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Study on new integral operators defined using confluent hypergeometric function

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Abstract

Two new integral operators are defined in this paper using the classical Bernardi and Libera integral operators and the confluent (or Kummer) hypergeometric function. It is proved that the new operators preserve certain classes of univalent functions, such as classes of starlike and convex functions, and that they extend starlikeness of order $\frac{1}{2}$ and convexity of order $\frac{1}{2}$ to starlikeness and convexity, respectively. For obtaining the original results, the method of admissible functions is used, and the results are also written as differential inequalities and interpreted using inclusion properties for certain subsets of the complex plane. The example provided shows an application of the original results.

MSC: 30C45; 30C80

1 Introduction

The method of differential subordinations or the method of admissible functions is one of the newest methods used in geometric function theory. It was introduced by Miller and Mocanu in two papers published in 1978 [10] and 1981 [9] and has the merit of being useful for easier proofs of known results and also for obtaining a new, interesting outcome. Another important aspect in geometric function theory is the study of different types of operators among which integral operators play an important role. Their study began in the early twentieth century when Alexander introduced the first integral operator in 1915 [1]. Libera integral operator was defined in 1965 [6], and it was proved that it preserves certain classes of univalent functions such as the class of starlike functions, convex functions, close-to-convex functions, starlike functions of order $-\frac{1}{2}$, and convex functions of order $-\frac{1}{2}$. In 1969, S.D. Bernardi generalized this operator and introduced what is now called Bernardi integral operator [3]. It was also proved that this operator preserves the same classes of univalent functions.

Studies on hypergeometric functions have been conducted especially because they have applications in many fields as it was so comprehensively presented in a recently published article [16] where the author shows many interesting developments emphasizing their applications related to univalent functions. A connection between hypergeometric functions and univalent functions theory was established through the proof given by de Branges for Bieberbach's conjecture in 1985 [4]. After this event, hypergeometric functions have been studied intensely using the means of geometric function theory. As part of this study, different types of hypergeometric functions were used for defining new operators. Among the well-known such operators, Dziok–Srivastava operator [5] and Srivastava–Wright operator [15] must be mentioned as they have given excellent tools for developing the study in geometric function theory.

The hypergeometric function considered in this study is confluent or Kummer hypergeometric function, referred to as *KHF* throughout the paper. In [13] it was proved that this function is convex for $a, c \in \mathbb{C}$, $c \neq 0, -1, -2, \ldots$, and also that it is univalent. According to the theorem of analytic characterization of convexity found as Theorem 4.2.1 in [12, p. 50], once the function is known to be convex, it is also a starlike function. Using Theorem 2.6a from [11, p. 57] known as Marx–Strohhäcker result [8, 17], knowing that the function is convex gives the certainty that it is starlike of order $\frac{1}{2}$. In [14] the theory of differential superordination was used to obtain that *KHF* is a Carathéodory function and differential inequalities associated to the results were interpreted as inclusions for certain subsets of the complex plane. A sandwich-type result was stated providing a link between [13] and [14]. *KHF* has already been used in defining a new operator using fractional integral in [7]. The definition of *KHF* that is also used in this paper is the following:

Definition 1.1 ([11], p. 5) Let a and c be complex numbers with $c \neq 0, -1, -2, \ldots$, and consider the function

$$ {}_1F_1(a, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \cdot \frac{z^k}{k!} = \frac{\Gamma(c)}{\Gamma(a)} \cdot \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(c+k)} \cdot \frac{z^k}{k!} \tag{1} $$

with

$$ (d)_k = \frac{\Gamma(d+k)}{\Gamma(d)} = d(d+1)(d+2)\cdots(d+k-1) \quad \text{and} \quad (d)_0 = 1. $$

This function is called confluent (Kummer) hypergeometric function (*KHF*).

The well-known definitions and notations familiar to the field of complex analysis are used.

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disc of the complex plane.

Denote by $H(U)$ the class of holomorphic functions in U, and let

$$ A_n = \{f \in H(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, z \in U\} \quad \text{with } A_1 = A. $$

Let S be the class of holomorphic and univalent functions in the open unit disc U which have the serial development $f(z) = z + a_2 z^2 + \cdots$ for $z \in U$.

For $a \in \mathbb{C}$, $n \in \mathbb{N}^*$, denote

$$ H[a, n] = \{f \in H(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, z \in U\} $$

with $H_0 = H[0, 1]$.

For $0 < \alpha < 1$, denote the class of starlike functions of order α by

$$ S^*(\alpha) = \left\{ f \in A : \operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha \right\}. $$

For $\alpha = 0$, the class of starlike functions is obtained and denoted by S^*.

For $0 < \alpha < 1$, denote the class of convex functions of order α by

$$K(\alpha) = \left\{ f \in A : \operatorname{Re}\left(\frac{\mathrm{zf}''(\mathrm{z})}{\mathrm{f}'(\mathrm{z})} + 1 \right) > \alpha \right\}.$$

For $\alpha = 0$, the class of convex functions is obtained and denoted by K.

In order to use the method of admissible functions, the next definition must be invoked.

Definition 1.2 ([12], p. 185) Let $\Omega = \Delta = \{w \in \mathbb{C} : \operatorname{Re} w > 0\}$ and denote by $\psi_n\{1\}$ the class of admissible functions $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ which satisfy the admissibility condition

$$\psi(\rho i, \sigma; z) \notin \Omega, \tag{A}$$

where $\rho, \sigma \in \mathbb{R}$, $\sigma \leq \frac{-n}{2}(1 + \rho^2)$, $z \in U$, $n \geq 1$.

The next lemma is an important tool in proving the original results of this paper.

Lemma A ([11], p. 35) *If* $\psi \in \psi_n\{1\}$, *then*

$$\operatorname{Re} \psi\big(p(z), zp'(z)\big) > 0 \quad \textit{implies} \quad \operatorname{Re} p(z) > 0, \quad z \in U. \tag{L-A}$$

The original subordination results presented in this paper are also given as differential inequalities in the complex plane which are interpreted in terms of inclusion relations involving the subsets of $\mathbb{C}$. This was already done for the results obtained in [14], and the technique can be seen in another very recent paper [2], which shows that it is a perspective in trend with an interesting outcome.

Using *KHF*, two integral operators are next defined, and some properties related to their ability of preserving starlikeness and convexity are stated and proved. The operators are given using Bernardi [3] and Libera [6] integral operators.

2 Main results

Definition 2.1 Let ${}_1F_1(a, c; z)$ be given by (1) and let $\gamma > 0$. The integral operator $B : H[1, 1] \to H[1, 1]$,

$$B\big[{}_1F_1(a, c; z)\big] = B(z) = \frac{\gamma}{z^\gamma} \int_0^z {}_1F_1(a, c; t) t^{\gamma - 1}\, dt \tag{2}$$

is called Kummer–Bernardi integral operator.

For $\gamma = 1$, the integral operator $L : H[1, 1] \to H[1, 1]$ is defined as

$$L\big[{}_1F_1(a, c; z)\big] = L(z) = \frac{1}{z} \int_0^z {}_1F_1(a, c; t)\, dt, \tag{3}$$

which is called Kummer–Libera integral operator.

Remark 2.1 Using (1) and (2), we have

$$B\big[{}_1F_1(a,c;0)\big] = B(0) = 1, \qquad B'(0) = \frac{a}{c} \cdot \frac{\gamma}{\gamma+1} \neq 0, \quad a \in \mathbb{C}, a \neq 0,$$

$$L\big[{}_1F_1(a,c;0)\big] = L(0) = 1, \qquad L'(0) = \frac{a}{c} \cdot \frac{1}{2} \neq 0, \quad a \in \mathbb{C}, a \neq 0.$$

Remark 2.2 The conditions

$$B'(0) = \frac{a}{c} \cdot \frac{\gamma}{\gamma+1} \neq 0, \qquad L'(0) = \frac{a}{c} \cdot \frac{1}{2} \neq 0$$

are necessary conditions for the operators $B[{}_1F_1(a,c;z)]$ and $L[{}_1F_1(a,c;z)]$ to be univalent.

In the next theorem the sufficient conditions for the operators $B[{}_1F_1(a,c;z)]$ and $L[{}_1F_1(a,c;z)]$ to be univalent and to preserve starlikeness are obtained. These conditions are also expressed in terms of differential inequalities in the complex plane and interpreted using inclusion properties for certain subsets of $\mathbb{C}$.

Theorem 2.1 *Let ${}_1F_1(a,c;z)$ be given by* (1) *with $B'(0) = \frac{a}{c} \cdot \frac{\gamma}{\gamma+1} \neq 0$, $\gamma > 0$, and*

$$\operatorname{Re} \frac{z{}_1F_1'(a,c;z)}{{}_1F_1(a,c;z)} > 0, \quad z \in U \text{ i.e. } {}_1F_1(a,c;z) \in S^*. \tag{4}$$

Then the Kummer–Bernardi integral operator given in (2) *is a starlike function and*

$$B\big[S^*\big] \subset S^*$$

or

$$B(U) \subset \{z \in C : z = x + iy, x > 0, y \in \mathbb{R}\}.$$

Proof Using the definition of Kummer–Bernardi integral operator given in (2), we obtain

$$z^\gamma B(z) = \gamma \int_0^z {}_1F_1(a,c;t)t^{\gamma-1}\,dt. \tag{5}$$

Differentiating (5) and doing some calculations, we obtain

$$\gamma B(z) + zB'(z) = \gamma\,{}_1F_1(a,c;z), \quad z \in U,$$

which is equivalent to

$$B(z)\left[\gamma + \frac{zB'(z)}{B(z)}\right] = \gamma\,{}_1F_1(a,c;z), \quad z \in U. \tag{6}$$

Since $B(z) \neq 0$, $z \in U$, we let

$$p(z) = \frac{zB'(z)}{B(z)}, \quad z \in U. \tag{7}$$

Using (7) in (6), we get

$$B(z)\big[\gamma + p(z)\big] = \gamma\,{}_1F_1(a,c;z), \quad z \in U. \tag{8}$$

Applying the logarithm to (8) and then differentiating the result, using (7), we obtain

$$p(z) + \frac{zp'(z)}{\gamma + p(z)} = \frac{z\,{}_1F_1'(a,c;z)}{{}_1F_1(a,c;z)}, \quad z \in U. \tag{9}$$

Using (4), relation (9) becomes

$$\operatorname{Re}\left[p(z) + \frac{zp'(z)}{\gamma + p(z)}\right] > 0, \quad z \in U. \tag{10}$$

□

For obtaining the result claimed by the theorem, Lemma A will be used. For that, it is necessary to show that the admissibility condition (A) is satisfied.

Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$,

$$\psi(r,s) = r + \frac{s}{\gamma + r}, \quad r,s \in \mathbb{C}, \gamma > 0. \tag{11}$$

For $r = p(z)$, $s = zp'(z)$, $z \in U$, relation (11) becomes

$$\psi\big(p(z), zp'(z)\big) = p(z) + \frac{zp'(z)}{\gamma + p(z)}, \quad z \in U. \tag{12}$$

Using (12), relation (10) becomes

$$\operatorname{Re}\psi\big(p(z), zp'(z)\big) > 0, \quad z \in U. \tag{13}$$

We evaluate

$$\begin{aligned}\operatorname{Re}\psi(\rho i,\sigma) &= \operatorname{Re}\left[\rho i + \frac{\sigma}{\gamma + \rho i}\right] = \operatorname{Re}\left[\frac{\sigma}{\gamma + \rho i}\right] = \operatorname{Re}\frac{\sigma(\gamma - \rho i)}{\gamma^2 + \rho^2} = \frac{\sigma\gamma}{\gamma^2 + \rho^2} \\ &\leq \frac{\gamma}{\gamma^2 + \rho^2} \cdot \frac{(-1)}{2}\big(1 + \rho^2\big) < 0.\end{aligned}$$

Since $\operatorname{Re}\psi(\rho i,\sigma) < 0$ and using Definition 1.2, we deduce that $\psi \in \psi_n\{1\}$.

Using now (13) and applying relation (L-A) from Lemma A, we obtain

$$\operatorname{Re} p(z) > 0, \quad z \in U. \tag{14}$$

Using (7) in (14), we conclude that

$$\operatorname{Re}\frac{zB'(z)}{B(z)} > 0, \quad z \in U \text{ i.e. } B \in S^* \text{ and } B\big[S^*\big] \subset S^*.$$

Remark 2.3 For $\gamma = 1$, from Theorem 2.1, we deduce the following corollary for Kummer–Libera integral operator $L[{}_1F_1(a,c;z)]$.

Corollary 2.1 *Let* ${}_1F_1(a,c;z)$ *be given by* (1) *with* ${}_1F_1(a,c;z) \neq 0$ *and*

$$\operatorname{Re} \frac{z_1F_1'(a,c;z)}{{}_1F_1(a,c;z)} > 0, \quad z \in U \text{ i.e. } {}_1F_1(a,c;z) \in S^*.$$

Then Kummer–Libera integral operator given in (3) *is a starlike function and* $L[S^*] \subset S^*$ *or*

$$B(U) \subset \{z \in C : z = x + iy, x > 0, y \in \mathbb{R}\}.$$

Using Theorem 4.4.4 [12, *p.* 76] *and Theorem* 2.1, *we prove in the next theorem the property that Kummer–Bernardi and Kummer–Libera integral operators have of extending stralikeness of order* $\frac{1}{2}$ *to stralikeness.*

Corollary 2.2 *Let* ${}_1F_1(a,c;z)$ *be given by* (1) *with* ${}_1F_1(a,c;z) \neq 0$ *and*

$$\operatorname{Re} \frac{z_1F_1'(a,c;z)}{{}_1F_1(a,c;z)} > \frac{1}{2}, \quad z \in U \text{ i.e. } {}_1F_1(a,c;z) \in S^*\left(\frac{1}{2}\right).$$

Then the Kummer–Bernardi integral operator given in (2) *is a starlike function and*

$$B\left[S^*\left(\frac{1}{2}\right)\right] \subset S^*$$

or

$$B(U) \subset \{z \in \mathbb{C} : z = x + iy, x > 0, y \in \mathbb{R}\}.$$

Corollary 2.3 *Let* ${}_1F_1(a,c;z)$ *be given by* (1) *with* ${}_1F_1(a,c;z) \neq 0$ *and*

$$\operatorname{Re} \frac{z_1F_1'(a,c;z)}{{}_1F_1(a,c;z)} > \frac{1}{2}, \quad z \in U \text{ i.e. } {}_1F_1(a,c;z) \in S^*\left(\frac{1}{2}\right).$$

Then the Libera–Bernardi integral operator given in (3) *is a starlike function and*

$$L\left[S^*\left(\frac{1}{2}\right)\right] \subset S^*$$

or

$$L(U) \subset \{z \in \mathbb{C} : z = x + iy, x > 0, y \in \mathbb{R}\}.$$

Knowing that the *KHF* given in (1) is convex, we prove in the next theorem the property that Kummer–Bernardi and Kummer–Libera integral operators have of preserving convexity, and this property is written in terms of sets inclusion related to certain subsets of the complex plane.

Theorem 2.2 *Let* ${}_1F_1(a,c;z)$ *be given by* (1) *with* ${}_1F_1'(a,c;0) \neq 0$, $\gamma > 0$, *and*

$$\operatorname{Re}\left[\frac{z_1F_1''(a,c;z)}{{}_1F_1'(a,c;z)} + 1\right] > 0, \quad z \in U \text{ i.e. } {}_1F_1(a,c;z) \in K. \tag{15}$$

Then the Kummer–Bernardi integral operator given in (2) *is convex in* U *and*

$$B[K] \subset K$$

or

$$B(U) \subset \{z \in \mathbb{C} : z = x + iy, x > 0, y \in \mathbb{R}\}.$$

Proof From relation (2), we get that $B'(z) \neq 0$, $z \in U$. We also can write

$$z^{\gamma} B(z) = \gamma \int_0^z {}_1F_1(a,c;t) t^{\gamma-1}\, dt,$$

and differentiating this relation, after a few calculations we obtain

$$\gamma B(z) + zB'(z) = \gamma\, {}_1F_1(a,c;z), \quad z \in U.$$

Differentiating this relation, we get

$$B'(z)\left[\gamma + 1 + \frac{zB''(z)}{B'(z)}\right] = \gamma\, {}_1F_1'(a,c;z), \quad z \in U, \tag{16}$$

and letting

$$p(z) = \frac{zB''(z)}{B'(z)} + 1, \quad z \in U, \tag{17}$$

we obtain

$$B'(z)\big[\gamma + p(z)\big] = \gamma\, {}_1F_1'(a,c;z), \quad z \in U. \tag{18}$$

Differentiating (18) and using (17), we can write

$$p(z) + \frac{zp'(z)}{\gamma + p(z)} = \frac{z\, {}_1F_1''(a,c;z)}{{}_1F_1'(a,c;z)} + 1. \tag{19}$$

Using (15) in (19), we have

$$\mathrm{Re}\left[p(z) + \frac{zp'(z)}{\gamma + p(z)}\right] > 0, \quad z \in U. \tag{20}$$

Relation (20) is equivalent to (10), which implies the conclusion that

$$\mathrm{Re}\, p(z) > 0, \quad z \in U. \tag{21}$$

Using now (17) in (21), we obtain

$$\mathrm{Re}\left[\frac{zB''(z)}{B'(z)} + 1\right] > 0, \quad z \in U \text{ i.e. } B \in K, B(K) \subset K. \qquad \square$$

Remark 2.4 For $\gamma = 1$, we obtain the following corollary for Kummer–Libera integral operator.

Corollary 2.4 *Let* ${}_1F_1(a,c;z)$ *be given by* (1) *with* ${}_1F_1'(a,c;0) \neq 0$, $\gamma > 0$ *and*

$$\mathrm{Re}\left[\frac{z_1F_1''(a,c;z)}{{}_1F_1'(a,c;z)} + 1\right] > 0, \quad z \in U \text{ i.e. } {}_1F_1(a,c;z) \in K.$$

Then the Kummer–Libera integral operator given in (3) *is convex in* U *and* $L[K] \subset K$ *or*

$$L(U) \subset \{z \in \mathbb{C} : z = x + iy, x > 0, y \in \mathbb{R}\}.$$

Remark 2.5 Using the Marx–Strohhäcker result [11, p. 55] and the convexity property of Kummer–Bernardi and Kummer–Libera integral operators, we can state the corollary giving the property of those operators to be starlike of order $\frac{1}{2}$.

Corollary 2.5 *Let* $\mathrm{Re}[\frac{zB''(z)}{B'(z)} + 1] > 0$. *Using the Marx–Strohhäcker result* [11, *p.* 55], *we get that* $\mathrm{Re}[\frac{zB'(z)}{B(z)} + 1] > \frac{1}{2}$ *i.e.* $B \in S^*(\frac{1}{2})$ *and* $B(K) \subset S^*(\frac{1}{2})$ *or*

$$B(U) \subset \left\{z \in \mathbb{C} : z = x + iy, x > \frac{1}{2}, y \in \mathbb{R}\right\}.$$

For $\gamma = 1$, we obtain the following corollary for Kummer–Libera integral operator.

Corollary 2.6 *If* $\mathrm{Re}[\frac{zL''(z)}{L'(z)} + 1] > 0$, *using the Marx–Strohhäcker result* [11, *p.* 55], *we get that* $\mathrm{Re}[\frac{zL'(z)}{L(z)} + 1] > \frac{1}{2}$ i.e. $L \in S^*(\frac{1}{2})$ *and* $L(K) \subset S^*(\frac{1}{2})$ *or*

$$L(U) \subset \left\{z \in \mathbb{C} : z = x + iy, x > \frac{1}{2}, y \in \mathbb{R}\right\}.$$

The study is concluded with an example of how the results presented in the paper are useful.

Example *Let* $a = -1$, $c = \frac{1+i}{4}$. *Then KHL is defined as*

$${}_1F_1\left(-1, \frac{1+i}{4}; z\right) = 1 + \frac{-1}{\frac{1+i}{4}}z = 1 + \frac{(-1)\cdot 4}{1+i} = 1 - \frac{4\cdot(1-i)}{2}z = 1 - 2(1-i)z.$$

Differentiating this, we get

$$\begin{aligned} {}_1F_1'\left(-1, \frac{1+i}{4}; z\right) &= -2(1-i)\frac{z_1F_1'(-1, \frac{1+i}{4}; z)}{{}_1F_1(-1, \frac{1+i}{4}; z)} \\ &= \frac{-2(1-i)z}{1-2(1-i)z} = 1 - \frac{1}{1-2(1-i)z}. \end{aligned}$$

We calculate

$$L\left[{}_1F_1\left(-1, \frac{1+i}{4}; z\right)\right] = \frac{1}{z}\int_0^z \left[1 - 2(1-i)t\right]dt = \frac{1}{z}\left[\int_0^z 1\cdot dt - 2(1-i)\int_0^z t\,dt\right]$$

$$= \frac{1}{z}\left[z - 2(1-i)\frac{z^2}{2}\right] = 1 - (1-i)z.$$

Using Corollary 2.1, we get: Let

$$\operatorname{Re} \frac{-2(1-i)z}{1-2(1-i)z} > 0, \quad z \in U.$$

Then we have

$$\operatorname{Re} \frac{z \cdot L'[{}_1F_1(-1, \frac{1+i}{4}; z)]}{L[{}_1F_1(-1, \frac{1+i}{4}; z)]} = \operatorname{Re} \frac{-(1-i)z}{1-(1-i)z} > 0, \quad z \in U.$$

Indeed,

$$\begin{aligned}
\operatorname{Re} \frac{z_1F_1'(-1, \frac{1+i}{4}; z)}{{}_1F_1(-1, \frac{1+i}{4}; z)} &= \operatorname{Re} \frac{-2(1-i)z}{1-2(1-i)z} \\
&= \operatorname{Re}\left[1 - \frac{1}{1-2(1-i)z}\right] \\
&= \operatorname{Re}\left[1 - \frac{1}{1-2(1-i)(\cos\alpha + i\sin\alpha)}\right] \\
&= \operatorname{Re}\left[1 - \frac{1}{1-2\cos\alpha - 2\sin\alpha + 2i(\cos\alpha - \sin\alpha)}\right] \\
&= \operatorname{Re}\left[1 - \frac{1-2\cos\alpha - 2\sin\alpha - 2i(\cos\alpha - \sin\alpha)}{(1-2\cos\alpha - 2\sin\alpha)^2 + 4(\cos\alpha - \sin\alpha)^2}\right] \\
&= 1 - \frac{1-2\cos\alpha - 2\sin\alpha}{(1-2\cos\alpha - 2\sin\alpha)^2 + 4(\cos\alpha - \sin\alpha)^2} \\
&= \frac{9 - 4\sin\alpha - 4\cos\alpha - 1 + 2\cos\alpha + 2\sin\alpha}{(1-2\cos\alpha - 2\sin\alpha)^2 + 4(\cos\alpha - \sin\alpha)^2} \\
&= \frac{4 + 2(1-\sin\alpha) + 2(1-\cos\alpha)}{(1-2\cos\alpha - 2\sin\alpha)^2 + 4(\cos\alpha - \sin\alpha)^2} > 0.
\end{aligned}$$

We evaluate now:

$$\begin{aligned}
\operatorname{Re} \frac{z \cdot L'[{}_1F_1(-1, \frac{1+i}{4}; z)]}{L[{}_1F_1(-1, \frac{1+i}{4}; z)]} &= \operatorname{Re} \frac{-(1-i)z}{1-(1-i)z} \\
&= \operatorname{Re} \frac{1-(1-i)z-1}{1-(1-i)z} \\
&= \operatorname{Re}\left[1 - \frac{1}{1-(1-i)z}\right] \\
&= \operatorname{Re}\left[1 - \frac{1}{1-(1-i)(\cos\alpha + i\sin\alpha)}\right] \\
&= \operatorname{Re}\left[1 - \frac{1}{1-\cos\alpha - \sin\alpha + i(\cos\alpha - \sin\alpha)}\right] \\
&= \operatorname{Re}\left[1 - \frac{1-\cos\alpha - \sin\alpha - i(\cos\alpha - \sin\alpha)}{(1-\cos\alpha - \sin\alpha)^2 + (\cos\alpha - \sin\alpha)^2}\right] \\
&= 1 - \frac{1-\cos\alpha - \sin\alpha}{(1-\cos\alpha - \sin\alpha)^2 + (\cos\alpha - \sin\alpha)^2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{3 - 2\cos\alpha - 2\sin\alpha - 1 + \cos\alpha + \sin\alpha}{(1 - \cos\alpha - \sin\alpha)^2 + (\cos\alpha - \sin\alpha)^2} \\
&= \frac{2 - \cos\alpha - \sin\alpha}{(1 - \cos\alpha - \sin\alpha)^2 + (\cos\alpha - \sin\alpha)^2} \\
&= \frac{(1 - \cos\alpha) + (1 - \sin\alpha)}{(1 - \cos\alpha - \sin\alpha)^2 + (\cos\alpha - \sin\alpha)^2} > 0.
\end{aligned}
$$

3 Discussion

Using a confluent or Kummer hypergeometric function, two integral operators are defined, and some properties related to their ability of preserving starlikeness and convexity are stated and proved. The original subordination results presented in this paper are also given as differential inequalities in the complex plane which are interpreted in terms of inclusion relations involving subsets of the complex plane. An example is included so that it is obvious how the original results are applied. The newly introduced operators could be used for many purposes, just as operators have generated interesting outcome in geometric function theory during time being studied in many aspects. Hopefully, the original results contained here would stimulate researchers' imagination and inspire them just as all the operators introduced before in studies related to functions of a complex variable have done. Other properties related to them could be investigated, and also they could prove useful in introducing special classes of functions based on those properties.

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