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Refinements of some fractional integral inequalities for refined $(\alpha, h - m)$ -convex function

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Abstract

This article investigates new inequalities for generalized Riemann–Liouville fractional integrals via the refined $(\alpha, h - m)$ -convex function. The established results give refinements of fractional integral inequalities for $(h - m)$ -convex, (α, m) -convex, (s, m) -convex, and related functions. Also, the k -fractional versions of given inequalities by using a parameter substitution are provided.

MSC: 26A51; 26A33; 33E12

Keywords: Refined $(\alpha, h - m)$ -convex function; Refined $(h - m)$ -convex function; Hermite–Hadamard inequality; Riemann–Liouville fractional integrals

1 Introduction

The origin of fractional calculus goes far back to the seventeenth century, when G.W. Leibniz and Marquis de l'Hospital started a discussion on semi-derivatives. This question became the source of inspiration for many well-known mathematicians to explore the contemporary conceptions of the field. Over the late nineteenth century, the theory of fractional calculus expanded significantly. Now it spans from mathematics, physics, viscoelasticity, rheology, chemical, and statistical physics to electrical and mechanical engineering, etc.

In mathematical analysis, the use of integral inequalities has undergone an exponential growth in publications. Many integral inequalities that have been established in recent years with a variety of definitions of fractional-order operators included Riemann–Liouville, Caputo, Katugampola, Caputo–Fabrizio. Using these integrals, researchers have obtained different versions of well-known inequalities of Hermite–Hadamard, Hardy, Opial, Ostrowski, and Grüss (see [1–6]).

Here, motivated and inspired by the ongoing research (see [7–24]), essentially the above-mentioned works, we intend to demonstrate a few novel as well as detailed generalizations using the Riemann–Liouville operator applied over established well-known Hermite–Hadamard inequalities. More precisely, we considered the Riemann–Liouville fractional integrals with monotonically increasing function that plays a crucial role in our study. The Hermite–Hadamard inequality is proved for this operator with a refined $(\alpha, h - m)$ -convex

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function. This inequality obtained here also pointed out to include various known results as their special cases. Also, k -fractional versions of the established inequalities are given. In connection with that we refine some known results existing in the literature. Further, associated with the findings of this paper, some new results are presented.

A function $f : [a, b] \rightarrow \mathbb{R}$ is said to be convex if, for $t \in [0, 1]$, $f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$, $x, y \in [a, b]$. A convex function $f : [a, b] \rightarrow \mathbb{R}$ is also defined by the Hermite–Hadamard inequality stated as follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a)+f(b)}{2}.$$

We recall the definition of refined $(\alpha, h - m)$ -convex function as follows.

Definition 1 ([25]) Let $J \subseteq \mathbb{R}$ be an interval containing $(0, 1)$, and let $h : J \rightarrow \mathbb{R}$ be a non-negative function. A function $f : [0, b] \rightarrow \mathbb{R}$ is called refined $(\alpha, h - m)$ convex function if f is nonnegative, and for all $x, y \in [0, b]$, $(\alpha, m) \in [0, 1]^2$ and $t \in (0, 1)$, one has

$$f(tx + m(1 - t)y) \leq h(t^\alpha)h(1 - t^\alpha)(f(x) + mf(y)). \quad (1.1)$$

If inequality (1.1) is reversed, then f is said to be refined $(\alpha, h - m)$ -concave.

For different suitable choices of h and taking particular values of parameters m and α , the above definition gives some already established definitions.

Remark 1 (i) If $\alpha = m = 1$ and $h(t) = t^s$, then Definition 1 reduces to the definition of s - tgs -convex function stated in [26, Definition 3].

(ii) If $\alpha = m = 1$ and $h(t) = t^{-s}$, then Definition 1 reduces to the definition of Godunova–Levin–Dragomir tgs -convex function stated in [26, Definition 4].

(iii) If $\alpha = m = 1$ and $h(t) = t$, then Definition 1 reduces to the definition of tgs -convex function stated in [20, Definition 2.1].

(iv) If $\alpha = m = h(t) = 1$, then Definition 1 reduces to the definition of P -function stated in [27].

(v) If $0 < h(t) < 1$, then Definition 1 gives the refinement of the definition of $(\alpha, h - m)$ -convex function stated in [28], that is, one can get refinements of all kinds of related convexities.

Many definitions can be obtained from Definition 1 for different suitable choices of h and involved parameters.

Definition 2 ([25]) A function $f : [0, b] \rightarrow \mathbb{R}$ is called refined (α, m) -convex function if for every $x, y \in [0, b]$, $(\alpha, m) \in [0, 1]^2$, and $t \in (0, 1)$ one has

$$f(xt + m(1 - t)y) \leq t^\alpha(1 - t^\alpha)(f(x) + mf(y)).$$

The above definition can be obtained by taking $h(t) = t$ in (1.1).

Definition 3 ([25]) A function $f : [0, b] \rightarrow \mathbb{R}$ is called refined α -convex function if for every $x, y \in [0, b]$, $\alpha \in [0, 1]$, and $t \in (0, 1)$ one has

$$f(tx + (1-t)y) \leq t^\alpha (1-t^\alpha)(f(x) + f(y)).$$

The above definition can be obtained by taking $h(t) = t$ and $m = 1$ in (1.1).

Definition 4 ([25]) Let $J \subseteq \mathbb{R}$ be an interval containing $(0, 1)$, and let $h : J \rightarrow \mathbb{R}$ be a non-negative function. A function $f : I \rightarrow \mathbb{R}$ is called refined $(h - m)$ -convex function if f is nonnegative and for all $x, y \in [0, b]$, $m \in [0, 1]$, and $t \in (0, 1)$ one has

$$f(tx + m(1-t)y) \leq h(t)h(1-t)(f(x) + mf(y)).$$

The above definition can be obtained by taking $\alpha = 1$ in (1.1).

Definition 5 ([25]) Let $J \subseteq \mathbb{R}$ be an interval containing $(0, 1)$, and let $h : J \rightarrow \mathbb{R}$ be a non-negative function. A function $f : [0, b] \rightarrow \mathbb{R}$ is called refined h -convex function if f is non-negative and for all $x, y \in [0, b]$, and $t \in (0, 1)$ one has

$$f(tx + (1-t)y) \leq h(t)h(1-t)(f(x) + f(y)).$$

The above definition can be obtained by taking $\alpha = m = 1$ in (1.1).

Definition 6 ([25]) A function $f : [0, b] \rightarrow \mathbb{R}$ is called refined (s, m) -convex function if for every $x, y \in [0, b]$, $(s, m) \in [0, 1]^2$, and $t \in (0, 1)$ one has

$$f(xt + m(1-t)y) \leq t^s(1-t)^s(f(x) + mf(y)).$$

The above definition can be obtained by taking $\alpha = 1$ and $h(t) = t^s$ in (1.1).

Definition 7 ([25]) A function $f : [0, b] \rightarrow \mathbb{R}$ is called refined s -convex function if for every $x, y \in [0, b]$, $s \in [0, 1]$, and $t \in (0, 1)$ one has

$$f(tx + (1-t)y) \leq t^s(1-t)^s(f(x) + f(y)).$$

The above definition can be obtained by taking $\alpha = m = 1$ and $h(t) = t^s$ in (1.1).

Definition 8 ([25]) A function $f : [0, b] \rightarrow \mathbb{R}$ is called refined m -convex function if for every $x, y \in [0, b]$, $m \in [0, 1]$, and $t \in (0, 1)$ one has

$$f(tx + m(1-t)y) \leq t(1-t)(f(x) + mf(y)).$$

The above definition can be obtained by taking $\alpha = 1$ and $h(t) = t$ in (1.1). The classical Riemann–Liouville fractional integral operator is defined as follows.

Definition 9 ([9]) Let $f \in L_1[a, b]$. Then left-sided and right-sided Riemann–Liouville fractional integrals of a function f of order μ , where $\Re(\mu) > 0$, are defined as follows:

$$I_{a+}^{\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} f(t) dt, \quad x > a, \quad (1.2)$$

and

$$I_{b-}^{\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_x^b (t-x)^{\mu-1} f(t) dt, \quad x < b. \quad (1.3)$$

The Hermite–Hadamard inequality was generalized by Riemann–Liouville fractional integrals of convex functions in [29, 30]. There exist many other versions of Hermite–Hadamard inequality in literature for different kinds of fractional integrals, see [2, 8, 13, 14, 17, 23, 31–34] and the references therein. In the following, we give fractional versions of Hermite–Hadamard inequalities for convex functions via Riemann–Liouville fractional integrals.

Theorem 1 ([29]) Let $f : [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. Also, suppose that f is a convex function on $[a, b]$, then the following fractional integral inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\mu+1)}{2(b-a)^{\mu}} [I_{a+}^{\mu} f(b) + I_{b-}^{\mu} f(a)] \leq \frac{f(a) + f(b)}{2}$$

with $\mu > 0$.

Theorem 2 ([30]) Let $f : [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. Also suppose that f is a convex function on $[a, b]$, then the following fractional integral inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{\mu-1} \Gamma(\mu+1)}{(b-a)^{\mu}} [I_{(\frac{a+b}{2})+}^{\mu} f(b) + I_{(\frac{a+b}{2})-}^{\mu} f(a)] \leq \frac{f(a) + f(b)}{2}$$

with $\mu > 0$.

The inequality for a tgs -convex function via Riemann–Liouville fractional integral is stated in the following theorem.

Theorem 3 ([20]) Let $f : [a, b] \rightarrow \mathbb{R}$ be a positive function with $a < b$ and $f \in L_1[a, b]$. If f is a tgs -convex function on $[a, b]$, then the following fractional integral inequality holds:

$$2f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\mu+1)}{2(b-a)^{\mu}} [I_{a+}^{\mu} f(b) + I_{b-}^{\mu} f(a)] \leq \frac{\mu[f(a) + f(b)]}{(\mu+1)(\mu+2)}$$

with $\mu > 0$.

The definition of generalized Riemann–Liouville fractional integrals by a monotonically increasing function is given here.

Definition 10 ([9]) Let $f \in L_1[a, b]$. Also let ψ be an increasing and positive monotone function on $(a, b]$ having a continuous derivative ψ' on (a, b) . The left-sided and right-sided fractional integrals of a function f with respect to another function ψ on $[a, b]$ of order μ , where $\Re(\mu) > 0$, are defined by

$$I_{a^+}^{\mu, \psi} f(x) = \frac{1}{\Gamma(\mu)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\mu-1} f(t) dt, \quad x > a, \quad (1.4)$$

and

$$I_{b^-}^{\mu, \psi} f(x) = \frac{1}{\Gamma(\mu)} \int_x^b \psi'(t) (\psi(t) - \psi(x))^{\mu-1} f(t) dt, \quad x < b. \quad (1.5)$$

Next, we give two versions of Hermite–Hadamard inequalities for generalized Riemann–Liouville fractional integrals.

Theorem 4 ([31]) Let $f : [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. Also, suppose that f is a convex function on $[a, b]$, ψ is an increasing and positive monotone function on $(a, b]$ having a continuous derivative ψ' on (a, b) . Then the following fractional integral inequality holds:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{\Gamma(\mu+1)}{2(b-a)^\mu} [I_{\psi^{-1}(a)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)^-}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(a))] \\ &\leq \frac{f(a) + f(b)}{2} \end{aligned}$$

with $\mu > 0$.

Theorem 5 ([32]) Let $f : [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. Also suppose that f is a convex function on $[a, b]$, ψ is an increasing and positive monotone function on $(a, b]$ having a continuous derivative ψ' on (a, b) . Then the following fractional integral inequality holds:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{2^{\mu-1} \Gamma(\mu+1)}{(b-a)^\mu} [I_{\psi^{-1}(\frac{a+b}{2})^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(\frac{a+b}{2})^-}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(a))] \\ &\leq \frac{f(a) + f(b)}{2} \end{aligned}$$

with $\mu > 0$.

The k -analogue of generalized Riemann–Liouville fractional integrals (1.4) and (1.5) is defined as follows.

Definition 11 ([35]) Let $f \in L_1[a, b]$. Also let ψ be an increasing and positive monotone function on $(a, b]$ having a continuous derivative ψ' on (a, b) . The left-sided and right-sided fractional integrals of a function f with respect to another function ψ on $[a, b]$ of order μ , where $\Re(\mu) > 0$, $k > 0$, are defined by

$${}_k I_{a^+}^{\mu, \psi} f(x) = \frac{1}{k \Gamma_k(\mu)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\frac{\mu}{k}-1} f(t) dt, \quad x > a, \quad (1.6)$$

and

$${}_k I_{b^-}^{\mu, \psi} f(x) = \frac{1}{k \Gamma_k(\mu)} \int_x^b \psi'(t) (\psi(t) - \psi(x))^{\frac{\mu}{k}-1} f(t) dt, \quad x < b, \quad (1.7)$$

where $\Gamma_k(\mu) = \int_0^\infty t^{\mu-1} e^{-\frac{t^k}{k}} dt$, $\Re(\mu) > 0$.

Using the fact $\Gamma_k(\mu) = k^{\frac{\mu}{k}-1} \Gamma(\frac{\mu}{k})$ in (1.4) and (1.5) after replacing μ with $\frac{\mu}{k}$, we get

$$k^{-\frac{\mu}{k}} I_{a^+}^{\frac{\mu}{k}, \psi} f(x) = {}_k I_{a^+}^{\mu, \psi} f(x), \quad (1.8)$$

$$k^{-\frac{\mu}{k}} I_{b^-}^{\frac{\mu}{k}, \psi} f(x) = {}_k I_{b^-}^{\mu, \psi} f(x). \quad (1.9)$$

For more details on the above defined fractional integrals, we refer the readers to see [36, 37]. The rest of the paper is organized in the following manner. In Sect. 2, we prove some fractional integral inequalities of Hermite–Hadamard type for refined $(\alpha, h - m)$ -convex functions via (1.4) and (1.5). Also, we have given refinements of a few fractional versions of Hermite–Hadamard inequalities proved in [20, 25, 29–32]. In Sect. 3, k -fractional versions of Hermite–Hadamard inequalities for refined $(\alpha, h - m)$ -convex functions are given.

2 Hermite–Hadamard inequalities for refined $(\alpha, h - m)$ -convex function

In this section, we give Hermite–Hadamard inequalities for the refined $(\alpha, h - m)$ -convex function.

Theorem 6 *Let $f : [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < mb$ and $f \in L_1[a, b]$. Also suppose that f is a refined $(\alpha, h - m)$ -convex function on $[a, b]$, ψ is an increasing and positive monotone function on $(a, b]$ having a continuous derivative ψ' on (a, b) . Then, for $(\alpha, m) \in (0, 1]^2$, the following fractional integral inequality holds:*

$$\begin{aligned} & \frac{f\left(\frac{a+mb}{2}\right)}{h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)} \\ & \leq \frac{\Gamma(\mu+1)}{(mb-a)^\mu} \left[I_{\psi^{-1}(a)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{\mu+1} I_{\psi^{-1}(b)^-}^{\mu, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \\ & \leq \mu \left[f(a) + 2mf(b) + m^2 f\left(\frac{a}{m^2}\right) \right] \int_0^1 h(t^\alpha) h(1-t^\alpha) t^{\mu-1} dt \end{aligned} \quad (2.1)$$

with $\mu > 0$.

Proof The following inequality holds for the refined $(\alpha, h - m)$ -convex function:

$$f\left(\frac{x+my}{2}\right) \leq h\left(\frac{1}{2^\alpha}\right) h\left(\frac{2^\alpha-1}{2^\alpha}\right) [f(x) + mf(y)]. \quad (2.2)$$

Setting $x = at + m(1-t)b$, $y = \frac{a}{m}(1-t) + bt$ in (2.2), we get the following inequality:

$$\frac{1}{h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)} f\left(\frac{a+mb}{2}\right) \leq f(at + m(1-t)b) + mf\left(\frac{a}{m}(1-t) + bt\right).$$

Using the refined $(\alpha, h - m)$ -convexity of f and integrating the resulting inequality over the interval $[0, 1]$ after multiplying with $t^{\mu-1}$, we get

$$\begin{aligned} \frac{f\left(\frac{a+mb}{2}\right)}{\mu h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^{\alpha-1}}{2^\alpha}\right)} &\leq \int_0^1 f(at + m(1-t)b)t^{\mu-1} dt + m \int_0^1 f\left(\frac{a}{m}(1-t) + bt\right)t^{\mu-1} dt \quad (2.3) \\ &\leq \left[f(a) + 2mf(b) + m^2f\left(\frac{a}{m^2}\right) \right] \int_0^1 h(t^\alpha)h(1-t^\alpha)t^{\mu-1} dt. \end{aligned}$$

Setting $\psi(u) = at + m(1-t)b$, that is, $t = \frac{mb-\psi(u)}{mb-a}$ and $\psi(v) = \frac{a}{m}(1-t) + bt$, that is, $t = \frac{\psi(v)-\frac{a}{m}}{b-\frac{a}{m}}$ in (2.3), then by applying Definition 10 and multiplying by μ , we get the inequality (2.1). $\square$

Remark 2 (i) If $m = \alpha = 1$, $h(t) = t$, and ψ is the identity function in (2.1), then we get Theorem 3.

(ii) If $\mu = \alpha = m = 1$, $h(t) = t$, and ψ is the identity function in (2.1), then we get the inequality stated in [20, Theorem 2.1].

(iii) If ψ is the identity function in (2.1), then the inequality stated in [25, Theorem 1] is obtained.

Corollary 1 Under the assumption of Theorem 6, the following fractional integral inequality holds for the refined (α, m) -convex function:

$$\begin{aligned} \frac{2^{2\alpha}}{(2^\alpha - 1)} f\left(\frac{a+mb}{2}\right) &\leq \frac{\Gamma(\mu+1)}{(b-a)^\mu} \left[I_{\psi^{-1}(a)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(mb)) \right. \\ &\quad \left. + m^{\mu+1} I_{\psi^{-1}(b)^-}^{\mu, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \\ &\leq \frac{\alpha\mu}{(\mu+\alpha)(\mu+2\alpha)} \left[f(a) + 2mf(b) + m^2f\left(\frac{a}{m^2}\right) \right]. \end{aligned} \quad (2.4)$$

Proof By taking $h(t) = t$ in (2.1), the required inequality (2.4) can be obtained. $\square$

Remark 3 (i) If $m = 1$ in (2.4), then the result for the refined α -convex function can be obtained.

(ii) If ψ is the identity function in (2.4), then the inequality stated in [25, Corollary 1] is obtained.

Corollary 2 Under the assumption of Theorem 6, the following fractional integral inequality holds for the refined $(h - m)$ -convex function:

$$\begin{aligned} \frac{1}{h^2\left(\frac{1}{2}\right)} f\left(\frac{a+mb}{2}\right) &\quad (2.5) \\ &\leq \frac{\Gamma(\mu+1)}{(mb-a)^\mu} \left[I_{\psi^{-1}(a)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{\mu+1} I_{\psi^{-1}(b)^-}^{\mu, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \\ &\leq \mu \left[f(a) + 2mf(b) + m^2f\left(\frac{a}{m^2}\right) \right] \int_0^1 h(t)h(1-t)t^{\mu-1} dt. \end{aligned}$$

Proof By setting $\alpha = 1$ in (2.1), the required inequality (2.5) can be obtained. $\square$

Remark 4 (i) If $m = 1$ in (2.5), then the result for the refined h -convex function can be obtained.

(ii) If ψ is the identity function in (2.5), then the inequality stated in [25, Corollary 2] can be obtained.

Corollary 3 Under the assumption of Theorem 6, the following fractional integral inequality holds for the refined (s, m) -convex function:

$$\begin{aligned} 2^{-2s} f\left(\frac{a+mb}{2}\right) &\leq \frac{\Gamma(\mu+1)}{(mb-a)^\mu} \left[I_{\psi^{-1}(a)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(mb)) \right. \\ &\quad \left. + m^{\mu+1} I_{\psi^{-1}(b)^-}^{\mu, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \\ &\leq \mu \left[f(a) + 2mf(b) + m^2 f\left(\frac{a}{m^2}\right) \right] B(1+s, s+\mu). \end{aligned} \quad (2.6)$$

Proof By setting $h(t) = t^s$ in (2.5), the required inequality (2.6) can be obtained. $\square$

Remark 5 (i) If $m = 1$ in (2.6), then the result for the s -tgs convex function can be obtained.

(ii) If ψ is the identity function in (2.6), then the inequality stated in [25, Corollary 4] can be obtained.

Corollary 4 Under the assumption of Theorem 6, the following inequality holds for the refined m -convex function:

$$\begin{aligned} 4f\left(\frac{a+mb}{2}\right) &\leq \frac{\Gamma(\mu+1)}{(b-a)^\mu} \left[I_{\psi^{-1}(a)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{\mu+1} I_{\psi^{-1}(b)^-}^{\mu, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \\ &\leq \frac{\mu}{(\mu+1)(\mu+2)} \left[f(a) + 2mf(b) + m^2 f\left(\frac{a}{m^2}\right) \right]. \end{aligned} \quad (2.7)$$

Proof By taking $h(t) = t$ in (2.5), the required inequality (2.7) can be obtained. $\square$

Remark 6 (i) If $m = 1$ in (2.7), then the result for the refined convex function can be obtained.

(ii) If ψ is the identity function in (2.7), then the inequality stated in [25, Corollary 5] can be obtained.

The following theorem is the extension of inequality (2.1).

Theorem 7 Under the assumption of Theorem 6, further if $h(t) \leq \frac{1}{\sqrt{2}}$, then the following inequality holds:

$$\begin{aligned} 2f\left(\frac{a+mb}{2}\right) &\leq \frac{1}{h(\frac{1}{2^\alpha})h(\frac{2^\alpha-1}{2^\alpha})} f\left(\frac{a+mb}{2}\right) \\ &\leq \frac{\Gamma(\mu+1)}{(mb-a)^\mu} \left[I_{\psi^{-1}(a)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(mb)) \right. \end{aligned} \quad (2.8)$$

$$\begin{aligned}
& + m^{\mu+1} I_{\psi^{-1}(b)^-}^{\mu, \psi} (f \circ \psi) \left(\psi^{-1} \left(\frac{a}{m} \right) \right) \Big] \\
& \leq \mu \left[f(a) + 2mf(b) + m^2 f \left(\frac{a}{m^2} \right) \right] \\
& \quad \times \int_0^1 h(t^\alpha) h(1-t^\alpha) t^{\mu-1} dt \\
& \leq \frac{1}{2} \left[f(a) + 2mf(b) + m^2 f \left(\frac{a}{m^2} \right) \right].
\end{aligned}$$

Remark 7 (i) If $m = 1$ in (2.8), then the refinement of Theorem 4 can be obtained.

(ii) If $m = 1$ and ψ is the identity function in (2.8), then the refinement of Theorem 1 can be obtained.

(iii) If ψ is the identity function in (2.8), then the inequality stated in [25, Theorem 2] can be obtained.

Theorem 8 Under the assumption of Theorem 6, the following fractional integral inequality holds:

$$\begin{aligned}
& \frac{1}{h(\frac{1}{2^\alpha})h(\frac{2^\alpha-1}{2^\alpha})} f \left(\frac{a+mb}{2} \right) \\
& \leq \frac{2^\mu \Gamma(\mu+1)}{(mb-a)^\mu} \left[I_{\psi^{-1}(\frac{a+mb}{2})^+}^{\mu, \psi} (f \circ \psi) (\psi^{-1}(mb)) + m^{\mu+1} \right. \\
& \quad \times \left. I_{\psi^{-1}(\frac{a+mb}{2m})^-}^{\mu, \psi} (f \circ \psi) \left(\psi^{-1} \left(\frac{a}{m} \right) \right) \right] \\
& \leq \mu \left[f(a) + 2mf(b) + m^2 f \left(\frac{a}{m^2} \right) \right] \\
& \quad \times \int_0^1 h \left(\frac{t^\alpha}{2^\alpha} \right) h \left(\frac{2^\alpha - t^\alpha}{2^\alpha} \right) t^{\mu-1} dt
\end{aligned} \tag{2.9}$$

with $\mu > 0$.

Proof Let $x = \frac{at}{2} + m \left(\frac{2-t}{2} \right) b$, $y = \frac{a}{m} \left(\frac{2-t}{2} \right) + \frac{bt}{2}$ in (2.2), we get the following inequality:

$$\frac{1}{h(\frac{1}{2^\alpha})h(\frac{2^\alpha-1}{2^\alpha})} f \left(\frac{a+mb}{2} \right) \leq f \left(\frac{at}{2} + m \left(\frac{2-t}{2} \right) b \right) + mf \left(\frac{a}{m} \left(\frac{2-t}{2} \right) + \frac{bt}{2} \right).$$

Using the refined $(\alpha, h-m)$ -convexity of f and integrating the resulting inequality over $[0, 1]$ after multiplying with $t^{\mu-1}$, we get

$$\begin{aligned}
& \frac{f(\frac{a+mb}{2})}{\mu h(\frac{1}{2^\alpha})h(\frac{2^\alpha-1}{2^\alpha})} \\
& \leq \int_0^1 f \left(\frac{at}{2} + m \left(\frac{2-t}{2} \right) b \right) t^{\mu-1} dt + m \int_0^1 f \left(\frac{a}{m} \left(\frac{2-t}{2} \right) + \frac{bt}{2} \right) t^{\mu-1} dt \\
& \leq \left[f(a) + 2mf(b) + m^2 f \left(\frac{a}{m^2} \right) \right] \int_0^1 h \left(\frac{t^\alpha}{2^\alpha} \right) h \left(\frac{2^\alpha - t^\alpha}{2^\alpha} \right) t^{\mu-1} dt.
\end{aligned} \tag{2.10}$$

Setting $\psi(u) = \frac{at}{2} + m(\frac{2-t}{2})b$, that is, $t = \frac{2(mb-\psi(u))}{mb-a}$, $\psi(v) = \frac{a}{m}(\frac{2-t}{2}) + \frac{bt}{2}$, that is, $t = \frac{2(\psi(v)-\frac{a}{m})}{b-\frac{a}{m}}$ in (2.10), then by applying Definition 10 and multiplying by μ , we get the inequality (2.9). $\square$

Remark 8 (i) If ψ is the identity function in (2.9), then the inequality stated in [25, Theorem 3] can be obtained.

(ii) If $\alpha = \mu = m = 1$, $h(t) = t$, and ψ is the identity function in (2.9), then the inequality stated in [20, Theorem 2.1] is obtained.

Corollary 5 Under the assumption of Theorem 8, the following fractional integral inequality holds for the refined (α, m) -convex function:

$$\begin{aligned} \frac{2^{2\alpha}}{(2^\alpha - 1)} f\left(\frac{a + mb}{2}\right) &\leq \frac{2^\mu \Gamma(\mu + 1)}{(b - a)^\mu} \left[I_{\psi^{-1}(\frac{a+mb}{2})^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{\mu+1} \right. \\ &\quad \times \left. I_{\psi^{-1}(\frac{a+mb}{2})^-}^{\mu, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \\ &\leq \frac{\mu(2^\alpha(\mu + 2\alpha) - (\mu + \alpha))}{2^{2\alpha}(\mu + \alpha)(\mu + 2\alpha)} \left[f(a) + 2mf(b) + m^2 f\left(\frac{a}{m^2}\right) \right]. \end{aligned} \quad (2.11)$$

Proof By setting $h(t) = t$ in (2.9), the required inequality (2.11) can be obtained. $\square$

Remark 9 (i) If $m = 1$ in (2.11), then the result for the refined α -convex function can be obtained.

(ii) If ψ is the identity function in (2.11), then the inequality stated in [25, Corollary 7] can be obtained.

Corollary 6 Under the assumption of Theorem 8, the following fractional integral inequality holds for the refined $(h - m)$ -convex function:

$$\begin{aligned} \frac{1}{h^2(\frac{1}{2})} f\left(\frac{a + mb}{2}\right) &\leq \frac{2^\mu \Gamma(\mu + 1)}{(mb - a)^\mu} \left[I_{\psi^{-1}(\frac{a+mb}{2})^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{\mu+1} \right. \\ &\quad \times \left. I_{\psi^{-1}(\frac{a+mb}{2})^-}^{\mu, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \\ &\leq \mu \left[f(a) + 2mf(b) + m^2 f\left(\frac{a}{m^2}\right) \right] \int_0^1 h\left(\frac{t}{2}\right) h\left(\frac{2-t}{2}\right) t^{\mu-1} dt. \end{aligned} \quad (2.12)$$

Proof By setting $\alpha = 1$ in (2.9), the required inequality (2.12) can be obtained. $\square$

Remark 10 (i) If $m = 1$ in (2.12), then the result for the refined h -convex function can be obtained.

(ii) If ψ is the identity function in (2.12), then the inequality stated in [25, Theorem 4] can be obtained.

Corollary 7 Under the assumption of Theorem 8, the following fractional integral inequality holds for the refined (s, m) -convex function:

$$2^{-2s} f\left(\frac{a + mb}{2}\right) \leq \frac{2^\mu \Gamma(\mu + 1)}{(mb - a)^\mu} \left[I_{\psi^{-1}(\frac{a+mb}{2})^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{\mu+1} \right] \quad (2.13)$$

$$\begin{aligned} & \times I_{\psi^{-1}\left(\frac{a+mb}{2m}\right)-}^{\mu, \psi} (f \circ \psi) \left(\psi^{-1} \left(\frac{a}{m} \right) \right) \Big] \\ & \leq 2^{\mu} \mu \left[f(a) + 2mf(b) + m^2 f \left(\frac{a}{m^2} \right) \right] B \left(\frac{1}{2}, s + \mu, 1 + s \right). \end{aligned}$$

Proof By setting $h(t) = t^s$ in (2.12), the required inequality (2.13) can be obtained. $\square$

Remark 11 (i) If $m = 1$ in (2.13), then the result for the s - tgs convex function can be obtained.

(i) If ψ is the identity function in (2.13), then the inequality stated in [25, Corollary 10] can be obtained.

Corollary 8 Under the assumption of Theorem 8, the following inequality holds for the refined m -convex function:

$$\begin{aligned} 4f \left(\frac{a+mb}{2} \right) & \leq \frac{2^{\mu} \Gamma(\mu+1)}{(b-a)^{\mu}} \left[I_{\psi^{-1}\left(\frac{a+b}{2}\right)+}^{\mu, \psi} (f \circ \psi) (\psi^{-1}(mb)) \right. \\ & \quad \left. + m^{\mu+1} I_{\psi^{-1}\left(\frac{a+b}{2}\right)-}^{\mu, \psi} (f \circ \psi) \left(\psi^{-1} \left(\frac{a}{m} \right) \right) \right] \\ & \leq \frac{\mu(\mu+3)}{4(\mu+1)(\mu+2)} \left[f(a) + 2mf(b) + m^2 f \left(\frac{a}{m^2} \right) \right]. \end{aligned} \quad (2.14)$$

Proof By setting $h(t) = t$ in (2.12), the required inequality (2.14) can be obtained. $\square$

Remark 12 (i) If $m = 1$ in (2.14), then the result for the refined convex function can be obtained.

(ii) If ψ is the identity function in (2.14), then the inequality stated in [25, Corollary 11] can be obtained.

The following theorem is the extension of inequality (2.9).

Theorem 9 Under the assumption of Theorem 8, further if $h(t) \leq \frac{1}{\sqrt{2}}$, then the following fractional integral inequality holds:

$$\begin{aligned} 2f \left(\frac{a+mb}{2} \right) & \leq \frac{1}{h\left(\frac{1}{2^{\alpha}}\right)h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)} f \left(\frac{a+mb}{2} \right) \\ & \leq \frac{2^{\mu} \Gamma(\mu+1)}{(mb-a)^{\mu}} \left[I_{\psi^{-1}\left(\frac{a+mb}{2}\right)+}^{\mu, \psi} (f \circ \psi) (\psi^{-1}(mb)) \right. \\ & \quad \left. + m^{\mu+1} I_{\psi^{-1}\left(\frac{a+mb}{2m}\right)-}^{\mu, \psi} (f \circ \psi) \left(\psi^{-1} \left(\frac{a}{m} \right) \right) \right] \\ & \leq \mu \left[f(a) + 2f(b) + mf \left(\frac{a}{m^2} \right) \right] \\ & \quad \times \int_0^1 h \left(\frac{t^{\alpha}}{2^{\alpha}} \right) h \left(\frac{2^{\alpha}-t^{\alpha}}{2^{\alpha}} \right) t^{\mu-1} dt \\ & \leq \frac{1}{2} \left[f(a) + 2mf(b) + m^2 f \left(\frac{a}{m^2} \right) \right]. \end{aligned} \quad (2.15)$$

Remark 13 (i) If $m = 1$ in (2.15), then the refinement of Theorem 5 can be obtained.

(ii) If $m = 1$ and ψ is the identity function in (2.15), then the refinement of Theorem 2 can be obtained.

(iii) If ψ is the identity function in (2.15), then the inequality stated in [25, Theorem 4] can be obtained.

3 k -Analogues of Hermite–Hadamard inequalities for refined $(\alpha, h - m)$ -convex function

In this section, we present k -fractional versions of Hermite–Hadamard type inequalities for the refined $(\alpha, h - m)$ -convex function discussed in Sect. 2.

Theorem 10 *Under the assumption of Theorem 6, for $k > 0$, the following k -fractional integral inequality holds:*

$$\begin{aligned} & \frac{f\left(\frac{a+mb}{2}\right)}{h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)} \\ & \leq \frac{\Gamma_k(\mu+k)}{(mb-a)^{\frac{\mu}{k}}} \left[{}_kI_{\psi^{-1}(a)^+}^{\mu,\psi}(f \circ \psi)(\psi^{-1}(mb)) + m^{\frac{\mu}{k}+1} {}_kI_{\psi^{-1}(b)^-}^{\mu,\psi}(f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \\ & \leq \frac{\mu}{k} \left[f(a) + 2mf(b) + m^2f\left(\frac{a}{m^2}\right) \right] \int_0^1 h(t^\alpha)h(1-t^\alpha)t^{\frac{\mu}{k}-1} dt \end{aligned} \quad (3.1)$$

with $\mu > 0$.

Proof Using (1.8) and (1.9) after replacing μ with $\frac{\mu}{k}$ in (2.1), we get the above inequality (3.1). $\square$

Remark 14 If ψ is the identity function in (3.1), then the inequality stated in [25, Theorem 5] can be obtained.

Corollary 9 *Under the assumption of Theorem 10, the following fractional integral inequality holds for the refined (α, m) -convex function:*

$$\begin{aligned} & \frac{2^{2\alpha}f\left(\frac{a+mb}{2}\right)}{(2^\alpha-1)} \\ & \leq \frac{\Gamma_k(\mu+k)}{(mb-a)^{\frac{\mu}{k}}} \left[{}_kI_{\psi^{-1}(a)^+}^{\mu,\psi}(f \circ \psi)(\psi^{-1}(mb)) + m^{\frac{\mu}{k}+1} {}_kI_{\psi^{-1}(b)^-}^{\mu,\psi}(f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \\ & \leq \frac{k\alpha\mu}{(\mu+\alpha k)(\mu+2\alpha k)} \left[f(a) + 2mf(b) + m^2f\left(\frac{a}{m^2}\right) \right]. \end{aligned} \quad (3.2)$$

Proof By setting $h(t) = t$ in (3.1), the above inequality can be obtained. $\square$

Remark 15 If ψ is the identity function in (3.2), then the inequality stated in [25, Corollary 13] can be obtained.

Corollary 10 Under the assumption of Theorem 10, for $k > 0$, the following k -fractional integral inequality holds for the refined $(h - m)$ -convex function:

$$\begin{aligned} & \frac{f\left(\frac{a+mb}{2}\right)}{h^2\left(\frac{1}{2}\right)} \\ & \leq \frac{\Gamma_k(\mu + k)}{(mb - a)^{\frac{\mu}{k}}} \left[{}_k I_{\psi^{-1}(a)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{\frac{\mu}{k}+1} {}_k I_{\psi^{-1}(b)^-}^{\mu, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \\ & \leq \frac{\mu}{k} \left[f(a) + 2mf(b) + m^2 f\left(\frac{a}{m^2}\right) \right] \int_0^1 h(t)h(1-t)t^{\frac{\mu}{k}-1} dt. \end{aligned} \quad (3.3)$$

Proof By setting $\alpha = 1$ in (3.1), the above inequality (3.3) can be obtained. $\square$

Remark 16 If ψ is the identity function in (3.3), then the inequality stated in [25, Corollary 13] can be obtained.

Corollary 11 Under the assumption of Theorem 10, the following fractional integral inequality holds for the refined (s, m) -convex function:

$$\begin{aligned} 2^{-2s} f\left(\frac{a+mb}{2}\right) & \leq \frac{\Gamma_k(\mu + k)}{(mb - a)^{\frac{\mu}{k}}} \left[{}_k I_{\psi^{-1}(a)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(mb)) \right. \\ & \quad \left. + m^{\frac{\mu}{k}+1} {}_k I_{\psi^{-1}(b)^-}^{\mu, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \\ & \leq \frac{\mu}{k} \left[f(a) + 2mf(b) + m^2 f\left(\frac{a}{m^2}\right) \right] B\left(1 + s, s + \frac{\mu}{k}\right). \end{aligned} \quad (3.4)$$

Proof By setting $h(t) = t^s$ in (3.3), the above inequality (3.4) can be obtained. $\square$

Remark 17 If ψ is the identity function in (3.4), then the inequality stated in [25, Corollary 14] can be obtained.

Corollary 12 Under the assumption of Theorem 10, the following inequality holds for the refined m -convex function:

$$\begin{aligned} 4f\left(\frac{a+mb}{2}\right) & \leq \frac{\Gamma_k(\mu + k)}{(b - a)^{\frac{\mu}{k}}} \left[{}_k I_{\psi^{-1}(a)^+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(mb)) \right. \\ & \quad \left. + m^{\frac{\mu}{k}+1} {}_k I_{\psi^{-1}(mb)^-}^{\mu, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \\ & \leq \frac{\mu k}{(\mu + k)(\mu + 2k)} \left[f(a) + 2mf(b) + m^2 f\left(\frac{a}{m^2}\right) \right]. \end{aligned} \quad (3.5)$$

Proof By setting $h(t) = t$ in (3.3), we get the above inequality (3.5). $\square$

Remark 18 If ψ is the identity function in (3.5), then the inequality stated in [25, Corollary 15] can be obtained.

Theorem 11 Under the assumption of Theorem 6, further if $h(t) \leq \frac{1}{\sqrt{2}}$, then the following inequality holds:

$$\begin{aligned} 2f\left(\frac{a+mb}{2}\right) &\leq \frac{1}{h(\frac{1}{2^\alpha})h(\frac{2^\alpha-1}{2^\alpha})}f\left(\frac{a+mb}{2}\right) \\ &\leq \frac{\Gamma_k(\mu+k)}{(mb-a)^{\frac{\mu}{k}}} \left[{}_kI_{\psi^{-1}(a)^+}^{\mu,\psi}(f \circ \psi)(\psi^{-1}(mb)) \right. \\ &\quad \left. + m^{\frac{\mu}{k}+1} {}_kI_{\psi^{-1}(b)^-}^{\mu,\psi}(f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \\ &\leq \frac{\mu}{k} \left[f(a) + 2f(b) + mf\left(\frac{a}{m^2}\right) \right] \\ &\quad \times \int_0^1 h(t^\alpha)h(1-t^\alpha)t^{\frac{\mu}{k}-1} dt \\ &\leq \frac{1}{2} \left[f(a) + 2mf(b) + m^2f\left(\frac{a}{m^2}\right) \right]. \end{aligned}$$

Theorem 12 Under the assumption of Theorem 6, for $k > 0$, the following k -fractional integral inequality holds:

$$\begin{aligned} \frac{1}{h(\frac{1}{2^\alpha})h(\frac{2^\alpha-1}{2^\alpha})}f\left(\frac{a+mb}{2}\right) &\leq \frac{2^{\frac{\mu}{k}}\Gamma(\mu+k)}{(mb-a)^{\frac{\mu}{k}}} \left[{}_kI_{\psi^{-1}(\frac{a+mb}{2})^+}^{\mu,\psi}(f \circ \psi)(\psi^{-1}(mb)) \right. \\ &\quad \left. + m^{\frac{\mu}{k}+1} {}_kI_{\psi^{-1}(\frac{a+mb}{2m})^-}^{\mu,\psi}(f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \\ &\leq \left[f(a) + 2mf(b) + m^2f\left(\frac{a}{m^2}\right) \right] \\ &\quad \times \int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right)h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right)t^{\frac{\mu}{k}-1} dt \end{aligned} \quad (3.6)$$

with $\mu > 0$.

Proof Using (1.8) and (1.9) after replacing μ with $\frac{\mu}{k}$ in (2.9), we get the above inequality (3.6). $\square$

Remark 19 If ψ is the identity function in (3.6), then the inequality stated in [25, Theorem 7] can be obtained.

Corollary 13 Under the assumption of Theorem 12, the following fractional integral inequality holds for the refined (α, m) -convex function:

$$\begin{aligned} \frac{2^{2\alpha}}{(2^\alpha-1)}f\left(\frac{a+mb}{2}\right) &\leq \frac{2^\mu\Gamma(\mu+k)}{(b-a)^{\frac{\mu}{k}}} \left[{}_kI_{\psi^{-1}(\frac{a+b}{2})^+}^{\mu,\psi}(f \circ \psi)(\psi^{-1}(mb)) + m^{\frac{\mu}{k}+1} \right. \\ &\quad \left. \times {}_kI_{\psi^{-1}(\frac{a+b}{2})^-}^{\mu,\psi}(f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \\ &\leq \frac{2^\alpha(\frac{\mu}{k}+2\alpha)-(\frac{\mu}{k}+\alpha)}{2^{2\alpha}(\frac{\mu}{k}+\alpha)(\mu+2\alpha)} \left[f(a) + 2mf(b) + m^2f\left(\frac{a}{m^2}\right) \right]. \end{aligned} \quad (3.7)$$

Proof By setting $h(t) = t$ in (3.6), we get the above inequality (3.7). $\square$

Remark 20 (i) If $\alpha = 1$ in (3.7), then the result for the refined m -convex function can be obtained.

(ii) If ψ is the identity function in (3.7), then the inequality stated in [25, Corollary 16] can be obtained.

Corollary 14 *Under the assumption of Theorem 12, the following inequality holds for the refined $(h - m)$ -convex function:*

$$\begin{aligned} \frac{1}{h^2(\frac{1}{2})} f\left(\frac{a+mb}{2}\right) &\leq \frac{2^{\frac{\mu}{k}} \Gamma_k(\mu+k)}{(mb-a)^{\frac{\mu}{k}}} \left[{}_k I_{\psi^{-1}(\frac{a+mb}{2})+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{\frac{\mu}{k}+1} \right. \\ &\quad \times \left. {}_k I_{\psi^{-1}(\frac{a+mb}{2m})-}^{\mu, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \\ &\leq \frac{\mu}{k} \left[f(a) + 2mf(b) + m^2 f\left(\frac{a}{m^2}\right) \right] \\ &\quad \times \int_0^1 h\left(\frac{t}{2}\right) h\left(\frac{2-t}{2}\right) t^{\mu-1} dt. \end{aligned} \quad (3.8)$$

Proof If $\alpha = 1$ in (3.6), then the above inequality (3.8) can be obtained. $\square$

Remark 21 If ψ is the identity function in (3.7), then the inequality stated in [25, Theorem 8] can be obtained.

Corollary 15 *Under the assumption of Theorem 12, the following fractional integral inequality holds for the refined (s, m) -convex function:*

$$\begin{aligned} 2^{-2s} f\left(\frac{a+mb}{2}\right) &\leq \frac{2^{\frac{\mu}{k}} \Gamma_k(\mu+k)}{(mb-a)^{\mu}} \left[{}_k I_{\psi^{-1}(\frac{a+mb}{2})+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{\frac{\mu}{k}+1} \right. \\ &\quad \times \left. {}_k I_{\psi^{-1}(\frac{a+mb}{2m})-}^{\mu, \psi} (f \circ \psi)\left(\psi^{-1}\left(\frac{a}{m}\right)\right) \right] \\ &\leq \frac{2^{\frac{\mu}{k}} \mu}{k} \left[f(a) + 2mf(b) + m^2 f\left(\frac{a}{m^2}\right) \right] \\ &\quad \times B\left(\frac{1}{2}, s + \frac{\mu}{k}, 1+s\right). \end{aligned} \quad (3.9)$$

Proof By setting $h(t) = t^s$ in (3.8), we get the above inequality (3.9). $\square$

Remark 22 (i) If $m = 1$ in (3.9), then the result for the k -fractional s -tgs convex function can be obtained.

(ii) If ψ is the identity function in (3.9), then the inequality stated in [25, Corollary 17] can be obtained.

Corollary 16 *Under the assumption of Theorem 12, the following inequality holds for the refined m -convex function:*

$$2f\left(\frac{a+mb}{2}\right) \leq \frac{2^{\frac{\mu}{k}} \Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}} \left[{}_k I_{\psi^{-1}(\frac{a+mb}{2})+}^{\mu, \psi} (f \circ \psi)(\psi^{-1}(mb)) + m^{\frac{\mu}{k}+1} \right] \quad (3.10)$$

$$\begin{aligned} & \times {}_k I_{\psi^{-1}\left(\frac{a+mb}{2m}\right)-}^{\mu, \psi} (f \circ \psi) \left(\psi^{-1} \left(\frac{a}{m} \right) \right) \Big] \\ & \leq \frac{\mu(\mu+3k)}{4(\mu+k)(\mu+2k)} \left[f(a) + 2mf(b) + m^2 f \left(\frac{a}{m^2} \right) \right]. \end{aligned}$$

Proof By setting $h(t) = t$ in (3.8), we get the above inequality (3.10). $\square$

Remark 23 (i) If $m = 1$ in (3.10), then the result for the refined convex function via generalized k -fractional integral can be obtained.

(ii) If ψ is the identity function in (3.10), then the inequality stated in [25, Corollary 18] can be obtained.

The following theorem is the extension of inequality (3.6).

Theorem 13 *Under the assumption of Theorem 8, further if $h(t) \leq \frac{1}{\sqrt{2}}$, then the following fractional integral inequality holds:*

$$\begin{aligned} 2f\left(\frac{a+mb}{2}\right) & \leq \frac{1}{h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)} f\left(\frac{a+mb}{2}\right) \\ & \leq \frac{2^{\frac{\mu}{k}} \Gamma_k(\mu+k)}{(mb-a)^{\frac{\mu}{k}}} \left[{}_k I_{\psi^{-1}\left(\frac{a+mb}{2}\right)+}^{\mu, \psi} (f \circ \psi) \left(\psi^{-1}(mb) \right) \right. \\ & \quad \left. + m^{\frac{\mu}{k}+1} {}_k I_{\psi^{-1}\left(\frac{a+mb}{2m}\right)-}^{\mu, \psi} (f \circ \psi) \left(\psi^{-1} \left(\frac{a}{m} \right) \right) \right] \\ & \leq \frac{\mu}{k} \left[f(a) + 2f(b) + mf\left(\frac{a}{m^2}\right) \right] \\ & \quad \times \int_0^1 h\left(\frac{t^\alpha}{2^\alpha}\right) h\left(\frac{2^\alpha-t^\alpha}{2^\alpha}\right) t^{\frac{\mu}{k}-1} dt \\ & \leq \frac{1}{2} \left[f(a) + 2mf(b) + m^2 f\left(\frac{a}{m^2}\right) \right]. \end{aligned} \tag{3.11}$$

4 Concluding remarks

This article deals with Hermite–Hadamard type inequalities for the refined $(\alpha, h-m)$ -convex function via generalized Riemann–Liouville fractional integrals. The outcomes of this research provide refinements of some Hermite–Hadamard type inequalities for different types of convexities. The k -fractional versions of these inequalities are also obtained for the differentiable refined $(\alpha, h-m)$ -convex function.

Acknowledgements

The work of Josip Pečarić was supported by the Ministry of Education and Science of the Russian Federation (the agreement no. 02.a03.21.0008).

Funding

There is no funding available for the publication of this paper.

Availability of data and materials

There is no additional data required for the finding of results of this paper.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have equal contribution in this article. All authors read and approved the final manuscript.

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Publisher's Note

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Received: 20 March 2021 Accepted: 6 August 2021 Published online: 21 August 2021

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