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Stability of solutions for generalized fractional differential problems by applying significant inequality estimates

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Abstract

In this research paper, we intend to study the stability of solutions of some nonlinear initial value fractional differential problems. These equations are studied within the generalized fractional derivative of various orders. In order to study the solutions' decay to zero as a power function, we establish sufficient conditions on the nonlinear terms. To this end, some versions of inequalities are combined and generalized via the so-called Bihari inequality. Moreover, we employ some properties of the generalized fractional derivative and appropriate regularization techniques. Finally, the paper involves examples to affirm the validity of the results.

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1 Introduction

Fractional calculus has gotten much consideration from analysts and engineers, as well as a tool in various areas of engineering, applied mathematics, and physics. Fractional-order differential equations are used to study plentiful phenomena such as fluid mechanics, plasma physics, optical fibers, nonlinear oscillations of an earthquake, flow in nonlinear electric circuits, biology, aerodynamics, mechanics, and regular variations in thermodynamics. Actually, the transform from theoretical to the application aspect of fractional calculus was appeared due to the work by Bagley and Torvik [1–3]. A generalized fractional derivative with respect to function t^ρ is a novel sort of fractional derivatives, which has been presented by Kilbas et al. [4], and then modified by Katugampola [5] and Almeida et al. [6]. In a series of papers [7–18] the authors studied the qualitative analysis for some classes of fractional differential equations involving this generalized fractional derivative.

Fractional-order derivatives can describe the asymptotic behavior of some nonlinear systems more comprehensively as compared to integer order. Further, integer-order derivatives are the special case of fractional-order derivative. The most significant peculiarity of the considered derivative here is that it provides a general platform that covers most classical fractional derivatives (e.g. Riemann–Liouville and Hadamard). That is why

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this fractional derivative is often referred to as a generalized fractional derivative. The said operator has been used in many articles. The said operator has been found useful in dealing with many real problems.

Fractional differential problems can describe the dynamics of several complex and non-local systems with memory. They emerge in numerous scientific and engineering areas. Especially, nonlinear systems describing various phenomena can be modeled with fractional derivatives. These fractional operators possess memory and this memory is efficient in describing and modeling complex systems nonlocally.

In this regard, the researchers studied many models and used a fractional-order derivative to describe the solution of them. Among them studying the asymptotic behavior of such models has meaningful interpretations like permanence, instability, and chaotic developments. The main aim of this article is to investigate the long-time behavior of solutions of the following problem:

$$
\begin{cases}
{}^{\rho}\mathfrak{D}_a^{\sigma}\varpi(\varkappa) = \mathfrak{F}(\varkappa, \varpi(\varkappa), {}^{\rho}\mathfrak{D}_a^{\gamma}\varpi(\varkappa)), & 0 < \gamma < \sigma < 1, \rho > 0, \varkappa > a > 0, \\
{}^{\rho}\mathfrak{J}_a^{1-\sigma}\varpi(\varkappa)|_{\varkappa=a} = b \in \mathbb{R},
\end{cases}
\tag{1.1}
$$

here ${}^{\rho}\mathfrak{J}_a^r$ and ${}^{\rho}\mathfrak{D}_a^r$ are the generalized fractional integral and derivative of order $r > 0$ ($r \in \{\sigma, \gamma, 1 - \sigma\}$), respectively. These types of operators are defined in the next sections. In [19], Furati and Tatar studied the problem

$$
\begin{cases}
\mathfrak{D}_a^{\vartheta_1}\varpi(\varkappa) = \mathfrak{F}(\varkappa, \varpi), & 0 < \vartheta_1 < 1, \varkappa > 0, \\
\varkappa^{(1-\vartheta_1)}\varpi(\varkappa)|_{\varkappa=0} = b \in \mathbb{R},
\end{cases}
\tag{1.2}
$$

and they proved that the solutions decay to zero. In this respect, Furati et al. [20] considered the problem

$$
\begin{cases}
\mathfrak{D}_0^{\vartheta_1,\vartheta_2}\varpi(\varkappa) = \mathfrak{F}(\varkappa, \varpi), & 0 < \vartheta_1 < 1, \varkappa > 0, \\
\varkappa^{(1-\vartheta_1)(1-\vartheta_2)}\varpi(\varkappa)|_{\varkappa=0} = b \in \mathbb{R},
\end{cases}
\tag{1.3}
$$

here $\mathfrak{D}_0^{\vartheta_1,\vartheta_2} = \mathfrak{J}_0^{\vartheta_2(1-\vartheta_1)}D\mathfrak{J}_0^{(1-\vartheta_2)(1-\vartheta_2)}$. They demonstrated that solutions of this problem decay as a power function. Plociniczak [21] studied the equation

$$
{}^{c}\mathfrak{D}^{\vartheta_1}\varpi(\varkappa) = \lambda q(\varkappa)\varpi(\varkappa), \quad 0 < \vartheta_1 < 1, \varkappa > 0,
\tag{1.4}
$$

where ${}^{c}\mathfrak{D}^{\vartheta_1}$ is the Caputo derivative and $q(\varkappa) \sim C_q \varkappa^{\nu} > 0$, $\nu > 0$. The author showed that the solutions of (1.4) obey the asymptotic properties according to values of λ. Medved et al. [22] considered the problem

$$
{}^{c}\mathfrak{D}_0^{\vartheta_1}\varpi(\varkappa) = \mathfrak{F}(\varkappa, \varpi(\varkappa), {}^{c}\mathfrak{D}_0^{\vartheta_2}\varpi(\varkappa)), \quad 0 < \vartheta_2 < \vartheta_1 < 1, \varkappa > 0.
\tag{1.5}
$$

They demonstrated that any solution of (1.5) has the asymptotic property $\varpi(\varkappa) = c\varkappa^{\vartheta_2} + o(\varkappa^{\vartheta_2})$ when $\varkappa \to \infty$ for some $c \in \mathbb{R}$. Kassim et al. [23] studied the equation

$$
\mathfrak{D}_0^{\vartheta_1}\varpi(\varkappa) = \mathfrak{F}(\varkappa, \varpi(\varkappa), \mathfrak{D}_0^{\vartheta_2}\varpi(\varkappa)), \quad 0 \le \vartheta_2 < \vartheta_1 < 1, \varkappa > 0.
\tag{1.6}
$$

They proved that solutions of (1.6) decay to zero. Recently, Kassim et al. [24] investigated the equation

$$\mathcal{D}_a^{\vartheta_1} \varpi(\varkappa) = \mathfrak{F}\big(\varkappa, \varpi(\varkappa), \mathcal{D}_a^{\vartheta_2} \varpi(\varkappa)\big), \quad 0 \le \vartheta_2 < \vartheta_1 < 1, \varkappa > a, \tag{1.7}$$

where $\mathcal{D}_a^\kappa$ is the Hadamard fractional derivative of order $\kappa \in \{\vartheta_1, \vartheta_2\}$ and they showed that solutions of (1.7) decay to zero.

Moreover, in 2020, the same authors [25] discussed the equation

$$^c\mathfrak{D}_0^{\vartheta_1} \varpi(\varkappa) = \mathfrak{F}\big(t, \varpi(\varkappa), {}^c\mathfrak{D}_0^{\vartheta_2} \varpi(\varkappa)\big), \quad \vartheta_2 < \vartheta_1. \tag{1.8}$$

They proved that the solutions of (1.8) approach power type functions for $1 < \vartheta_1 < 2$ and bounded for $0 < \vartheta_2 < 1$. For more results related to asymptotic behavior, we refer to [19, 20, 23, 26–32], and the references therein. In this letter, generalization of the results in [19, 20, 22–24] to problem (1.1) was established. In particular, the problems (1.6) and (1.7) studied in [23, 24] become special cases of (1.1) when $\rho = 1$ and $\rho \to 0$, respectively.

In the present work, we prove under certain conditions the solutions of (1.1) decay towards zero as $(\frac{\varkappa^\rho - a^\rho}{\rho})^{\vartheta_1 - 1}$. The investigated fractional derivatives belong to a general class of fractional operators including Riemann–Liouville, Caputo, and in the limiting case to zero the Hadamard and Caputo–Hadamard fractional derivatives. The used techniques are novel and new and applied in a clever way. The studied fractional equation is studied in this category for the first time and this will help researchers to proceed more in this direction.

In Sect. 2, we prepare some materials. In Sect. 3, we state and prove the main result. Illustrative examples are given in Sect. 4. Concluding remarks are discussed in the last section.

2 Preliminaries

In this section, we briefly recall some definitions, lemmas, properties and notations and well-known estimations we will use later.

Definition 2.1 ([12]) Let $\rho > 0$ and $0 \le \gamma < 1$, we introduce the spaces $C_{\rho,\gamma}[a,b]$ and $C_{\rho,\gamma}^n[a,b]$ as follows:

$$C_{\rho,\gamma}[a,b] = \left\{ g(\varkappa) \in C(a,b], \left(\frac{\varkappa^\rho - a^\rho}{\rho} \right)^\gamma g(\varkappa) \in C[a,b] \right\},$$

$$C_{\rho,\gamma}^n[a,b]$$
$$= \left\{ \left(\varkappa^{1-\rho} \frac{d}{d\varkappa} \right)^n g(\varkappa) \in C_{\rho,\gamma}[a,b], \left(\varkappa^{1-\rho} \frac{d}{d\varkappa} \right)^k g(\varkappa) \in C[a,b], k = 0, \ldots, n-1 \right\},$$

with the norms

$$\|g\|_{C_{\rho,\gamma}} = \left\| \left(\frac{\varkappa^\rho - a^\rho}{\rho} \right)^\gamma g(\varkappa) \right\|_C$$

and

$$\|g\|_{C_{\rho,\gamma}^n} = \left\|\left(\varkappa^{1-\rho}\frac{d}{d\varkappa}\right)^n g\right\|_{C_{\rho,\gamma}} + \sum_{j=0}^{n-1}\left\|\left(\varkappa^{1-\rho}\frac{d}{d\varkappa}\right)^j g\right\|_C,$$

respectively.

Here we present some requisite definitions, notation and properties of the generalized fractional integral and derivative.

Definition 2.2 ([5]) The generalized fractional integral and derivative are defined, respectively, by

$${}^{\rho}\mathfrak{J}_a^{\vartheta_1}g(\varkappa) = \frac{\rho^{1-\vartheta_1}}{\Gamma(\vartheta_1)}\int_a^\varkappa \left(\varkappa^\rho - t^\rho\right)^{\vartheta_1-1} t^{\rho-1}g(t)\,dt, \quad \vartheta_1 > 0, \rho > 0,$$

and

$$\begin{aligned}
{}^{\rho}\mathfrak{D}_a^{\vartheta_1}g(\varkappa) &= \delta_\rho^n\big({}^{\rho}\mathfrak{J}_a^{n-\vartheta_1}g\big)(\varkappa) \\
&= \frac{\rho^{1+\vartheta_1-n}}{\Gamma(n-\vartheta_1)}\left(\varkappa^{1-\rho}\frac{d}{d\varkappa}\right)^n \int_a^\varkappa \left(\varkappa^\rho - t^\rho\right)^{n-\vartheta_1-1} t^{\rho-1}g(t)\,dt, \quad \vartheta_1 > 0, \rho > 0,
\end{aligned}$$

where

$$n = -[-\vartheta_1], \qquad \delta_\rho^n = \left(\varkappa^{1-\rho}\frac{d}{d\varkappa}\right)^n.$$

The generalized fractional integral and derivative (Definition 2.2) satisfy the following properties.

Property 2.3 ([33]) If $\vartheta_1 \geq 0$, $\rho > 0$ and $\vartheta_2 > 0$, then

$${}^{\rho}\mathfrak{J}_a^{\vartheta_1}\left(\frac{\varkappa^\rho - a^\rho}{\rho}\right)^{\vartheta_2} = \frac{\Gamma(\vartheta_2 + 1)}{\Gamma(\vartheta_2 + \vartheta_1 + 1)}\left(\frac{\varkappa^\rho - a^\rho}{\rho}\right)^{\vartheta_2+\vartheta_1}, \quad \varkappa > a,$$

$${}^{\rho}\mathfrak{D}_a^{\vartheta_1}\left(\frac{\varkappa^\rho - a^\rho}{\rho}\right)^{\vartheta_2} = \frac{\Gamma(\vartheta_2 + 1)}{\Gamma(\vartheta_2 - \vartheta_1 + 1)}\left(\frac{\varkappa^\rho - a^\rho}{\rho}\right)^{\vartheta_2-\vartheta_1}, \quad \varkappa > a.$$

Property 2.4 ([33]) Let ϑ_1, ρ, $\vartheta_2 > 0$ and $0 \leq \mu < 1$. If $g \in C_{\mu,\rho}[a,b]$, then

$${}^{\rho}\mathfrak{J}_a^{\vartheta_1}\,{}^{\rho}\mathfrak{J}_a^{\vartheta_2}g(\varkappa) = {}^{\rho}\mathfrak{J}_a^{\vartheta_1+\vartheta_2}g(\varkappa), \quad \varkappa > a.$$

Property 2.5 ([5]) Let $\vartheta_1 > \vartheta_2 > 0$. If $g \in C_{\mu,\rho}[a,b]$, then

$${}^{\rho}\mathfrak{D}_a^{\vartheta_2}\,{}^{\rho}\mathfrak{J}_a^{\vartheta_1}g(\varkappa) = {}^{\rho}\mathfrak{J}_a^{\vartheta_1-\vartheta_2}g(\varkappa), \quad \varkappa > a.$$

Property 2.6 ([5]) If $g \in C_{\mu,\rho}[a,b]$ and $\vartheta_1 > 0$, then

$${}^{\rho}\mathfrak{D}_a^{\vartheta_1}\,{}^{\rho}\mathfrak{J}_a^{\vartheta_1}g = g, \quad \varkappa \in (a,b).$$

Theorem 2.7 ([12]) *Let $0 < \vartheta_1 < 1$, $\rho > 0$ and $0 \leq \gamma < 1$. If $g \in C_{\rho,\gamma}[a,b]$ and $^\rho\mathfrak{J}_a^{1-\vartheta_1}g \in C_{\rho,\gamma}^1[a,b]$, then*

$$^\rho\mathfrak{J}_a^{\vartheta_1}{}^\rho\mathfrak{D}_a^{\vartheta_1}g(\varkappa) = g(\varkappa) - \frac{(^\rho\mathfrak{J}_a^{1-\vartheta_1}g)(a)}{\Gamma(\vartheta_1)}\left(\frac{\varkappa^\rho - a^\rho}{\rho}\right)^{\vartheta_1-1}.$$

Next, we prove the following useful lemma.

Lemma 2.8 *If $g \in C_{1-\vartheta_1,\rho}[a,b]$, $^\rho\mathfrak{J}_a^{1-\vartheta_1}g \in C_{1-\vartheta_1,\rho}^1[a,b]$, $0 < \vartheta_1 < 1$, then, for $0 \leq \vartheta_2 < \vartheta_1 < 1$, we have*

$$^\rho\mathfrak{D}_a^{\vartheta_2}g(\varkappa) = {}^\rho\mathfrak{J}_a^{\vartheta_1-\vartheta_2}{}^\rho\mathfrak{D}_a^{\vartheta_1}g(\varkappa) + \frac{^\rho\mathfrak{J}_a^{1-\vartheta_1}g(a)}{\Gamma(\vartheta_1-\vartheta_2)}\left(\frac{\varkappa^\rho - a^\rho}{\rho}\right)^{\vartheta_1-\vartheta_2-1}, \quad \varkappa \in (a,b].$$

Proof By Theorem 2.7, we have

$$^\rho\mathfrak{J}_a^{\vartheta_1}{}^\rho\mathfrak{D}_a^{\vartheta_1}g(\varkappa) = g(\varkappa) - \frac{(^\rho\mathfrak{J}_a^{1-\vartheta_1}g)(a)}{\Gamma(\vartheta_1)}\left(\frac{\varkappa^\rho - a^\rho}{\rho}\right)^{\vartheta_1-1}, \quad \varkappa > a. \tag{2.1}$$

Applying $^\rho\mathfrak{D}_a^{\vartheta_2}$ to (2.1), using Properties 2.3 and 2.5, we obtain

$$^\rho\mathfrak{D}_a^{\vartheta_2}g(\varkappa) = {}^\rho\mathfrak{J}_a^{\vartheta_1-\vartheta_2}{}^\rho\mathfrak{D}_a^{\vartheta_1}g(\varkappa) + \frac{^\rho\mathfrak{J}_a^{1-\vartheta_1}g(a)}{\Gamma(\vartheta_1-\vartheta_2)}\left(\frac{\varkappa^\rho - a^\rho}{\rho}\right)^{\vartheta_1-\vartheta_2-1}, \quad \varkappa \in (a,b]. \qquad \square$$

We recall some basic well-known results.

Theorem 2.9 (Bihari inequality, [34]) *Let v and g be continuous nonnegative functions defined on $[0,\infty)$. Let $z(v)$ be a continuous nondecreasing function defined on $[0,\infty)$ and $z(v) > 0$ on $(0,\infty)$. If*

$$v(\varkappa) \leq c + \int_0^\varkappa g(s)z(v(s))\,ds,$$

for $\varkappa \in [0,\infty)$, where $c \geq 0$, then

$$v(\varkappa) \leq \mathfrak{F}^{-1}\left(\mathfrak{F}(c) + \int_0^\varkappa g(s)\,ds\right),$$

where $\mathfrak{F}^{-1}$ is the inverse function of

$$\mathfrak{F}(\varkappa) = \int_{\varkappa_0}^\varkappa \frac{ds}{z(s)}, \quad \varkappa > 0, \varkappa_0 > 0.$$

Lemma 2.10 ([35, 36]) *For a nonnegative a and b, we have*

$$a^r + b^r \leq (a+b)^r \leq 2^{r-1}(a^r + b^r), \quad r \geq 1,$$

and

$$2^{r-1}(a^r + b^r) \leq (a+b)^r \leq a^r + b^r, \quad 1 \geq r \geq 0.$$

3 Stability

We discuss the problem (1.1) with the following suppositions.

(A1) $\mathfrak{F}(\varkappa, v, w) : (a, \infty) \times \mathbb{R}^2 \to \mathbb{R}$ is a function such that $\mathfrak{F}(., v(\cdot), w(\cdot)) \in C_{1-\vartheta_1, \rho}[a, \infty)$ for every $v, w \in C_{1-\vartheta_1, \rho}[a, \infty)$.

(A2) There is continuous functions $\varphi_k, k = 1, 2, h : [a, \infty) \to [0, \infty)$, such that

$$\left| \mathfrak{F}(\varkappa, v, w) \right| \leq \left(\frac{\varkappa^\rho - a^\rho}{\rho} \right)^\gamma e^{-\delta(\varkappa^\rho - a^\rho)} h(\varkappa)$$
$$\times \varphi_1 \left(\left(\frac{\varkappa^\rho - a^\rho}{\rho} \right)^{1-\vartheta_1} |v| \right) \varphi_2 \left(\left(\frac{\varkappa^\rho - a^\rho}{\rho} \right)^{1-(\vartheta_1-\vartheta_2)} |w| \right), \quad (3.1)$$

where $\varphi_k, k = 1, 2$, are nondecreasing functions and $\gamma \in \mathbb{R}$.

(A3) There are two continuous functions $\varphi_k, h_k : [a, \infty) \to [0, \infty)$, $k = 1, 2$, such that

$$\left| \mathfrak{F}(\varkappa, v, w) \right| \leq \left(\frac{\varkappa^\rho - a^\rho}{\rho} \right)^{\gamma_1} e^{-\delta_1(\varkappa^\rho - a^\rho)} h_1(\varkappa) \varphi_1 \left(\left(\frac{\varkappa^\rho - a^\rho}{\rho} \right)^{1-\vartheta_1} |v| \right)$$
$$+ \left(\frac{\varkappa^\rho - a^\rho}{\rho} \right)^{\gamma_2} e^{-\delta_2(\varkappa^\rho - a^\rho)} h_2(\varkappa)$$
$$\times \varphi_2 \left(\left(\frac{\varkappa^\rho - a^\rho}{\rho} \right)^{1-(\vartheta_1-\vartheta_2)} |w| \right), \quad (3.2)$$

where $\varphi_k, k = 1, 2$, are nondecreasing functions and $\gamma_k \in \mathbb{R}, k = 1, 2$.

The following general interesting inequality is proved.

Lemma 3.1 *For $\lambda, v, \rho > 0, \omega > 0$, we have*

$$\left(\varkappa^\rho - a^\rho \right)^{1-v} \int_a^\varkappa \left(\varkappa^\rho - s^\rho \right)^{v-1} \left(s^\rho - a^\rho \right)^{\lambda-1} e^{-\omega(s^\rho - a^\rho)} s^{\rho-1} \, ds \leq C, \quad \varkappa > a,$$

where

$$C = \rho^{-1} \max\left(1, 2^{1-v}\right) \omega^{-\lambda} \Gamma(\lambda) \left(1 + \frac{\lambda(\lambda + 1)}{v} \right).$$

Proof Put

$$L(\varkappa) := \left(\varkappa^\rho - a^\rho \right)^{1-v} \int_a^\varkappa \left(\varkappa^\rho - s^\rho \right)^{v-1} \left(s^\rho - a^\rho \right)^{\lambda-1} e^{-\omega(s^\rho - a^\rho)} s^{\rho-1} \, ds, \quad \varkappa > a.$$

Let $r = \frac{s^\rho - a^\rho}{\varkappa^\rho - a^\rho}$. Then $s^\rho = r(\varkappa^\rho - a^\rho) + a^\rho$ and $s^{\rho-1} \, ds = \frac{1}{\rho}(\varkappa^\rho - a^\rho) \, dr$. Therefore

$$L(\varkappa) = \left(\varkappa^\rho - a^\rho \right)^{1-v} \int_0^1 \left(\varkappa^\rho - a^\rho - \left(\varkappa^\rho - a^\rho \right) r \right)^{v-1} \left(\left(\varkappa^\rho - a^\rho \right) r \right)^{\lambda-1}$$
$$\times e^{-\omega r(\varkappa^\rho - a^\rho)} \frac{1}{\rho} \left(\varkappa^\rho - a^\rho \right) dr$$
$$= \frac{1}{\rho} \left(\varkappa^\rho - a^\rho \right)^\lambda \int_0^1 (1-r)^{v-1} r^{\lambda-1} e^{-\omega r(\varkappa^\rho - a^\rho)} \, dr$$
$$= \frac{1}{\rho} \left(\varkappa^\rho - a^\rho \right)^\lambda \int_0^{1/2} (1-r)^{v-1} r^{\lambda-1} e^{-\omega r(\varkappa^\rho - a^\rho)} \, dr$$

$$+ \frac{1}{\rho} \left(\varkappa^\rho - a^\rho \right)^\lambda \int_{1/2}^1 (1-r)^{\nu-1} r^{\lambda-1} e^{-\omega r(\varkappa^\rho - a^\rho)} \, dr$$

$$\leq \rho^{-1} \max\left(1, 2^{1-\nu}\right) \left(\varkappa^\rho - a^\rho \right)^\lambda \int_0^{1/2} r^{\lambda-1} e^{-\omega r(\varkappa^\rho - a^\rho)} \, dr$$

$$+ \rho^{-1} \left(\varkappa^\rho - a^\rho \right)^\lambda \int_{1/2}^1 (1-r)^{\nu-1} r^{\lambda-1} e^{-\omega r(\varkappa^\rho - a^\rho)} \, dr. \tag{3.3}$$

Let $u = \omega r(\varkappa^\rho - a^\rho)$, then $du = \omega(\varkappa^\rho - a^\rho)\, dr$ and

$$\left(\varkappa^\rho - a^\rho \right)^\lambda \int_0^{1/2} r^{\lambda-1} e^{-\omega r(\varkappa^\rho - a^\rho)} \, dr \leq \omega^{-\lambda} \int_0^\infty u^{\lambda-1} e^{-u} \, du = \omega^{-\lambda} \Gamma(\lambda). \tag{3.4}$$

If $\omega r(\varkappa^\rho - a^\rho) \geq 1$, then

$$e^{\omega r(\varkappa^\rho - a^\rho)} \geq \frac{(\omega r(\varkappa^\rho - a^\rho))^{[\lambda]+1}}{([\lambda]+1)!} = \frac{(\omega r(\varkappa^\rho - a^\rho))^{[\lambda]+1}}{\Gamma([\lambda]+2)} \geq \frac{(\omega r(\varkappa^\rho - a^\rho))^\lambda}{\Gamma(\lambda+2)}.$$

Therefore, when $1/2 < r \leq 1$

$$r^{\lambda-1} e^{-\omega r(\varkappa^\rho - a^\rho)} \leq \frac{r^{\lambda-1} \Gamma(\lambda+2)}{(\omega r(\varkappa^\rho - a^\rho))^\lambda} \leq \frac{2\omega^{-\lambda} \Gamma(\lambda+2)}{(\varkappa^\rho - a^\rho)^\lambda}$$

and consequently

$$\rho^{-1} \left(\varkappa^\rho - a^\rho \right)^\lambda \int_{1/2}^1 (1-r)^{\nu-1} r^{\lambda-1} e^{-\omega r(\varkappa^\rho - a^\rho)} \, dr$$

$$\leq 2\omega^{-\lambda} \rho^{-1} \Gamma(\lambda+2) \int_{1/2}^1 (1-r)^{\nu-1} \, dr = \frac{2^{1-\nu} \omega^{-\lambda} \Gamma(\lambda+2)}{\rho \nu}. \tag{3.5}$$

When $\omega r(\varkappa^\rho - a^\rho) < 1$, we have $e^{\omega r(\varkappa^\rho - a^\rho)} \geq 1 > (\omega r(\varkappa^\rho - a^\rho))^\lambda$ and thus

$$\rho^{-1} \left(\varkappa^\rho - a^\rho \right)^\lambda \int_{1/2}^1 (1-r)^{\nu-1} r^{\lambda-1} e^{-\omega r(\varkappa^\rho - a^\rho)} \, dr$$

$$< \rho^{-1} \left(\varkappa^\rho - a^\rho \right)^\lambda \int_{1/2}^1 (1-r)^{\nu-1} r^{\lambda-1} \left(\omega r \left(\varkappa^\rho - a^\rho \right) \right)^{-\lambda} \, dr$$

$$< 2\rho^{-1} \omega^{-\lambda} \int_{1/2}^1 (1-r)^{\nu-1} \, dr = 2^{1-\nu} \frac{\omega^{-\lambda}}{\rho \nu}. \tag{3.6}$$

Take into consideration (3.3)–(3.6) we conclude that

$$L(\varkappa) \leq \rho^{-1} \max\left(1, 2^{1-\nu}\right) \omega^{-\lambda} \Gamma(\lambda) + \frac{2^{1-\nu} \omega^{-\lambda} \Gamma(\lambda+2)}{\rho \nu}$$

$$\leq \rho^{-1} \max\left(1, 2^{1-\nu}\right) \omega^{-\lambda} \Gamma(\lambda) \left(1 + \frac{\lambda(\lambda+1)}{\nu} \right). \qquad \square$$

We prove a useful inequality enjoyed by solutions of Problem (1.1).

Lemma 3.2 *Suppose that $\mathfrak{F}$ satisfies* (A1), (A2) *and* $\varpi(\varkappa)$ *is solution of* (1.1). *Then*

$$\max\left\{ \left(\frac{\varkappa^\rho - a^\rho}{\rho} \right)^{1-\vartheta_1} |\varpi(\varkappa)|, \left(\frac{\varkappa^\rho - a^\rho}{\rho} \right)^{1-(\vartheta_1-\vartheta_2)} |{}^\rho\mathfrak{D}_a^{\vartheta_2} \varpi(\varkappa)| \right\}$$

$$\leq z(\varkappa), \quad \varkappa > a, \tag{3.7}$$

where

$$z(\varkappa) = K_1 + K_2 \left\{ \int_a^\varkappa h^q(s) \varphi_1^q \left(\left(\frac{s^\rho - a^\rho}{\rho} \right)^{1-\vartheta_1} |\varpi(s)| \right) \varphi_2^q \right.$$

$$\left. \times \varphi_2^q \left(\left(\frac{s^\rho - a^\rho}{\rho} \right)^{1-(\vartheta_1 - \vartheta_2)} \left| {}^\rho \mathfrak{D}_a^{\vartheta_2} \varpi(s) \right| \right) \frac{ds}{s^{1-\rho}} \right\}^{1/q} \tag{3.8}$$

when $h \in L_q(a, \infty)$ *for some* $q > \frac{1}{\vartheta_1 - \vartheta_2}$, $\gamma > \frac{1}{q} - 1$,

$$K_1 = |b| \max\left\{ \frac{1}{\Gamma(\vartheta_1)}, \frac{1}{\Gamma(\vartheta_1 - \vartheta_2)} \right\} \quad \text{and} \quad K_2 = \rho^{-\gamma} \max\left\{ \frac{C_1'}{\Gamma(\vartheta_1)}, \frac{C_2'}{\Gamma(\vartheta_1 - \vartheta_2)} \right\},$$

where $p + q = pq$,

$$C_1' = \left[\frac{1}{\rho} \max\{1, 2^{p(1-\vartheta_1)}\}(p\delta)^{-(p\gamma+1)} \Gamma(p\gamma + 1) \left(1 + \frac{(p\gamma + 1)(p\gamma + 2)}{p(1 - \vartheta_1) + 1} \right) \right]^{1/p}$$

and

$$C_2' = \left[\frac{1}{\rho} \max\{1, 2^{p(1-\vartheta_1+\vartheta_2)}\} \left(1 + \frac{(p\gamma + 1)(p\gamma + 2)}{p(\vartheta_1 - \vartheta_2 - 1) + 1} \right) (p\delta)^{-(p\gamma+1)} \Gamma(p\gamma + 1) \right]^{1/p}.$$

Proof Applying ${}^\rho \mathfrak{I}_a^{\vartheta_1}$ to (1.1) and using Theorem 2.7, we find that

$$\varpi(\varkappa) = \frac{b}{\Gamma(\vartheta_1)} \left(\frac{\varkappa^\rho - a^\rho}{\rho} \right)^{\vartheta_1 - 1}$$

$$+ \frac{1}{\Gamma(\vartheta_1)} \int_a^\varkappa \left(\frac{\varkappa^\rho - s^\rho}{\rho} \right)^{\vartheta_1 - 1} \mathfrak{F}\left(s, \varpi(s), {}^\rho \mathfrak{D}_a^{\vartheta_2} \varpi(s)\right) \frac{ds}{s^{1-\rho}} \tag{3.9}$$

and using (3.1), we obtain

$$\left(\frac{\varkappa^\rho - a^\rho}{\rho} \right)^{1-\vartheta_1} |\varpi(\varkappa)|$$

$$\leq K_1 + \frac{1}{\Gamma(\vartheta_1)} \left(\frac{\varkappa^\rho - a^\rho}{\rho} \right)^{1-\vartheta_1}$$

$$\times \int_a^\varkappa \left\{ \left(\frac{\varkappa^\rho - s^\rho}{\rho} \right)^{\vartheta_1 - 1} \left(\frac{s^\rho - a^\rho}{\rho} \right)^\gamma e^{-\delta(s^\rho - a^\rho)} h(s) \right.$$

$$\times \varphi_1 \left[\left(\frac{s^\rho - a^\rho}{\rho} \right)^{1-\vartheta_1} |\varpi(s)| \right]$$

$$\left. \times \varphi_2 \left[\left(\frac{s^\rho - a^\rho}{\rho} \right)^{1-(\vartheta_1 - \vartheta_2)} \left| {}^\rho \mathfrak{D}_a^{\vartheta_2} \varpi(s) \right| \right] \frac{ds}{s^{1-\rho}} \right\}, \quad \varkappa > a. \tag{3.10}$$

Now, the Hölder inequality yields

$$J_1 := \int_a^\varkappa \left\{ \left(\frac{\varkappa^\rho - s^\rho}{\rho} \right)^{\vartheta_1 - 1} \left(\frac{s^\rho - a^\rho}{\rho} \right)^\gamma e^{-\delta(s^\rho - a^\rho)} \right.$$

$$\times h(s)\varphi_1\left[\left(\frac{s^\rho - a^\rho}{\rho}\right)^{1-\vartheta_1}|\varpi(s)|\right]\varphi_2\left[\left(\frac{s^\rho - a^\rho}{\rho}\right)^{1-(\vartheta_1-\vartheta_2)}\left|{}^\rho\mathfrak{D}_a^{\vartheta_2}\varpi(s)\right|\right]\frac{ds}{s^{1-\rho}}\bigg\}$$

$$\leq \left[\int_a^\varkappa \left(\frac{\varkappa^\rho - s^\rho}{\rho}\right)^{p(\vartheta_1-1)}\left(\frac{s^\rho - a^\rho}{\rho}\right)^{p\gamma}e^{-p\delta(s^\rho-a^\rho)}\frac{ds}{s^{1-\rho}}\right]^{\frac{1}{p}}$$

$$\times \left\{\int_a^\varkappa h^q(s)\varphi_1^q\left[\left(\frac{s^\rho - a^\rho}{\rho}\right)^{1-\vartheta_1}|\varpi(s)|\right]\right.$$

$$\times \left.\varphi_2^q\left[\left(\frac{s^\rho - a^\rho}{\rho}\right)^{1-(\vartheta_1-\vartheta_2)}\left|{}^\rho\mathfrak{D}_a^{\vartheta_2}\varpi(s)\right|\right]\frac{ds}{s^{1-\rho}}\right\}^{\frac{1}{q}}$$

$$\leq \rho^{1-\vartheta_1-\gamma}\left[\int_a^\varkappa \left(\varkappa^\rho - s^\rho\right)^{p(\vartheta_1-1)}\left(s^\rho - a^\rho\right)^{p\gamma}e^{-p\delta(s^\rho-a^\rho)}\frac{ds}{s^{1-\rho}}\right]^{\frac{1}{p}}$$

$$\times \left\{\int_a^\varkappa h^q(s)\varphi_1^q\left[\left(\frac{s^\rho - a^\rho}{\rho}\right)^{1-\vartheta_1}|\varpi(s)|\right]\right.$$

$$\times \left.\varphi_2^q\left[\left(\frac{s^\rho - a^\rho}{\rho}\right)^{1-(\vartheta_1-\vartheta_2)}\left|\mathfrak{D}_a^{\vartheta_2}\varpi(s)\right|\right]\frac{ds}{s^{1-\rho}}\right\}^{\frac{1}{q}}, \quad \varkappa > a.$$

Now, as $q > \frac{1}{\vartheta_1-\vartheta_2}$ implies $p(\vartheta_1-1) > -1$ and $p\gamma > -1$, we apply Lemma 3.1 to get

$$J_1 \leq \rho^{1-\vartheta_1-\gamma}C_1'\left(\varkappa^\rho - a^\rho\right)^{\vartheta_1-1}$$

$$\times \left\{\int_a^\varkappa h^q(s)\varphi_1^q\left[\left(\frac{s^\rho - a^\rho}{\rho}\right)^{1-\vartheta_1}|\varpi(s)|\right]\right.$$

$$\times \left.\varphi_2^q\left[\left(\frac{s^\rho - a^\rho}{\rho}\right)^{1-(\vartheta_1-\vartheta_2)}\left|\mathfrak{D}_a^{\vartheta_2}\varpi(s)\right|\right]\frac{ds}{s^{1-\rho}}\right\}^{\frac{1}{q}}, \quad \varkappa > a. \tag{3.11}$$

Combining (3.10) and (3.11) we conclude that

$$\left(\frac{\varkappa^\rho - a^\rho}{\rho}\right)^{1-\vartheta_1}|\varpi(\varkappa)| \leq K_1 + K_2\left\{\int_a^\varkappa h^q(s)\varphi_1^q\left[\left(\frac{s^\rho - a^\rho}{\rho}\right)^{1-\vartheta_1}|\varpi(s)|\right]\right.$$

$$\times \left.\varphi_2^q\left[\left(\frac{s^\rho - a^\rho}{\rho}\right)^{1-(\vartheta_1-\vartheta_2)}\left|\mathfrak{D}_a^{\vartheta_2}\varpi(s)\right|\right]\frac{ds}{s^{1-\rho}}\right\}^{\frac{1}{q}}, \tag{3.12}$$

for $\varkappa > a$. By using Lemma 2.8, we have

$${}^\rho\mathfrak{D}_a^{\vartheta_2}\varpi(\varkappa)$$

$$= {}^\rho\mathfrak{J}_a^{\vartheta_1-\vartheta_2}{}^\rho\mathfrak{D}_a^{\vartheta_1}\varpi(\varkappa) + \frac{b}{\Gamma(\vartheta_1-\vartheta_2)}\left(\frac{\varkappa^\rho - a^\rho}{\rho}\right)^{\vartheta_1-\vartheta_2-1}$$

$$= \frac{b}{\Gamma(\vartheta_1-\vartheta_2)}\left(\frac{\varkappa^\rho - a^\rho}{\rho}\right)^{\vartheta_1-\vartheta_2-1}$$

$$+ \frac{1}{\Gamma(\vartheta_1-\vartheta_2)}\int_a^\varkappa \left(\frac{\varkappa^\rho - s^\rho}{\rho}\right)^{\vartheta_1-\vartheta_2-1}{}^\rho\mathfrak{D}_a^{\vartheta_1}\varpi(s)\frac{ds}{s^{1-\rho}}$$

$$= \frac{b}{\Gamma(\vartheta_1-\vartheta_2)}\left(\frac{\varkappa^\rho - a^\rho}{\rho}\right)^{\vartheta_1-\vartheta_2-1}$$

$$+ \frac{1}{\Gamma(\vartheta_1-\vartheta_2)}\int_a^\varkappa \left(\frac{\varkappa^\rho - s^\rho}{\rho}\right)^{\vartheta_1-\vartheta_2-1}\mathfrak{F}\left(s, \varpi(s), {}^\rho\mathfrak{D}_a^{\vartheta_2}\varpi(s)\right)\frac{ds}{s^{1-\rho}}, \quad \varkappa > a \tag{3.13}$$

and in view of (3.1)

$$
\left(\frac{\varkappa^\rho - a^\rho}{\rho}\right)^{1-(\vartheta_1-\vartheta_2)} |{}^\rho\mathfrak{D}_a^{\vartheta_2}\varpi(\varkappa)|
$$

$$
\leq K_1 + \frac{1}{\Gamma(\vartheta_1-\vartheta_2)}\left(\frac{\varkappa^\rho - a^\rho}{\rho}\right)^{1-(\vartheta_1-\vartheta_2)} \int_a^\varkappa \left\{ \left(\frac{\varkappa^\rho - s^\rho}{\rho}\right)^{\vartheta_1-\vartheta_2-1} \left(\frac{s^\rho - a^\rho}{\rho}\right)^\gamma \right.
$$

$$
\times e^{-\delta(s^\rho-a^\rho)}h(s)\varphi_1\left[\left(\frac{s^\rho-a^\rho}{\rho}\right)^{1-\vartheta_1}|\varpi(s)|\right]
$$

$$
\left. \times \varphi_2\left[\left(\frac{s^\rho-a^\rho}{\rho}\right)^{1-(\vartheta_1-\vartheta_2)}|{}^\rho\mathfrak{D}_a^{\vartheta_2}\varpi(s)|\right]\frac{ds}{s^{1-\rho}} \right\}, \tag{3.14}
$$

for $\varkappa > a$. By using the Hölder inequality, we see that

$$
J_2 := \int_a^\varkappa \left\{ \left(\frac{\varkappa^\rho - s^\rho}{\rho}\right)^{\vartheta_1-\vartheta_2-1} \left(\frac{s^\rho - a^\rho}{\rho}\right)^\gamma \right.
$$

$$
\times e^{-\delta(s^\rho-a^\rho)}h(s)\varphi_1\left[\left(\frac{s^\rho-a^\rho}{\rho}\right)^{1-\vartheta_1}|\varpi(s)|\right]
$$

$$
\left. \times \varphi_2\left[\left(\frac{s^\rho-a^\rho}{\rho}\right)^{1-(\vartheta_1-\vartheta_2)}|{}^\rho\mathfrak{D}_a^{\vartheta_2}\varpi(s)|\right]\frac{ds}{s^{1-\rho}} \right\}
$$

$$
\leq \left[\int_a^\varkappa \left(\frac{\varkappa^\rho - s^\rho}{\rho}\right)^{p(\vartheta_1-\vartheta_2-1)} \left(\frac{s^\rho - a^\rho}{\rho}\right)^{p\gamma} e^{-p\delta(s^\rho-a^\rho)}\frac{ds}{s^{1-\rho}}\right]^{\frac{1}{p}}
$$

$$
\times \left\{\int_a^\varkappa h^q(s)\varphi_1^q\left[\left(\frac{s^\rho-a^\rho}{\rho}\right)^{1-\vartheta_1}|\varpi(s)|\right]\right.
$$

$$
\left. \times \varphi_2^q\left[\left(\frac{s^\rho-a^\rho}{\rho}\right)^{1-(\vartheta_1-\vartheta_2)}|{}^\rho\mathfrak{D}_a^{\vartheta_2}\varpi(s)|\right]\frac{ds}{s^{1-\rho}}\right\}^{\frac{1}{q}}
$$

$$
\leq \rho^{1-(\vartheta_1-\vartheta_2)-\gamma}\left[\int_a^\varkappa (\varkappa^\rho - s^\rho)^{p(\vartheta_1-\vartheta_2-1)}(s^\rho - a^\rho)^{p\gamma} e^{-p\delta(s^\rho-a^\rho)}\frac{ds}{s^{1-\rho}}\right]^{\frac{1}{p}}
$$

$$
\times \left\{\int_a^\varkappa h^q(s)\varphi_1^q\left[\left(\frac{s^\rho-a^\rho}{\rho}\right)^{1-\vartheta_1}|\varpi(s)|\right]\right.
$$

$$
\left. \times \varphi_2^q\left[\left(\frac{s^\rho-a^\rho}{\rho}\right)^{1-(\vartheta_1-\vartheta_2)}|{}^\rho\mathfrak{D}_a^{\vartheta_2}\varpi(s)|\right]\frac{ds}{s^{1-\rho}}\right\}^{\frac{1}{q}},
$$

for $\varkappa > a$. Again by Lemma 3.1 (with $p(\vartheta_1 - \vartheta_2 - 1) > -1$ and $p\gamma > -1$), we obtain

$$
J_2 \leq \rho^{-\gamma} C_2'\left(\frac{s^\rho - a^\rho}{\rho}\right)^{\vartheta_1-\vartheta_2-1}\left\{\int_a^\varkappa h^q(s)\varphi_1^q\left[\left(\frac{s^\rho-a^\rho}{\rho}\right)^{1-\vartheta_1}|\varpi(s)|\right]\right.
$$

$$
\left. \times \varphi_2^q\left[\left(\frac{s^\rho-a^\rho}{\rho}\right)^{1-(\vartheta_1-\vartheta_2)}|{}^\rho\mathfrak{D}_a^{\vartheta_2}\varpi(s)|\right]\frac{ds}{s^{1-\rho}}\right\}^{\frac{1}{q}}, \tag{3.15}
$$

Combining (3.14) and (3.15), we arrive at

$$
\left(\frac{\varkappa^\rho - a^\rho}{\rho}\right)^{1-(\vartheta_1-\vartheta_2)} |{}^\rho\mathfrak{D}_a^{\vartheta_2}\varpi(\varkappa)|
$$

$$\leq K_1 + K_2 \left\{ \int_a^\varkappa h^q(s)\varphi_1^q\left[\left(\frac{s^\rho - a^\rho}{\rho}\right)^{1-\vartheta_1}|\varpi(s)|\right]\right.$$

$$\left. \times \varphi_2^q\left[\left(\frac{s^\rho - a^\rho}{\rho}\right)^{1-(\vartheta_1-\vartheta_2)}\left|{}^\rho\mathfrak{D}_a^{\vartheta_2}\varpi(s)\right|\right]\frac{ds}{s^{1-\rho}}\right\}^{\frac{1}{q}}, \tag{3.16}$$

for $\varkappa > a$. The relation (3.7) is an instant consequence of (3.12) and (3.16). $\qquad\square$

Lemma 3.3 *Suppose that $\mathfrak{F}$ satisfies* (A1), (A3) *and $\varpi(\varkappa)$ is solution of* (1.1). *Then*

$$\max\left\{\left(\frac{\varkappa^\rho - a^\rho}{\rho}\right)^{1-\vartheta_1}|\varpi(\varkappa)|, \left(\frac{\varkappa^\rho - a^\rho}{\rho}\right)^{1-(\vartheta_1-\vartheta_2)}\left|{}^\rho\mathfrak{D}_a^{\vartheta_2}\varpi(\varkappa)\right|\right\}$$

$$\leq z(\varkappa), \quad \varkappa > a, \tag{3.17}$$

where

$$z(\varkappa) = K_1 + K_2\left\{\left[\int_a^\varkappa h_1^q(s)\varphi_1^q\left[\left(\frac{s^\rho - a^\rho}{\rho}\right)^{1-\vartheta_1}|\varpi(s)|\right]\frac{ds}{s^{1-\rho}}\right]^{1/q}\right.$$

$$\left. + \left[\int_a^\varkappa h_2^q(s)\varphi_2^q\left(\left(\frac{s^\rho - a^\rho}{\rho}\right)^{1-(\vartheta_1-\vartheta_2)}\left|{}^\rho\mathfrak{D}_a^{\vartheta_2}\varpi(s)\right|\right)\frac{ds}{s^{1-\rho}}\right]^{1/q}\right\}, \quad \varkappa > a, \tag{3.18}$$

where $h_k \in L_q(a,\infty)$ for some $q > \frac{1}{\vartheta_1-\vartheta_2}$, $\gamma_k > \frac{1}{q} - 1$, and $\delta_k > 0$, $k = 1, 2$,

$$K_1 = |b|\max\left\{\frac{1}{\Gamma(\vartheta_1)}, \frac{1}{\Gamma(\vartheta_1-\vartheta_2)}\right\} \quad and \quad K_2 = \max\{C_3, C_3'\},$$

$$C_3 = \frac{\rho^{-\gamma_k}}{\Gamma(\vartheta_1)}\max\{C_1, C_2\},$$

$$C_k = \left(\frac{1}{\rho}\max\{1, 2^{p(1-\vartheta_1)}\}\Gamma(1+p\gamma_k)\left(\frac{(p\gamma_k+1)(p\gamma_k+2)}{p(\vartheta_1-1)+1}+1\right)(p\delta_k)^{-(1+p\gamma_k)}\right)^{\frac{1}{p}},$$

$$C_3' = \frac{\rho^{-\gamma_k}}{\Gamma(\vartheta_1-\vartheta_2)}\max\{C_1', C_2'\},$$

$$C_k' = \left(\frac{1}{\rho}\max\{1, 2^{p(1-(\vartheta_1-\vartheta_2))}\}\Gamma(p\gamma_k+1)\left(1+\frac{(p\gamma_k+1)(p\gamma_k+2)}{p(\vartheta_1-\vartheta_2-1)+1}\right)(p\delta_k)^{-(p\gamma_k+1)}\right)^{\frac{1}{p}}.$$

Proof Applying ${}^\rho\mathfrak{I}_a^{\vartheta_1}$ to (1.1) and using Theorem 2.7, we find that

$$\varpi(\varkappa) = \frac{b}{\Gamma(\vartheta_1)}\left(\frac{\varkappa^\rho - a^\rho}{\rho}\right)^{\vartheta_1-1} + \frac{1}{\Gamma(\vartheta_1)}\int_a^\varkappa\left(\frac{\varkappa^\rho - s^\rho}{\rho}\right)^{\vartheta_1-1}$$

$$\times \mathfrak{F}\left(s, \varpi(s), {}^\rho\mathfrak{D}_a^{\vartheta_2}\varpi(s)\right)\frac{ds}{s^{1-\rho}}, \tag{3.19}$$

for $\varkappa > a$. Multiplying (3.19) by $(\frac{\varkappa^\rho - a^\rho}{\rho})^{1-\vartheta_1}$ and using (3.2), we get

$$\left(\frac{\varkappa^\rho - a^\rho}{\rho}\right)^{1-\vartheta_1}|\varpi(\varkappa)|$$

$$\leq K_1 + \frac{1}{\Gamma(\vartheta_1)}\left(\frac{\varkappa^\rho - a^\rho}{\rho}\right)^{1-\vartheta_1}$$

$$\times \int_a^{\varkappa} \left(\frac{\varkappa^\rho - s^\rho}{\rho} \right)^{\vartheta_1 - 1} \left(\frac{s^\rho - a^\rho}{\rho} \right)^{\gamma_1} e^{-\delta_1(s^\rho - a^\rho)}$$

$$\times h_1(s)\varphi_1 \left[\left(\frac{s^\rho - a^\rho}{\rho} \right)^{1 - \vartheta_1} \left| \varpi(s) \right| \right] \frac{ds}{s^{1-\rho}}$$

$$+ \frac{1}{\Gamma(\vartheta_1)} \left(\frac{\varkappa^\rho - a^\rho}{\rho} \right)^{1 - \vartheta_1} \int_a^{\varkappa} \left(\frac{\varkappa^\rho - s^\rho}{\rho} \right)^{\vartheta_1 - 1} \left(\frac{s^\rho - a^\rho}{\rho} \right)^{\gamma_2} e^{-\delta_2(s^\rho - a^\rho)}$$

$$\times h_2(s)\varphi_2 \left[\left(\frac{s^\rho - a^\rho}{\rho} \right)^{1 - (\vartheta_1 - \vartheta_2)} \left| {}^\rho \mathfrak{D}_a^{\vartheta_1} \varpi(s) \right| \right] \frac{ds}{s^{1-\rho}}.$$

From the Hölder inequality we have

$$\left(\frac{\varkappa^\rho - a^\rho}{\rho} \right)^{1 - \vartheta_1} \left| \varpi(\varkappa) \right| \le K_1 + \frac{1}{\Gamma(\vartheta_1)} \left(\frac{\varkappa^\rho - a^\rho}{\rho} \right)^{1 - \vartheta_1}$$

$$\times \left(\int_a^{\varkappa} \left(\frac{\varkappa^\rho - s^\rho}{\rho} \right)^{p(\vartheta_1 - 1)} \left(\frac{s^\rho - a^\rho}{\rho} \right)^{p\gamma_1} e^{-p\delta_1(s^\rho - a^\rho)} \frac{ds}{s^{1-\rho}} \right)^{\frac{1}{p}}$$

$$\times \left(\int_a^{\varkappa} h_1^q(s)\varphi_1^q \left(\left(\frac{s^\rho - a^\rho}{\rho} \right)^{1 - \vartheta_1} \left| \varpi(s) \right| \right) \frac{ds}{s^{1-\rho}} \right)^{\frac{1}{q}}$$

$$+ \frac{1}{\Gamma(\vartheta_1)} \left(\frac{\varkappa^\rho - a^\rho}{\rho} \right)^{1 - \vartheta_1}$$

$$\times \left(\int_a^{\varkappa} \left(\frac{\varkappa^\rho - s^\rho}{\rho} \right)^{p(\vartheta_1 - 1)} \left(\frac{s^\rho - a^\rho}{\rho} \right)^{p\gamma_2} e^{-p\delta_2(s^\rho - a^\rho)} \frac{ds}{s^{1-\rho}} \right)^{\frac{1}{p}}$$

$$\times \left(\int_a^{\varkappa} h_2^q(s)\varphi_2^q \left(\left(\frac{s^\rho - a^\rho}{\rho} \right)^{1 - (\vartheta_1 - \vartheta_2)} \left| {}^\rho \mathfrak{D}_a^{\vartheta_1} \varpi(s) \right| \right) \frac{ds}{s^{1-\rho}} \right)^{\frac{1}{q}}.$$

Since $q > \frac{1}{\vartheta_1 - \vartheta_2}$, $\gamma_k > \frac{1}{q} - 1$, $\delta_k > 0$, we have $p(\vartheta_1 - 1) + 1 > 0$ and $1 + p\gamma_k > 0$, $k = 1, 2$, so we can apply Lemma 3.1 to get

$$\left(\frac{\varkappa^\rho - a^\rho}{\rho} \right)^{1 - \vartheta_1} \left| \varpi(\varkappa) \right|$$

$$\le K_1 + C_3 \left[\left(\int_a^{\varkappa} h_1^q(s)\varphi_1^q \left(\left(\frac{s^\rho - a^\rho}{\rho} \right)^{1 - \vartheta_1} \left| \varpi(s) \right| \right) \frac{ds}{s^{1-\rho}} \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\int_a^{\varkappa} h_2^q(s)\varphi_2^q \left(\left(\frac{s^\rho - a^\rho}{\rho} \right)^{1 - (\vartheta_1 - \vartheta_2)} \left| {}^\rho \mathfrak{D}_a^{\vartheta_1} \varpi(s) \right| \right) \frac{ds}{s^{1-\rho}} \right)^{\frac{1}{q}} \right]. \qquad (3.20)$$

Also we have

$$ {}^\rho \mathfrak{D}_a^{\vartheta_2} \varpi(\varkappa) = \frac{b}{\Gamma(\vartheta_1 - \vartheta_2)} \left(\frac{\varkappa^\rho - a^\rho}{\rho} \right)^{\vartheta_1 - \vartheta_2 - 1}$$

$$+ \frac{1}{\Gamma(\vartheta_1 - \vartheta_2)} \int_a^{\varkappa} \left(\frac{\varkappa^\rho - s^\rho}{\rho} \right)^{\vartheta_1 - \vartheta_2 - 1} {}^\rho \mathfrak{D}_a^{\vartheta_1} \varpi(s) \frac{ds}{s^{1-\rho}}$$

$$= \frac{b}{\Gamma(\vartheta_1 - \vartheta_2)} \left(\frac{\varkappa^\rho - a^\rho}{\rho} \right)^{\vartheta_1 - \vartheta_2 - 1} + \frac{1}{\Gamma(\vartheta_1 - \vartheta_2)}$$

$$\times \int_a^{\varkappa} \left(\frac{\varkappa^\rho - s^\rho}{\rho} \right)^{\vartheta_1 - \vartheta_2 - 1} \mathfrak{F}\left(s, \varpi(s), {}^\rho \mathfrak{D}_a^{\vartheta_2} \varpi(s)\right) \frac{ds}{s^{1-\rho}}. \qquad (3.21)$$

Multiplying (3.21) by $(\frac{\varkappa^\rho - a^\rho}{\rho})^{1-(\vartheta_1 - \vartheta_2)}$ and using (3.2), we find that

$$
\left(\frac{\varkappa^\rho - a^\rho}{\rho} \right)^{1-(\vartheta_1 - \vartheta_2)} |{}^\rho\mathfrak{D}_a^{\vartheta_2} \varpi(\varkappa)|
$$

$$
\leq K_1 + \frac{(\frac{\varkappa^\rho - a^\rho}{\rho})^{1-(\vartheta_1 - \vartheta_2)}}{\Gamma(\vartheta_1 - \vartheta_2)} \int_a^\varkappa \left(\frac{\varkappa^\rho - s^\rho}{\rho} \right)^{\vartheta_1 - \vartheta_2 - 1} \left(\frac{s^\rho - a^\rho}{\rho} \right)^{\gamma_1} e^{-\delta_1(s^\rho - a^\rho)}
$$

$$
\times h_1(s)\varphi_1\left(\left(\frac{s^\rho - a^\rho}{\rho} \right)^{1-\vartheta_1} |\varpi(s)| \right) \frac{ds}{s^{1-\rho}}
$$

$$
+ \frac{(\frac{\varkappa^\rho - a^\rho}{\rho})^{1-(\vartheta_1 - \vartheta_2)}}{\Gamma(\vartheta_1 - \vartheta_2)} \int_a^\varkappa \left(\frac{\varkappa^\rho - s^\rho}{\rho} \right)^{\vartheta_1 - \vartheta_2 - 1} \left(\frac{s^\rho - a^\rho}{\rho} \right)^{\gamma_2} e^{-\delta_2(s^\rho - a^\rho)}
$$

$$
\times h_2(s)\varphi_2\left(\left(\frac{s^\rho - a^\rho}{\rho} \right)^{1-(\vartheta_1 - \vartheta_2)} |{}^\rho\mathfrak{D}_a^{\vartheta_2} \varpi(s)| \right) \frac{ds}{s^{1-\rho}}.
$$

From the Hölder inequality, we have

$$
\left(\frac{\varkappa^\rho - a^\rho}{\rho} \right)^{1-(\vartheta_1 - \vartheta_2)} |{}^\rho\mathfrak{D}_a^{\vartheta_2} \varpi(\varkappa)|
$$

$$
\leq K_1 + \frac{(\frac{\varkappa^\rho - a^\rho}{\rho})^{1-(\vartheta_1 - \vartheta_2)}}{\Gamma(\vartheta_1 - \vartheta_2)}
$$

$$
\times \left(\int_a^\varkappa \left(\frac{\varkappa^\rho - s^\rho}{\rho} \right)^{p(\vartheta_1 - \vartheta_2 - 1)} \left(\frac{s^\rho - a^\rho}{\rho} \right)^{p\gamma_1} e^{-p\delta_1(s^\rho - a^\rho)} \frac{ds}{s^{1-\rho}} \right)^{\frac{1}{p}}
$$

$$
\times \left(\int_a^\varkappa h_1^q(s)\varphi_1^q\left(\left(\frac{s^\rho - a^\rho}{\rho} \right)^{1-\vartheta_1} |\varpi(s)| \right) \frac{ds}{s^{1-\rho}} \right)^{\frac{1}{q}}
$$

$$
+ \frac{(\frac{\varkappa^\rho - a^\rho}{\rho})^{1-(\vartheta_1 - \vartheta_2)}}{\Gamma(\vartheta_1 - \vartheta_2)} \left(\int_a^\varkappa \left(\frac{\varkappa^\rho - s^\rho}{\rho} \right)^{p(\vartheta_1 - \vartheta_2 - 1)} \left(\frac{s^\rho - a^\rho}{\rho} \right)^{p\gamma_2} e^{-p\delta_2(s^\rho - a^\rho)} \frac{ds}{s^{1-\rho}} \right)^{\frac{1}{p}}
$$

$$
\times \left(\int_a^\varkappa h_2^q(s)\varphi_2^q\left(\left(\frac{s^\rho - a^\rho}{\rho} \right)^{1-(\vartheta_1 - \vartheta_2)} |{}^\rho\mathfrak{D}_a^{\vartheta_2} \varpi(s)| \right) \frac{ds}{s^{1-\rho}} \right)^{\frac{1}{q}}.
$$

Applying Lemma 3.1, we obtain

$$
\left(\frac{\varkappa^\rho - a^\rho}{\rho} \right)^{1-(\vartheta_1 - \vartheta_2)} |{}^\rho\mathfrak{D}_a^{\vartheta_2} \varpi(\varkappa)|
$$

$$
\leq K_1 + C_3' \left[\left(\int_a^\varkappa h_1^q(s)\varphi_1^q\left(\left(\frac{s^\rho - a^\rho}{\rho} \right)^{1-\vartheta_1} |\varpi(s)| \right) \frac{ds}{s^{1-\rho}} \right)^{\frac{1}{q}} \right.
$$

$$
\left. + \left(\int_a^\varkappa h_2^q(s)\varphi_2^q\left(\left(\frac{s^\rho - a^\rho}{\rho} \right)^{1-(\vartheta_1 - \vartheta_2)} |{}^\rho\mathfrak{D}_a^{\vartheta_2} \varpi(s)| \right) \frac{ds}{s^{1-\rho}} \right)^{\frac{1}{q}} \right]. \qquad (3.22)
$$

Equation (3.17) is an immediate consequence of (3.18), (3.20) and (3.22). $\qquad \square$

Theorem 3.4 *Assume that the assumptions of Lemma* 3.2 *hold, then the solutions of* (1.1) *satisfy*

$$
|\varpi(\varkappa)| \leq C\left(\frac{\varkappa^\rho - a^\rho}{\rho} \right)^{\vartheta_1 - 1} \quad \text{and}
$$

$$\left| {}^{\rho}\mathfrak{D}_a^{\vartheta_2} \varpi(\varkappa) \right| < C \left(\frac{\varkappa^\rho - a^\rho}{\rho} \right)^{\vartheta_1 - \vartheta_2 - 1}, \quad C > 0, \varkappa > a$$

provided that

$$\int_{\varkappa_0}^{\infty} \frac{ds}{\varphi_1^q(s^{\frac{1}{q}})\varphi_2^q(s^{\frac{1}{q}})} = \infty, \quad \varkappa_0 > 0.$$

Proof Thanks to Lemma 3.2 and the fact that φ_k, $k = 1, 2$, are nondecreasing, we have

$$\begin{cases} \varphi_1[(\frac{\varkappa^\rho - a^\rho}{\rho})^{1-\vartheta_1} |\varpi(\varkappa)|] \le \varphi_1[z(\varkappa)], & \varkappa > a \\ \varphi_2[(\frac{\varkappa^\rho - a^\rho}{\rho})^{1-(\vartheta_1-\vartheta_2)} |{}^{\rho}\mathfrak{D}_a^{\vartheta_2}\varpi(\varkappa)|] \le \varphi_2[z(\varkappa)], & \varkappa > a. \end{cases} \tag{3.23}$$

Therefore (3.8) and (3.23), lead to

$$z(\varkappa) \le K_1 + K_2 \left(\int_a^{\varkappa} h^q(s)\varphi_1^q\big(z(s)\big)\varphi_2^q\big(z(s)\big)\,ds \right)^{\frac{1}{q}}, \quad \varkappa > a. \tag{3.24}$$

Applying Lemma 2.10 to (3.24), we get

$$z^q(\varkappa) \le B_1 + B_2 \int_a^{\varkappa} h^q(s)\varphi_1^q\big(z(s)\big)\varphi_2^q\big(z(s)\big)\,ds, \quad \varkappa > a, \tag{3.25}$$

where

$$B_1 = 2^{q-1}K_1 \quad \text{and} \quad B_2 = 2^{q-1}K_2.$$

Now, put $u(\varkappa) = z^q(\varkappa)$, then (3.25) becomes

$$u(\varkappa) \le B_1 + B_2 \int_a^{\varkappa} h^q(s)\varphi_1^q\big(u^{\frac{1}{q}}(s)\big)\varphi_2^q\big(u^{\frac{1}{q}}(s)\big)\,ds, \quad \varkappa > a. \tag{3.26}$$

Let

$$w(r) = \varphi_1^q\big(r^{\frac{1}{q}}\big)\varphi_2^q\big(r^{\frac{1}{q}}\big). \tag{3.27}$$

Then w is a nondecreasing continuous function and

$$u(\varkappa) \le B_1 + B_2 \int_a^{\varkappa} h^q(s)w(u)\,ds, \quad \varkappa > a. \tag{3.28}$$

Applying Theorem 2.9 to (3.28), we obtain

$$u(\varkappa) \le G^{-1}\left(G(B_1) + B_2 \int_a^{\varkappa} h^q(s)\,ds \right), \quad \varkappa > a, \tag{3.29}$$

where

$$G(\varkappa) = \int_{\varkappa_0}^{\varkappa} \frac{ds}{w(s)} = \int_{\varkappa_0}^{\varkappa} \frac{ds}{\varphi_1^q(s^{\frac{1}{q}})\varphi_2^q(s^{\frac{1}{q}})}, \quad \varkappa_0 > 0, \varkappa > 0.$$

Clearly

$$G(B_1) + B_2 \int_a^{\varkappa} h^q(s)\, ds \in \mathrm{Dom}\big(G^{-1}\big), \varkappa > a.$$

As $h \in L_q \in (a, \infty)$, we have

$$H_1 = G(B_1) + B_2 \int_a^{\infty} h^q(s)\, ds < \infty \tag{3.30}$$

and

$$u(\varkappa) \leq H_2 := G^{-1}(H_1) < \infty.$$

Therefore $z(\varkappa) \leq C := H_2^{\frac{1}{q}}$ and the result follows. $\qquad\square$

Theorem 3.5 *Assume that the assumptions of Lemma* 3.3 *hold, then the solutions of* (1.1) *satisfy*

$$\big|\varpi(\varkappa)\big| \leq C \left(\frac{\varkappa^{\rho} - a^{\rho}}{\rho} \right)^{\vartheta_1 - 1} \quad and$$

$$\big|{}^{\rho}\mathfrak{D}_a^{\vartheta_2} \varpi(\varkappa)\big| < C \left(\frac{\varkappa^{\rho} - a^{\rho}}{\rho} \right)^{\vartheta_1 - \vartheta_2 - 1}, \quad C > 0, \varkappa > a,$$

provided that

$$\int_{\varkappa_0}^{\infty} \frac{ds}{\varphi_1^q(s^{\frac{1}{q}}) + \varphi_2^q(s^{\frac{1}{q}})} = \infty, \quad \varkappa_0 > 0.$$

Proof By using Lemma 3.3, we deduce

$$
\begin{aligned}
\varphi_1 \left(\left(\frac{\varkappa^{\rho} - a^{\rho}}{\rho} \right)^{1-\vartheta_1} \big|\varpi(\varkappa)\big| \right) &\leq \varphi_1\big(z(\varkappa)\big), \quad \varkappa > a, \\
\varphi_2 \left(\left(\frac{\varkappa^{\rho} - a^{\rho}}{\rho} \right)^{1-(\vartheta_1 - \vartheta_2)} \big|{}^{\rho}\mathfrak{D}_a^{\vartheta_2} \varpi(\varkappa)\big| \right) &\leq \varphi_2\big(z(\varkappa)\big), \quad \varkappa > a,
\end{aligned}
\tag{3.31}
$$

where $z(\varkappa)$ is as in (3.18). Take into consideration (3.18) and (3.31) we find that

$$z(\varkappa) \leq K_1 + K_2 \left[\left(\int_a^{\varkappa} h_1^q(s) \varphi_1^q\big(z(s)\big)\, ds \right)^{\frac{1}{q}} + \left(\int_a^{\varkappa} h_2^q(s) \varphi_2^q\big(z(s)\big)\, ds \right)^{\frac{1}{q}} \right]. \tag{3.32}$$

By using Lemma (2.10), we have

$$z^q(\varkappa) \leq B_1 + B_2 \left[\int_a^{\varkappa} h_1^q(s) \varphi_1^q\big(z(s)\big)\, ds + \int_a^{\varkappa} h_2^q(s) \varphi_2^q\big(z(s)\big)\, ds \right], \tag{3.33}$$

where

$$B_1 = 2^{q-1} K_1^q \quad and \quad B_2 = 2^{2(q-1)} K_2^q.$$

Furthermore, due to the inequality

$$h_1^q(s)\varphi_1^q(z(s)) + h_2^q(s)\varphi_2^q(z(s)) \leq \left[h_1^q(s) + h_2^q(s)\right]\left[\varphi_1^q(z(s)) + \varphi_2^q(z(s))\right], \tag{3.34}$$

we have by (3.33) and (3.34)

$$z^q(\varkappa) \leq B_1 + B_2 \int_a^\varkappa \left[h_1^q(s) + h_2^q(s)\right]\left[\varphi_1^q(z(s)) + \varphi_2^q(z(s))\right] ds. \tag{3.35}$$

Now, let $u(\varkappa) = z^q(\varkappa)$. Then (3.35) becomes

$$u(\varkappa) \leq B_1 + B_2 \int_a^\varkappa \left[h_1^q(s) + h_2^q(s)\right]\left[\varphi_1^q\left(u^{\frac{1}{q}}(s)\right) + \varphi_2^q\left(u^{\frac{1}{q}}(s)\right)\right] ds, \quad \varkappa > a. \tag{3.36}$$

Let

$$g(r) = \varphi_1^q\left(r^{\frac{1}{q}}\right) + \varphi_2^q\left(r^{\frac{1}{q}}\right). \tag{3.37}$$

Then g is nondecreasing continuous function, since φ_1 and φ_2 are nondecreasing continuous functions.

Hence, from (3.36) and (3.37), we get

$$u(\varkappa) \leq B_1 + B_2 \int_a^\varkappa \left[h_1^q(s) + h_2^q(s)\right]g(u(s))\, ds, \quad \varkappa > a. \tag{3.38}$$

Applying Theorem 2.9 to (3.38), we have

$$u(\varkappa) \leq G^{-1}\left(G(B_1) + B_2 \int_a^\varkappa \left[h_1^q(s) + h_2^q(s)\right] ds\right), \quad \varkappa > a, \tag{3.39}$$

where

$$G(\varkappa) = \int_{\varkappa_0}^\varkappa \frac{ds}{g(s)} = \int_{\varkappa_0}^\varkappa \frac{ds}{\varphi_1^q(s^{\frac{1}{q}}) + \varphi_2^q(s^{\frac{1}{q}})}, \quad \varkappa_0 > 0, \varkappa > 0.$$

As $h_k \in L_q \in (a, \infty)$, we let

$$H_1 = G(B_1) + B_2 \int_a^\infty \left[h_1^q(s) + h_2^q(s)\right] ds. \tag{3.40}$$

Therefore

$$u(\varkappa) \leq H_2 := G^{-1}(H_1) < \infty.$$

Next, $u(\varkappa) = z^q(\varkappa)$ implies that $z(\varkappa) \leq C := H_2^{\frac{1}{q}}$. Then we get from (3.17)

$$\left|\varpi(\varkappa)\right| \leq C\left(\frac{\varkappa^\rho - a^\rho}{\rho}\right)^{\vartheta_1 - 1} \quad \text{and} \quad \left|{}^\rho\mathfrak{D}_a^{\vartheta_2}\varpi(\varkappa)\right| < C\left(\frac{\varkappa^\rho - a^\rho}{\rho}\right)^{\vartheta_1 - \vartheta_2 - 1}, \quad \varkappa > a. \quad \square$$

4 Examples

Example 4.1 Consider the problem

$$
\begin{cases}
{}^{\rho}\mathfrak{D}_a^{1/2}\varpi(\varkappa) = (\frac{\varkappa^{\rho}-a^{\rho}}{\rho})^3(\cos\varpi^2)(\sin\varkappa^{\rho-1}) \\
\qquad\qquad \times\, e^{-2(\varkappa^{\rho}-a^{\rho})}[\varpi(\varkappa)]^{1/4}[{}^{\rho}\mathfrak{D}_a^{1/3}\varpi(\varkappa)]^{1/5}, \quad \varkappa > a, \\
{}^{\rho}\mathfrak{I}_a^{1/2}\varpi(\varkappa)|_{\varkappa=a} = b.
\end{cases}
\tag{4.1}
$$

Here we have

$$
\left|\mathfrak{F}\big(\varkappa,\varpi(\varkappa),{}^{\rho}\mathfrak{D}_a^{1/4}\varpi(\varkappa)\big)\right|
$$

$$
= \left|\left(\frac{\varkappa^{\rho}-a^{\rho}}{\rho}\right)^3(\cos\varpi^2)(\sin\varkappa^{\rho-1})e^{-2(\varkappa^{\rho}-a^{\rho})}\big[\varpi(\varkappa)\big]^{1/4}\big[{}^{\rho}\mathfrak{D}_a^{1/3}\varpi(\varkappa)\big]^{1/5}\right|
$$

$$
\leq \left|\varkappa^{\rho-1}\left(\frac{\varkappa^{\rho}-a^{\rho}}{\rho}\right)^{3-1/8-1/6}e^{-2(\varkappa^{\rho}-a^{\rho})}\right.
$$

$$
\left. \times\left[\left(\frac{\varkappa^{\rho}-a^{\rho}}{\rho}\right)^{1/2}\varpi(\varkappa)\right]^{1/4}\left[\left(\frac{\varkappa^{\rho}-a^{\rho}}{\rho}\right)^{5/6}{}^{\rho}\mathfrak{D}_a^{1/3}\varpi(\varkappa)\right]^{1/5}\right|
$$

$$
\leq \left(\frac{\varkappa^{\rho}-a^{\rho}}{\rho}\right)^{\gamma}e^{-(\varkappa^{\rho}-a^{\rho})}h(\varkappa)\varphi_1\left[\left(\frac{\varkappa^{\rho}-a^{\rho}}{\rho}\right)^{1-1/2}\varpi(\varkappa)\right]
$$

$$
\times\,\varphi_2\left[\left(\frac{\varkappa^{\rho}-a^{\rho}}{\rho}\right)^{1-(1/2-1/3)}\mathfrak{D}_a^{1/3}\varpi(\varkappa)\right],
$$

where $\gamma = \frac{17}{24}$, $h(\varkappa) = \varkappa^{\rho-1}(\frac{\varkappa^{\rho}-a^{\rho}}{\rho})^2 e^{-(\varkappa^{\rho}-a^{\rho})}$, $\varphi_1(\varkappa) = \varkappa^{1/4}$ and $\varphi_2(\varkappa) = \varkappa^{1/5}$. All the conditions of Theorem 3.4 are satisfied. Then

$$
\big|\varpi(\varkappa)\big| \leq C\left(\frac{\varkappa^{\rho}-a^{\rho}}{\rho}\right)^{-1/2} \quad \text{and} \quad \big|{}^{\rho}\mathfrak{D}_a^{1/3}\varpi(\varkappa)\big| \leq C\left(\frac{\varkappa^{\rho}-a^{\rho}}{\rho}\right)^{-5/6}, \quad \varkappa > a.
$$

Example 4.2 Consider the problem

$$
\begin{cases}
{}^{\rho}\mathfrak{D}_a^{1/3}\varpi(\varkappa) = (\frac{\varkappa^{\rho}-a^{\rho}}{\rho})^2(\cos\varpi^2)e^{-2(\varkappa^{\rho}-a^{\rho})-3\varkappa}[\varpi(\varkappa)]^{1/2} \\
\qquad\qquad +\, (\frac{\varkappa^{\rho}-a^{\rho}}{\rho})^2\varkappa^{-2}e^{-3(\varkappa^{\rho}-a^{\rho})}(\cos\varkappa^2)[{}^{\rho}\mathfrak{D}_a^{1/4}\varpi(\varkappa)]^{1/2}, \quad \varkappa > a, \\
{}^{\rho}\mathfrak{I}_a^{2/3}\varpi(\varkappa)|_{\varkappa=a} = b.
\end{cases}
\tag{4.2}
$$

We can rewrite the right hand side of (4.2) as follows:

$$
\left|\left(\frac{\varkappa^{\rho}-a^{\rho}}{\rho}\right)^2(\cos\varpi^2)e^{-2(\varkappa^{\rho}-a^{\rho})}e^{-3\varkappa}\big[\varpi(\varkappa)\big]^{1/2}\right.
$$

$$
\left. +\left(\frac{\varkappa^{\rho}-a^{\rho}}{\rho}\right)^2 e^{-3(\varkappa^{\rho}-a^{\rho})}(\cos\varkappa^2)\varkappa^{-2}\big[{}^{\rho}\mathfrak{D}_a^{1/4}\varpi(\varkappa)\big]^{1/2}\right|
$$

$$
\leq \left(\frac{\varkappa^{\rho}-a^{\rho}}{\rho}\right)^{\gamma_1}e^{-2(\varkappa^{\rho}-a^{\rho})}h_1(\varkappa)\varphi_1\left[\left(\frac{\varkappa^{\rho}-a^{\rho}}{\rho}\right)^{1-1/3}\varpi(\varkappa)\right]
$$

$$
+\left(\frac{\varkappa^{\rho}-a^{\rho}}{\rho}\right)^{\gamma_2}e^{-3(\varkappa^{\rho}-a^{\rho})}h_2(\varkappa)\varphi_2\left[\left(\frac{\varkappa^{\rho}-a^{\rho}}{\rho}\right)^{1-(1/3-1/4)}{}^{\rho}\mathfrak{D}_a^{1/4}\varpi(\varkappa)\right],
$$

where $\gamma_1 = 8/3$, $\gamma_2 = 37/24$, $h_1(\varkappa) = e^{-3\varkappa}$, $h_2(\varkappa) = \varkappa^{-2}$ and $\varphi_1(\varkappa) = \varphi_2(\varkappa) = \varkappa^{1/2}$. Obviously, all conditions of Theorem 3.5 are satisfied. Then

$$\left| \varpi(\varkappa) \right| \le C \left(\frac{\varkappa^\rho - a^\rho}{\rho} \right)^{-1/2} \quad \text{and}$$

$$\left| {}^\rho \mathfrak{D}_a^{1/4} \varpi(\varkappa) \right| \le C \left(\frac{\varkappa^\rho - a^\rho}{\rho} \right)^{-3/4}, \quad \varkappa > a.$$

Remark 4.3

  (i) The reported fractional operator here generalizes both the Riemann–Liouville and Hadamard fractional operators in one form, and it is also most regarding the Erdélyi–Kober fractional operator, especially, when $\rho \to 1$, we get a Riemann–Liouville fractional derivative and doing $\rho \downarrow 0$, we get a Caputo–Hadamard fractional derivative. Also, we get (Liouville and Weyl) for $\rho \to 1$, ($a = 0$ and $a = -\infty$), respectively.

  (ii) The results obtained in this work will remain valid if we take into account the aforementioned special cases.

  (iii) Our current problem (1.1) provides a general platform that covers most of the classic problems from (1.2) into (1.8) mentioned in the introduction Sect.

5 Concluding remarks

In the present work, we have investigated the stability of solutions for some fractional differential problems. These types of equations involved the generalized fractional derivative of different orders. In fact, we have established sufficient conditions on the nonlinear terms, via making use of some modified generalized versions of inequalities, to study the decay of solutions to zero in terms of a power function. Besides, some characteristics of the generalized fractional derivative and appropriate regularization techniques have been employed. Ultimately, this paper concludes with relevant examples to confirm the legitimacy of the acquired results.

In future work, many cases can be established when one takes a more generalised operator that contains another function instead of $\varkappa^\rho$ in the structure of its kernel. For instance, the generalized Caputo [37] (or Hilfer [38]) fractional operators. Further, it will be of interest to study the existing problem in this article for the Mittag-Leffler power law [39] and for fractal fractional operators [40].

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