K}\big(\mathfrak{a}, \varphi(\mathfrak{a})\big) + I_{0+}^{b_1+b_2;\vartheta}\omega(\mathfrak{a})\big)\Bigg] \\ &+ \frac{1}{\Lambda}\Bigg[\frac{\phi_T^{b_2+b_3}}{\Gamma(b_2+b_3+1)}\big(I_{0+}^{b_2;\vartheta}\mathcal{K}\big(\mathfrak{a}, \varphi(\mathfrak{a})\big) + I_{0+}^{b_1+b_2;\vartheta}\omega(\mathfrak{a})\big) \\ &- \frac{\phi_{\mathfrak{a}}^{b_2}}{\Gamma(b_2+1)}\big(I_{0+}^{b_2+b_3;\vartheta}\mathcal{K}\big(T, \varphi(T)\big) + I_{0+}^{b_1+b_2+b_3;\vartheta}\omega(T)\big)\Bigg]. \end{aligned}$$

Hence $u \in \tilde{\mathcal{S}}(\varphi)$.

Next, we show that there is a $\widetilde{\theta} \in (0,1)$, $(\widetilde{\theta} = \|\tilde{\mathfrak{m}}\|\zeta_1 + n_{\mathcal{K}}\zeta_2)$ such that

$$H^d\big(\tilde{\mathcal{S}}(\varphi), \tilde{\mathcal{S}}(\overline{\varphi})\big) \leq \widetilde{\theta}\|\varphi - \overline{\varphi}\| \quad \text{for each } \varphi, \overline{\varphi} \in \mathfrak{C}.$$

Let $\varphi, \overline{\varphi} \in \mathfrak{C}$ and $\widetilde{\phi}_1 \in \tilde{\mathcal{S}}(\varphi)$. There exists $\omega_1(\mathfrak{t}) \in \mathcal{H}(\mathfrak{t}, \varphi(\mathfrak{t}))$ provided that for each $\mathfrak{t} \in \mathfrak{J}_T$,

$$\begin{aligned}
\widetilde{\phi}_1(\mathfrak{t}) &= I_{0+}^{b_2;\vartheta}\mathcal{K}\big(\mathfrak{t}, \varphi(\mathfrak{t})\big) + I_{0+}^{b_1+b_2;\vartheta}\omega_1(\mathfrak{t}) \\
&\quad + \frac{(\vartheta(\mathfrak{t}) - \vartheta(0))^{b_2}}{\Lambda\Gamma(b_2+1)}\bigg[I_{0+}^{b_2+b_3;\vartheta}\mathcal{K}\big(T, \varphi(T)\big) + I_{0+}^{b_1+b_2+b_3;\vartheta}\omega_1(T) \\
&\quad - \frac{\phi_T^{b_3}}{\Gamma(b_3+1)}\big(I_{0+}^{b_2;\vartheta}\mathcal{K}\big(\mathfrak{a}, \varphi(\mathfrak{a})\big) + I_{0+}^{b_1+b_2;\vartheta}\omega_1(\mathfrak{a})\big)\bigg] \\
&\quad + \frac{1}{\Lambda}\bigg[\frac{\phi_T^{b_2+b_3}}{\Gamma(b_2+b_3+1)}\big(I_{0+}^{b_2;\vartheta}\mathcal{K}\big(\mathfrak{a}, \varphi(\mathfrak{a})\big) + I_{0+}^{b_1+b_2;\vartheta}\omega_1(\mathfrak{a})\big) \\
&\quad - \frac{\phi_{\mathfrak{a}}^{b_2}}{\Gamma(b_2+1)}\big(I_{0+}^{b_2+b_3;\vartheta}\mathcal{K}\big(T, \varphi(T)\big) + I_{0+}^{b_1+b_2+b_3;\vartheta}\omega_1(T)\big)\bigg].
\end{aligned}$$

By (Hyp7),

$$H^d\big(\mathcal{H}(\mathfrak{t}, \varphi), \mathcal{H}(\mathfrak{t}, \overline{\varphi})\big) \leq \tilde{\mathfrak{m}}(\mathfrak{t})\big|\varphi(\mathfrak{t}) - \overline{\varphi}(\mathfrak{t})\big|.$$

Thus, there is $\varkappa(\mathfrak{t}) \in \mathcal{H}(\mathfrak{t}, \overline{\varphi})$ such that

$$\big|\omega_1(\mathfrak{t}) - \varkappa\big| \leq \tilde{\mathfrak{m}}(\mathfrak{t})\big|\varphi(\mathfrak{t}) - \overline{\varphi}(\mathfrak{t})\big|, \quad \forall \mathfrak{t} \in \mathfrak{J}_T.$$

We define a multi–function $\mathcal{O} : \mathfrak{J}_T \to \mathfrak{P}(\mathbb{R})$ by

$$\mathcal{O}(\mathfrak{t}) = \big\{\varkappa \in \mathbb{R} : \big|\omega_1(\mathfrak{t}) - \varkappa\big| \leq \tilde{\mathfrak{m}}(\mathfrak{t})\big|\varphi(\mathfrak{t}) - \overline{\varphi}(\mathfrak{t})\big|\big\}.$$

Notice that ω_1 and $\sigma = \tilde{\mathfrak{m}}|\varphi - \overline{\varphi}|$ are measurable, so we can infer that $\mathcal{O}(\mathfrak{t}) \cap \mathcal{H}(\mathfrak{t}, \overline{\varphi})$ is measurable. Now we choose the function $\omega_2(\mathfrak{t}) \in \mathcal{H}(\mathfrak{t}, \overline{\varphi})$ with

$$\big|\omega_1(\mathfrak{t}) - \omega_2(\mathfrak{t})\big| \leq \tilde{\mathfrak{m}}(\mathfrak{t})\big|\varphi(\mathfrak{t}) - \overline{\varphi}(\mathfrak{t})\big|, \quad \forall \mathfrak{t} \in \mathfrak{J}_T.$$

Define

$$\begin{aligned}
\widetilde{\phi}_2(\mathfrak{t}) &= I_{0+}^{b_2;\vartheta}\mathcal{K}\big(\mathfrak{t}, \overline{\varphi}(\mathfrak{t})\big) + I_{0+}^{b_1+b_2;\vartheta}\omega_2(\mathfrak{t}) \\
&\quad + \frac{(\vartheta(\mathfrak{t}) - \vartheta(0))^{b_2}}{\Lambda\Gamma(b_2+1)}\bigg[I_{0+}^{b_2+b_3;\vartheta}\mathcal{K}\big(T, \overline{\varphi}(T)\big) + I_{0+}^{b_1+b_2+b_3;\vartheta}\omega_2(T) \\
&\quad - \frac{\phi_T^{b_3}}{\Gamma(b_3+1)}\big(I_{0+}^{b_2;\vartheta}\mathcal{K}\big(\mathfrak{a}, \overline{\varphi}(\mathfrak{a})\big) + I_{0+}^{b_1+b_2;\vartheta}\omega_2(\mathfrak{a})\big)\bigg] \\
&\quad + \frac{1}{\Lambda}\bigg[\frac{\phi_T^{b_2+b_3}}{\Gamma(b_2+b_3+1)}\big(I_{0+}^{b_2;\vartheta}\mathcal{K}\big(\mathfrak{a}, \overline{\varphi}(\mathfrak{a})\big) + I_{0+}^{b_1+b_2;\vartheta}\omega_2(\mathfrak{a})\big) \\
&\quad - \frac{\phi_{\mathfrak{a}}^{b_2}}{\Gamma(b_2+1)}\big(I_{0+}^{b_2+b_3;\vartheta}\mathcal{K}\big(T, \overline{\varphi}(T)\big) + I_{0+}^{b_1+b_2+b_3;\vartheta}\omega_2(T)\big)\bigg].
\end{aligned}$$

As a sequel, we obtain

$$\left|\widetilde{\phi}_1(\mathfrak{t})-\widetilde{\phi}_2(\mathfrak{t})\right| \le \left(\|\tilde{\mathfrak{m}}\|\zeta_1+n_{\mathcal{K}}\zeta_2\right)\|\varphi-\overline{\varphi}\|.$$

Therefore

$$\|\widetilde{\phi}_1-\widetilde{\phi}_2\| \le \left(\|\tilde{\mathfrak{m}}\|\zeta_1+n_{\mathcal{K}}\zeta_2\right)\|\varphi-\overline{\varphi}\|.$$

Similarly, interchanging the roles of φ and $\overline{\varphi}$, we get

$$H^d\left(\tilde{\mathcal{S}}(\varphi),\tilde{\mathcal{S}}(\overline{\varphi})\right) \le \left(\|\tilde{\mathfrak{m}}\|\zeta_1+n_{\mathcal{K}}\zeta_2\right)\|\varphi-\overline{\varphi}\|.$$

Because $\tilde{\mathcal{S}}$ is a contraction, we deduce that it has a fixed-point, which is a solution of (3) by the Covitz–Nadler theorem, and the proof is completed. □

4 Examples

In this section, we consider some particular cases of BVPs consisting of fractional differential inclusions to validate the existence results.

Consider the fractional differential inclusions of the form

$$\begin{cases} {}^{C}D_{0+}^{b_1;\vartheta}({}^{C}D_{0+}^{b_2;\vartheta}\varphi(\mathfrak{t})-\mathcal{K}(\mathfrak{t},\varphi(\mathfrak{t})))\in\mathcal{H}(\mathfrak{t},\varphi(\mathfrak{t})), \\ \varphi(\mathfrak{a})=0, \qquad I_{0+}^{b_3;\vartheta}\varphi(T)=0, \quad \mathfrak{a}\in(0,T), \end{cases} \tag{12}$$

for $\mathfrak{t}\in(0,T)$. The following examples are instances of fractional differential inclusions in the particular cases of (12).

Example 4.1 Consider the fractional differential inclusion (12). Taking $\vartheta(\mathfrak{t})=\mathfrak{t}$, $b_1=\frac{1}{2}$, $b_2=\frac{1}{3}$, $b_3=\frac{3}{2}$, $T=1$, and $\mathfrak{a}=\frac{1}{4}$, we convert problem (12) to

$$\begin{cases} {}^{C}D_{0+}^{\frac{1}{2};\mathfrak{t}}({}^{C}D_{0+}^{\frac{1}{3};\mathfrak{t}}\varphi(\mathfrak{t})-\mathcal{K}(\mathfrak{t},\varphi(\mathfrak{t})))\in[\frac{1}{(\mathfrak{t}^3+6\exp(\mathfrak{t}^2))}\frac{\varphi^2}{3(\varphi^2+2)},\frac{1}{\sqrt{\mathfrak{t}+9}}\frac{|\varphi|}{|\varphi|+1}], \quad \mathfrak{t}\in(0,1), \\ \varphi(\frac{1}{4})=0, \qquad I_{0+}^{\frac{3}{2};\mathfrak{t}}\varphi(1)=0. \end{cases} \tag{13}$$

With these data, we find $\Lambda=-0.47457\neq 0$. We define the function $\mathcal{K}$ and multifunction $\mathcal{H}:[0,1]\times\mathbb{R}\to\mathfrak{P}(\mathbb{R})$ as follows

$$\mathcal{K}(\mathfrak{t},\varphi)=\frac{\sin(\mathfrak{t})}{\exp(\mathfrak{t}^2)+20}\left(\frac{|\varphi|}{|\varphi|+1}\right), \quad \forall(\mathfrak{t},\varphi)\in[0,1]\times\mathbb{R}, \tag{14}$$

and

$$\mathcal{H}(\mathfrak{t},\varphi)=\left[\frac{1}{(\mathfrak{t}^3+6\exp(\mathfrak{t}^2))}\frac{\varphi^2}{3(\varphi^2+2)},\frac{1}{\sqrt{\mathfrak{t}+9}}\frac{|\varphi|}{|\varphi|+1}\right]. \tag{15}$$

For $\varphi,\overline{\varphi}\in\mathbb{R}$, we have

$$\begin{aligned} \left|\mathcal{K}(\mathfrak{t},\varphi)-\mathcal{K}(\mathfrak{t},\overline{\varphi})\right| &= \left|\frac{\sin(\mathfrak{t})}{\exp(\mathfrak{t}^2)+20}\left(\frac{|\varphi|}{|\varphi|+1}-\frac{|\overline{\varphi}|}{|\overline{\varphi}|+1}\right)\right| \\ &\le \frac{1}{\exp(\mathfrak{t}^2)+20}\left(\frac{|\varphi-\overline{\varphi}|}{(1+|\varphi|)(1+|\overline{\varphi}|)}\right)\le\frac{1}{21}|\varphi-\overline{\varphi}|; \end{aligned} \tag{16}$$

also, we get

$$\mathcal{K}(\mathrm{t},\varphi) \le \frac{1}{\exp(\mathrm{t}^2)+20} = \psi_{\mathcal{K}}(\mathrm{t}), \quad \forall(\mathrm{t},\varphi)\in[0,1]\times\mathbb{R}.$$

Thus assumptions (Hyp3)–(Hyp4) are satisfied. Obviously, the set-valued map $\mathcal{H}$ satisfies hypothesis (Hyp1), and

$$\|\mathcal{H}(\mathrm{t},\varphi)\|_{\mathfrak{P}} = \sup\{|\eta| : \eta\in\mathcal{H}(\mathrm{t},\varphi)\} \le \frac{1}{\sqrt{\mathrm{t}+9}} = \widetilde{R}_1(\mathrm{t})\widetilde{R}_2(\|\varphi\|),$$

where $\|\widetilde{R}_1\| = \frac{1}{3}$ and $\widetilde{R}_2(\|\varphi\|) = 1$. Thus (Hyp2) holds, and by (Hyp5)

$$\mathcal{L} > 2.8842.$$

So all conditions of Theorem 3.2 are satisfied. Hence problem (13) has a solution on $[0,1]$.

Example 4.2 Based on the fractional differential inclusion (12), by taking $\vartheta(\mathrm{t}) = \mathrm{t}^2$, $b_1 = \frac{1}{3}$, $b_2 = \frac{1}{4}$, $b_3 = \frac{5}{4}$, $T = 1$, $\mathfrak{a} = \frac{1}{2}$, problem (12) is given by

$$\begin{cases} {}^{C}D_{0+}^{\frac{1}{3};\mathrm{t}^2}({}^{C}D_{0+}^{\frac{1}{4};\mathrm{t}^2}\varphi(\mathrm{t}) - \mathcal{K}(\mathrm{t},\varphi(\mathrm{t}))) \in [0, \frac{2\sin(\varphi)}{(\mathrm{t}^2+30)} + \frac{1}{20}], & \mathrm{t}\in(0,1), \\ \varphi(\frac{1}{2}) = 0, \qquad I_{0+}^{\frac{5}{4};\mathrm{t}^2}\varphi(1) = 0, \end{cases} \tag{17}$$

where with these data, we find $\Lambda = -0.58012 \neq 0$. Define $\mathcal{H} : [0,1]\times\mathbb{R} \to \mathfrak{P}(\mathbb{R})$ by

$$\varphi \to \mathcal{H}(\mathrm{t},\varphi) = \left[0, \frac{2\sin(\varphi)}{(\mathrm{t}^2+30)} + \frac{1}{20}\right], \tag{18}$$

and the function $\mathcal{K}$ similar to above in (14). From (16) we see that assumption (Hyp3) is satisfied with $n_{\mathcal{K}} = \frac{1}{21}$. Next, we have $H^d(\mathcal{H}(\mathrm{t},\varphi),\mathcal{H}(\mathrm{t},\overline{\varphi})) \le \tilde{\mathfrak{m}}(\mathrm{t})|\varphi-\overline{\varphi}|$, where $\tilde{\mathfrak{m}}(\mathrm{t}) = \frac{2}{(\mathrm{t}^2+30)}$ and $d(0,\mathcal{H}(\mathrm{t},0)) = \frac{1}{20} \le \tilde{\mathfrak{m}}(\mathrm{t})$ for almost all $\mathrm{t}\in[0,1]$. Furthermore, we obtain $\|\tilde{\mathfrak{m}}\| = \frac{1}{15}$, implying $\|\tilde{\mathfrak{m}}\|\zeta_1 + n_{\mathcal{K}}\zeta_2 \approx 0.79 < 1$. Accordingly, all conditions of Theorem 3.4 are fulfilled. Then it guarantees the existence of a solution to problem (17) on $[0,1]$.

5 Conclusive remarks

Generalized fractional operators are a generalization of the standard operators with special kernels. Besides, fixed point theorems play a key role in studying the qualitative properties of the solutions to certain fractional dynamical equations representing complex systems and chaotic systems. In this paper, we investigated the existence results by assuming two cases where the set-valued map has convex or nonconvex values of (3) in the frame of power law with generalized kernel. We employed some nonlinear analysis techniques. Along with the use of generalized fractional operators, we established a nonlinear alternative for contractive maps in the case of the convex multifunctions and the Nadler–Covitz fixed point theorem in relation to contractions in the case of nonconvex-valued multifunctions. We gave simulative examples to illustrate the theoretical results.

As a future work, we will try to extend the existing FBVP in the present paper to a general structure with the Mittag-Leffler power law [47] and for ψ-Hilfer fractional operator [48].

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Author details
[1]Department of Mathematics, University of Annaba, P.O. Box 12, Annaba, 23000, Algeria. [2]Department of Mathematics, Hodeidah University, Al-Hodeidah, Yemen. [3]Department of Mathematics and Informatics, University of Souk Ahras, P.O. Box 1553, Souk Ahras, 41000, Algeria. [4]Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran. [5]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan.

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