Ansari et al. *Advances in Difference Equations* (2021) 2021:374
https://doi.org/10.1186/s13662-021-03528-0

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Estimation of divergence measures on time scales via Taylor's polynomial and Green's function with applications in q-calculus

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Abstract

Taylor's polynomial and Green's function are used to obtain new generalizations of an inequality for higher order convex functions containing Csiszár divergence on time scales. Various new inequalities for some divergence measures in quantum calculus and h-discrete calculus are also established.

Keywords: Taylor's polynomial; Quantum calculus; Time scales calculus; Green's function

1 Introduction

The theory of convexity plays a significant role in the development of inequalities. In spite of that the importance of inequalities containing convex functions is magnificent as it tackles numerous problems in various fields of mathematics at a substantial rate. Consequently, the study of these inequalities has gained tons of attention (see [3, 9, 24, 36, 43] and the references cited therein).

The inequalities that include higher order convexity have been utilized by several physicists in higher dimension problems since the founding of higher order convexity by T. Popoviciu (see [43, p. 15]). Over the recent years, the inequalities for n-convex functions have been generalized by numerous researchers. In [19], Butt *et al.* obtained useful identities via Taylor polynomial and generalized Popoviciu inequality for n-convex functions. In [42], Pečarić *et al.* introduced a new class of n-convex functions. They proposed an interesting theory to evaluate linear operator inequalities utilizing n-convex functions. This approach leads to various impressive and insightful results with a number of developments in statistics and operator theory. In [29], Khan *et al.* generalized new inequalities of Rényi Shannon entropies and provided the refinement of Jensen's inequality for higher order convex functions by utilizing the Montgomery identity. In [20], Butt *et al.* generalized Popoviciu's inequality for higher order convex functions by employing Fink's identity in combination with a new Green's function. Using Taylor's polynomial along with new Green's functions, Latif *et al.* in [32] obtained generalized results concerning majorization inequality. In [37], Niaz *et al.* estimated various entropies by utilizing Jensen type functionals. In addition, the authors generalized new inequalities for higher order convex functions employing Taylor's formula. Levinson's inequality has been generalized for 3-convex function utilizing Green's functions by Adeel *et al.* [1]. Moreover, the obtained results are used in information theory via f-divergence, Shannon entropy, and Rényi divergence. Further, in [2], the authors used Taylor's polynomial and generalized Levinson type inequalities for the class of m-convex functions. The obtained results are applied in information theory. In [45], Siddique *et al.* used Fink's identity and Green's functions and obtained generalized results related to majorization type inequalities. They also gave a generalized majorization theorem for higher order convex functions. The obtained results are applied with regard to Kullback–Leibler divergence and Shannon entropy.

For the past quarter-century, mathematicians have been fascinated by the idea of time scales. In mathematical analysis, time scales play a significant role. In 1988, Stefan Hilger initiated the theory of calculus on time scales. Difference calculus, differential calculus, and quantum calculus are the three most famous examples of calculus on time scales. Many of the fundamental aspects of time scales were covered by the books of Bohner and Peterson [15, 16]. During the past decade, several researchers have worked on this subject and established excellent results (see [4, 5, 10–12, 22, 40, 44, 47] and the references cited therein). Dynamic equations and inequalities have a number of applications in other disciplines besides mathematics. For example, population dynamics, physical problems, quantum mechanics, wave equations, optical problems, heat transfer, and finance problems [18, 27, 50]. The modern name for a type of calculus that works without the concept of limits is quantum calculus. It is also known as q-calculus and is originally based on the idea of finite difference re-scaling. The concept of q-calculus was stated in 1740s, as Euler introduced the theory of partitions, also known as analytic number theory. At the beginning of the twentieth century, Jackson [26] presented the notion of q-definite integrals and generalized the concept of q-calculus. Due to significant importance of mathematics, referred to as modeling of quantum computing, the q-calculus has established a connection between mathematics and physics. Many of the fundamental aspects of quantum calculus are covered by the book of Kac and Cheung [27]. The concept of q-derivatives over the finite intervals was given by Tariboon *et al.* [48, 49] who discussed numerous quantum analogues of classical mathematical inequalities. In the last few decades, there has been a considerable development in q-calculus, see [17, 25, 35, 39, 51–54, 56] and the references therein. Various Hermite–Hadamard type quantum integral inequalities for convex functions have been established by Sudsutad *et al.* [46]. Chen and Yang in [21] and Liu and Yang in [34] established various Chebyshev and Grüss type inequalities on finite intervals via quantum integrals, respectively. In [8], Alomari proved q-analogue of Bernoulli inequality. In [33], Li *et al.* established a novel quantum integral identity and obtained some new estimates of Hermite–Hadamard inequalities for quantum integrals. In [23], Erden *et al.* used convex functions and established various quantum integral inequalities. A novel generalized q-integral identity containing q-differentiable function has been established by Awan *et al.* [13]. Further considering the class of preinvex functions the obtained result were used to determine several associated quantum bounds. In [30], Khan *et al.* established quantum Hermite–Hadamard inequality via the Green's function approach. In [7], Ali *et al.* proved some new Ostrowski type integral inequalities for q-differentiable bounded functions. In [31], Kunt *et al.* determined a new version of the celebrated Montgomery identity via quantum integral operators. The obtained result is used to establish some quantum integral inequalities of Ostrowski type. In [14], Ben *et al.* established q-fractional integral inequalities of Henry–Gronwall type.

Despite its resemblance to q-calculus, h-calculus is quite different. It is, in fact, the calculus of finite differences, but a more precise similarity with classical calculus makes it clear. For example, Newton's interpolation formula is similar to h-Taylor formula, and Abel transform is just like the h-integration by parts. The definite h-integral is similar to Riemann sum; consequently, the fundamental theorem of h-calculus permits one to estimate finite sums.

Motivated by the above discussion, we generalize an inequality involving Csiszár divergence on time scales for n-convex functions by using Green's function along with Taylor's polynomial as a unification of both discrete and integral cases. In addition, we estimate Kullback–Leibler divergence, differential entropy, Shannon entropy, Jeffreys distance, and triangular discrimination on time scales, q-calculus, and h-discrete calculus.

2 Preliminaries

Let us take a quick look at time scales, as well as the essential definitions and notations. The details can be followed from [15]:

For $\vartheta \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined as

$$\sigma(\vartheta) := \inf\{\alpha \in \mathbb{T} : \alpha > \vartheta\}.$$

Right-dense continuous (rd-continuous) function Assume that $\mathfrak{g} : \mathbb{T} \to \mathbb{R}$ is a function. Then $\mathfrak{g}$ is rd-continuous if it is continuous at right-dense points of $\mathbb{T}$ and its left-sided limit is finite at left-dense points of $\mathbb{T}$. The set of rd-continuous functions $\mathfrak{g} : \mathbb{T} \to \mathbb{R}$ will be denoted in this paper by C_{rd}.

The derived set $\mathbb{T}^k$ is defined as follows: Given $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - m$; else, $\mathbb{T}^k = \mathbb{T}$.

Delta derivative Let $\mathfrak{g} : \mathbb{T} \to \mathbb{R}$ and $\vartheta \in \mathbb{T}^k$. Then $\mathfrak{g}^\Delta(\vartheta)$ is defined to be the number (if it exists) with the property that for any $\epsilon > 0$ there exists a neighborhood $\mathfrak{U}$ of ϑ such that

$$\bigl|\mathfrak{g}\bigl(\sigma(\vartheta)\bigr) - \mathfrak{g}(\alpha) - \mathfrak{g}^\Delta(\vartheta)\bigl(\sigma(\vartheta) - \alpha\bigr)\bigr| \le \epsilon\bigl|\sigma(\vartheta) - \alpha\bigr|, \quad \forall \alpha \in \mathfrak{U}.$$

Then $\mathfrak{g}$ is known as delta differentiable at ϑ. If $\mathbb{T} = \mathbb{R}$, then $\mathfrak{g}^\Delta$ reduces to the usual derivative $\mathfrak{g}'$, and $\mathfrak{g}^\Delta$ becomes the forward difference operator $\Delta\mathfrak{g}(\vartheta) = \mathfrak{g}(\vartheta + 1) - \mathfrak{g}(\vartheta)$ for $\mathbb{T} = \mathbb{Z}$. If $\mathbb{T} = q^{\mathbb{N}_0} = \{q^n : n \in \mathbb{N}_0\}$, q-difference operator ($q > 1$) is given by

$$\mathfrak{g}^\Delta(\vartheta) = \frac{\mathfrak{g}(q\vartheta) - \mathfrak{g}(\vartheta)}{(q-1)\vartheta}, \qquad \mathfrak{g}^\Delta(0) = \lim_{\alpha \to 0} \frac{\mathfrak{g}(\alpha) - \mathfrak{g}(0)}{\alpha}.$$

Existence of antiderivatives Every rd-continuous function has an antiderivative. If $x_0 \in \mathbb{T}$, then F is given by

$$F(\vartheta) := \int_{x_0}^{x} f(\vartheta)\Delta\vartheta \quad \text{for } x \in \mathbb{T}^k$$

is an antiderivative of f.

If $\mathbb{T}=\mathbb{R}$, then $\int_a^b f(\vartheta)\Delta\vartheta = \int_a^b f(\vartheta)\,d\vartheta$, and $\int_a^b f(\vartheta)\Delta\vartheta = \sum_{\vartheta=a}^{b-1} f(\vartheta)$ for $\mathbb{T}=\mathbb{N}$, where $a,b\in\mathbb{T}$ with $a\le b$. If $\mathbb{T}=h\mathbb{Z}$, $h>0$ then $\int_a^b f(\vartheta)\Delta\vartheta = \sum_{j=\frac{a}{h}}^{\frac{b}{h}-1} f(jh)h$ and $\int_a^b f(\vartheta)\Delta\vartheta = \sum_{j=m}^{n-1} q^{j+1}f(q^j)$ for $\mathbb{T}=q^{\mathbb{N}_0}$, $q>1$, $a=q^m$, and $b=q^n$ with $m<n$, where $\mathbb{N}_0=\mathbb{N}\bigcup\{0\}$.

3 Improvement of the inequality involving Csiszár divergence

Let $\vartheta_1,\vartheta_2\in\mathbb{R}$ with $\vartheta_1<\vartheta_2$ and consider the Green's function $G:[\vartheta_1,\vartheta_2]\times[\vartheta_1,\vartheta_2]\to\mathbb{R}$ defined as follows:

$$G(x,s)=\begin{cases}\frac{(x-\vartheta_2)(s-\vartheta_1)}{\vartheta_2-\vartheta_1} & \text{for } \vartheta_1\le s\le x,\\ \frac{(s-\vartheta_2)(x-\vartheta_1)}{\vartheta_2-\vartheta_1} & \text{for } x\le s\le\vartheta_2,\end{cases}\tag{1}$$

where G is convex and continuous corresponding to both variables. It is notable that (see for example [28, 38, 41, 55]) any function $\Phi\in C^2([\vartheta_1,\vartheta_2],\mathbb{R})$ can be written as

$$\Phi(x)=\frac{\vartheta_2-x}{\vartheta_2-\vartheta_1}\Phi(\vartheta_1)+\frac{x-\vartheta_1}{\vartheta_2-\vartheta_1}\Phi(\vartheta_2)+\int_{\vartheta_1}^{\vartheta_2}G(x,s)\Phi''(s)\,ds,\tag{2}$$

where $G(x,s)$ is defined in (1).

Consider the following set:

$$\Omega:=\left\{\mathbf{q}\,\middle|\,\mathbf{q}:\mathbb{T}\to[0,\infty),\mathbf{q}(\xi)\ge 0,\int_a^b\mathbf{q}(\xi)\Delta\xi=1\right\}.$$

The following result is given by Ansari *et al.* [11].

Theorem A *Suppose that $\Phi:[0,\infty)\to\mathbb{R}$ is a convex function on $[\vartheta_1,\vartheta_2]\subset[0,\infty)$ and $\vartheta_1\le 1\le\vartheta_2$. If $\mathbf{q}_1,\mathbf{q}_2\in\Omega$ with $\vartheta_1\le\frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)}\le\vartheta_2$ for all $\xi\in\mathbb{T}$, then*

$$\int_a^b\mathbf{q}_2(\xi)\Phi\left(\frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)}\right)\Delta\xi\le\frac{\vartheta_2-1}{\vartheta_2-\vartheta_1}\Phi(\vartheta_1)+\frac{1-\vartheta_1}{\vartheta_2-\vartheta_1}\Phi(\vartheta_2).\tag{3}$$

Motivated by inequality (3), we begin by the following theorem.

Theorem 1 *Assume the hypothesis of Theorem A, then (3) is equivalent to the following inequality:*

$$\int_a^b\mathbf{q}_2(\xi)G\left(\frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)},s\right)\Delta\xi\le\frac{\vartheta_2-1}{\vartheta_2-\vartheta_1}G(\vartheta_1,s)+\frac{1-\vartheta_1}{\vartheta_2-\vartheta_1}G(\vartheta_2,s),\tag{4}$$

where $G(\cdot,s)$ is defined in (1) and $s\in[\vartheta_1,\vartheta_2]$. Moreover, if we reverse the inequality in both (3) and (4), then again (3) and (4) are equivalent.

Proof Let (3) hold. As the function $G(\cdot,s)$ ($s\in[\vartheta_1,\vartheta_2]$) is convex and continuous, therefore (4) is valid.

Let (4) hold and $\Phi\in C^2([\vartheta_1,\vartheta_2],\mathbb{R})$. Then by using (2) one can get

$$\begin{aligned}&\frac{\vartheta_2-1}{\vartheta_2-\vartheta_1}\Phi(\vartheta_1)+\frac{1-\vartheta_1}{\vartheta_2-\vartheta_1}\Phi(\vartheta_2)-\int_a^b\mathbf{q}_2(\xi)\Phi\left(\frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)}\right)\Delta\xi\\&\quad=\frac{\vartheta_2-1}{\vartheta_2-\vartheta_1}\left[\Phi(\vartheta_1)+\int_{\vartheta_1}^{\vartheta_2}G(\vartheta_1,s)\Phi''(s)\,ds\right]\end{aligned}$$

$$+\frac{1-\vartheta_1}{\vartheta_2-\vartheta_1}\left[\Phi(\vartheta_2)+\int_{\vartheta_1}^{\vartheta_2} G(\vartheta_2,s)\Phi''(s)\,ds\right]$$
$$-\int_a^b \mathbf{q}_2(\xi)\left[\frac{\vartheta_2-\frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)}}{\vartheta_2-\vartheta_1}\Phi(\vartheta_1)+\frac{\frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)}-\vartheta_1}{\vartheta_2-\vartheta_1}\Phi(\vartheta_2)\right.$$
$$\left.+\int_{\vartheta_1}^{\vartheta_2} G\left(\frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)},s\right)\Phi''(s)\,ds\right]\Delta\xi. \tag{5}$$

Execute Fubini's theorem with $\int_a^b \mathbf{q}_1(\xi)\Delta\xi=\int_a^b \mathbf{q}_2(\xi)\Delta\xi=1$ in (5) to obtain

$$\frac{\vartheta_2-1}{\vartheta_2-\vartheta_1}\Phi(\vartheta_1)+\frac{1-\vartheta_1}{\vartheta_2-\vartheta_1}\Phi(\vartheta_2)-\int_a^b \mathbf{q}_2(\xi)\Phi\left(\frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)}\right)\Delta\xi$$
$$=\frac{\vartheta_2-1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} G(\vartheta_1,s)\Phi''(s)\,ds+\frac{1-\vartheta_1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} G(\vartheta_2,s)\Phi''(s)\,ds$$
$$-\int_{\vartheta_1}^{\vartheta_2}\Phi''(s)\left[\int_a^b \mathbf{q}_2(\xi)G\left(\frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)},s\right)\Delta\xi\right]ds.$$

If Φ is convex, then $\Phi''(s)\geq 0$ for all $s\in[\vartheta_1,\vartheta_2]$, and thus inequality (3) holds for every convex function $\Phi\in C^2([\vartheta_1,\vartheta_2],\mathbb{R})$. One can prove the last part of the theorem similarly. □

Remark 1 Assume the hypothesis of Theorem 1, the following two statements are equivalent:

(l_1') If $\Phi\in C([\vartheta_1,\vartheta_2],\mathbb{R})$ is concave, the inequality in (3) holds in reverse direction.

(l_2') For all $s\in[\vartheta_1,\vartheta_2]$, the reverse inequality in (4) holds.

In addition, if we reverse the inequality in both statements (l_1') and (c_2'), then again (l_1') and (l_2') are equivalent.

Theorem 2 *Assume the hypothesis of Theorem 1 and define the functional involving Csiszár divergence*:

$$\mathfrak{J}_1(\Phi)=\begin{cases}\frac{\vartheta_2-1}{\vartheta_2-\vartheta_1}\Phi(\vartheta_1)+\frac{1-\vartheta_1}{\vartheta_2-\vartheta_1}\Phi(\vartheta_2)-\int_a^b \mathbf{q}_2(\xi)\Phi(\frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)})\Delta\xi\\ \quad \textit{if the inequality in (4) holds for all } s\in[\vartheta_1,\vartheta_2],\\ \int_a^b \mathbf{q}_2(\xi)\Phi(\frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)})\Delta\xi-\frac{\vartheta_2-1}{\vartheta_2-\vartheta_1}\Phi(\vartheta_1)-\frac{1-\vartheta_1}{\vartheta_2-\vartheta_1}\Phi(\vartheta_2)\\ \quad \textit{for all } s\in[\vartheta_1,\vartheta_2], \textit{if the reverse inequality in (4) holds.}\end{cases} \tag{6}$$

Remark 2 Assume the hypothesis of Theorem 2 with Φ is convex and continuous, then $\mathfrak{J}_1(\Phi)\geq 0$.

4 Generalization of an inequality containing Csiszár divergence by Taylor's formula

Let us begin by defining the real-valued function

$$(s-t)_+=\begin{cases}s-t, & t\leq s,\\ 0, & t>s.\end{cases} \tag{7}$$

In [6], Taylor's formula is given as follows:

Assume $n \in \mathbb{Z}^+$ and $h : [a,b] \to \mathbb{R}$ is such that $h^{(n-1)}$ is absolutely continuous, then Taylor's formula at the point $c \in [a,b]$ is

$$h(s) = T_{n-1}(h;c,s) + R_{n-1}(h;c,s) \quad \text{for all } s \in [a,b], \tag{8}$$

where the degree of Taylor polynomial $T_{n-1}(h;c,x)$ is $n-1$, i.e.,

$$T_{n-1}(h;c,s) = \sum_{k=0}^{n-1} \frac{h^k(c)}{k!}(s-c)^k,$$

and $R_{n-1}(h;c,s)$ is defined as

$$R_{n-1}(h;c,s) = \frac{1}{(n-1)!}\int_c^s h^n(t)(s-t)^{n-1}\,dt.$$

Use Taylor's formula at the endpoints to obtain

$$h(s) = \sum_{k=0}^{n-1} \frac{h^k(a)}{k!}(s-a)^k + \frac{1}{(n-1)!}\int_a^b h^n(t)(s-t)_+^{n-1}\,dt, \tag{9}$$

$$h(s) = \sum_{k=0}^{n-1} \frac{h^k(b)}{k!}(b-s)^k(-1)^k - \frac{1}{(n-1)!}\int_a^b (-1)^{n-1}h^n(t)(t-s)_+^{n-1}\,dt. \tag{10}$$

We establish the following identities by using (9) and (10).

Theorem 3 *Assume $n \in \mathbb{Z}^+$ and the function $\Phi : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ with $\Phi^{(n-1)}$ is absolutely continuous and $\vartheta_1 \leq 1 \leq \vartheta_2$. If $\mathbf{q}_1, \mathbf{q}_2 \in \Omega$ with $\vartheta_1 \leq \frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)} \leq \vartheta_2$ for all $\xi \in \mathbb{T}$, then*

$$\begin{aligned}\mathfrak{J}_1\big(\Phi(x)\big) = {}& \int_{\vartheta_1}^{\vartheta_2} \mathfrak{J}_1\big(G(\cdot,s)\big)\left(\sum_{u=2}^{n-1}\frac{\Phi^{(u)}(\vartheta_1)}{(u-2)!}(s-\vartheta_1)_+^{u-2}\right)ds \\ & + \frac{1}{(n-3)!}\int_{\vartheta_1}^{\vartheta_2} \Phi^{(n)}(t)\left(\int_t^{\vartheta_2} \mathfrak{J}_1\big(G(\cdot,s)\big)(s-t)_+^{n-3}\,ds\right)dt,\end{aligned} \tag{11}$$

where

$$\mathfrak{J}_1\big(G(\cdot,s)\big) = \frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1}G(\vartheta_1,s) + \frac{1-\vartheta_1}{\vartheta_2-\vartheta_1}G(\vartheta_2,s) - \int_a^b \mathbf{q}_2(\xi)G\left(\frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)},s\right)\Delta\xi \tag{12}$$

and

$$\begin{aligned}\mathfrak{J}_1\big(\Phi(x)\big) = {}& \int_{\vartheta_1}^{\vartheta_2} \mathfrak{J}_1\big(G(\cdot,s)\big)\left(\sum_{u=2}^{n-1}\frac{\Phi^{(u)}(\vartheta_2)}{(u-2)!}(s-\vartheta_2)_+^{u-2}\right)ds \\ & + \frac{1}{(n-3)!}\int_{\vartheta_1}^{\vartheta_2} \Phi^{(n)}(t)\left(\int_{\vartheta_1}^{t} \mathfrak{J}_1\big(G(\cdot,s)\big)(s-t)_+^{n-3}\,ds\right)dt,\end{aligned} \tag{13}$$

where $\mathfrak{J}_1(G(\cdot,s))$ is given in (12).

Proof Use (2) in (6) and the linearity of $\mathfrak{J}_1(\cdot)$ to obtain

$$\mathfrak{J}_1\big(\Phi(x)\big) = \int_{\vartheta_1}^{\vartheta_2} \mathfrak{J}_1\big(G(\cdot,s)\big)\Phi''(s)\,ds. \tag{14}$$

On the function Φ, apply Taylor's formula (8) at the point ϑ_1 and replace n with $n-2$ ($n \geq 3$) or take the second derivative of (8) with $c = \vartheta_1$ to obtain

$$\Phi''(s) = \sum_{u=2}^{n-1} \frac{\Phi^{(u)}(\vartheta_1)}{(u-2)!}(s-\vartheta_1)_+^{u-2} + \frac{1}{(n-3)!}\int_{\vartheta_1}^{s} \Phi^{(n)}(t)(s-t)_+^{n-3}\,dt, \tag{15}$$

and for $c = \vartheta_2$ one gets

$$\Phi''(s) = \sum_{u=2}^{n-1} \frac{\Phi^{(u)}(\vartheta_2)}{(u-2)!}(s-\vartheta_2)_+^{u-2} + \frac{1}{(n-3)!}\int_{s}^{\vartheta_2} \Phi^{(n)}(t)(s-t)_+^{n-3}\,dt. \tag{16}$$

Use (15) in (14) to obtain

$$\begin{aligned}\mathfrak{J}_1\big(\Phi(x)\big) &= \sum_{u=2}^{n-1} \frac{\Phi^{(u)}(\vartheta_1)}{(u-2)!}\int_{\vartheta_1}^{\vartheta_2} \mathfrak{J}_1\big(G(\cdot,s)\big)(s-\vartheta_1)_+^{u-2}\,ds \\ &\quad + \frac{1}{(n-3)!}\int_{\vartheta_1}^{\vartheta_2} \mathfrak{J}_1\big(G(\cdot,s)\big)\int_{\vartheta_1}^{s} \Phi^{(n)}(t)(s-t)_+^{n-3}\,dt\,ds. \end{aligned} \tag{17}$$

Utilize Fubini's theorem on the final term of (17) to obtain (11). Furthermore, use (16) in (14) and execute Fubini's theorem to get (13). □

Corollary 1 *Choose $\mathbb{T} = \mathbb{R}$ in Theorem 3 to get the following new identities*:

$$\begin{aligned}\mathfrak{J}_1\big(\Phi(\cdot)\big) &= \int_{\vartheta_1}^{\vartheta_2} \mathfrak{J}_1\big(G(\cdot,s)\big)\left(\sum_{u=2}^{n-1} \frac{\Phi^{(u)}(\vartheta_1)}{(u-2)!}(s-\vartheta_1)_+^{u-2}\right)ds \\ &\quad + \frac{1}{(n-3)!}\int_{\vartheta_1}^{\vartheta_2} \Phi^{(n)}(t)\left(\int_{t}^{\vartheta_2} \mathfrak{J}_1\big(G(\cdot,s)\big)(s-t)_+^{n-3}\,ds\right)dt,\end{aligned}$$

where

$$\mathfrak{J}_1\big(\Phi(\cdot)\big) = \frac{\vartheta_2-1}{\vartheta_2-\vartheta_1}\Phi(\vartheta_1) + \frac{1-\vartheta_1}{\vartheta_2-\vartheta_1}\Phi(\vartheta_2) - \int_a^b \mathbf{q}_2(\xi)\Phi\left[\frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)}\right]d\xi \tag{18}$$

and

$$\mathfrak{J}_1\big(G(\cdot,s)\big) = \frac{\vartheta_2-1}{\vartheta_2-\vartheta_1}G(\vartheta_1,s) + \frac{1-\vartheta_1}{\vartheta_2-\vartheta_1}G(\vartheta_2,s) - \int_a^b \mathbf{q}_2(\xi)G\left(\frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)},s\right)d\xi. \tag{19}$$

Also

$$\begin{aligned}\mathfrak{J}_1\big(\Phi(\cdot)\big) &= \int_{\vartheta_1}^{\vartheta_2} \mathfrak{J}_1\big(G(\cdot,s)\big)\left(\sum_{u=2}^{n-1} \frac{\Phi^{(u)}(\vartheta_2)}{(u-2)!}(s-\vartheta_2)_+^{u-2}\right)ds \\ &\quad + \frac{1}{(n-3)!}\int_{\vartheta_1}^{\vartheta_2} \Phi^{(n)}(t)\left(\int_{\vartheta_1}^{t} \mathfrak{J}_1\big(G(\cdot,s)\big)(s-t)_+^{n-3}\,ds\right)dt,\end{aligned}$$

where $\mathfrak{J}_1(\Phi(\cdot))$ and $\mathfrak{J}_1(G(\cdot,s))$ are given in (18) and (19) respectively.

Corollary 2 *Put $\mathbb{T} = h\mathbb{Z}$ $(h > 0)$ in Theorem 3 to obtain the following new identities in h-discrete calculus*:

$$\mathfrak{J}_1\big(\Phi(\cdot)\big) = \int_{\vartheta_1}^{\vartheta_2} \mathfrak{J}_1\big(G(\cdot,s)\big)\left(\sum_{u=2}^{n-1} \frac{\Phi^{(u)}(\vartheta_1)}{(u-2)!}(s-\vartheta_1)_+^{u-2}\right) ds + \frac{1}{(n-3)!}\int_{\vartheta_1}^{\vartheta_2} \Phi^{(n)}(t)\left(\int_t^{\vartheta_2} \mathfrak{J}_1\big(G(\cdot,s)\big)(s-t)_+^{n-3}\, ds\right) dt,$$

where

$$\mathfrak{J}_1\big(\Phi(\cdot)\big) = \frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1}\Phi(\vartheta_1) + \frac{1-\vartheta_1}{\vartheta_2 - \vartheta_1}\Phi(\vartheta_2) - \sum_{j=\frac{a}{h}}^{\frac{b}{h}-1} \mathbf{q}_2(jh)h\Phi\left(\frac{\mathbf{q}_1(jh)}{\mathbf{q}_2(jh)}\right), \tag{20}$$

$$\mathfrak{J}_1\big(G(\cdot,s)\big) = \frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1}G(\vartheta_1,s) + \frac{1-\vartheta_1}{\vartheta_2 - \vartheta_1}G(\vartheta_2,s) - \sum_{j=\frac{a}{h}}^{\frac{b}{h}-1} \mathbf{q}_2(jh)hG\left(\frac{\mathbf{q}_1(jh)}{\mathbf{q}_2(jh)},s\right) \tag{21}$$

and

$$\mathfrak{J}_1\big(\Phi(x)\big) = \int_{\vartheta_1}^{\vartheta_2} \mathfrak{J}_1\big(G(\cdot,s)\big)\left(\sum_{u=2}^{n-1} \frac{\Phi^{(u)}(\vartheta_2)}{(u-2)!}(s-\vartheta_2)_+^{u-2}\right) ds + \frac{1}{(n-3)!}\int_{\vartheta_1}^{\vartheta_2} \Phi^{(n)}(t)\left(\int_{\vartheta_1}^{t} \mathfrak{J}_1\big(G(\cdot,s)\big)(s-t)_+^{n-3}\, ds\right) dt,$$

where $\mathfrak{J}_1(\Phi(\cdot))$ and $\mathfrak{J}_1(G(\cdot,s))$ are given in (20) and (21) respectively.

Remark 3 Choose $h = 1$ in Example 2. Suppose that $a = 0$, $b = n$, $\mathbf{q}_1(j) = (\mathbf{q}_1)_j$ and $\mathbf{q}_2(j) = (\mathbf{q}_2)_j$ to get the following new identities in the discrete case:

$$\mathfrak{J}_1\big(\Phi(\cdot)\big) = \int_{\vartheta_1}^{\vartheta_2} \mathfrak{J}_1\big(G(\cdot,s)\big)\left(\sum_{u=2}^{n-1} \frac{\Phi^{(u)}(\vartheta_1)}{(u-2)!}(s-\vartheta_1)_+^{u-2}\right) ds + \frac{1}{(n-3)!}\int_{\vartheta_1}^{\vartheta_2} \Phi^{(n)}(t)\left(\int_t^{\vartheta_2} \mathfrak{J}_1\big(G(\cdot,s)\big)(s-t)_+^{n-3}\, ds\right) dt,$$

where

$$\mathfrak{J}_1\big(\Phi(\cdot)\big) = \frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1}\Phi(\vartheta_1) + \frac{1-\vartheta_1}{\vartheta_2 - \vartheta_1}\Phi(\vartheta_2) - \sum_{j=1}^{n} (\mathbf{q}_2)_j\Phi\left(\frac{(\mathbf{q}_1)_j}{(\mathbf{q}_2)_j}\right) \tag{22}$$

and

$$\mathfrak{J}_1\big(G(\cdot,s)\big) = \frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1}G(\vartheta_1,s) + \frac{1-\vartheta_1}{\vartheta_2 - \vartheta_1}G(\vartheta_2,s) - \sum_{j=1}^{n} (\mathbf{q}_2)_jG\left(\frac{(\mathbf{q}_1)_j}{(\mathbf{q}_2)_j},s\right). \tag{23}$$

Also

$$\mathfrak{J}_1\big(\Phi(x)\big) = \int_{\vartheta_1}^{\vartheta_2} \mathfrak{J}_1\big(G(\cdot,s)\big)\left(\sum_{u=2}^{n-1} \frac{\Phi^{(u)}(\vartheta_2)}{(u-2)!}(s-\vartheta_2)_+^{u-2}\right) ds$$
$$+ \frac{1}{(n-3)!}\int_{\vartheta_1}^{\vartheta_2} \Phi^{(n)}(t)\left(\int_{\vartheta_1}^{t} \mathfrak{J}_1\big(G(\cdot,s)\big)(s-t)_+^{n-3}\, ds\right) dt,$$

where $\mathfrak{J}_1(\Phi(\cdot))$ and $\mathfrak{J}_1(G(\cdot,s))$ are given in (22) and (23) respectively.

Corollary 3 *Let $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, $a = q^m$, and $b = q^n$ with $m < n$ to get the following new identities in q-calculus*:

$$\mathfrak{J}_1\big(\Phi(\cdot)\big) = \int_{\vartheta_1}^{\vartheta_2} \mathfrak{J}_1\big(G(\cdot,s)\big)\left(\sum_{u=2}^{n-1} \frac{\Phi^{(u)}(\vartheta_1)}{(u-2)!}(s-\vartheta_1)_+^{u-2}\right) ds$$
$$+ \frac{1}{(n-3)!}\int_{\vartheta_1}^{\vartheta_2} \Phi^{(n)}(t)\left(\int_{t}^{\vartheta_2} \mathfrak{J}_1\big(G(\cdot,s)\big)(s-t)_+^{n-3}\, ds\right) dt,$$

where

$$\mathfrak{J}_1\big(\Phi(\cdot)\big) = \frac{\vartheta_2-1}{\vartheta_2-\vartheta_1}\Phi(\vartheta_1) + \frac{1-\vartheta_1}{\vartheta_2-\vartheta_1}\Phi(\vartheta_2) - \sum_{j=m}^{n-1} q^{j+1}\mathbf{q}_2\big(q^j\big)\Phi\left(\frac{\mathbf{q}_1(q^j)}{\mathbf{q}_2(q^j)}\right), \tag{24}$$

$$\mathfrak{J}_1\big(G(\cdot,s)\big) = \frac{\vartheta_2-1}{\vartheta_2-\vartheta_1}G(\vartheta_1,s) + \frac{1-\vartheta_1}{\vartheta_2-\vartheta_1}G(\vartheta_2,s) - \sum_{j=m}^{n-1} q^{j+1}\mathbf{q}_2\big(q^j\big)G\left(\frac{\mathbf{q}_1(q^j)}{\mathbf{q}_2(q^j)},s\right) \tag{25}$$

and

$$\mathfrak{J}_1\big(\Phi(x)\big) = \int_{\vartheta_1}^{\vartheta_2} \mathfrak{J}_1\big(G(\cdot,s)\big)\left(\sum_{u=2}^{n-1} \frac{\Phi^{(u)}(\vartheta_2)}{(u-2)!}(s-\vartheta_2)_+^{u-2}\right) ds$$
$$+ \frac{1}{(n-3)!}\int_{\vartheta_1}^{\vartheta_2} \Phi^{(n)}(t)\left(\int_{\vartheta_1}^{t} \mathfrak{J}_1\big(G(\cdot,s)\big)(s-t)_+^{n-3}\, ds\right) dt,$$

where $\mathfrak{J}_1(\Phi(\cdot))$ and $\mathfrak{J}_1(G(\cdot,s))$ are given in (24) and (25) respectively.

Theorem 4 *Assume the hypothesis of Theorem 3. Also suppose that Φ is an n-convex function with $\Phi^{(n-1)}$ is absolutely continuous. If*

$$\int_{s}^{\vartheta_2} \mathfrak{J}_1\big(G(\cdot,s)\big)(s-t)_+^{n-3}\, ds \geq 0, \quad t \in [\vartheta_1,\vartheta_2], \tag{26}$$

then

$$\mathfrak{J}_1\big(\Phi(x)\big) \geq \int_{\vartheta_1}^{\vartheta_2} \mathfrak{J}_1\big(G(\cdot,s)\big)\left(\sum_{u=2}^{n-1} \frac{\Phi^{(u)}(\vartheta_1)}{(u-2)!}(s-\vartheta_1)_+^{u-2}\right) ds \tag{27}$$

holds, and if

$$\int_{\vartheta_1}^{s} \mathfrak{J}_1\big(G(\cdot,s)\big)(s-t)_+^{n-3}\, ds \leq 0, \quad t \in [\vartheta_1,\vartheta_2], \tag{28}$$

then

$$\mathfrak{I}_1\big(\Phi(x)\big) \geq \int_{\vartheta_1}^{\vartheta_2} \mathfrak{I}_1\big(G(\cdot,s)\big)\left(\sum_{u=2}^{n-1} \frac{\Phi^{(u)}(\vartheta_2)}{(u-2)!}(s-\vartheta_2)_+^{u-2}\right) ds \tag{29}$$

holds.

Proof Since $\Phi^{(n-1)}$ is absolutely continuous on $[\vartheta_1, \vartheta_2]$, $\Phi^{(n)}$ exists almost everywhere. Given that Φ is n-convex, for all $x \in [\vartheta_1, \vartheta_2]$, we have $\Phi^{(n)}(x) \geq 0$ (see [43, p. 16]). Use Theorem 3 to get (27) and (29) respectively. □

The next result is an instant significance of Theorem 4 as $\mathfrak{I}_1(\Phi) \geq 0$.

Corollary 4 *Under the assumptions of Theorem 3 with $\mathbf{q} \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ positive such that $\int_a^b \mathbf{q}(t)\Delta t = 1$. Then*

(i) *If Φ is an n-convex function ($n \geq 3$), then (27) holds for $n \in \{3, \dots\}$. Further, if*

$$\sum_{u=2}^{n-1} \frac{\Phi^{(u)}(\vartheta_1)}{(u-2)!}(s-\vartheta_1)_+^{u-2} \geq 0,$$

then

$$\mathfrak{I}_1\big(\Phi(x)\big) \geq 0 \tag{30}$$

holds.

(ii) *For even n, (29) is valid. In addition, if*

$$\sum_{u=2}^{n-1} \frac{\Phi^{(u)}(\vartheta_2)}{(u-2)!}(s-\vartheta_2)_+^{u-2} \geq 0,$$

then (30) is valid too.

Remark 4 It is also possible to compute Grüss, Cebyšev, and Ostrowski type bounds corresponding to following functional:

$$\mathfrak{I}_1(\Phi) := \frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1}\Phi(\vartheta_1) + \frac{1-\vartheta_1}{\vartheta_2 - \vartheta_1}\Phi(\vartheta_2) - \int_a^b \mathbf{q}_2(\xi)\Phi\left(\frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)}\right)\Delta\xi.$$

5 Applications to information theory

Shannon entropy is the fundamental term in information theory, and it is often dealt with measure of uncertainty. The random variable, entropy, is characterized regarding its probability distribution, and it can appear as a better measure of uncertainty or predictability. Shannon entropy allows the estimation of the normal least number of bits essential to encode a string of symbols based on alphabet size and frequency of symbols.

5.1 Differential entropy on time scales

On time scale, Ansari *et al.* [10] introduced the differential entropy which is given as

$$h_{\bar{b}}(X) := \int_a^b \mathbf{q}(\xi) \log \frac{1}{\mathbf{q}(\xi)} \Delta\xi, \tag{31}$$

where $\bar{b} > 1$ and X is a continuous random variable and $\mathbf{q}$ is a positive density function on $\mathbb{T}$ to X such that $\int_a^b \mathbf{q}(\xi)\Delta\xi = 1$, whenever the integral exists.

Theorem 5 *Let X be a continuous random variable and $\mathbf{q}_1, \mathbf{q}_2 \in \Omega$ with $\vartheta_1 \leq \frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)} \leq \vartheta_2$ for all $\xi \in \mathbb{T}$. If n is odd $(n = 3, 5, \ldots)$, then*

$$\mathfrak{J}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2} \mathfrak{J}_1\big(G(\cdot, s)\big) \left(\sum_{u=2}^{n-1} \frac{(-1)^{u-1}(u-1)(s-\vartheta_1)_+^{u-2}}{\vartheta_1^u} \right) ds \tag{32}$$

and

$$\mathfrak{J}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2} \mathfrak{J}_1\big(G(\cdot, s)\big) \left(\sum_{u=2}^{n-1} \frac{(-1)^{u-1}(u-1)(s-\vartheta_2)_+^{u-2}}{\vartheta_2^u} \right) ds, \tag{33}$$

where

$$\mathfrak{J}_1(\cdot) = \frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1} \log \vartheta_1 + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1} \log \vartheta_2 - \int_a^b \mathbf{q}_2(\xi) \log \mathbf{q}_1(\xi) \Delta\xi - \tilde{h}_{\bar{b}}(X)$$

and

$$\mathfrak{J}_1\big(G(\cdot, s)\big) = \frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1} G(\vartheta_1, s) + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1} G(\vartheta_2, s) - \int_a^b \mathbf{q}_2(\xi) G\left(\frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)}, s \right) \Delta\xi.$$

Proof Since the Green's function $G(\cdot, s)$ given in (1) is convex, by using Remark 2, $\mathfrak{J}_1 G(\cdot, s) \geq 0$ and $(s-t)_+^{n-3} \geq 0$ for $n = 3, 5, \ldots$. The function $\Phi(x) := \log x$ is n-convex for $n = 3, 5, \ldots$. Use $\Phi(x) = \log x$ in Theorem 4, then (27) and (29) become (32) and (33), respectively, where

$$\tilde{h}_{\bar{b}}(X) = \int_a^b \mathbf{q}_2(\xi) \log \frac{1}{\mathbf{q}_2(\xi)} \Delta\xi. \qquad \square$$

Example 1 Choose $\mathbb{T} = \mathbb{R}$, in Theorem 5, inequalities(32) and (33) take the following form, respectively:

$$\begin{aligned} \mathfrak{J}_1(\cdot) \geq{}& \int_{\vartheta_1}^{\vartheta_2} \left[\frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1} G(\vartheta_1, s) + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1} G(\vartheta_2, s) - \int_a^b \mathbf{q}_2(\xi) G\left(\frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)}, s \right) d\xi \right] \\ & \times \left(\sum_{u=2}^{n-1} \frac{(-1)^{u-1}(u-1)(s-\vartheta_1)_+^{u-2}}{\vartheta_1^u} \right) ds \end{aligned} \tag{34}$$

and

$$\mathfrak{J}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2} \left[\frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1} G(\vartheta_1, s) + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1} G(\vartheta_2, s) - \int_a^b \mathbf{q}_2(\xi) G\left(\frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)}, s \right) d\xi \right] \times \left(\sum_{u=2}^{n-1} \frac{(-1)^{u-1}(u-1)(s-\vartheta_2)_+^{u-2}}{\vartheta_2^u} \right) ds, \tag{35}$$

where

$$\mathfrak{J}_1(\cdot) = \frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1} \log \vartheta_1 + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1} \log \vartheta_2 - \int_a^b \mathbf{q}_2(\xi) \log \mathbf{q}_1(\xi)\, d\xi - \tilde{h}_{\bar{b}}(X)$$

and

$$\tilde{h}_{\bar{b}}(X) := \int_a^b \mathbf{q}_2(\xi) \log \frac{1}{\mathbf{q}_2(\xi)}\, d\xi.$$

Example 2 Choose $\mathbb{T} = h\mathbb{Z}$, $h > 0$ in Theorem 5, inequalities (32) and (33) take the following form in h-discrete calculus, respectively:

$$\mathfrak{J}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2} \left[\frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1} G(\vartheta_1, s) + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1} G(\vartheta_2, s) - \sum_{j=\frac{a}{h}}^{\frac{b}{h}-1} \mathbf{q}_2(jh) h G\left(\frac{\mathbf{q}_1(jh)}{\mathbf{q}_2(jh)}, s \right) \right] \times \left(\sum_{u=2}^{n-1} \frac{(-1)^{u-1}(u-1)(s-\vartheta_1)_+^{u-2}}{\vartheta_1^u} \right) ds \tag{36}$$

and

$$\mathfrak{J}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2} \left[\frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1} G(\vartheta_1, s) + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1} G(\vartheta_2, s) - \sum_{j=\frac{a}{h}}^{\frac{b}{h}-1} \mathbf{q}_2(jh) h G\left(\frac{\mathbf{q}_1(jh)}{\mathbf{q}_2(jh)}, s \right) \right] \times \left(\sum_{u=2}^{n-1} \frac{(-1)^{u-1}(u-1)(s-\vartheta_2)_+^{u-2}}{\vartheta_2^u} \right) ds, \tag{37}$$

where

$$\mathfrak{J}_1(\cdot) = \frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1} \big(-\log(\vartheta_1) \big) + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1} \big(-\log(\vartheta_2) \big) + \sum_{j=\frac{a}{h}}^{\frac{b}{h}-1} \mathbf{q}_2(jh) h \log\big[\mathbf{q}_1(jh) h \big] + \tilde{S}$$

and

$$\tilde{S} := \sum_{j=\frac{a}{h}}^{\frac{b}{h}-1} \mathbf{q}_2(jh) h \log \frac{1}{\mathbf{q}_2(jh) h}.$$

Remark 5 Choose $h = 1$ in Example 2. Suppose that $a = 0$, $b = n$, $\mathbf{q}_1(j) = (\mathbf{q}_1)_j$, and $\mathbf{q}_2(j) = (\mathbf{q}_2)_j$ to get the following new forms of inequalities (32) and (33), respectively:

$$\begin{aligned}\mathfrak{J}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2} &\left[\frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1} G(\vartheta_1, s) + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1} G(\vartheta_2, s) - \sum_{j=1}^{n} (\mathbf{q}_2)_j G\left(\frac{(\mathbf{q}_1)_j}{(\mathbf{q}_2)_j}, s\right)\right] \\ &\times \left(\sum_{u=2}^{n-1} \frac{(-1)^{u-1}(u-1)(s-\vartheta_1)_+^{u-2}}{\vartheta_1^u}\right) ds \end{aligned} \tag{38}$$

and

$$\begin{aligned}\mathfrak{J}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2} &\left[\frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1} G(\vartheta_1, s) + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1} G(\vartheta_2, s) - \sum_{j=1}^{n} (\mathbf{q}_2)_j G\left(\frac{(\mathbf{q}_1)_j}{(\mathbf{q}_2)_j}, s\right)\right] \\ &\times \left(\sum_{u=2}^{n-1} \frac{(-1)^{u-1}(u-1)(s-\vartheta_2)_+^{u-2}}{\vartheta_2^u}\right) ds, \end{aligned} \tag{39}$$

where

$$\mathfrak{J}_1(\cdot) = \frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1}\bigl(\log(\vartheta_1)\bigr) + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1}\bigl(\log(\vartheta_2)\bigr) - \sum_{j=1}^{n} (\mathbf{q}_2)_j \log(\mathbf{q}_1)_j - S$$

and

$$S := \sum_{j=1}^{n} (\mathbf{q}_2)_j \log \frac{1}{(\mathbf{q}_2)_j}$$

is discrete Shannon entropy.

Example 3 Let $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, $a = q^m$, and $b = q^n$ with $m < n$, inequalities (32) and (33) take the following form in quantum calculus, respectively:

$$\begin{aligned}\mathfrak{J}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2} &\left[\frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1} G(\vartheta_1, s) + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1} G(\vartheta_2, s) - \sum_{j=m}^{n-1} q^{j+1} \mathbf{q}_2(q^j) G\left(\frac{\mathbf{q}_1(q^j)}{\mathbf{q}_2(q^j)}, s\right)\right] \\ &\times \left(\sum_{u=2}^{n-1} \frac{(-1)^{u-1}(u-1)(s-\vartheta_1)_+^{u-2}}{\vartheta_1^u}\right) ds \end{aligned} \tag{40}$$

and

$$\begin{aligned}\mathfrak{J}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2} &\left[\frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1} G(\vartheta_1, s) + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1} G(\vartheta_2, s) - \sum_{j=m}^{n-1} q^{j+1} \mathbf{q}_2(q^j) G\left(\frac{\mathbf{q}_1(q^j)}{\mathbf{q}_2(q^j)}, s\right)\right] \\ &\times \left(\sum_{u=2}^{n-1} \frac{(-1)^{u-1}(u-1)(s-\vartheta_2)_+^{u-2}}{\vartheta_2^u}\right) ds, \end{aligned} \tag{41}$$

where

$$\mathfrak{J}_1(\cdot) = \frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1} \log(\vartheta_1) + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1} \log(\vartheta_2) - \sum_{j=m}^{n-1} q^{j+1} \mathbf{q}_2\bigl(q^j\bigr) \log\bigl[\mathbf{q}_1\bigl(q^j\bigr)\bigr] - S_q$$

and

$$S_q := \sum_{j=m}^{n-1} q^{j+1}\mathbf{q}_2\big(q^j\big)\log\frac{1}{\mathbf{q}_2(q^j)}.$$

5.2 Kullback–Leibler divergence

Ansari *et al.* [11] defined the Kullback–Leibler divergence on time scale by

$$D(\mathbf{q}_1,\mathbf{q}_2) = \int_a^b \mathbf{q}_1(\xi)\ln\left[\frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)}\right]\Delta\xi. \tag{42}$$

Theorem 6 *Let X be a continuous random variable and $\mathbf{q}_1,\mathbf{q}_2 \in \Omega$ with $\vartheta_1 \leq \frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)} \leq \vartheta_2$ for all $\xi \in \mathbb{T}$. If n is odd ($n = 3,5,\ldots$), then*

$$\mathfrak{J}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2} \mathfrak{J}_1\big(G(\cdot,s)\big)\left(\sum_{u=2}^{n-1}\frac{(-1)^{u-1}(s-\vartheta_1)_+^{u-2}}{\vartheta_1^{u-1}}\right)ds \tag{43}$$

and

$$\mathfrak{J}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2} \mathfrak{J}_1\big(G(\cdot,s)\big)\left(\sum_{u=2}^{n-1}\frac{(-1)^{u-1}(s-\vartheta_2)_+^{u-2}}{\vartheta_2^{u-1}}\right)ds, \tag{44}$$

where

$$\mathfrak{J}_1(\cdot) = \frac{\vartheta_2-1}{\vartheta_2-\vartheta_1}(-\vartheta_1\ln\vartheta_1) + \frac{1-\vartheta_1}{\vartheta_2-\vartheta_1}(-\vartheta_2\ln\vartheta_2) + D(\mathbf{q}_1,\mathbf{q}_2)$$

and

$$\mathfrak{J}_1\big(G(\cdot,s)\big) = \frac{\vartheta_2-1}{\vartheta_2-\vartheta_1}G(\vartheta_1,s) + \frac{1-\vartheta_1}{\vartheta_2-\vartheta_1}G(\vartheta_2,s) - \int_a^b \mathbf{q}_2(\xi)G\left(\frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)},s\right)\Delta\xi,$$

where $D(\mathbf{q}_1,\mathbf{q}_2)$ given in (42).

Proof The function $\Phi(x) := -x\ln x$ is n-convex for $n = 3,5,\ldots$. Use $\Phi(x) = -x\ln x$ and observe the analogous method as in the proof of Theorem 5 to obtain (43) and (44). □

Example 4 Choose $\mathbb{T} = \mathbb{R}$ in Theorem 6, inequalities(43) and (44) take the following form, respectively:

$$\begin{aligned}\mathfrak{J}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2}&\left[\frac{\vartheta_2-1}{\vartheta_2-\vartheta_1}G(\vartheta_1,s) + \frac{1-\vartheta_1}{\vartheta_2-\vartheta_1}G(\vartheta_2,s) - \int_a^b \mathbf{q}_2(\xi)G\left(\frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)},s\right)d\xi\right]\\ &\times\left(\sum_{u=2}^{n-1}\frac{(-1)^{u-1}(s-\vartheta_1)_+^{u-2}}{\vartheta_1^{u-1}}\right)ds\end{aligned} \tag{45}$$

and

$$\begin{aligned}\mathfrak{J}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2} & \left[\frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1} G(\vartheta_1, s) + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1} G(\vartheta_2, s) - \int_a^b \mathbf{q}_2(\xi) G\left(\frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)}, s\right) d\xi \right] \\ & \times \left(\sum_{u=2}^{n-1} \frac{(-1)^{u-1}(s - \vartheta_2)_+^{u-2}}{\vartheta_2^{u-1}} \right) ds, \end{aligned} \tag{46}$$

where

$$\mathfrak{J}_1(\cdot) = \frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1}(-\vartheta_1 \ln \vartheta_1) + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1}(-\vartheta_2 \ln \vartheta_2) + D_{\mathrm{KL}}(\mathbf{q}_1, \mathbf{q}_2)$$

and

$$D_{\mathrm{KL}}(\mathbf{q}_1, \mathbf{q}_2) := \int_a^b \mathbf{q}_1(\xi) \ln \frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)} \, d\xi.$$

Example 5 Choose $\mathbb{T} = h\mathbb{Z}$ ($h > 1$) in Theorem 6, inequalities (43) and (44) take the following form in h-discrete calculus, respectively:

$$\begin{aligned}\mathfrak{J}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2} & \left[\frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1} G(\vartheta_1, s) + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1} G(\vartheta_2, s) - \sum_{j=\frac{a}{h}}^{\frac{b}{h}-1} \mathbf{q}_2(jh) h G\left(\frac{\mathbf{q}_1(jh)}{\mathbf{q}_2(jh)}, s\right) \right] \\ & \times \left(\sum_{u=2}^{n-1} \frac{(-1)^{u-1}(s - \vartheta_1)_+^{u-2}}{\vartheta_1^{u-1}} \right) ds \end{aligned} \tag{47}$$

and

$$\begin{aligned}\mathfrak{J}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2} & \left[\frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1} G(\vartheta_1, s) + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1} G(\vartheta_2, s) - \sum_{j=\frac{a}{h}}^{\frac{b}{h}-1} \mathbf{q}_2(jh) h G\left(\frac{\mathbf{q}_1(jh)}{\mathbf{q}_2(jh)}, s\right) \right] \\ & \times \left(\sum_{u=2}^{n-1} \frac{(-1)^{u-1}(s - \vartheta_2)_+^{u-2}}{\vartheta_2^{u-1}} \right) ds, \end{aligned} \tag{48}$$

where

$$\mathfrak{J}_1(\cdot) = \frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1}(-\vartheta_1 \ln \vartheta_1) + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1}(-\vartheta_2 \ln \vartheta_2) + \tilde{D}_{\mathrm{KL}}(\mathbf{q}_1, \mathbf{q}_2)$$

and

$$\tilde{D}_{\mathrm{KL}}(\mathbf{q}_1, \mathbf{q}_2) := \sum_{j=\frac{a}{h}}^{\frac{b}{h}-1} \mathbf{q}_1(jh) h \ln \frac{\mathbf{q}_1(jh)}{\mathbf{q}_2(jh)}.$$

Remark 6 Choose $h = 1$ in Example 5. Suppose that $a = 0$, $b = n$, $\mathbf{q}_1(j) = (\mathbf{q}_1)_j$, and $\mathbf{q}_2(j) = (\mathbf{q}_2)_j$ to get the following new forms of inequalities (43) and (44), respectively:

$$\mathfrak{J}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2} \left[\frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1} G(\vartheta_1, s) + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1} G(\vartheta_2, s) - \sum_{j=1}^{n} (\mathbf{q}_2)_j G\left(\frac{(\mathbf{q}_1)_j}{(\mathbf{q}_2)_j}, s \right) \right] \times \left(\sum_{u=2}^{n-1} \frac{(-1)^{u-1}(s - \vartheta_1)_+^{u-2}}{\vartheta_1^{u-1}} \right) ds \tag{49}$$

and

$$\mathfrak{J}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2} \left[\frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1} G(\vartheta_1, s) + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1} G(\vartheta_2, s) - \sum_{j=1}^{n} (\mathbf{q}_2)_j G\left(\frac{(\mathbf{q}_1)_j}{(\mathbf{q}_2)_j}, s \right) \right] \times \left(\sum_{u=2}^{n-1} \frac{(-1)^{u-1}(s - \vartheta_2)_+^{u-2}}{\vartheta_2^{u-1}} \right) ds, \tag{50}$$

where

$$\mathfrak{J}_1(\cdot) = \frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1}(-\vartheta_1 \ln \vartheta_1) + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1}(-\vartheta_2 \ln \vartheta_2) + \mathrm{KL}(\mathbf{q}_1, \mathbf{q}_2)$$

and

$$\mathrm{KL}(\mathbf{q}_1, \mathbf{q}_2) := \sum_{j=1}^{n} (\mathbf{q}_1)_j \ln \frac{(\mathbf{q}_1)_j}{(\mathbf{q}_2)_j}$$

is discrete Kullback–Leibler divergence.

Example 6 Choose $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, $a = q^m$, and $b = q^n$ with $m < n$, inequalities (43) and (44) take the following form in q-calculus, respectively:

$$\mathfrak{J}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2} \left[\frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1} G(\vartheta_1, s) + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1} G(\vartheta_2, s) - \sum_{j=m}^{n-1} q^{j+1} \mathbf{q}_2(q^j) G\left(\frac{\mathbf{q}_1(q^j)}{\mathbf{q}_2(q^j)}, s \right) \right] \times \left(\sum_{u=2}^{n-1} \frac{(-1)^{u-1}(s - \vartheta_1)_+^{u-2}}{\vartheta_1^{u-1}} \right) ds \tag{51}$$

and

$$\mathfrak{J}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2} \left[\frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1} G(\vartheta_1, s) + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1} G(\vartheta_2, s) - \sum_{j=m}^{n-1} q^{j+1} \mathbf{q}_2(q^j) G\left(\frac{\mathbf{q}_1(q^j)}{\mathbf{q}_2(q^j)}, s \right) \right] \times \left(\sum_{u=2}^{n-1} \frac{(-1)^{u-1}(s - \vartheta_2)_+^{u-2}}{\vartheta_2^{u-1}} \right) ds, \tag{52}$$

where

$$\mathfrak{J}_1(\cdot) = \frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1}(-\vartheta_1 \ln \vartheta_1) + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1}(-\vartheta_2 \ln \vartheta_2) + \mathrm{KL}_q(\mathbf{q}_1, \mathbf{q}_2)$$

and

$$\mathrm{KL}_q(\mathbf{q}_1,\mathbf{q}_2) := \sum_{j=m}^{n-1} q^{j+1}\mathbf{q}_1\big(q^j\big)\ln\frac{\mathbf{q}_1(q^j)}{\mathbf{q}_2(q^j)}.$$

5.3 Jeffreys distance

Ansari *et al.* [11] defined the Jeffreys distance on time scale by

$$D_J(\mathbf{q}_1,\mathbf{q}_2) := \int_a^b \big(\mathbf{q}_1(\xi)-\mathbf{q}_2(\xi)\big)\ln\left[\frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)}\right]\Delta\xi. \tag{53}$$

Theorem 7 *Let X be a continuous random variable and $\mathbf{q}_1,\mathbf{q}_2\in\Omega$ with $\vartheta_1\le\frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)}\le\vartheta_2$ for all $\xi\in\mathbb{T}$. If n is odd ($n=3,5,\ldots$), then*

$$\mathfrak{J}_1(\cdot)\ge\int_{\vartheta_1}^{\vartheta_2}\mathfrak{J}_1\big(G(\cdot,s)\big)\left(\sum_{u=2}^{n-1}(-1)^{u+1}(s-\vartheta_1)_+^{u-2}\left[\frac{1}{\vartheta_1^{u-1}}+\frac{u-1}{\vartheta_1^{u}}\right]\right)ds \tag{54}$$

and

$$\mathfrak{J}_1(\cdot)\ge\int_{\vartheta_1}^{\vartheta_2}\mathfrak{J}_1\big(G(\cdot,s)\big)\left(\sum_{u=2}^{n-1}(-1)^{u+1}(s-\vartheta_2)_+^{u-2}\left[\frac{1}{\vartheta_2^{u-1}}+\frac{u-1}{\vartheta_2^{u}}\right]\right)ds, \tag{55}$$

where

$$\mathfrak{J}_1(\cdot)=\frac{\vartheta_2-1}{\vartheta_2-\vartheta_1}(1-\vartheta_1)\ln\vartheta_1+\frac{1-\vartheta_1}{\vartheta_2-\vartheta_1}(1-\vartheta_2)\ln\vartheta_2+D_J(\mathbf{q}_1,\mathbf{q}_2),$$

$D_J(\mathbf{q}_1,\mathbf{q}_2)$ is defined in (53), and

$$\mathfrak{J}_1\big(G(\cdot,s)\big)=\frac{\vartheta_2-1}{\vartheta_2-\vartheta_1}G(\vartheta_1,s)+\frac{1-\vartheta_1}{\vartheta_2-\vartheta_1}G(\vartheta_2,s)-\int_a^b\mathbf{q}_2(\xi)G\left(\frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)},s\right)\Delta\xi.$$

Proof The function $\Phi(x):=(1-x)\ln x$ is n-convex for $n=3,5,\ldots$. Use $\Phi(x)=(1-x)\ln x$ and observe the analogous method like in the proof of Theorem 5 to obtain (54) and (55). □

Example 7 Choose $\mathbb{T}=\mathbb{R}$, in Theorem 7, inequalities (54) and (55) take the following form, respectively:

$$\begin{aligned}\mathfrak{J}_1(\cdot)\ge\int_{\vartheta_1}^{\vartheta_2}&\left[\frac{\vartheta_2-1}{\vartheta_2-\vartheta_1}G(\vartheta_1,s)+\frac{1-\vartheta_1}{\vartheta_2-\vartheta_1}G(\vartheta_2,s)-\int_a^b\mathbf{q}_2(\xi)G\left(\frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)},s\right)d\xi\right]\\&\times\left(\sum_{u=2}^{n-1}(-1)^{u+1}(s-\vartheta_1)_+^{u-2}\left[\frac{1}{\vartheta_1^{u-1}}+\frac{u-1}{\vartheta_1^{u}}\right]\right)ds\end{aligned} \tag{56}$$

and

$$\begin{aligned}\mathfrak{J}_1(\cdot)\ge\int_{\vartheta_1}^{\vartheta_2}&\left[\frac{\vartheta_2-1}{\vartheta_2-\vartheta_1}G(\vartheta_1,s)+\frac{1-\vartheta_1}{\vartheta_2-\vartheta_1}G(\vartheta_2,s)-\int_a^b\mathbf{q}_2(\xi)G\left(\frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)},s\right)d\xi\right]\\&\times\left(\sum_{u=2}^{n-1}(-1)^{u+1}(s-\vartheta_2)_+^{u-2}\left[\frac{1}{\vartheta_2^{u-1}}+\frac{u-1}{\vartheta_2^{u}}\right]\right)ds,\end{aligned} \tag{57}$$

where

$$\mathfrak{J}_1(\cdot) = \frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1}(1-\vartheta_1)\ln\vartheta_1 + \frac{1-\vartheta_1}{\vartheta_2 - \vartheta_1}(1-\vartheta_2)\ln\vartheta_2 + D_{Ja}(\mathbf{q}_1,\mathbf{q}_2)$$

and

$$D_{Ja}(\mathbf{q}_1,\mathbf{q}_2) := \int_a^b \big[\mathbf{q}_1(\xi) - \mathbf{q}_2(\xi)\big]\ln\frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)}\,d\xi.$$

Example 8 Choose $\mathbb{T} = h\mathbb{Z}$ ($h > 1$) in Theorem 7, inequalities (54) and (55) take the following form in h-discrete calculus, respectively:

$$\begin{aligned}\mathfrak{J}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2}&\left[\frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1}G(\vartheta_1,s) + \frac{1-\vartheta_1}{\vartheta_2 - \vartheta_1}G(\vartheta_2,s) - \sum_{j=\frac{a}{h}}^{\frac{b}{h}-1}\mathbf{q}_2(jh)hG\left(\frac{\mathbf{q}_1(jh)}{\mathbf{q}_2(jh)},s\right)\right]\\ &\times\left(\sum_{u=2}^{n-1}(-1)^{u+1}(s-\vartheta_1)_+^{u-2}\left[\frac{1}{\vartheta_1^{u-1}} + \frac{u-1}{\vartheta_1^{u}}\right]\right)ds \end{aligned}\tag{58}$$

and

$$\begin{aligned}\mathfrak{J}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2}&\left[\frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1}G(\vartheta_1,s) + \frac{1-\vartheta_1}{\vartheta_2 - \vartheta_1}G(\vartheta_2,s) - \sum_{j=\frac{a}{h}}^{\frac{b}{h}-1}\mathbf{q}_2(jh)hG\left(\frac{\mathbf{q}_1(jh)}{\mathbf{q}_2(jh)},s\right)\right]\\ &\times\left(\sum_{u=2}^{n-1}(-1)^{u+1}(s-\vartheta_2)_+^{u-2}\left[\frac{1}{\vartheta_2^{u-1}} + \frac{u-1}{\vartheta_2^{u}}\right]\right)ds, \end{aligned}\tag{59}$$

where

$$\mathfrak{J}_1(\cdot) = \frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1}(1-\vartheta_1)\ln(\vartheta_1) + \frac{1-\vartheta_1}{\vartheta_2 - \vartheta_1}(1-\vartheta_2)\ln(\vartheta_2) + \tilde{D}_{J_a}(\mathbf{q}_1,\mathbf{q}_2),$$

and

$$\tilde{D}_{J_a}(\mathbf{q}_1,\mathbf{q}_2) := \sum_{j=\frac{a}{h}}^{\frac{b}{h}-1}(\mathbf{q}_1 - \mathbf{q}_2)(jh)h\ln\frac{\mathbf{q}_1(jh)}{\mathbf{q}_2(jh)}.$$

Remark 7 Put $h = 1$ in Example 8. Suppose that $a = 0$, $b = n$, $\mathbf{q}_1(j) = (\mathbf{q}_1)_j$, and $\mathbf{q}_2(j) = (\mathbf{q}_2)_j$ to get the following new forms of inequalities (54) and (55), respectively:

$$\begin{aligned}\mathfrak{J}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2}&\left[\frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1}G(\vartheta_1,s) + \frac{1-\vartheta_1}{\vartheta_2 - \vartheta_1}G(\vartheta_2,s) - \sum_{j=1}^{n}(\mathbf{q}_2)_j G\left(\frac{(\mathbf{q}_1)_j}{(\mathbf{q}_2)_j},s\right)\right]\\ &\times\left(\sum_{u=2}^{n-1}(-1)^{u+1}(s-\vartheta_1)_+^{u-2}\left[\frac{1}{\vartheta_1^{u-1}} + \frac{u-1}{\vartheta_1^{u}}\right]\right)ds \end{aligned}\tag{60}$$

and

$$\mathfrak{J}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2} \left[\frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1} G(\vartheta_1, s) + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1} G(\vartheta_2, s) - \sum_{j=1}^{n} (\mathbf{q}_2)_j G\left(\frac{(\mathbf{q}_1)_j}{(\mathbf{q}_2)_j}, s \right) \right] \times \left(\sum_{u=2}^{n-1} (-1)^{u+1} (s - \vartheta_2)_+^{u-2} \left[\frac{1}{\vartheta_2^{u-1}} + \frac{u-1}{\vartheta_2^{u}} \right] \right) ds, \tag{61}$$

where

$$\mathfrak{J}_1(\cdot) = \frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1} (1 - \vartheta_1) \ln(\vartheta_1) + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1} (1 - \vartheta_2) \ln(\vartheta_2) + J_a(\mathbf{q}_1, \mathbf{q}_2)$$

and

$$J_a(\mathbf{q}_1, \mathbf{q}_2) := \sum_{j=1}^{n} (\mathbf{q}_1 - \mathbf{q}_2)_j \ln \frac{(\mathbf{q}_1)_j}{(\mathbf{q}_2)_j}$$

is discrete Jeffreys distance.

Example 9 Choose $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, $a = q^m$, and $b = q^n$ with $m < n$, inequalities (54) and (55) take the following form in q-calculus:

$$\mathfrak{J}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2} \left[\frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1} G(\vartheta_1, s) + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1} G(\vartheta_2, s) - \sum_{j=m}^{n-1} q^{j+1} \mathbf{q}_2(q^j) G\left(\frac{\mathbf{q}_1(q^j)}{\mathbf{q}_2(q^j)}, s \right) \right] \times \left(\sum_{u=2}^{n-1} (-1)^{u+1} (s - \vartheta_1)_+^{u-2} \left[\frac{1}{\vartheta_1^{u-1}} + \frac{u-1}{\vartheta_1^{u}} \right] \right) ds \tag{62}$$

and

$$\mathfrak{J}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2} \left[\frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1} G(\vartheta_1, s) + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1} G(\vartheta_2, s) - \sum_{j=m}^{n-1} q^{j+1} \mathbf{q}_2(q^j) G\left(\frac{\mathbf{q}_1(q^j)}{\mathbf{q}_2(q^j)}, s \right) \right] \times \left(\sum_{u=2}^{n-1} (-1)^{u+1} (s - \vartheta_2)_+^{u-2} \left[\frac{1}{\vartheta_2^{u-1}} + \frac{u-1}{\vartheta_2^{u}} \right] \right) ds, \tag{63}$$

where

$$\mathfrak{J}_1(\cdot) = \frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1} (1 - \vartheta_1) \ln(\vartheta_1) + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1} (1 - \vartheta_1) \ln(\vartheta_2) + D_{J_q}(\mathbf{q}_1, \mathbf{q}_2)$$

and

$$D_{J_q}(\mathbf{q}_1, \mathbf{q}_2) := \sum_{j=m}^{n-1} q^{j+1} \left[\mathbf{q}_1(q^j) - \mathbf{q}_2(q^j) \right] \ln \frac{\mathbf{q}_1(q^j)}{\mathbf{q}_2(q^j)}.$$

5.4 Triangular discrimination

Ansari *et al.* [11] defined the triangular discrimination on time scales by

$$D_{\triangle}(\mathbf{q}_1,\mathbf{q}_2) = \int_a^b \frac{[\mathbf{q}_2(\xi) - \mathbf{q}_1(\xi)]^2}{\mathbf{q}_2(\xi) + \mathbf{q}_1(\xi)} \Delta\xi. \tag{64}$$

Theorem 8 *Let X be a continuous random variable and $\mathbf{q}_1,\mathbf{q}_2 \in \Omega$ with $\vartheta_1 \leq \frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)} \leq \vartheta_2$ for all $\xi \in \mathbb{T}$. If n is odd ($n = 3,5,\ldots$), then*

$$\mathfrak{I}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2} \mathfrak{I}_1\big(G(\cdot,s)\big)\left(\sum_{u=2}^{n-1} 4(-1)^{u+1}(s-\vartheta_1)_+^{u-2} \frac{u(u-1)}{(1+\vartheta_1)^{u+1}}\right) ds \tag{65}$$

and

$$\mathfrak{I}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2} \mathfrak{I}_1\big(G(\cdot,s)\big)\left(\sum_{u=2}^{n-1} 4(-1)^{u+1}(s-\vartheta_2)_+^{u-2} \frac{u(u-1)}{(1+\vartheta_2)^{u+1}}\right) ds, \tag{66}$$

where

$$\mathfrak{I}_1(\cdot) = -\frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1}\frac{(\vartheta_1 - 1)^2}{\vartheta_1 + 1} - \frac{1-\vartheta_1}{\vartheta_2 - \vartheta_1}\frac{(\vartheta_2 - 1)^2}{\vartheta_2 + 1} + D_{\triangle}(\mathbf{q}_1,\mathbf{q}_2),$$

$D_{\triangle}(\mathbf{q}_1,\mathbf{q}_2)$ is given in (64), and

$$\mathfrak{I}_1\big(G(\cdot,s)\big) = \frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1}G(\vartheta_1,s) + \frac{1-\vartheta_1}{\vartheta_2 - \vartheta_1}G(\vartheta_2,s) - \int_a^b \mathbf{q}_2(\xi) G\left(\frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)},s\right)\Delta\xi.$$

Proof The function $\Phi(x) := -\frac{(x-1)^2}{x+1}$ is n-convex for $n = 3,5,\ldots$. Use $\Phi(x) = -\frac{(x-1)^2}{x+1}$ and observe the same method like in the proof of Theorem 5 to obtain (65) and (66). □

Example 10 Choose $\mathbb{T} = \mathbb{R}$ in Theorem 8, inequalities (65) and (66) take the following form, respectively:

$$\begin{aligned} \mathfrak{I}_1(\cdot) \geq{}& \int_{\vartheta_1}^{\vartheta_2} \left[\frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1}G(\vartheta_1,s) + \frac{1-\vartheta_1}{\vartheta_2 - \vartheta_1}G(\vartheta_2,s) - \int_a^b \mathbf{q}_2(\xi) G\left(\frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)},s\right) d\xi\right] \\ &\times \left(\sum_{u=2}^{n-1} 4(-1)^{u+1}(s-\vartheta_1)_+^{u-2} \frac{u(u-1)}{(1+\vartheta_2)^{u+1}}\right) ds \end{aligned} \tag{67}$$

and

$$\begin{aligned} \mathfrak{I}_1(\cdot) \geq{}& \int_{\vartheta_1}^{\vartheta_2} \left[\frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1}G(\vartheta_1,s) + \frac{1-\vartheta_1}{\vartheta_2 - \vartheta_1}G(\vartheta_2,s) - \int_a^b \mathbf{q}_2(\xi) G\left(\frac{\mathbf{q}_1(\xi)}{\mathbf{q}_2(\xi)},s\right) d\xi\right] \\ &\times \left(\sum_{u=2}^{n-1} 4(-1)^{u+1}(s-\vartheta_2)_+^{u-2} \frac{u(u-1)}{(1+\vartheta_2)^{u+1}}\right) ds, \end{aligned} \tag{68}$$

where

$$\mathfrak{I}_1(\cdot) = -\frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1}\frac{(\vartheta_1 - 1)^2}{\vartheta_1 + 1} - \frac{1-\vartheta_1}{\vartheta_2 - \vartheta_1}\frac{(\vartheta_2 - 1)^2}{\vartheta_2 + 1} + D_{\triangle_a}(\mathbf{q}_1,\mathbf{q}_2)$$

and

$$D_{\Delta_a}(\mathbf{q}_1,\mathbf{q}_2) := \int_a^b \frac{[\mathbf{q}_2(\xi)-\mathbf{q}_1(\xi)]^2}{\mathbf{q}_1(\xi)+\mathbf{q}_2(\xi)}\,d\xi$$

is triangular discrimination.

Example 11 Choose $\mathbb{T} = h\mathbb{Z}$ $(h > 1)$ in Theorem 8, inequalities (65) and (66) take the following form in h-discrete calculus, respectively:

$$\begin{aligned}\mathfrak{J}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2} &\Bigg[\frac{\vartheta_2-1}{\vartheta_2-\vartheta_1}G(\vartheta_1,s)+\frac{1-\vartheta_1}{\vartheta_2-\vartheta_1}G(\vartheta_2,s)-\sum_{j=\frac{a}{h}}^{\frac{b}{h}-1}\mathbf{q}_2(jh)hG\bigg(\frac{\mathbf{q}_1(jh)}{\mathbf{q}_2(jh)},s\bigg)\Bigg]\\ &\times\left(\sum_{u=2}^{n-1}4(-1)^{u+1}(s-\vartheta_1)_+^{u-2}\frac{u(u-1)}{(1+\vartheta_1)^{u+1}}\right)ds \end{aligned}\tag{69}$$

and

$$\begin{aligned}\mathfrak{J}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2} &\Bigg[\frac{\vartheta_2-1}{\vartheta_2-\vartheta_1}G(\vartheta_1,s)+\frac{1-\vartheta_1}{\vartheta_2-\vartheta_1}G(\vartheta_2,s)-\sum_{j=\frac{a}{h}}^{\frac{b}{h}-1}\mathbf{q}_2(jh)hG\bigg(\frac{\mathbf{q}_1(jh)}{\mathbf{q}_2(jh)},s\bigg)\Bigg]\\ &\times\left(\sum_{u=2}^{n-1}4(-1)^{u+1}(s-\vartheta_2)_+^{u-2}\frac{u(u-1)}{(1+\vartheta_2)^{u+1}}\right)ds, \end{aligned}\tag{70}$$

where

$$\mathfrak{J}_1(\cdot) = -\frac{\vartheta_2-1}{\vartheta_2-\vartheta_1}\frac{(\vartheta_1-1)^2}{\vartheta_1+1}-\frac{1-\vartheta_1}{\vartheta_2-\vartheta_1}\frac{(\vartheta_2-1)^2}{\vartheta_2+1}+\tilde{D}_{\Delta_a}(\mathbf{q}_1,\mathbf{q}_2)$$

and

$$\tilde{D}_{\Delta_a}(\mathbf{q}_1,\mathbf{q}_2) := \sum_{j=\frac{a}{h}}^{\frac{b}{h}-1}\frac{h[\mathbf{q}_2(hj)-\mathbf{q}_1(hj)]^2}{\mathbf{q}_1(hj)+\mathbf{q}_2(hj)}.$$

Remark 8 Take $h = 1$ in Example 11 and consider $a = 0$, $b = n$, $\mathbf{q}_1(j) = (\mathbf{q}_1)_j$, and $\mathbf{q}_2(j) = (\mathbf{q}_2)_j$ to get the following new forms of inequalities (65) and (66), respectively:

$$\begin{aligned}\mathfrak{J}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2} &\Bigg[\frac{\vartheta_2-1}{\vartheta_2-\vartheta_1}G(\vartheta_1,s)+\frac{1-\vartheta_1}{\vartheta_2-\vartheta_1}G(\vartheta_2,s)-\sum_{j=1}^{n}(\mathbf{q}_2)_jG\bigg(\frac{(\mathbf{q}_1)_j}{(\mathbf{q}_2)_j},s\bigg)\Bigg]\\ &\times\left(\sum_{u=2}^{n-1}4(-1)^{u+1}(s-\vartheta_1)_+^{u-2}\frac{u(u-1)}{(1+\vartheta_1)^{u+1}}\right)ds \end{aligned}\tag{71}$$

and

$$\mathfrak{J}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2} \left[\frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1} G(\vartheta_1, s) + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1} G(\vartheta_2, s) - \sum_{j=1}^{n} (\mathbf{q}_2)_j G\left(\frac{(\mathbf{q}_1)_j}{(\mathbf{q}_2)_j}, s \right) \right] \times \left(\sum_{u=2}^{n-1} 4(-1)^{u+1} (s - \vartheta_2)_+^{u-2} \frac{u(u-1)}{(1+\vartheta_2)^{u+1}} \right) ds, \tag{72}$$

where

$$\mathfrak{J}_1(\cdot) = -\frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1} \frac{(\vartheta_1 - 1)^2}{\vartheta_1 + 1} - \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1} \frac{(\vartheta_2 - 1)^2}{\vartheta_2 + 1} + \Delta(\mathbf{q}_1, \mathbf{q}_2)$$

and

$$\Delta(\mathbf{q}_1, \mathbf{q}_2) := \sum_{j=1}^{n} \frac{[(\mathbf{q}_2)_j - (\mathbf{q}_1)_j]^2}{(\mathbf{q}_2)_j + (\mathbf{q}_1)_j}$$

is discrete triangular discrimination.

Example 12 Choose $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, $a = q^m$, and $b = q^n$ with $m < n$, inequalities (65) and (66) take the following form in q-calculus, respectively:

$$\mathfrak{J}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2} \left[\frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1} G(\vartheta_1, s) + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1} G(\vartheta_2, s) - \sum_{j=m}^{n-1} q^{j+1} \mathbf{q}_2(q^j) G\left(\frac{\mathbf{q}_1(q^j)}{\mathbf{q}_2(q^j)}, s \right) \right] \times \left(\sum_{u=2}^{n-1} 4(-1)^{u+1} (s - \vartheta_1)_+^{u-2} \frac{u(u-1)}{(1+\vartheta_1)^{u+1}} \right) ds \tag{73}$$

and

$$\mathfrak{J}_1(\cdot) \geq \int_{\vartheta_1}^{\vartheta_2} \left[\frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1} G(\vartheta_1, s) + \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1} G(\vartheta_2, s) - \sum_{j=m}^{n-1} q^{j+1} \mathbf{q}_2(q^j) G\left(\frac{\mathbf{q}_1(q^j)}{\mathbf{q}_2(q^j)}, s \right) \right] \times \left(\sum_{u=2}^{n-1} 4(-1)^{u+1} (s - \vartheta_2)_+^{u-2} \frac{u(u-1)}{(1+\vartheta_2)^{u+1}} \right) ds, \tag{74}$$

where

$$\mathfrak{J}_1(\cdot) = -\frac{\vartheta_2 - 1}{\vartheta_2 - \vartheta_1} \frac{(\vartheta_1 - 1)^2}{\vartheta_1 + 1} - \frac{1 - \vartheta_1}{\vartheta_2 - \vartheta_1} \frac{(\vartheta_2 - 1)^2}{\vartheta_2 + 1} + D_{\Delta_q}(\mathbf{q}_1, \mathbf{q}_2)$$

and

$$D_{\Delta_q}(\mathbf{q}_1, \mathbf{q}_2) := \sum_{j=m}^{n-1} q^{j+1} \frac{[\mathbf{q}_2(q^j) - \mathbf{q}_1(q^j)]^2}{\mathbf{q}_1(q^j) + \mathbf{q}_2(q^j)}.$$

Acknowledgements
The authors wish to thank the anonymous referees for their very careful reading of the manuscript and fruitful comments and suggestions. The research of the 5th author (Josip Pečarić) is supported by the Ministry of Education and Science of the Russian Federation (Agreement number 02.a03.21.0008).

Funding
There is no funding for this work.

Availability of data and materials
Data sharing is not applicable to this paper as no data sets were generated or analyzed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors jointly worked on the results and they read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 8 June 2021 Accepted: 28 July 2021 Published online: 10 August 2021

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