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Finite-time synchronization problem of a class of discontinuous Cohen–Grossberg neural networks with mixed delays via new switching design

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Abstract

This paper investigates a class of generalized Cohen–Grossberg neural networks (CGNNs) with discontinuous activations and mixed delays. Based on the nonsmooth analysis theory, the drive-response concept, differential inclusions theory, we give several basic assumptions to gain the finite-time synchronization issue of CGNNs. Sufficient conditions are provided without the boundedness or monotonicity of discontinuous activation functions. Moreover, one can estimate the settling time's upper bounds of the system. At last, two numerical examples and their simulations are given to further show the benefits of the obtained control approach.

Keywords: Finite-time synchronization; Cohen–Grossberg neural networks; Discontinuous activations; Mixed delays

1 Introduction

Recently, the research of neural networks with discontinuous activation functions has gradually attracted the attention of many researchers, including systems oscillating under earthquake, power circuits, chaos phenomenon, and dry friction (see [1–4]). Because the non-Lipschitz phenomenon has many special advantages, the emergence of nonsmooth has greatly improved the research of neural networks. In addition, the analysis of neural networks with discontinuous activations is accompanied by many interesting practical phenomena to explore important dynamic behavior characteristics. This arouses researchers' interest in the generalized neural networks by using discontinuous activation functions [5–8].

In 1983, Cohen and Grossberg firstly introduced the CGNNs system, which was a useful recurrent neural networks system, including evolutionary theory, population biology, neurobiology [9]. After that, a large number of results have emerged [10–17] such as the existence, dissipation, and exponential stability of the CGNNs model. However, there are few works on discontinuous CGNNs system with mixed delays. In [18–20], the authors investigated the exponential stability and exponential synchronization of a class of CGNNs.

Abdurahman and his team in [21] studied the exponential lag synchronization for both discrete time-delays and distributed delays CGNNs. It worthy to know that, in 2003, Forti introduced the global stability of a discontinuous right-hand side neural network system via the framework of the theory of Filippov differential inclusions [22, 23]. In [24] they pointed out that the sliding mode method was used to solve constrained optimization problems, because high-gain neuron activations were often encountered in the neural networks system. In [25], the authors analyzed the fixed-time synchronization of a class of discontinuous fuzzy inertial neural networks with time-varying delays based on the new improved fixed-time stability lemmas.

As we know, the synchronization phenomenon has been widely used in software engineering, ecological structure, security storage, information processing system, and many other fields, which gets a lot of social attention. There are some micro motions from the view of the mathematical model, which we call qualitative or stability problems. Meanwhile, the researchers pay attention to the macroscopic topology based on the synchronization problem. The literature [26] was the first paper to consider finite-time control of discontinuous chaotic systems, and [27] studied the finite-time synchronization of time-delayed neural networks. There were many classifications of synchronization, such as anti-synchronization [28, 29], exponential synchronization [30], robust synchronization [31], chaos synchronization [32], and so on. The synchronization technologies mentioned above have many defects in a real practical environment. For instance, the existence results of the above synchronization usually are guaranteed over the infinite horizon. In addition, when the finite initial value is required and the control accuracy has great influence on the system, it is always difficult to estimate. Moreover, even if stringent convergence time is given, the neural networks model may not be available in a real experimental environment. For the sake of convergence time, one needs to propose a concept named finite-time synchronization, which means that the settling time function of any finite initial value is bounded. Song and his team provided a novel and effective techniques method in [33], then in [34] they investigated the finite-time synchronization problem of a class of discontinuous neural networks with nonlinear coupling and mixed delays. Peng and his team in [35] investigated the finite-time synchronization control methodology for the CGNNs system. Yang in [36] verified that the considered neural networks can gain the synchronization in a finite time. In [37], the authors ensured that the target model realized the finite-time synchronization goal of the coupled neural networks. In [38], the author solved the challenging issues in the field of finite-time synchronization of the cellular neural networks.

Motivated by the aforementioned works on finite-time synchronization of CGNNs system, this paper aims to realize the finite-time synchronization issue for the considered system CGNNs. Our main contributions of this paper include the following three aspects.

- The CGNNs discussed in this brief are state-dependent discontinuous systems; based on the properties of differential inclusion and set-analysis theory, the drive-response CGNNs can be transformed into a synchronization error system. Theoretical analysis can be extended to other fields.
- When both mixed delays and discontinuities exist in the dynamical CGNNs, how to deal with the discrepancy within the scope of the Filippov solutions of the drive system and the response system?

- Because the system has special discontinuous characteristics, in order to shorten the settling time of the drive-response CGNNs system, the more ingenious switching controller should be devised.

2 Model description and some basic definitions

This section considers a general class of discontinuous CGNNs with mixed delays. Based on the previous works [39–45], one can describe the model by the following equation:

$$\begin{aligned}\frac{\mathrm{d}\pi_i(t)}{\mathrm{d}t} &= -\varpi_i\big(\pi_i(t)\big)\Bigg[a_i\big(\pi_i(t)\big) - \sum_{j=1}^{n} b_{ij}(t)f_j\big(\pi_j(t)\big) - \sum_{j=1}^{n} c_{ij}(t)f_j\big(\pi_j\big(t-\tau_{ij}(t)\big)\big)\\ &\quad - \sum_{j=1}^{n}\int_0^{+\infty} K_{ij}(t,s)f_j\big(\pi_j(s)\big)\,\mathrm{d}s - I_i(t)\Bigg], \quad i=1,2,\ldots,n, \end{aligned}\tag{1}$$

the state vector is $\pi(t) = (\pi_1(t),\pi_2(t),\ldots,\pi_n(t))^T \in \mathbb{R}$; $\varpi_i(\cdot)$ is the amplification function of the system; $a_i(\cdot)$ is the function with proper behavior; $\tau_{ij}(t)$ and $K_{ij} : [0,\infty) \to [0,\infty)$ denote the discrete delay and the distributed delay, respectively; $b_{ij}(t)$ and $c_{ij}(t)$ are the connection strength and the delayed feedbacks of two different neurons; $f_j(\cdot)$ is the neuron input–output activation of the ith neuron; $I_i(\cdot)$ is an input signal function of the external factors;

The neuron activation functions $f_j(\cdot)$ in the above model satisfy the following conditions:

(H1) $f_i : \mathbb{R} \to \mathbb{R}$ is discontinuous on a countable set of isolate point $\{\rho_k^i\}$ for each $i = 1,2,\ldots,n$.

(H2) There are two nonnegative constants L_i and h_i that satisfy the following inequality:

$$\big\|\mathbb{F}\big[f_i(x)-f_i(y)\big]\big\| = \sup_{\zeta_i\in\mathbb{F}[f_i(x)-f_i(y)]}\|\zeta_i\| \le L_i\|x-y\| + h_i, \quad i=1,2,\ldots,n,$$

where $\mathbb{F}(f_i(x)) = K[f_i(x)] = [\min\{f_i(x_i^-),f_i(x_i^+)\},\max\{f_i(x_i^-),f_i(x_i^+)\}]$.

(H3) There exist nonnegative constants K_{ij} satisfying

$$\int_0^{+\infty} K_{ij}(\cdot)\,\mathrm{d}s \le K_{ij}, \quad i,j = 1,2\ldots,n.$$

For every $i,j = 1,2,\ldots,n$, we assume that $a_i(t)$, $b_{ij}(t)$, $c_{ij}(t)$, $\tau_{ij}(t)$ are continuous ω-periodic functions and $I_i(t)$ are almost periodic functions; $0 \le \tau_{ij}(t) \le \tau_{ij}$, $\dot{\tau}_{ij}(t) \le \sigma_{ij} < 1$; $\varpi_i(\cdot)$ is continuous and $0 < \underline{\omega} \le \varpi_i(\cdot) \le \overline{\omega}$; $\dot{a}_i(\cdot) \ge a_i$, where τ_{ij}, σ_{ij}, $\underline{\omega}$, $\overline{\omega}$, and a_i are nonnegative constants. Moreover, we denote $a^{\max} = \max_{1\le i\le n}\sup_{t\in\mathbb{R}}|a_i(t)|$, $b^{\max} = \max_{1\le i\le n,1\le j\le n}\sup_{t\in\mathbb{R}}|b_{ij}(t)|$, $c^{\max} = \max_{1\le i\le n,1\le j\le n}\sup_{t\in\mathbb{R}}|c_{ij}(t)|$.

Choose a transformation function $\Phi_i^{-1}(\cdot)$, which satisfies

$$\frac{\mathrm{d}}{\mathrm{d}u}\big(\Phi_i^{-1}(u)\big) = \frac{1}{\varpi_i(u)}.$$

From the above discussion we know that $\frac{1}{\varpi_i(u)} > 0$, which yields that $\Phi_i^{-1}(\cdot)$ is strictly monotone increasing, then the inverse function of $\Phi_i^{-1}(\cdot)$ exists, we denote $(\Phi_i^{-1}(\cdot))^{-1} =$

$\Phi_i(\cdot)$. Let $x_i(t) = \Phi_i^{-1}(\pi_i(t))$, one can have $\pi_i(t) = \Phi_i(x(t))$ and $\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = \frac{\mathrm{d}\Phi_i^{-1}(\pi_i(t))}{\mathrm{d}\pi_i(t)}\dot{\pi}_i(t) = \frac{1}{\varpi_i(\pi_i(t))}\dot{\pi}_i(t)$, then we can obtain that

$$\begin{aligned}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &= -a_i\big(\Phi_i\big(x_i(t)\big)\big) + \sum_{j=1}^{n} b_{ij}(t)f_j\big(\Phi_i\big(x_i(t)\big)\big) + \sum_{j=1}^{n} c_{ij}(t)f_j\big(\Phi_i\big(x_i\big(t-\tau_{ij}(t)\big)\big)\big)\\ &\quad + \sum_{j=1}^{n}\int_0^{+\infty} K_{ij}(t,s)f_j\big(\Phi_i\big(x_i(s)\big)\big)\,\mathrm{d}s + I_i(t), \quad i = 1,2,\ldots,n. \end{aligned} \tag{2}$$

Obviously, in the framework of differential inclusions, system (2) can be rewritten as follows:

$$\begin{aligned}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &\in -a_i\big(\Phi_i\big(x_i(t)\big)\big) + \sum_{j=1}^{n} b_{ij}(t)K\big[f_j\big(\Phi_i\big(x_i(t)\big)\big)\big] + \sum_{j=1}^{n} c_{ij}(t)K\big[f_j\big(\Phi_i\big(x_i\big(t-\tau_{ij}(t)\big)\big)\big)\big]\\ &\quad + \sum_{j=1}^{n}\int_0^{+\infty} K_{ij}(t,s)K\big[f_j\big(\Phi_i\big(x_i(s)\big)\big)\big]\,\mathrm{d}s + I_i(t).\end{aligned}$$

For any compact interval of $[0,\tau)$, the vector function $x = (x_1, x_2, \ldots, x_n)^T$ is continuous and absolutely continuous. According to the Filippov framework, one can find a measurable function $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)^T : (-\infty, \tau) \to \mathbb{R}^n$ such that $\gamma_i(t) \in K[f_i(\Phi_i(x_i(t)))]$, then one can obtain that x is a state solution of CGNNs and

$$\begin{aligned}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &= -a_i\big(\Phi_i\big(x_i(t)\big)\big) + \sum_{j=1}^{n} b_{ij}(t)\gamma_j(t) + \sum_{j=1}^{n} c_{ij}(t)\gamma_j(t-\tau)\\ &\quad + \sum_{j=1}^{n}\int_0^{+\infty} K_{ij}(t,s)\gamma_j(s)\,\mathrm{d}s + I_i(t).\end{aligned} \tag{3}$$

Through the above discussion, consider CGNNs system (1) as the drive system. By giving the initial value of CGNNs $\phi(s) = (\phi_1(s), \phi_2(s), \ldots, \phi_n(s))^T$, we can obtain the corresponding response system as follows:

$$\begin{aligned}\frac{\mathrm{d}\xi_i(t)}{\mathrm{d}t} &= -\varpi_i\big(\xi_i(t)\big)\Bigg[a_i\big(\xi_i(t)\big) - \sum_{j=1}^{n} b_{ij}(t)f_j\big(\xi_j(t)\big) - \sum_{j=1}^{n} c_{ij}(t)f_j\big(\xi_j\big(t-\tau_{ij}(t)\big)\big)\\ &\quad - \sum_{j=1}^{n}\int_0^{+\infty} K_{ij}(t,s)f_j\big(\xi_j(s)\big)\,\mathrm{d}s - I_i(t)\Bigg] + u_i(t), \quad i = 1,2,\ldots,n,\end{aligned} \tag{4}$$

where $u_i(t)$ is the appropriate controller.

Similarly, let $y_i(t) = \Phi_i^{-1}(\xi_i(t))$, $i = 1,2,\ldots,n$, we can derive that

$$\begin{aligned}\frac{\mathrm{d}y_i(t)}{\mathrm{d}t} &= -a_i\big(\Phi_i\big(y_i(t)\big)\big) + \sum_{j=1}^{n} b_{ij}(t)\widetilde{\gamma}_j(t) + \sum_{j=1}^{n} c_{ij}(t)\widetilde{\gamma}_j(t-\tau)\\ &\quad + \sum_{j=1}^{n}\int_0^{+\infty} K_{ij}(t,s)\widetilde{\gamma}_j(s)\,\mathrm{d}s + I_i(t) + \frac{u_i(t)}{\varpi_i(\Phi_i(y_i(t)))},\end{aligned} \tag{5}$$

where $\widetilde{\gamma}_i(t) \in K[f_i(\Phi_i(y_i(t)))]$ and $\widetilde{\gamma}(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t))^T$.

Lemma 2.1 (See [23, 39]) *If $V(y(t)) : \mathbb{R}^n \times \mathbb{R}$ is a C-regular function, for any compact interval of $[0,+\infty)$, $y(t) : [0,+\infty) \to \mathbb{R}^n$ is an absolutely continuous function. For a continuous function $\Upsilon : (0,\infty) \to \mathbb{R}$ with $\Upsilon(\varrho) > 0$ for $\varrho \in (0,+\infty)$, if it satisfies that*

$$\frac{dV(t)}{dt} \le -\Upsilon\bigl(V(t)\bigr) \quad \textit{for a.e. } t \ge 0$$

and

$$\int_0^{V(0)} \frac{1}{\Upsilon(\varrho)} = t^* < +\infty,$$

then we have $V(t) = 0$ for $t \ge t^*$; especially, we have:

(1) *If $\Upsilon = K_1\varrho + K_2\varrho^\mu$ for all $\varrho > 0$, where $\mu \in (0,1)$ and $K_1, K_2 > 0$, then one can estimate the settling time as*

$$t^* = \frac{1}{K_1(1-\mu)} \ln \frac{K_1 V^{1-\mu}(0) + K_2}{K_2}.$$

(2) *If $\Upsilon(\varrho) = K\varrho^\mu$ and $K > 0$, then one can estimate the settling time as*

$$t^* = \frac{V^{1-\mu}(0)}{K(1-\mu)}.$$

Remark 2.2 Since $\Phi(\cdot)$ is strictly monotone increasing and $a_i(\Phi_i(t))$ is an abstract function which contains linear functions as special cases. In other words, we can express $a_i(\Phi_i(t))$ as a common function $-D_i(t)\Phi_i(t)$. Compared with the existing papers, our model which considers CGNNs is more general and common of previous results.

3 Main results

Firstly, we focus on ensuring the finite-time synchronization issue between the above response model (4) and the drive model (1). Let $e_i(t) = y_i(t) - x_i(t)$, $i = 1,2,\dots,n$, one can obtain

$$\begin{aligned} \frac{\mathrm{d}e_i(t)}{\mathrm{d}t} ={}& -\bigl(a_i\bigl(\Phi_i\bigl(y_i(t)\bigr)\bigr) - a_i\bigl(\Phi_i\bigl(x_i(t)\bigr)\bigr)\bigr) + \sum_{j=1}^n b_{ij}(t)\gamma_j^*(t) + \sum_{j=1}^n c_{ij}(t)\gamma_j^*(t-\tau) \\ &+ \sum_{j=1}^n \int_0^{+\infty} K_{ij}(t,s)\gamma_j^*(s)\,\mathrm{d}s + \frac{u_i(t)}{\varpi_i(\Phi_i(y_i(t)))}, \end{aligned} \tag{6}$$

where $\gamma_j^*(t) = \widetilde{\gamma}_j(t) - \gamma_j(t)$.

Then we consider the following two kinds of important controllers to achieve the finite-time synchronization issue.

Case (1). The state-feedback controller $u_i(t)$:

$$u_i(t) = -k_1\bigl(\pi_i(t) - \xi_i(t)\bigr) - k_2 \operatorname{sign}\bigl(\pi_i(t) - \xi_i(t)\bigr), \tag{7}$$

where $i = 1,2,\dots,N$, $k_1, k_2 > 0$.

Case (2). The corresponding adaptive controller $u_i(t)$s of Case (1):

$$u_i(t) = -p_i\bigl(\pi_i(t) - \xi_i(t)\bigr) - q_i \operatorname{sign}\bigl(\pi_i(t) - \xi_i(t)\bigr), \tag{8}$$

where $P = \operatorname{diag}(p_1, p_2, \dots, p_n)$, $Q = \operatorname{diag}(q_1, q_2, \dots, q_n)$, and the controller rules of p_i and q_i are as follows:

$$\begin{aligned} \dot{p}_i &= e_i^T(t)\frac{\rho_i}{\varpi_i(y_i(t))}\bigl(\pi_i(t) - \xi_i(t)\bigr) \quad \text{and} \\ \dot{q}_i &= e_i^T(t)\frac{\varrho_i}{\varpi_i(y_i(t))}\operatorname{sign}\bigl(\pi_i(t) - \xi_i(t)\bigr), \quad i = 1, 2, \dots, n, \end{aligned}$$

ρ_i and ϱ_i are arbitrary positive constants.

Theorem 3.1 *If conditions* (H1)–(H3) *are supported, the response system* (4) *with state-feedback controller* (7) *can synchronize to the corresponding drive system* (1) *in a finite-time if the following assumption holds*:

(H4) $\underline{\omega} + \frac{k_1 \cdot \underline{\omega}}{\overline{\omega}} > (b^{\max} + c^{\max} + K^{\max}) \cdot nL^{\max}$ *and* $\frac{k_2}{\overline{\omega}} > (b^{\max} + c^{\max} + K^{\max}) \cdot nh^{\max}$.

Proof Let

$$V(t) = \frac{1}{2}e^T(t)e(t) + \frac{n}{2} \cdot L^{\max}c^{\max}\int_{t-\tau}^{t} e^T(s) \cdot e(s)\,\mathrm{d}s,$$

where $e(t) = (e_1(t), e_2(t), \dots, e_n(t))^T$.

Obviously $V(t)$ is C-regular. The derivative of Lyapunov function $V(t)$ can be obtained along the error system (6) as follows:

$$\begin{aligned} \frac{\mathrm{d}V(t)}{\mathrm{d}t} &= e^T(t)\dot{e}(t) = \sum_{i=1}^{n} e_i^T(t)\dot{e}_i(t) \\ &= \sum_{i=1}^{n} e_i^T(t)\Biggl(-\bigl(a_i\bigl(\Phi_i\bigl(y_i(t)\bigr)\bigr) - a_i\bigl(\Phi_i\bigl(x_i(t)\bigr)\bigr)\bigr) + \sum_{j=1}^{n} b_{ij}(t)\gamma_j^*(t) \\ &\quad + \sum_{j=1}^{n} c_{ij}(t)\gamma_j^*(t-\tau) + \sum_{j=1}^{n}\int_0^{+\infty} K_{ij}(t,s)\gamma_j^*(s)\,\mathrm{d}s \\ &\quad - \frac{k_1}{\varpi_i(y_i(t))}\bigl(\pi_i(t) - \xi_i(t)\bigr) - \frac{k_2}{\varpi_i(y_i(t))}\operatorname{sign}\bigl(\pi_i(t) - \xi_i(t)\bigr)\Biggr) \\ &\quad + \frac{n}{2} \cdot L^{\max}c^{\max}e^T(t)e(t) - \frac{n}{2} \cdot L^{\max}c^{\max}e^T(t-\tau)e(t-\tau) \\ &= -\sum_{i=1}^{n} e_i^T(t)\bigl(a_i\bigl(\Phi_i\bigl(y_i(t)\bigr)\bigr) - a_i\bigl(\Phi_i\bigl(x_i(t)\bigr)\bigr)\bigr) + \sum_{i=1}^{n}\sum_{j=1}^{n} e_i^T(t)b_{ij}(t)\gamma_j^*(t) \\ &\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} e_i^T(t)c_{ij}(t)\gamma_j^*(t-\tau) + \sum_{i=1}^{n}\sum_{j=1}^{n} e_i^T(t)\int_0^{+\infty} K_{ij}(t,s)\gamma_j^*(s)\,\mathrm{d}s \\ &\quad - \sum_{i=1}^{n} e_i^T(t)k_1\frac{1}{\varpi_i(y_i(t))}\bigl(\Phi_i\bigl(y_i(t)\bigr) - \Phi_i\bigl(x_i(t)\bigr)\bigr) \end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=1}^{n} e_i^T(t) k_2 \frac{1}{\varpi_i(y_i(t))} \operatorname{sign}\bigl[\Phi_i\bigl(y_i(t)\bigr) - \Phi_i\bigl(x_i(t)\bigr)\bigr] \\
&+ \frac{n}{2} \cdot L^{\max} c^{\max} e^T(t) e(t) - \frac{n}{2} \cdot L^{\max} c^{\max} e^T(t-\tau) e(t-\tau).
\end{aligned}
\tag{9}
$$

Based on the definition of function $\Phi(\cdot)$ and generalized mean value theorem, one can have

$$
-\sum_{i=1}^{n} e_i^T(t)\bigl(a_i\bigl(\Phi_i\bigl(y_i(t)\bigr)\bigr) - a_i\bigl(\Phi_i\bigl(x_i(t)\bigr)\bigr)\bigr) \le -\sum_{i=1}^{n} e_i^T(t) \underline{\omega} e_i(t). \tag{10}
$$

From assumptions (H1)–(H2), we can obtain that

$$
\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} e_i^T(t) b_{ij}(t) \gamma_j^*(t) &\le \sum_{i=1}^{n}\sum_{j=1}^{n} \bigl|e_i^T(t)\bigr| \cdot \bigl|b_{ij}(t)\bigr| \cdot \bigl|\gamma_j^*(t)\bigr| \\
&\le \sum_{i=1}^{n}\sum_{j=1}^{n} \bigl|e_i^T(t)\bigr| \cdot \bigl|b_{ij}(t)\bigr| \cdot \bigl(L_j\bigl|e_j(t)\bigr| + h_j\bigr) \\
&\le n b^{\max} L^{\max} \sum_{i=1}^{n} e_i^T(t) e_i(t) + n b^{\max} h^{\max} \sum_{i=1}^{n}\sum_{k=1}^{n} \bigl|e_{ik}(t)\bigr|,
\end{aligned}
\tag{11}
$$

and

$$
\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} e_i^T(t) c_{ij}(t) \gamma_j^*(t-\tau) &\le \sum_{i=1}^{n}\sum_{j=1}^{n} \bigl|e_i^T(t)\bigr| \cdot \bigl|c_{ij}(t)\bigr| \cdot \bigl|\gamma_j^*(t-\tau)\bigr| \\
&\le \sum_{i=1}^{n}\sum_{j=1}^{n} \bigl|e_i^T(t)\bigr| \cdot \bigl|c_{ij}(t)\bigr| \cdot \bigl(L_j\bigl|e_j(t-\tau)\bigr| + h_j\bigr) \\
&\le c^{\max} L^{\max} \sum_{i=1}^{n}\sum_{j=1}^{n} e_i^T(t) e_j(t-\tau) + n c^{\max} h^{\max} \sum_{i=1}^{n}\sum_{i=1}^{n} \bigl|e_{ik}(t)\bigr| \\
&\le c^{\max} L^{\max} \left(\frac{n}{2}\sum_{i=1}^{n} e_i^T(t) e_i(t) + \frac{n}{2}\sum_{j=1}^{n} e_j^T(t-\tau) e_j(t-\tau)\right) \\
&\quad + n c^{\max} h^{\max} \sum_{i=1}^{n}\sum_{k=1}^{n} \bigl|e_{ik}(t)\bigr|.
\end{aligned}
\tag{12}
$$

From condition (H3), similarly we have

$$
\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} e_i^T(t) \int_0^{+\infty} K_{ij}(t,s) \gamma_j^*(s)\,\mathrm{d}s &\le \sum_{i=1}^{n}\sum_{j=1}^{n} K_{ij} e_i^T(t) \gamma_j^*(s) \\
&\le n K^{\max} L^{\max} \sum_{i=1}^{n} e_i^T(t) e_i(t) \\
&\quad + n K^{\max} h^{\max} \sum_{i=1}^{n}\sum_{k=1}^{n} \bigl|e_{ik}(t)\bigr|.
\end{aligned}
\tag{13}
$$

Moreover, because $\Phi_i(\cdot)$ is strictly monotone increasing, which implies that $\operatorname{sign}(\Phi_i(y_i(t)) - \Phi_i(x_i(t))) = \operatorname{sign}(y_i(t) - x_i(t)) = \operatorname{sign} e_i(t)$, then we obtain

$$-\sum_{i=1}^{n} e_i^T(t) k_1 \frac{1}{\varpi_i(y_i(t))}\big(\Phi_i\big(y_i(t)\big) - \Phi_i\big(x_i(t)\big)\big) \le -\sum_{i=1}^{n} \frac{k_1 \cdot \underline{\omega}}{\overline{\varpi}} e_i^T(t) e_i(t), \tag{14}$$

$$-\sum_{i=1}^{n} e_i^T(t) k_2 \frac{1}{\varpi_i(y_i(t))} \operatorname{sign}\big[\Phi_i\big(y_i(t)\big) - \Phi_i\big(x_i(t)\big)\big] \le -\sum_{i=1}^{n}\sum_{k=1}^{n} \frac{k_2}{\overline{\varpi}}\big|e_{ik}(t)\big|. \tag{15}$$

Recalling controller (9) and combined with equations (10)–(15), based on the basic inequalities of Jensen's inequality, one can gain

$$\begin{aligned}\frac{\mathrm{d}V(t)}{\mathrm{d}t} &\le -\left[\underline{\omega} + \frac{k_1 \cdot \underline{\omega}}{\overline{\varpi}} - \big(b^{\max} + c^{\max} + K^{\max}\big) \cdot nL^{\max}\right] \cdot \sum_{i=1}^{n} e_i^T(t) e_i(t) \\ &\quad - \left[\frac{k_2}{\overline{\varpi}} - \big(b^{\max} + c^{\max} + K^{\max}\big) \cdot nh^{\max}\right] \cdot \sum_{i=1}^{n}\sum_{k=1}^{n}\big|e_{ik}(t)\big| \\ &\le -\kappa_1 V(t) - \kappa_2 V^{\frac{1}{2}}(t).\end{aligned} \tag{16}$$

Then, according to assumption (H4) in this theorem, one can see $\kappa_1 > 0$ and $\kappa_2 > 0$, then we know that the origin of error system (6) is finite-time stable with feedback controller (7), and the settling time is obtained by

$$t_1^* \le \frac{2}{\kappa_1} \ln \frac{\kappa_1 V^{\frac{1}{2}}(0) + \kappa_2}{\kappa_2} = \frac{2}{\kappa_1} \ln \frac{\kappa_1 \|e(0)\|_2 + \kappa_2}{\kappa_2}. \qquad \square$$

Theorem 3.2 *If conditions* (H1)–(H3) *are supported, then the response system* (4) *with adaptive controller* (8) *can synchronize to the corresponding drive system* (1).

Proof In this proof, we let the new Lyapunov functional be as follows:

$$\begin{aligned}V(t) &= \frac{1}{2}\sum_{i=1}^{n} e_i^T(t) e_i(t) + \frac{n}{2} \cdot L^{\max} c^{\max} \int_{t-\tau}^{t} e^T(s) \cdot e(s)\,\mathrm{d}s \\ &\quad + \sum_{i=1}^{n} \frac{1}{2\rho_{ik}}(p_i - \theta)^2 + \sum_{i=1}^{n} \frac{1}{2\varrho_i}(q_i - \vartheta)^2,\end{aligned}$$

where θ and ϑ are positive constants to be determined.

The derivative of Lyapunov function $V(t)$ can be obtained along the error system (6) as follows:

$$\begin{aligned}\dot{V}(t) &= \sum_{i=1}^{n} e_i^T(t)\Bigg[-\big(a_i\big(\Phi_i\big(y_i(t)\big)\big) - a_i\big(\Phi_i\big(x_i(t)\big)\big)\big) + \sum_{j=1}^{n} b_{ij}(t)\gamma_j^*(t) + \sum_{j=1}^{n} c_{ij}(t)\gamma_j^*(t-\tau) \\ &\quad + \sum_{j=1}^{n}\int_0^{+\infty} K_{ij}(t,s)\gamma_j^*(s)\,\mathrm{d}s - \frac{p_i}{\varpi_i(y_i(t))}\big(\pi_i(t) - \xi_i(t)\big) \\ &\quad - \frac{q_i}{\varpi_i(y_i(t))}\operatorname{sign}\big(\pi_i(t) - \xi_i(t)\big)\Bigg]\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}(p_i-\theta)e_i^T(t)\frac{1}{\varpi_i(y_i(t))}\big(\pi_i(t)-\xi_i(t)\big)\\
&+\sum_{i=1}^{n}(q_i-\vartheta)e_i^T(t)\frac{1}{\varpi_i(y_i(t))}\operatorname{sign}\big(\pi_i(t)-\xi_i(t)\big)\\
&+\frac{n}{2}\cdot L^{\max}c^{\max}e^T(t)e(t)-\frac{n}{2}\cdot L^{\max}c^{\max}e^T(t-\tau)e(t-\tau).
\end{aligned}
\tag{17}
$$

Since

$$
-\sum_{i=1}^{n}e_i^T(t)\frac{\theta}{\varpi_i(y_i(t))}\big(\Phi_i\big(y_i(t)\big)-\Phi_i\big(x_i(t)\big)\big)\le-\sum_{i=1}^{n}\frac{\theta\cdot\underline{\omega}}{\overline{\varpi}}e_i^T(t)e_i(t),
\tag{18}
$$

$$
-\sum_{i=1}^{n}e_i^T(t)\frac{\vartheta}{\varpi_i(y_i(t))}\operatorname{sign}\big[\Phi_i\big(y_i(t)\big)-\Phi_i\big(x_i(t)\big)\big]\le-\sum_{i=1}^{n}\sum_{k=1}^{n}\frac{\vartheta}{\overline{\varpi}}\big|e_{ik}(t)\big|.
\tag{19}
$$

It follows from (18)–(19), recalling (10)–(13), one can deduce that

$$
\begin{aligned}
\frac{\mathrm{d}V(t)}{\mathrm{d}t}&\le-\left[\underline{\omega}+\frac{\theta\cdot\underline{\omega}}{\overline{\varpi}}-\big(b^{\max}+c^{\max}+K^{\max}\big)\cdot nL^{\max}\right]\cdot\sum_{i=1}^{n}e_i^T(t)e_i(t)\\
&\quad-\left[\frac{\vartheta}{\overline{\varpi}}-\big(b^{\max}+c^{\max}+K^{\max}\big)\cdot nh^{\max}\right]\cdot\sum_{i=1}^{n}\sum_{k=1}^{n}\big|e_{ik}(t)\big|\\
&\le-\kappa_3V(t)-\kappa_4V^{\frac{1}{2}}(t).
\end{aligned}
\tag{20}
$$

Based on the definition of θ and ϑ, we can choose suitable values of θ and ϑ to make $\kappa_3=\underline{\omega}+\frac{\theta\cdot\underline{\omega}}{\overline{\varpi}}-(b^{\max}+c^{\max}+K^{\max})\cdot nL^{\max}>0$ and $\kappa_4=\frac{\vartheta}{\overline{\varpi}}-(b^{\max}+c^{\max}+K^{\max})\cdot nh^{\max}>0$. Then we prove that the origin of error system (6) is finite-time stable with adaptive controller (8), and the settling time is obtained by

$$
t_1^*\le\frac{2}{\kappa_3}\ln\frac{\kappa_3V^{\frac{1}{2}}(0)+\kappa_4}{\kappa_4}=\frac{2}{\kappa_3}\ln\frac{\kappa_3\|e(0)\|_2+\kappa_4}{\kappa_4}.
\qquad\square
$$

Remark 3.3 This paper considers that the distributed delays are unbounded, which is more difficult to verify than the bounded case. In the pervious results, the delay kernels satisfy

$$
K_{ij}=\begin{cases}1, & 0\le t\le\tau_{ij},\\ 0, & t>\tau_{ij},\end{cases}
$$

where $\tau_{ij}>0$ are constants, then the CGNNs can be rewritten as a special case in this paper.

Remark 3.4 In fact, the finite time synchronization problem is very complex and difficult to calculate when there exist the discontinuity phenomenon, mixed delays, and switching controllers in the traditional neural network model. This paper overcomes these difficulties and has some innovation. By using Theorem 3.1 and Theorem 3.2, the finite-time synchronization problem can be generalized, that is, by choosing the appropriate controller, the stability time of the synchronization error system can be estimated more easily. On the other hand, the controller selected in this paper is more widely used as the estimation of stability time. It provides a theoretical basis for solving complex problems in engineering application.

4 Examples

Example 4.1 Consider the following two-dimensional discontinuous CGNNs with and mixed delays:

$$\begin{aligned}\frac{\mathrm{d}\pi_i(t)}{\mathrm{d}t} &= -\varpi_i\big(\pi_i(t)\big)\Bigg[a_i\big(\pi_i(t)\big) - \sum_{j=1}^{n} b_{ij}(t)f_j\big(\pi_j(t)\big) - \sum_{j=1}^{n} c_{ij}(t)f_j\big(\pi_j\big(t-\tau_{ij}(t)\big)\big) \\ &\quad - \sum_{j=1}^{n}\int_0^{+\infty} K_{ij}(t,s)f_j\big(\pi_j(s)\big)\,\mathrm{d}s - I_i(t)\Bigg], \quad i = 1,2. \end{aligned} \tag{21}$$

Let $\varpi_1(\pi_1(t)) = 0.5 + 0.1\cos(\pi_1(t))$, $\varpi_2(\pi_2(t)) = 0.5 - 0.1\sin(\pi_2(t))$, $a_1(\pi_1(t)) = -0.4\pi_1(t)$, $a_2(\pi_2(t)) = -0.4\pi_2(t)$, $b_{11}(t) = b_{22}(t) = 0.1$, $b_{12}(t) = b_{21}(t) = 0$, $c_{11}(t) = c_{22}(t) = 0.2$, $c_{12}(t) = c_{21}(t) = 0$, $K_{11} = K_{12} = K_{21} = K_{22} = 1$, and $I_1(t) = 0.2\sin\sqrt{2}t + 0.1\sin\sqrt{5}t$, $I_2(t) = 0.3 \times \cos\sqrt{3}t - 0.2\sin 2t$.

The two neuron activation functions which satisfy the two conditions (H1)–(H2) are designed as follows:

$$f_1(\cdot) = f_2(\cdot) = \begin{cases} x - 0.1, & x < 0, \\ x + 0.1, & x \geq 0. \end{cases}$$

Let $L^{\max} = h^{\max} = 1$, then we consider the control rule $u_i(t) = -k_1(\pi_i(t) - \xi(t)) - k_2\operatorname{sign}(\pi_i(t) - \xi(t))$ with $k_1 = k_2 = 3.5$. It is easy to check that

$$\begin{aligned} &2.7 = \underline{\omega} + \frac{k_1 \cdot \underline{\omega}}{\overline{\omega}} > \big(b^{\max} + c^{\max} + K^{\max}\big) \cdot nL^{\max} = 2.6, \\ &5.8 = \frac{k_2}{\overline{\omega}} > \big(b^{\max} + c^{\max} + K^{\max}\big) \cdot nh^{\max} = 2.6, \end{aligned}$$

which show that the assumption in Theorem 3.1 is satisfied. Figure 1 and Fig. 2(c) indicate the simulation results.

Example 4.2 Consider the following discontinuous CGNNs with mixed delays:

$$\begin{aligned}\frac{\mathrm{d}\pi_i(t)}{\mathrm{d}t} &= -\varpi_i\big(\pi_i(t)\big)\Bigg[a_i\big(\pi_i(t)\big) - \sum_{j=1}^{n} b_{ij}(t)f_j\big(\pi_j(t)\big) - \sum_{j=1}^{n} c_{ij}(t)f_j\big(\pi_j\big(t-\tau_{ij}(t)\big)\big) \\ &\quad - \sum_{j=1}^{n}\int_0^{+\infty} K_{ij}(t,s)f_j\big(\pi_j(s)\big)\,\mathrm{d}s - I_i(t)\Bigg], \quad i = 1,2, \end{aligned} \tag{22}$$

where the parameters have the same meanings as in equation (21).

We consider the novel adaptive controller $u_i(t) = -p_i(\pi_i(t) - \xi(t)) - q_i\operatorname{sign}(\pi_i(t) - \xi(t))$ with $\dot{p}_i = e_i^T(t)\frac{\varrho_i}{\varpi_i(y_i(t))}(\pi_i(t) - \xi_i(t))$, $\dot{q}_i = e_i^T(t)\frac{\varrho_i}{\varpi_i(y_i(t))}\operatorname{sign}(\pi_i(t) - \xi_i(t))$. For CGNNs (22), we

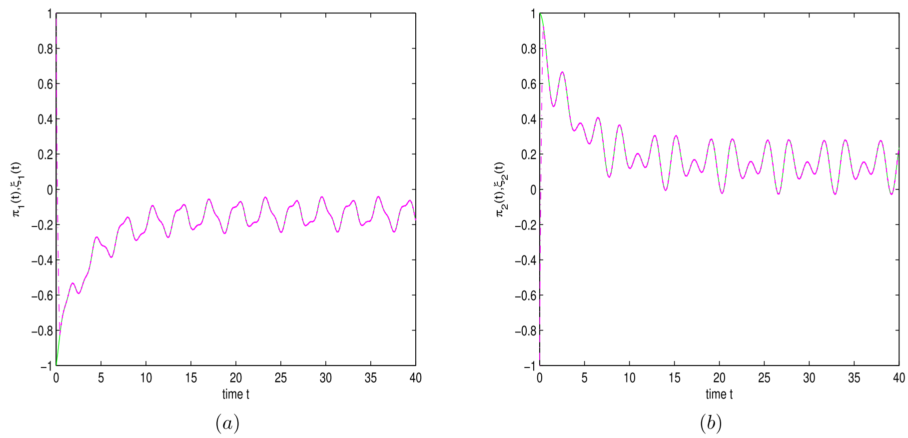

Figure 1 (**a**) and (**b**) show the state trajectories $\pi_1(t)$, $\xi_1(t)$ and state trajectories $\pi_2(t)$, $\xi_2(t)$ in Example 4.1 and Example 4.2, respectively

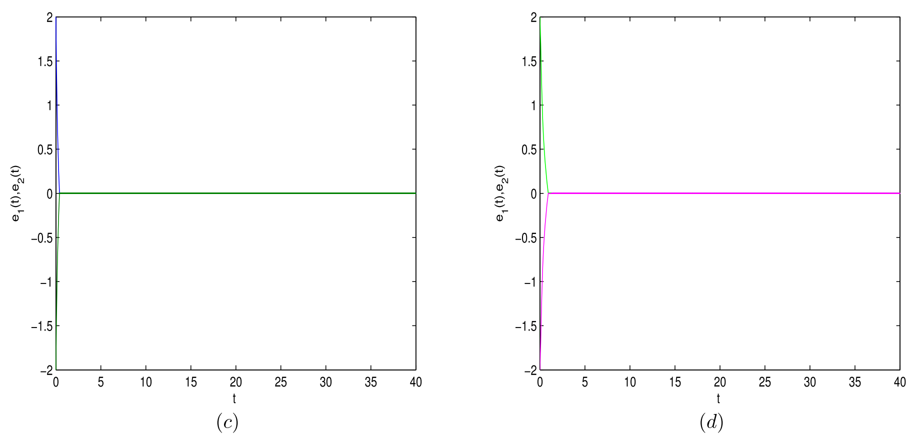

Figure 2 (**c**) Synchronization error trajectories of the drive system and the response system with feedback controller can be achieved in finite-time in Example 4.1; (**d**) Synchronization error trajectories of the drive system and the response system with adaptive controller can be achieved in finite-time in Example 4.2

can choose suitable parameters $\theta = 3$ and $\vartheta = 3$ to make the condition in Theorem 3.2 be satisfied. With given random initial state, Fig. 2(d) shows that the two state trajectories approach the zero solution, then the simulation result is presented to illustrate the obtained theoretical findings.

5 Conclusions

The finite-time synchronization problem of discontinuous CGNNs with mixed delay is studied in this brief. By using the Lyapunov functional framework, new mathematical analysis techniques, Filippov theory, and inequality techniques, a new state feedback controller and an adaptive controller are constructed to realize finite-time synchronization of complex neural networks. Compared with previous results, we overcome the problem of non-Lipschitz continuity system, so how to deal with the right discontinuous system is a challenge. Finally, two numerical simulations verify the advantages of the proposed switching control method and mathematical calculation method. Future research will focus on the analysis of neural networks with time delay and the design of more effective coupling schemes between different neurons in the system.

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Availability of data and materials
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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
ZX and RL initiated and discussed the research problem; ZX verified the correctness of the experimental results; RL performed the experiments and took figures, analyzed the data; ZX drafted the paper. All authors have read and approved the paper.

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