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# Impulsive synchronization of fractional-order complex-variable dynamical network

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## Abstract

The impulsive synchronization of a fractional-order complex-variable network is investigated. Firstly, static impulsive controllers are designed and the corresponding synchronization criteria are derived. From the criteria, the impulsive gains can be calculated. Secondly, adaptive impulsive controllers are designed. Noticeably, the impulsive gains can be adjusted to the needed values adaptively. Finally, numerical examples are provided to verify the results.

**Keywords:** Synchronization; Complex variable; Fractional-order network; Impulsive control

## 1 Introduction

In recent years, fractional-order differential systems have gained increasing attentions due to the fact that they can better describe the memory and hereditary properties, such as elastic systems, dielectric polarization, heat conduction, electromagnetic waves, and financial systems [1–12]. In [7–9], the authors designed some kinds of memristive hyperchaotic system and discussed their applications. In [11], the author studied a fractional-order financial system. For those large-scale fractional-order systems, they usually contain large number of interactive individuals and are modeled by fractional-order dynamical network. The nodes denote the individuals and the edges denote the interactions among individuals. In [13], the authors considered fractional-order neural networks. In [14], the authors investigated a time-delay neural network.

Synchronization of dynamical networks has been extensively studied [15–37], with a view on power grids, unmanned aircraft operation, parallel image processing, and so on. However, due to the complexities of a dynamic network, achieving synchronization by inner adjustment is difficult and even impossible. Therefore, appropriate external controllers need to be designed. So far, many control schemes have been adopted to design suitable controllers, such as impulsive control [16, 21–32], intermittent control [38–41], pinning control [42, 43], and feedback control [44, 45].

In the real world, many complex systems cannot be controlled by continuous control and endure continuous disturbance. Impulsive control, as a typical discontinuous control

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scheme, has been widely adopted to design proper controllers, i.e., the controllers are applied to the systems only at certain moments. That is, the impulsive controllers have a relatively simple structure and are easy to implement and have low cost. Researchers have obtained many valuable results about impulsive control and synchronization in integer-order dynamical networks [21–26, 29, 30, 32]. From a practical point of view, a fractional-order network can describe some practical phenomena more accurate than an integer-order model. Therefore, impulsive control is adopted to study the synchronization of fractional-order dynamical networks as well. In [27], the authors investigated impulsive stabilization and synchronization of fractional-order complex-valued neural networks. In [28], the authors investigated synchronization for a class of fractional-order linear complex networks via impulsive control. In [27, 28], some useful synchronization conditions are obtained, from which the impulsive gains and intervals can be calculated for a given network. However, for different networks, it is necessary to calculate the required impulsive intervals and gains repetitively. Therefore, how to design the universal impulsive controllers deserves further studies.

In this paper, we introduce the fractional-order complex-variable dynamical network model and present some preliminaries in Sect. 2. In Sect. 3, we design static and adaptive impulsive controllers, respectively. For static impulsive controllers, we derive the sufficient conditions for achieving synchronization. For adaptive impulsive controllers, we provide the updating laws of the impulsive gains. We perform three numerical examples to verify the results in Sect. 4. In Sect. 5, we give the conclusions.

## 2 Model description and preliminaries

In this section, some definitions and lemma are recalled.

**Definition 1** ([3, 6]) For an integrable function  $f(t) : [t_0, +\infty) \rightarrow \mathbb{R}$ , its  $\alpha$ th-order fractional integral is defined as

$${}_{t_0}I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} f(s) ds, \quad t \geq t_0,$$

where  $\Gamma(\cdot)$  stands for the Gamma function and  $\alpha > 0$ .

**Definition 2** ([3, 6]) For function  $f \in C^m([t_0, +\infty), \mathbb{R})$ , its  $\alpha$ th-order Caputo derivative is defined by

$${}_{t_0}D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^t (t-s)^{m-\alpha-1} f^{(m)}(s) ds, \quad t > t_0,$$

where  $m$  is a positive integer such that  $m-1 < \alpha < m$ .

**Lemma 1** ([27]) For  $\forall t_0 \in \mathbb{R}$  and a real-valued continuous function  $V(t)$  on  $[t_0, +\infty)$ , if there exists a constant  $\theta$  such that

$${}_{t_0}D_t^\alpha V(t) \leq \theta V(t), \quad 0 < \alpha < 1,$$

then

$$V(t) \leq V(t_0) e^{\frac{\theta}{\Gamma(\alpha+1)}(t-t_0)^\alpha}.$$

Consider a fractional-order complex-variable dynamical network, described by

$${}_{t_\sigma}D_t^\alpha x_k(t) = f(x_k(t)) + b \sum_{l=1}^N a_{kl} \Gamma x_l(t), \quad t \in (t_\sigma, t_{\sigma+1}], \sigma = 0, 1, 2, \dots, \tag{1}$$

where  $k = 1, 2, \dots, N$ ,  $0 < \alpha < 1$ ,  $x_k(t) = (x_{k1}(t), x_{k2}(t), \dots, x_{kn}(t))^T \in \mathbb{C}^n$  is the state variable of node  $k$ ,  $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$  is a nonlinear complex-valued vector function,  $b > 0$  is the coupling strength,  $\Gamma = \text{diag}(\gamma_1, \gamma_2, \dots, \gamma_n)$  is the inner coupling matrix.  $A = (a_{kl}) \in \mathbb{R}^{N \times N}$  is the zero-row-sum outer coupling matrix representing the network topology, defined as: if node  $k$  is affected by node  $l$  ( $k \neq l$ ), then  $a_{kl} \neq 0$ ; otherwise,  $a_{kl} = 0$ . The time series  $\{t_\sigma\}$  satisfies  $0 = t_0 < t_1 < t_2 < \dots < t_\sigma < t_{\sigma+1} < \dots$  and  $t_\sigma \rightarrow +\infty$  as  $\sigma \rightarrow +\infty$ .

The network (1) is said to achieve synchronization, if  $\lim_{t \rightarrow \infty} \|x_k(t) - s(t)\| = 0$ , where  $s(t)$  is a solution of an isolated node, i.e.,  ${}_{t_\sigma}D_t^\alpha s(t) = f(s(t))$  for  $t \in (t_\sigma, t_{\sigma+1}]$ .

The controlled network with impulsive controllers is written as

$${}_{t_\sigma}D_t^\alpha x_k(t) = f(x_k(t)) + b \sum_{l=1}^N a_{kl} \Gamma x_l(t), \quad t \in (t_\sigma, t_{\sigma+1}), \tag{2}$$

$$x_k(t_{\sigma+1}^+) = x_k(t_{\sigma+1}^-) + B(t_{\sigma+1})(x_k(t_{\sigma+1}^-) - s(t_{\sigma+1}^-)),$$

where  $x_k(t_{\sigma+1}^+) = \lim_{t \rightarrow t_{\sigma+1}^+} x_k(t)$  and  $x_k(t_{\sigma+1}^-) = \lim_{t \rightarrow t_{\sigma+1}^-} x_k(t)$ . Any solution of (2) satisfies  $x_k(t_{\sigma+1}^-) = x_k(t_{\sigma+1})$ .  $B(t_{\sigma+1}) \in (-2, -1) \cup (-1, 0)$  is the impulsive gain at  $t = t_{\sigma+1}$ ,  $B(t_0) = 0$  and  $B(t) = 0$  for  $t \neq t_{\sigma+1}$ .

**Assumption 1** Suppose that there exists a positive constant  $L$  such that

$$\begin{aligned} & (x(t) - s(t))^T \overline{(f(x(t)) - f(s(t)))} + (f(x(t)) - f(s(t)))^T \overline{(x(t) - s(t))} \\ & \leq L(x(t) - s(t))^T \overline{(x(t) - s(t))} \end{aligned}$$

holds for any  $x(t), s(t) \in \mathbb{C}^n$  and  $t > 0$ .

Throughout this paper, we make Assumption 1. Since the coupling matrix is a zero-row-sum matrix and the impulsive gains  $B(t_{\sigma+1}) \in (-2, -1) \cup (-1, 0)$ , according to the discussions in Refs. [12] and [46], the existence of the solutions of (3) is guaranteed.

Let  $e_k(t) = x_k(t) - s(t)$ , we have the following error system:

$${}_{t_\sigma}D_t^\alpha e_k(t) = f(x_k(t)) - f(s(t)) + b \sum_{l=1}^N a_{kl} \Gamma e_l(t), \quad t \in (t_\sigma, t_{\sigma+1}), \tag{3}$$

$$e_k(t_{\sigma+1}^+) = e_k(t_{\sigma+1}^-) + B(t_{\sigma+1})e_k(t_{\sigma+1}^-), \quad t = t_{\sigma+1}.$$

### 3 Main results

In what follows, let  $e(t) = ((e_1(t))^T, (e_2(t))^T, \dots, (e_N(t))^T)^T$ ,  $\tau_\sigma = t_{\sigma+1} - t_\sigma$  be the impulsive intervals,  $\lambda$  be the largest eigenvalue of matrix  $b(A + A^T) \otimes \Gamma$ ,  $\theta = \frac{L+\lambda}{\Gamma(\alpha+1)}$ , and  $\delta(t) = (1 + B(t))^2$ . From the definition of  $B(t)$ , one has  $\delta(t) = 1$  for  $t \neq t_\sigma$ .

**Theorem 1** *Suppose that Assumption 1 holds. If there exists a constant  $\xi > 0$  such that*

$$\ln \delta(t) + \xi + \theta \tau_\sigma^\alpha < 0, \quad \sigma = 1, 2, \dots, \tag{4}$$

*hold, then network (2) achieves synchronization.*

*Proof* Consider the following Lyapunov functional candidate:

$$V(e(t)) = \sum_{k=1}^N e_k^T(t) \overline{e_k(t)}, \quad t \in (t_\sigma, t_{\sigma+1}].$$

When  $t \in (t_\sigma, t_{\sigma+1})$ , the derivative of  $V(e(t))$  is

$$\begin{aligned} {}_{t_\sigma} D_t^\alpha V(e(t)) &\leq \sum_{k=1}^N e_k^T(t) ({}_{t_\sigma} D_t^\alpha \overline{e_k(t)}) + \sum_{k=1}^N ({}_{t_\sigma} D_t^\alpha e_k^T(t)) \overline{e_k(t)} \\ &= \sum_{k=1}^N (e_k^T(t) \overline{f(x_k(t) - f(s(t)))} + (f(x_k(t) - f(s(t)))^T \overline{e_k(t)}) \\ &\quad + b \sum_{k=1}^N \sum_{l=1}^N (e_l^T(t) a_{kl} \Gamma \overline{e_k(t)} + e_k^T(t) a_{kl} \Gamma \overline{e_l(t)}) \\ &\leq L e^T(t) \overline{e(t)} + b e^T(t) (A + A^T) \otimes \Gamma \overline{e(t)} \\ &\leq (L + \lambda) V(e(t)), \end{aligned}$$

which gives

$$\begin{aligned} V(e(t)) &\leq V(e(t_\sigma^+)) e^{\frac{L+\lambda}{\Gamma(\alpha+1)}(t-t_\sigma)^\alpha} \\ &= V(e(t_\sigma^+)) e^{\theta(t-t_\sigma)^\alpha}. \end{aligned} \tag{5}$$

When  $t = t_{\sigma+1}$ , one has

$$\begin{aligned} V(e(t_{\sigma+1}^+)) &= \sum_{k=1}^N e_k^T(t_{\sigma+1}^+) \overline{e_k(t_{\sigma+1}^+)} \\ &= (1 + B(t_{\sigma+1}))^2 \sum_{k=1}^N e_k^T(t_{\sigma+1}^-) \overline{e_k(t_{\sigma+1}^-)} \\ &= \delta(t_{\sigma+1}) V(e(t_{\sigma+1}^-)). \end{aligned} \tag{6}$$

When  $\sigma = 0$ , from (5) and (6),

$$\begin{aligned} V(e(t_1^-)) &\leq V(e(t_0)) \exp(\theta \tau_1^\alpha), \\ V(e(t_1^+)) &\leq \delta(t_1) V(e(t_1^-)) \leq \delta(t_1) V(e(t_0)) \exp(\theta \tau_1^\alpha). \end{aligned}$$

When  $\sigma = 1$ ,

$$V(e(t_2^-)) \leq V(e(t_1)) \exp(\theta \tau_2^\alpha) \leq \delta(t_1) V(e(t_0)) \exp(\theta(\tau_1^\alpha + \tau_2^\alpha)),$$

$$\begin{aligned} V(e(t_2^+)) &\leq \delta(t_2)V(e(t_2^-)) \\ &\leq \delta(t_2)\delta(t_1)V(e(t_0)) \exp(\theta(\tau_1^\alpha + \tau_2^\alpha)) \\ &= V(e(t_0)) \prod_{\epsilon=1}^2 \delta(t_\epsilon) \exp(\theta\tau_\epsilon^\alpha). \end{aligned}$$

By induction,

$$V(e(t_\sigma^+)) \leq V(e(t_0)) \prod_{\epsilon=1}^\sigma \delta(t_\epsilon) \exp(\theta\tau_\epsilon^\alpha), \quad \sigma = 1, 2, \dots$$

From inequalities (4),

$$\delta(t_\epsilon) \exp(\theta\tau_\epsilon^\alpha) \leq \exp(-\xi), \quad \epsilon = 1, 2, \dots,$$

and

$$V(e(t_\sigma^+)) \leq V(e(t_0)) \exp(-\sigma\xi).$$

That is,  $V(e(t_\sigma^+)) \rightarrow 0$  when  $\sigma \rightarrow +\infty$ . Therefore, when  $t \in (t_\sigma, t_{\sigma+1}]$ ,

$$V(e(t)) \leq V(e(t_\sigma^+)) \exp(\theta(t - t_\sigma)^\alpha),$$

i.e.,  $V(e(t)) \rightarrow 0$  and  $\|e_k(t)\| \rightarrow 0$  as  $t \rightarrow +\infty$ . This completes the proof. □

*Remark 1* By simple calculations, we can estimate the positive constant  $\theta$  in Theorem 1, and then calculate the impulsive gains from conditions (4). However, for different networks, we must repeatedly calculate the impulsive gains. Therefore, we design adaptive impulsive controllers to avoid this situation.

**Theorem 2** *Suppose that Assumption 1 holds. If there exists a constant  $\xi > 0$  such that the following conditions:*

$$\ln \delta(t_\sigma) + \xi + \widehat{\theta}(t_\sigma)\tau^\alpha < 0, \quad \sigma = 0, 1, 2, \dots, \tag{7}$$

hold, where  $\widehat{\theta}(t)$  is the estimated value of  $\theta$ ,  ${}_{t_\sigma}D_t^\alpha \widehat{\theta}(t) = \Gamma(\alpha + 1)\omega \sum_{k=1}^N e_k^T(t)\overline{e_k(t)}$ ,  $t \in (t_\sigma, t_{\sigma+1}]$  and  $\omega > 0$  is a positive constant, then the controlled network (2) achieves synchronization.

*Proof* Consider the following Lyapunov function:

$$V(e(t)) = \sum_{k=1}^N e_k^T(t)\overline{e_k(t)} + \frac{\delta(t)}{2\omega} (\widehat{\theta}(t) - \theta)^2, \quad t \in (t_\sigma, t_{\sigma+1}].$$

When  $t \in (t_\sigma, t_{\sigma+1})$ , the function  $V(e(t))$  can be written as

$$V(e(t)) = \sum_{k=1}^N e_k^T(t)\overline{e_k(t)} + \frac{1}{2\omega} (\widehat{\theta}(t) - \theta)^2,$$

and the derivative of  $V(e(t))$  can be calculated as

$$\begin{aligned} {}_{t_\sigma}D_t^\alpha V(e(t)) &\leq \sum_{k=1}^N ({}_{t_\sigma}D_t^\alpha e_k^T(t)) \overline{e_k(t)} + \sum_{k=1}^N e_k^T(t) ({}_{t_\sigma}D_t^\alpha \overline{e_k(t)}) \\ &\quad + \frac{1}{\omega} (\widehat{\theta}(t) - \theta) {}_{t_\sigma}D_t^\alpha \widehat{\theta}(t) \\ &= \sum_{k=1}^N (e_k^T(t) \overline{(f(x_k(t)) - f(s(t)))}) + (f(x_k(t)) - f(s(t)))^T \overline{e_k(t)}) \\ &\quad + b \sum_{k=1}^N \sum_{l=1}^N (e_l^T(t) a_{kl} \Gamma \overline{e_k(t)} + e_k^T(t) a_{kl} \Gamma \overline{e_l(t)}) \\ &\quad + \Gamma(\alpha + 1) (\widehat{\theta}(t) - \theta) \sum_{k=1}^N e_k^T(t) \overline{e_k(t)} \\ &\leq \Gamma(\alpha + 1) \widehat{\theta}(t) e^T(t) \overline{e(t)} \\ &\leq \Gamma(\alpha + 1) \widehat{\theta}(t_{\sigma+1}) V(e(t)), \end{aligned}$$

which gives

$$V(e(t)) \leq V(e(t_\sigma^+)) \exp(\widehat{\theta}(t_{\sigma+1})(t - t_\sigma)^\alpha). \tag{8}$$

When  $t = t_{\sigma+1}$ , one has

$$\begin{aligned} V(e(t_{\sigma+1}^+)) &= \sum_{k=1}^N e_k^T(t_{\sigma+1}^+) \overline{e_k(t_{\sigma+1}^+)} + \frac{\delta(t_{\sigma+1})}{2\omega} (\widehat{\theta}(t) - \theta)^2 \\ &= (1 + B(t_{\sigma+1}))^2 \sum_{k=1}^N e_k^T(t_{\sigma+1}^-) \overline{e_k(t_{\sigma+1}^-)} + \frac{\delta(t_{\sigma+1})}{2\omega} (\widehat{\theta}(t) - \theta)^2 \\ &= \delta(t_{\sigma+1}) \left( \sum_{k=1}^N e_k^T(t_{\sigma+1}^-) \overline{e_k(t_{\sigma+1}^-)} + \frac{1}{2\omega} (\widehat{\theta}(t) - \theta)^2 \right) \\ &= \delta(t_{\sigma+1}) V(e(t_{\sigma+1}^-)). \end{aligned} \tag{9}$$

Therefore, similar to the proof of Theorem 1, the proof is completed. □

*Remark 2* When  $\tau_\sigma$  and  $\xi$  are fixed, we choose

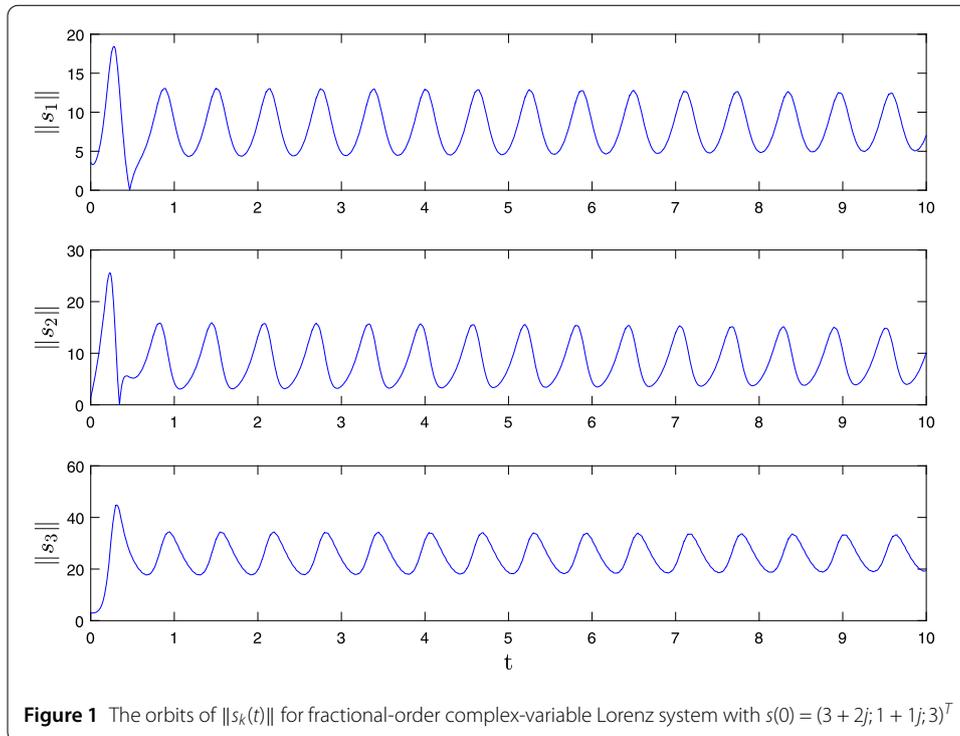
$$-\exp\left(-\frac{\xi + \widehat{\theta}(t_\sigma)\tau_\sigma}{2}\right) - 1 + \varepsilon \leq B(t_\sigma) \leq \exp\left(-\frac{\xi + \widehat{\theta}(t_\sigma)\tau_\sigma}{2}\right) - 1 - \varepsilon,$$

such that the conditions (7) is satisfied, where  $\varepsilon > 0$  is an arbitrary constant.

#### 4 Numerical illustrations

*Example 1* Choose the node dynamics as the fractional-order complex-variable Lorenz system [47]

$${}_{t_\sigma}D_t^\alpha x_{k1}(t) = a(x_{k2}(t) - x_{k1}(t)),$$



$$\begin{aligned}
 {}_{t_0}D_t^\alpha x_{k2}(t) &= bx_{k1}(t) - x_{k2}(t) - x_{k1}(t)x_{k3}(t), \\
 {}_{t_0}D_t^\alpha x_{k3}(t) &= \frac{1}{2}(\overline{x_{k1}(t)}x_{k2}(t) + x_{k1}(t)\overline{x_{k2}(t)}) - cx_{k3}(t),
 \end{aligned}$$

where  $x_{k1}$  and  $x_{k2}$  are complex variables,  $x_{k3}$  is real variable, which is chaotic when the system parameters are chosen as  $a = 10, b = 28, c = 8/3, \alpha = 0.995$  and  $\tau_\sigma = 0.05\sigma$ . Figure 1 shows the orbits of  $\|s_k(t)\|$  with  $s(0) = (3 + 2j, 1 + j, 3)^T$  and  $j = \sqrt{-1}$ . From Fig. 1, there exist three constants  $M_1 = 19, M_2 = 26, M_3 = 47$  such that  $\|s_1\| \leq M_1, \|s_2\| \leq M_2, \|s_3\| \leq M_3$ . Therefore, one has

$$\begin{aligned}
 & (x_k(t) - s(t))^T (\overline{f(x_k) - f(s)}) + (f(x_k) - f(s))^T (\overline{x_k(t) - s(t)}) \\
 &= -2ae_{k1}\overline{e_{k1}} - 2e_{k2}\overline{e_{k2}} - 2ce_{k3}^2 + (a + b - s_3)(e_{k1}\overline{e_{k2}} + \overline{e_{k1}}e_{k2}) \\
 & \quad + s_2(\overline{e_{k1}}e_{k3} + e_{k1}e_{k3}) \\
 & \leq -2ae_{k1}\overline{e_{k1}} - 2e_{k2}\overline{e_{k2}} - 2ce_{k3}^2 + (a + b + M_3)(e_{k1}\overline{e_{k2}} + \overline{e_{k1}}e_{k2}) \\
 & \quad + M_2(\overline{e_{k1}}e_{k3} + e_{k1}e_{k3}) \\
 & \leq (-2a + (a + b + M_3)\mu + M_2\nu)e_{k1}\overline{e_{k1}} + (-2 + (a + b + M_3)\mu^{-1})e_{k2}\overline{e_{k2}} \\
 & \quad + (-2c + M_2\nu^{-1})e_{k3}^2,
 \end{aligned}$$

where  $\mu > 0, \nu > 0$ . Choosing  $\mu = 1.05, \nu = 0.31$  gives  $L = 79$  in Assumption 1.

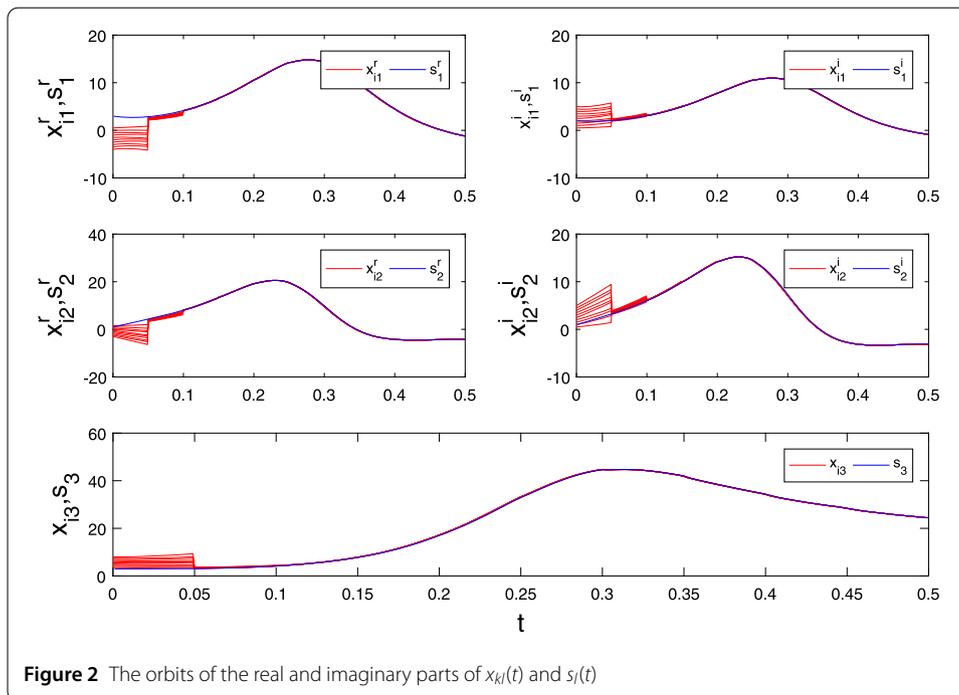
Consider the synchronization of network (2) with 10 nodes. Choose  $b = 0.5$ ,  $\Gamma = \text{diag}(1, 1, 1)$  and

$$A = \begin{bmatrix} -3 & 1 & 0 & 2 & 0 & -2 & 1 & 0 & 0 & 1 \\ 0 & -6 & 0 & 1 & 0 & 1 & 2 & 1 & 0 & 1 \\ -2 & 1 & -5 & 0 & 1 & 2 & 0 & 2 & 0 & 1 \\ 1 & 1 & 0 & -4 & 1 & 0 & -2 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 & -6 & 0 & 1 & 2 & 1 & 2 \\ 0 & 1 & 2 & 1 & 0 & -5 & 0 & -1 & 0 & 2 \\ 2 & 0 & 0 & 2 & 0 & 2 & -7 & 0 & 0 & 1 \\ 0 & -2 & 3 & 0 & 2 & 0 & 1 & -4 & 0 & 0 \\ 1 & 0 & 3 & 0 & -1 & 1 & 0 & 0 & -4 & 0 \\ 0 & 1 & 0 & 1 & -2 & 0 & 0 & 2 & 1 & -3 \end{bmatrix},$$

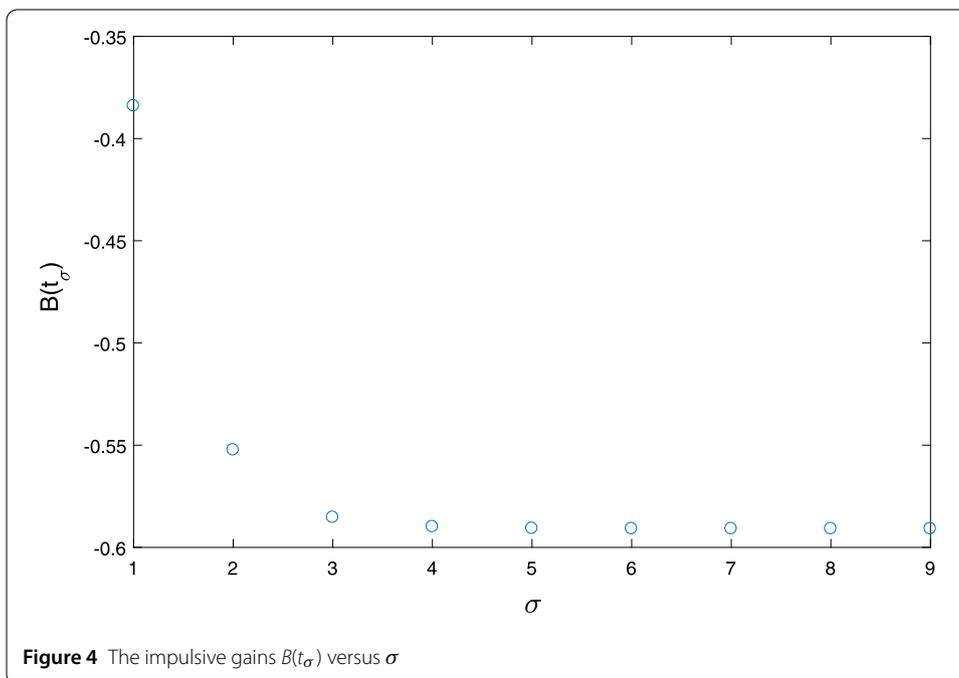
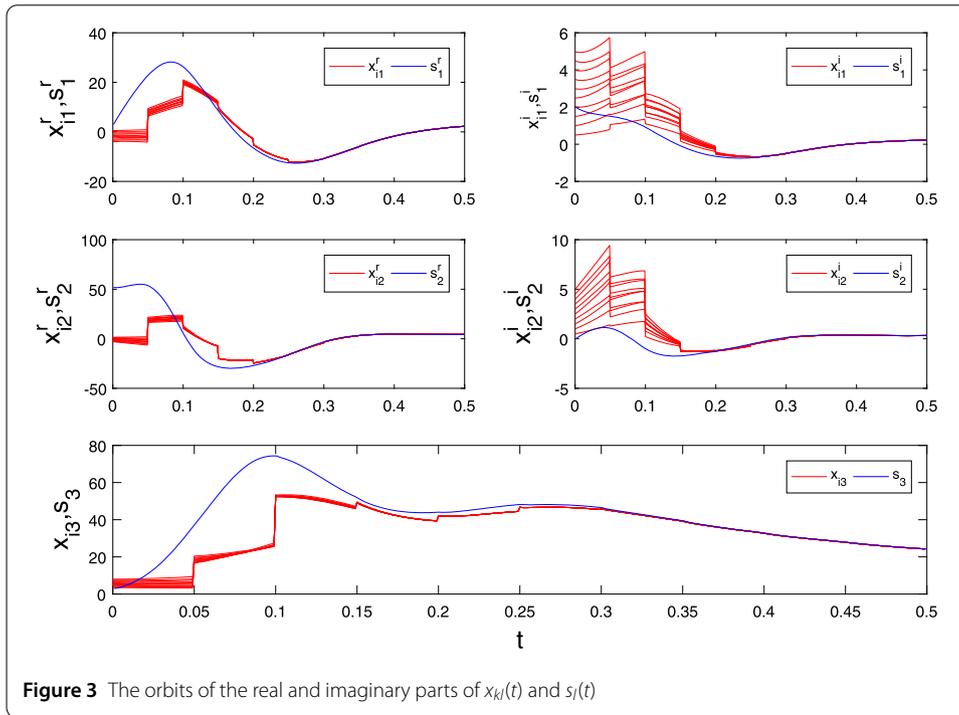
which gives  $\lambda = 0.4693$ . In numerical simulations, choose  $\xi = 0.001$ , the impulsive gains  $B(t_\sigma) = -0.9$ ,  $\sigma = 1, 2, \dots$ , one has  $\delta(t_\sigma) = 0.01$ . By simple calculations, we have  $\ln \delta(t) + \xi + \theta \tau_\sigma^\alpha = -0.5707 < 0$ , i.e., the conditions (4) hold and the synchronization can be achieved. The initial values of  $s(t)$  and  $x_k(t)$  are chosen randomly. Figure 2 shows the orbits of the real and imaginary parts of  $x_{kl}(t)$  and  $s_l(t)$ ,  $k = 1, 2, \dots, 10$ ,  $l = 1, 2, 3$ . The superscripts  $r$  and  $i$  denote the real parts and the imaginary parts, respectively.

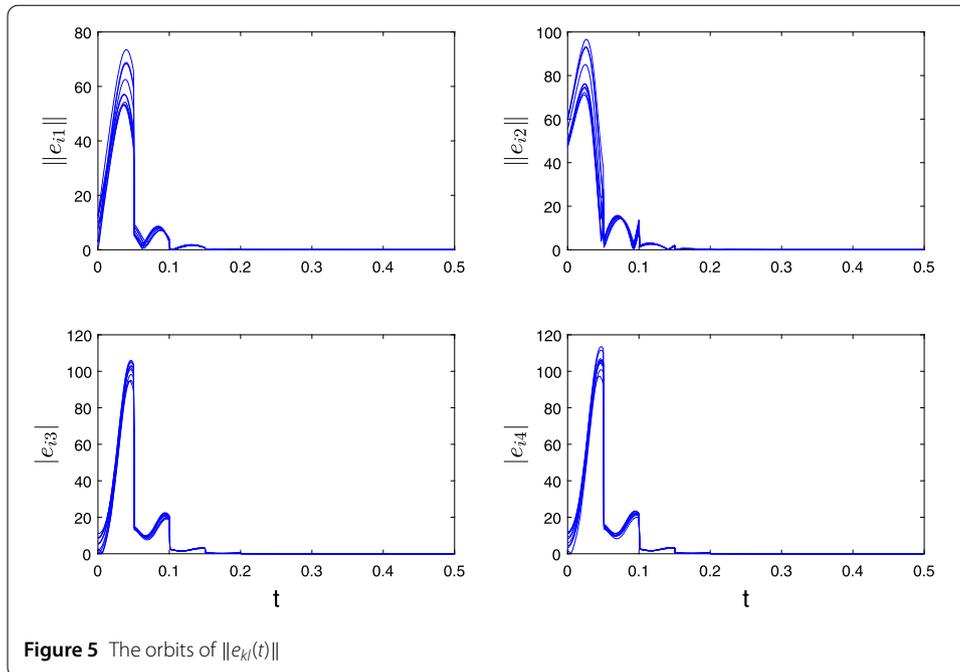
*Example 2* Consider the above network in Example 1 via the adaptive impulsive controllers. Choose  $\omega = 0.01$ ,  $\xi = 0.001$  and  $\hat{\theta}(0) = 1$ . According to Remark 2, choose

$$B(t_\sigma) = \exp\left(-\frac{\xi + \hat{\theta}(t_\sigma)\tau_\sigma}{2}\right) - 1 - \varepsilon,$$



with  $\varepsilon = 0.001$ . Figure 3 shows the orbits of the real and imaginary parts of  $x_{kl}(t)$  and  $s_l(t)$ ,  $k = 1, 2, \dots, 10$ ,  $l = 1, 2, 3$ . Figure 4 shows the impulsive gains  $B(t_\sigma)$ .





*Example 3* Choose the node dynamics as fractional-order hyperchaotic complex Lü system [48]

$$\begin{aligned}
 D_{t_0}^\alpha x_{k1} &= 42(x_{k2} - x_{k1}) + x_{k4}, \\
 D_{t_0}^\alpha x_{k2} &= 25x_{k2} + x_{k4} - x_{k1}x_{k3}, \\
 D_{t_0}^\alpha x_{k3} &= \frac{1}{2}(\overline{x_{k1}}x_{k2} + x_{k1}\overline{x_{k2}}) - 6x_{k3}, \\
 D_{t_0}^\alpha x_{k4} &= \frac{1}{2}(\overline{x_{k1}}x_{k2} + x_{k1}\overline{x_{k2}}) - 5x_{k4},
 \end{aligned}$$

where  $k = 1, 2, \dots, N$ ,  $x_{k1}$  and  $x_{k2}$  are complex variables,  $x_{k3}$  and  $x_{k4}$  are real variables,  $\alpha = 0.995$ . Choose the same  $A$  as Example 1, the initial values  $s(0) = (3 + 2j, 1 + j, 3, 4)^T$  and  $x_k(t)$  are chosen randomly.

Choose  $\tau_\sigma = 0.2$ ,  $\omega = 0.001$ ,  $\xi = 0.001$  and  $\widehat{\theta}(0) = 0.1$ . According to Remark 2, choose

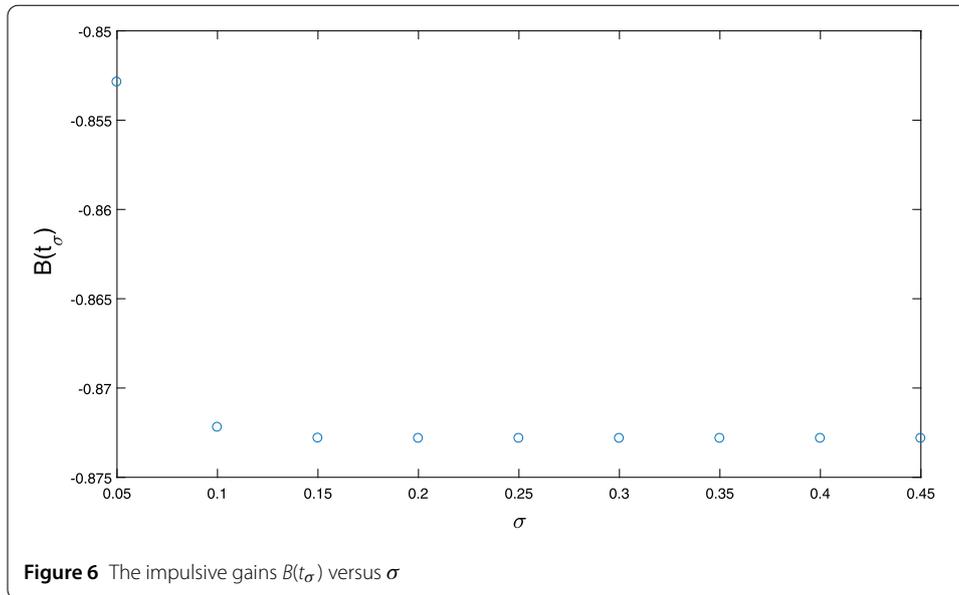
$$B(t_\sigma) = \exp\left(-\frac{\xi + \widehat{\theta}(t_\sigma)\tau_\sigma}{2}\right) - 1 - \varepsilon,$$

with  $\varepsilon = 0.001$ . Figure 5 shows the orbits of the real and imaginary parts of  $x_{kl}(t)$  and  $s_l(t)$ ,  $k = 1, 2, \dots, 10$ ,  $l = 1, 2, 3$ . Figure 6 shows the impulsive gains  $B(t_\sigma)$ .

From Examples 2 and 3, the impulsive gains need not be calculated in advance for different networks. And they can adjust themselves to the required values according to the updating laws. That is, the adaptive impulsive controllers are universal to some extent.

### 5 Conclusions

Both static and adaptive impulsive controllers were designed. Two corresponding synchronization conditions were derived as well. Particularly, for the adaptive impulsive con-



trollers, the updating law for the impulsive gains was provided. Examples 2 and 3 demonstrated the points well and implied that the adaptive impulsive controllers are universal for different networks.

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#### Availability of data and materials

Not applicable.

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All the authors contributed equally to this work. They all read and approved the final version of the manuscript.

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