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# On a new linear operator formulated by Airy functions in the open unit disk

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## Abstract

In this note, we formulate a new linear operator given by Airy functions of the first type in a complex domain. We aim to study the operator in view of geometric function theory based on the subordination and superordination concepts. The new operator is suggested to define a class of normalized functions (the class of univalent functions) calling the Airy difference formula. As a result, the suggested difference formula joining the linear operator is modified to different classes of analytic functions in the open unit disk.

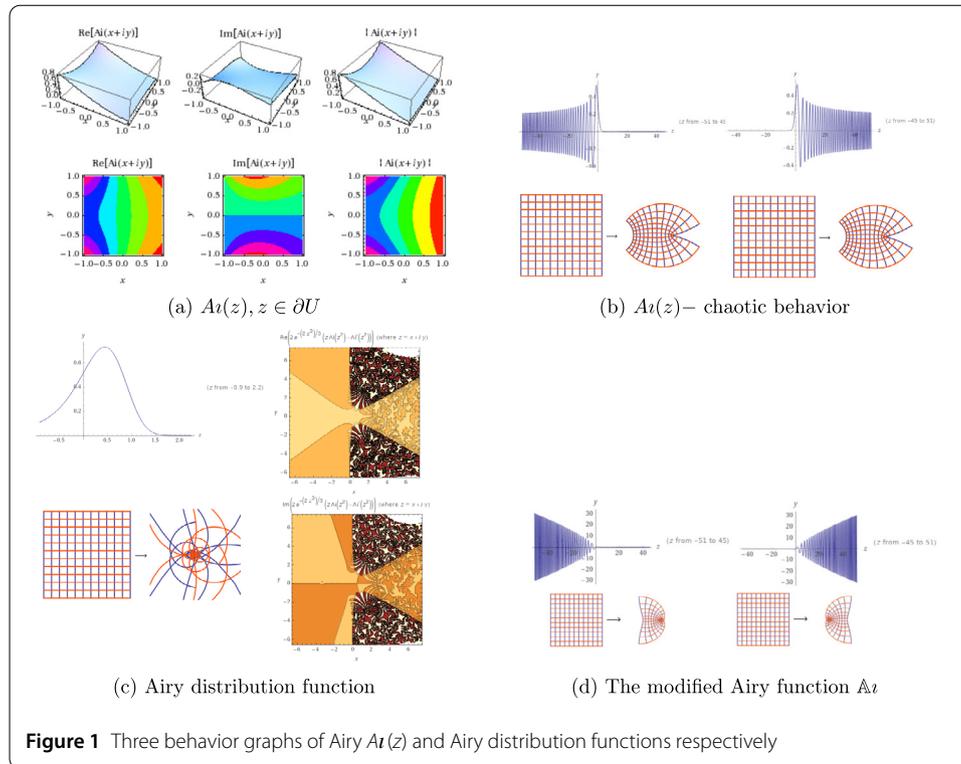
**Keywords:** Analytic function; Univalent function; Subordination; Open unit disk

## 1 Introduction

The field of geometric function theory is rich with different types of linear, differential, integral, and mixed operators. A few linear operators have been formulated in this field, such as the Carlson–Shaffer operator [1], hypergeometric linear operator [2, 3], and Fox–Write linear operator [4]. In this note, we present a linear operator formulated by the Airy functions [5], which are special functions determined by the hypergeometric function of a complex variable. These functions are solutions for the Airy equation  $f''(z) - zf(z) = 0$ . The class of these differential equations plays an important role in applied sciences such as optics, economy, and astronomy. The greatest benefit of Airy functions in mathematical studies is development in the fields of special functions and statistical studies [6]. The formula of the Airy function of a complex variable is given by

$$\begin{aligned} A_1(z) &= \frac{1}{2\pi i} \int_U \exp\left(\frac{\zeta^3}{3} - z\zeta\right) d\zeta \\ &= \frac{1}{\pi 3^{2/3}} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n+1}{3}\right)}{n!} \sin\left(\frac{2\pi(n+1)}{3}\right) (3^{1/3}z)^n \\ &= \frac{1}{\pi 3^{2/3}} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n+1}{3}\right)}{n!} \left(\frac{\pi}{\Gamma\left(1 - \frac{2(n+1)}{3}\right)\Gamma\left(\frac{2(n+1)}{3}\right)}\right) (3^{1/3}z)^n \end{aligned} \quad (1)$$

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**Figure 1** Three behavior graphs of Airy  $Ai(z)$  and Airy distribution functions respectively

$$\begin{aligned}
 &\approx \sum_{n=0}^{\infty} \left( \frac{\Gamma(3/2)}{\Gamma(1/2)} \right) \left( \frac{\Gamma(\frac{n+1}{3})}{\Gamma(\frac{1-2n}{3})\Gamma(\frac{2(n+1)}{3})} \right) \left( \frac{1}{\Gamma(1 + \frac{n}{3})} \right) \frac{z^n}{n!} \\
 &= {}_2\Psi_4 \left[ \begin{matrix} (3/2, 0) & (1/3, 1/3) \\ (1/2, 0) & (1/3, -2/3) & (2/3, 2/3) & (1, 1/3) \end{matrix} ; z \right],
 \end{aligned}$$

where the integral is over the open unit disk  $U := \{z \in \mathbb{C} : |z| < 1\}$ ,  $3^{n/3} \approx \frac{1}{\Gamma(1+\frac{n}{3})}$ ,  $\frac{1}{3^{2/3}} \approx \frac{\Gamma(3/2)}{\Gamma(1/2)}$ , and  ${}_p\Psi_q$  is the Fox–Wright function having the series

$${}_p\Psi_q \left[ \begin{matrix} (a_1, \aleph_1) & (a_2, \aleph_2) & \cdots & (a_p, \aleph_p) \\ (b_1, B_1) & (b_2, B_2) & \cdots & (b_q, B_q) \end{matrix} ; z \right] = \sum_{n=0}^{\infty} \frac{\Gamma(a_1 + \aleph_1 n) \cdots \Gamma(a_p + \aleph_p n)}{\Gamma(b_1 + B_1 n) \cdots \Gamma(b_q + B_q n)} \frac{z^n}{n!}.$$

Moreover, the Airy distribution function of the random variable  $\chi$  is given by the formula (see Fig. 1)

$$P(\chi) = 2e^{-2/3\chi^3} [\chi Ai(\chi^2) - (Ai)'(\chi^2)]. \tag{2}$$

By using the complex probability [7, 8], Eq. (2) can be extended to the complex domain as follows:

$$P(z) = 2e^{-2/3z^3} [\chi Ai(z^2) - (Ai)'(z^2)]. \tag{3}$$

## 2 Methods

Let  $\Lambda$  be the class of normalized functions in  $U$  having the series

$$f(z) = z + \sum_{n=2}^{\infty} \varphi_n z^n, \quad z \in U. \tag{4}$$

And let  $\mathbb{S}^*, \mathbb{C}$  be the classes of starlike and convex functions respectively. The Hadamard product (convolution product) is defined by the series

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} \varphi_n \psi_n z^n, \quad z \in U,$$

where  $g(z) = z + \sum_{n=2}^{\infty} \psi_n z^n$ . An analytic function  $f \in U$  is on subordination with the analytic function  $g \in U$  represented by  $f \prec g$  if there occurs an analytic function  $w$  with  $|w(z)| \leq |z|$  such that  $f = (g(w))$ . In the sequel, we shall use the class of normalized functions  $\Lambda$  satisfying  $f(0) = 0$  and  $f'(0) = 1$  having the series (see [9])

$$f(z) = z + \sum_{n=2}^{\infty} \phi_n z^n, \quad z \in U.$$

Moreover, two analytic functions  $f$  and  $g$  in  $U$ , the function  $f$  is majored by  $g$  ( $f \ll g$ ) if there is an analytic function  $\varpi, |\varpi| < 1$  such that  $f(z) = \varpi(z)g(z)$ . Note that there is a connection between majorization and subordination concepts (see [10, 11]). Under some conditions, we have  $f \ll g \Leftrightarrow f \prec g$ .

### 2.1 Linear operator

We shall use the Hadamard product to define the new linear operator using the Airy function of a complex variable  $z \in U$ . Construct the modified Airy function as follows:

$$\begin{aligned} \mathbb{A}_I(z) &:= (3^{2/3} \Gamma(2/3) z) A_I(z) \\ &= z - \left( \frac{3^{1/3} \Gamma(2/3)}{\Gamma(1/3)} \right) z^2 + \dots + O(z^6). \end{aligned} \tag{5}$$

Define a linear operator  $\Delta : \Lambda \rightarrow \Lambda$  as follows:

$$\begin{aligned} \Delta f(z) &= \mathbb{A}_I(z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \delta_n \varphi_n z^n \in \Lambda, \end{aligned} \tag{6}$$

where  $\delta_n$  indicates the coefficient of  $\mathbb{A}_I(z)$ . The linear operator (6) is called the Airy linear operator of normalized analytic functions. It is well known that for  $\Re(z) > 0$  the Airy function is convex with  $\Re(A_I(z)) > 0$ . We have the following proposition, which indicates that the linear operator can be formulated by a set of special functions and other properties, which are easily proved. Therefore, we omit the proof.

**Proposition 1** *Consider the linear operator  $\Delta f(z), f \in \Lambda$ . Then it can be formulated by the following special functions:*

•

$$\Delta f(z) = \left( \frac{z A_I(z) G(5/3) 3^{2/3}}{G(2/3)} \right) * f(z);$$

•

$$\Delta f(z) = \left( \frac{z G(5/3) 3^{2/3} (I_{-1/3}(\frac{2z^{3/2}}{3}) \sqrt{z} - \frac{z I_{1/3}(\frac{2z^{3/2}}{3})}{\sqrt{z}})}{3G(2/3)} \right) * f(z);$$

•

$$\Delta f(z) = \left( \frac{z G(5/3) 3^{2/3} (J_{-1/3}(\frac{2}{3} i z^{3/2}) \sqrt{z e^{i\pi/3}} - \frac{z J_{1/3}(\frac{2}{3} i z^{3/2})}{\sqrt{z e^{i\pi/3}}})}{(3G(2/3))} \right) * f(z);$$

•

$$\Delta f(z) = \frac{3^{\frac{1}{4}} \pi^{\frac{2}{3}} \left( \frac{(2+\sqrt{3})(1 + \frac{2(\sqrt{2}+\sqrt{3})(5+3\sqrt{3})}{(3+\sqrt{2}+\sqrt{3})^2})}{(1 - \frac{2(\sqrt{2}+\sqrt{3})(5+3\sqrt{3})}{(3+\sqrt{2}+\sqrt{3})^2})^2} \right)^{1/3} z A_I(z)}{(2 + \frac{1}{\sqrt{2+\sqrt{3}}}) K(\frac{2(\sqrt{2}+\sqrt{3})(5+3\sqrt{3})}{(3+\sqrt{2}+\sqrt{3})^2})} * f(z);$$

•

$$(\mathbb{A}_I(z))' = 3^{2/3} \Gamma\left(\frac{2}{3}\right) (A_I(z) + z A_I'(z));$$

•

$$\Re(\mathbb{A}_I(z)) \approx 0.385116, \quad \Re(z) = 0.88405;$$

•

$$\int_U \mathbb{A}_I(z) dz = 3^{2/3} \Gamma(2/3) A_I'(z), \quad \left| \int_U \mathbb{A}_I(z) dz \right| \approx 0.419648,$$

where  $G(\omega)$  is the Barnes  $G$ -function,  $I_n(\omega)$  is the modified Bessel function of the first kind,  $J_n(\omega)$  is the Bessel function of the first kind, and  $K$  is the complete elliptic integral of the first kind.

### 2.2 The difference formula

We proceed to defining our class of normalized analytic functions based on the Airy equation. The Airy equation can be reformulated by the structure

$$\frac{z f''(z)}{f'(z)} - \frac{z^3}{f(z)} = 0, \quad f \in \Lambda.$$

Our structure of the class of analytic functions is given by the Airy difference formula

$$\Psi(z) := \frac{z f''(z)}{f'(z)} - \frac{z^3}{f(z)}.$$

By utilizing the linear operator  $\Delta f$ , we have the following class.

**Definition 1** Let  $f \in \Lambda$ . Define the class of analytic functions  $\Lambda_{t,s}$  satisfying the following subordination:

$$\Psi_s(z) := \frac{1}{2s} \left( \frac{zf''(z)}{f'(z)} - \frac{z^3}{f(z)} \right) \prec \Delta f(z)$$

$$(z \in U, f \in \Lambda, s \in \mathbb{R} \setminus \{0\}).$$

*Example 1* Let  $f(z) = \frac{z}{(1-z)^s}$ , we have the formula

$$\begin{aligned} \frac{1}{2s} \left( \frac{zf''(z)}{f'(z)} - \frac{z^3}{f(z)} \right) &= \frac{\frac{sz((s-1)z+2)}{(1-z)((s-1)z+1)} - \frac{(1-z)z^3}{(s-1)z+1}}{2s} \\ &= z + \frac{1}{2}(3-s)z^2 + \frac{(s^3 - 3s^2 + 4s - 1)z^3}{(2s)} + (-s^3/2 + 2s^2 - 3s + 3)z^4 \\ &\quad + \frac{1}{2}(s^4 - 5s^3 + 10s^2 - 11s + 7)z^5 + O(z^6). \end{aligned}$$

Moreover, we have

$$\begin{aligned} \Delta f(z) &= \left( z - \frac{3^{1/3}z^2\Gamma(2/3)}{\Gamma(1/3)} + \frac{z^4}{6} - \frac{z^5\Gamma(2/3)}{4(3^{2/3}\Gamma(1/3))} + O(z^6) \right) \\ &\quad * \left( z + sz^2 + \frac{1}{2!}s(s+1)z^3 + \frac{1}{3!}s(s+1)(s+2)z^4 \right. \\ &\quad \left. + \frac{1}{4!}s(s+1)(s+2)(s+3)z^5 + O(z^6) \right) \\ &= z - \frac{3^{1/3}sz^2\Gamma(2/3)}{\Gamma(1/3)} + \frac{1}{3!}s(s+1)(s+2)\frac{z^4}{6} \\ &\quad - \left( \frac{1}{4!}s(s+1)(s+2)(s+3) \right) \frac{z^5\Gamma(2/3)}{4(3^{2/3}\Gamma(1/3))} + O(z^6). \end{aligned}$$

It is clear that the formula

$$\Psi_s(z)|_{z=0} = \frac{1}{2s} \left( \frac{zf''(z)}{f'(z)} - \frac{z^3}{f(z)} \right) \Big|_{z=0} = \Delta f(z)|_{z=0} = 0.$$

And by comparing the coefficients of  $\Delta f(z)$  and  $\Psi_s(z)$ , we have that the unique real root of  $s^3 - 3s^2 + 4s - 1 = 0$  is

$$s = 1 - \left( \frac{2}{3(-9 + \sqrt{93})} \right)^{1/3} + \frac{(\frac{1}{2}(-9 + \sqrt{93}))^{1/3}}{3^{2/3}} \approx 0.31767.$$

As a conclusion, we have

$$\Psi_s(z) \ll \Delta \left( \frac{z}{(1-z)^s} \right), \quad s \leq 0.31767, z \in U.$$

Note that the function  $\frac{z}{(1-z)^s}$  is called the generalized Koebe function, which is an extreme function in  $U$  for some values of  $s$ .

Our investigation is based on the following result which can be located in [9].

**Lemma 2** *Suppose that  $\rho_1(z)$  is analytic in  $U$  and  $\rho_2(z)$  is convex univalent in  $U$  with  $\rho_1(0) = \rho_2(0)$ . If*

$$\rho_1(z) + \frac{1}{\kappa}(\rho_1'(z)) \prec \rho_2(z)$$

*for a nonzero complex constant number  $\kappa$  with  $\Re(\kappa) \geq 0$ , then  $\rho_1(z) \prec \rho_2(z)$ .*

### 3 Results

In the result section, we present the sufficient condition for functions to be in the class  $\Lambda_{I_s}$ .

**Theorem 3** *Let  $f \in \Lambda$ , and for some constants  $s \in \mathbb{R} \setminus \{0\}$  define the functional*

$$\Psi_s(z) = \frac{1}{2s} \left( \frac{zf''(z)}{f'(z)} - \frac{z^3}{f(z)} \right).$$

*Then  $\Psi_s(z) \prec \Delta f(z)$  if one of the subordinations occurs:*

- $1 + s(z\Psi_s'(z)) \prec (1 + z)^{1/2}, s \geq \max\{|s_0|, |s_1|\}$ , where

$$s_0 = \frac{(2(-1 + \sqrt{2} + \log(2) - \log(1 + \sqrt{2})))}{(3^{2/3}\Gamma(2/3)A_I(1) - 1)}, \quad s_1 = \frac{(2(\log(2) - 1))}{(3^{2/3}\Gamma(2/3)A_I(-1) + 1)};$$

- $1 + s(z\frac{\Psi_s'(z)}{\Psi_s(z)}) \prec (1 + z)^{1/2}, s \geq \max\{|s_2|, |s_3|\}$ , where for some  $m \in \mathbb{N}$

$$s_2 = \frac{6(-1 + \sqrt{2} + \log(2) - \log(1 + \sqrt{2}))}{2\log(3) + 3\log(\Gamma(2/3)) + 3\log(A_I(1))},$$

$$s_3 = \frac{6(1 - \log(2))}{|(6m + 3)\pi - i(\log(9) + 3\log(A_I(-1)\Gamma(2/3)))|};$$

- $1 + s(z\frac{\Psi_s'(z)}{\Psi_s^2(z)}) \prec (1 + z)^{1/2}, s \geq \max\{|s_4|, |s_5|\}$ , where

$$s_4 = \frac{113 \times 3^{2/3}\Gamma(2/3)A_I(1)}{250(3^{2/3}\Gamma(2/3)A_I(1) - 1)}, \quad s_5 = \frac{3 \times 3^{2/3}\Gamma(2/3)A_I(-1)}{5(-1 - 3^{2/3}\Gamma(2/3)A_I(-1))}.$$

**Proof 1** Case I:  $1 + s(z\Psi_s'(z)) \prec (1 + z)^{1/2}$ .

Define a function  $F_s : U \rightarrow \mathbb{C}$  formulating by

$$F_s(z) = 1 + \frac{2}{s}((1 + z)^{1/2} - \log(1 + (1 + z)^{1/2}) + \log(2) - 1).$$

Obviously, the analytic function  $F_s(z)$  achieves  $F_s(0) = 1$  and satisfies

$$1 + s(zF_s'(z)) = (1 + z)^{1/2}. \tag{7}$$

Thus, we obtain  $\mathfrak{F}(z) := s(zF_s'(z)) = (1 + z)^{1/2} - 1$  is starlike in  $U$ . Consequently, by Lemma 2, it yields

$$1 + s(z\Psi_s'(z)) \prec 1 + s(zF_s'(z)) \Rightarrow \Psi_s(z) \prec F_s(z).$$

To complete this argument, we must prove that  $F_s(z) \prec \Delta f(z)$ , or equivalently,  $F_s(z) \prec \mathbb{A}_l(z)$ . Evidently, the function  $F_s(z)$  is increasing in the interval  $(-1, 1)$ , which fulfils the inequality  $F_s(-1) \leq F_s(1)$ . Since

$$\begin{aligned} -\frac{J_{-1/3}(\frac{2}{3}) + J_{1/3}(\frac{2}{3})}{3} &\approx -\left(3^{2/3}\Gamma\left(\frac{2}{3}\right)\right)A_l(-1) \leq F_s(-1) \\ &\leq F_s(1) \leq 3^{2/3}\Gamma\left(\frac{2}{3}\right)A_l(1) \approx \frac{I_{-1/3}(\frac{2}{3}) - I_{1/3}(\frac{2}{3})}{3}, \end{aligned}$$

where  $s \geq \max\{|s_0|, |s_1|\}$  such that

$$s_0 = \frac{(2(-1 + \sqrt{2} + \log(2) - \log(1 + \sqrt{2})))}{(3^{2/3}\Gamma(2/3)A_l(1) - 1)}, \quad s_1 = \frac{(2(\log(2) - 1))}{(3^{2/3}\Gamma(2/3)A_l(-1) + 1)}$$

then we obtain

$$\Psi_s(z) \prec F_s(z) \prec \Delta f(z) \quad \Rightarrow \quad \Psi_s(z) \prec \Delta f(z).$$

This indicates that  $f \in \Lambda_{l_s}$ .

Case II:  $1 + s\left(\frac{z\Psi'_s(z)}{\Psi_s(z)}\right) \prec (1 + z)^{1/2}$ . Construct the function  $\Omega_s : U \rightarrow \mathbb{C}$  as follows:

$$\Omega_s(z) = \exp\left(\frac{2}{s}\left((1 + z)^{1/2} - \log(1 + (1 + z)^{1/2}) + \log(2) - 1\right)\right).$$

The function  $\Omega_s(z)$  is analytic in  $U$  having  $\Omega_s(0) = 1$ , and it is a solution of the differential equation

$$1 + s\left(\frac{z\Omega'_s(z)}{\Omega_s(z)}\right) = (1 + z)^{1/2}, \quad z \in U. \tag{8}$$

By considering  $\mathfrak{F}(z) = (1 + z)^{1/2} - 1$ , which is starlike in  $U$  and  $\mathfrak{G}(z) = \mathfrak{F}(z) + 1$ , we have

$$\Re\left(\frac{z\mathfrak{F}'(z)}{\mathfrak{F}(z)}\right) = \Re\left(\frac{z\mathfrak{G}'(z)}{\mathfrak{F}(z)}\right) > 0.$$

Again by Lemma 2, we have

$$1 + s\left(\frac{z\Psi'_s(z)}{\Psi_s(z)}\right) \prec 1 + s\left(\frac{z\Omega'_s(z)}{\Omega_s(z)}\right) \quad \Rightarrow \quad \Psi_s(z) \prec \Omega_s(z).$$

Consequently, we get

$$\begin{aligned} -\frac{J_{-1/3}(\frac{2}{3}) + J_{1/3}(\frac{2}{3})}{3} &\approx -\left(3^{2/3}\Gamma\left(\frac{2}{3}\right)\right)A_l(-1) \leq \Omega_s(-1) \\ &\leq \Omega_s(1) \leq (3^{2/3}\Gamma\left(\frac{2}{3}\right)A_l(-1)) \approx \frac{I_{-1/3}(\frac{2}{3}) - I_{1/3}(\frac{2}{3})}{3}, \end{aligned}$$

whenever  $s \geq \max\{|s_2|, |s_3|\}$ , where

$$s_2 = \frac{6(-1 + \sqrt{2} + \log(2) - \log(1 + \sqrt{2}))}{2 \log(3) + 3 \log(\Gamma(2/3)) + 3 \log(A_I(1))},$$

$$s_3 = \frac{6(1 - \log(2))}{|(6m + 3)\pi - i(\log(9) + 3 \log(A_I(-1)\Gamma(2/3)))|}.$$

This indicates that the subordination inequalities

$$\Psi_s(z) \prec \Omega_s(z) \prec \Delta f(z) \implies \Psi_s(z) \prec \Delta f(z).$$

Hence,  $f \in \Lambda_{I_s}$ .

Case III:  $1 + s\left(\frac{z\Psi'_s(z)}{\Psi_s^2(z)}\right) \prec (1 + z)^{1/2}$ . Consider the function  $\tilde{\delta}_s : U \rightarrow \mathbb{C}$  by

$$\tilde{\delta}_s(z) = \frac{1}{\left(1 - \frac{2}{s}((1 + z)^{1/2} - \log(1 + (1 + z)^{1/2}) \log(2) - 1)\right)}.$$

Clearly,  $\tilde{\delta}_s(z)$  is analytic in  $U$  such that  $\tilde{\delta}_s(0) = 1$ , and it satisfies

$$1 + s\left(\frac{z\tilde{\delta}'_s(z)}{\tilde{\delta}_s(z)}\right) = (1 + z)^{1/2}. \tag{9}$$

By Lemma 2, one can have

$$1 + s\left(\frac{z\Psi'_s(z)}{\Psi_s^2(z)}\right) \prec 1 + s\left(\frac{z\tilde{\delta}'_s(z)}{\tilde{\delta}_s^2(z)}\right) \implies \Psi_s(z) \prec \tilde{\delta}_s(z).$$

This implies

$$-\frac{J_{-1/3}(\frac{2}{3}) + J_{1/3}(\frac{2}{3})}{3} \approx -\left(3^{2/3}\Gamma\left(\frac{2}{3}\right)\right)A_I(-1) \leq \tilde{\delta}_s(-1)$$

$$\leq \tilde{\delta}_s(1) \leq 3^{2/3}\Gamma\left(\frac{2}{3}\right)A_I(1) \approx \frac{I_{-1/3}(\frac{2}{3}) - I_{1/3}(\frac{2}{3})}{3},$$

whenever  $s \geq \max\{|s_4|, |s_5|\}$ , where

$$s_4 = \frac{113 \times 3^{2/3}\Gamma(2/3)A_I(1)}{250(3^{2/3}\Gamma(2/3)A_I(1) - 1)}, \quad s_5 = \frac{3 \times 3^{2/3}\Gamma(2/3)A_I(-1)}{5(-1 - 3^{2/3}\Gamma(2/3)A_I(-1))}.$$

As a conclusion, we have the consequences

$$\Psi_s(z) \prec \tilde{\delta}_s(z) \prec \Delta f(z) \implies \Psi_s(z) \prec \Delta f(z).$$

This leads to  $f \in \Lambda_{I_s}$ . □

Other results are given in the next theorem.

**Theorem 4** Let  $f \in \Lambda$  and

$$\Psi_s(z) = \frac{1}{2s} \left( \frac{zf''(z)}{f'(z)} - \frac{z^3}{f(z)} \right).$$

If

- $\Psi_s(z) \ll \Delta f(z)$  and  $f \in \mathcal{C}$  (the class of convex analytic functions in  $U$ ), then  $f \in \Lambda_{1_s}$  for  $|z| \in (0.28, \sqrt{2} - 1]$ ;
- $\Psi_s(z) \ll \Delta f(z)$  and  $f \in \mathcal{S}^*$  (the class of starlike analytic functions in  $U$ ), then  $f \in \Lambda_{1_s}$  for  $|z| \in (0.21, 0.3)$ ;
- $f \in \Lambda_{1_s}$  and  $f \in U_\wp$ ,  $\wp \geq 1.65$  (the set of all locally univalent functions of order  $\wp$ ), then

$$\Psi'_s(z) \ll (\Delta f(z))', \quad |z| \leq (\wp + 1) - \sqrt{\wp^2 + 2\wp}.$$

**Proof 2** For the first conclusion, since  $f$  is convex and  $A_1$  is convex in  $U$  whenever  $\Re(z) > 0$ , then  $\Delta f \in \mathcal{C}$  (see [12]). By [10, Corollary 1], we have  $\Psi_s(z) \prec \Delta f(z)$ , and hence  $f \in \Lambda_{1_s}$  for  $|z| \in (0.28, \sqrt{2} - 1]$ . The second part comes from the fact that  $\Delta f \in \mathcal{S}^*$ , and hence by [10, Corollary 2] we get  $\Psi_s(z) \prec \Delta f(z) \Rightarrow f \in \Lambda_{1_s}$  for  $|z| \in (0.21, 0.3)$ . Lastly, in view of [10, Theorem 3], we have the desired assertion.  $\square$

#### 4 Conclusion

From the above study, we formulated a new linear operator utilizing the Airy function. By using the new operator, we defined a new class of analytic functions and investigated its properties. We showed that the operator can be approximated by well-known special functions. Sufficient conditions are studied to be sure that the normalized function  $f$  is recognized in the new class. For future works, one can suggest new classes of analytic functions involving the linear operator.

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#### Authors' contributions

All authors contributed equally and significantly to writing this article. All authors read and agreed to the published version of the manuscript.

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