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On generalized Ostrowski, Simpson and Trapezoidal type inequalities for co-ordinated convex functions via generalized fractional integrals

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Abstract

In this study, we prove an identity for twice partially differentiable mappings involving the double generalized fractional integral and some parameters. By using this established identity, we offer some generalized inequalities for differentiable co-ordinated convex functions with a rectangle in the plane $\mathbb{R}^2$. Furthermore, by special choice of parameters in our main results, we obtain several well-known inequalities such as the Ostrowski inequality, trapezoidal inequality, and the Simpson inequality for Riemann and Riemann–Liouville fractional integrals.

MSC: 26D07; 26D10; 26D15; 26B15; 26B25

Keywords: Simpson inequality; Ostrowski inequality; Co-ordinated convex function; Generalized fractional integrals

1 Introduction

The inequality of Ostrowski gives us an estimate for the deviation of the values of a function from its mean value. More precisely, let $f:[a,b]\to\mathbb{R}$ be a differentiable function with bounded derivative. Then the following integral inequality:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{4}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)\|f'\|_\infty \tag{1.1}$$

is valid for every $x\in[a,b]$, which is proved by Ostrowski in 1938. In addition to this, the constant $1/4$ is the best possible.

Simpson's inequality plays significant role in many areas of mathematics. To be more precise, the classical Simpson's inequality is expressed as follows for four times continuously differentiable functions.

Theorem 1 ([12]) *Suppose for a mapping $f:[a,b]\to\mathbb{R}$ which is four times continuously differentiable on (a,b), and suppose also $\|f^{(4)}\|_\infty=\sup_{x\in(a,b)}|f^{(4)}(x)|<\infty$. Then one has the* *inequality*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{1}{2880}\left\|f^{(4)}\right\|_\infty(b-a)^4.$$

Over the years, many variations of Ostrowski and Simpson type inequalities have been studied for various function classes, such as convex functions, bounded functions, functions of bounded variation, and so on. Specifically, since convexity theory is an effective and powerful way to solve a large number of problems from different branches of pure and applied mathematics, many papers have been dedicated to the Simpson inequality for convex functions. For instance, Alomari et al. proved some Ostrowski type inequalities for s-convex functions in [4]. Moreover, Ostrowski type inequalities were studied for different kinds of convexities such as m-convex functions [20], (α,m)-convex functions [29], and h-convex functions [26, 44]. In [41], Set first obtained the Riemann–Liouville fractional version of the Ostrowski inequality for s-convex functions. In addition to this, many researchers focused on establishing Ostrowski type inequalities for certain fractional integral operators, such as k-Riemann–Liouville fractional integrals [16], local fractional integrals [34], Raina fractional integrals [2], generalized k-g-fractional integrals [10] and ψ-Hilfer fractional integrals [5]. On the other hand, several Ostrowski inequalities for co-ordinated convex mapping in involving double Riemann integrals and double Riemann–Liouville fractional integrals are introduced in [24] and [23], respectively.

Some authors established several Simpson type inequalities for differentiable and twice differentiable convex functions [3, 36–38]. Ozdemir et al. proved Simpson type inequalities for co-ordinated convex functions in [28]. In [8, 19, 42], authors obtained some new Simpson inequalities for Riemann–Liouville fractional integrals. Furthermore, a number of papers were devoted to Simpson inequalities for other fractional integrals or for functions belong to other convex classes such as p-convex function [1], h-convex function [27], preinvex functions [31], (m,h_1,h_2)-preinvexity [31], and generalized harmonic convex functions [43].

The inequalities, introduced by C. Hermite and J. Hadamard for convex functions, are of considerable significance in the literature. These inequalities state that, if $\digamma:I\to\mathbb{R}$ is a convex function on the interval I of real numbers and $a,b\in I$ with $a<b$, then

$$\digamma\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b\digamma(x)\,dx\leq\frac{\digamma(a)+\digamma(b)}{2}. \tag{1.2}$$

If $\digamma$ is concave, then both inequalities in (1.2) hold in the reverse direction.

Over the years, considerable number of studies have been focused on obtaining trapezoidal and midpoint type inequalities which give bounds for the right-hand side and left-hand side of the inequality (1.2), respectively. For example, Dragomir and Agarwal first obtained trapezoidal inequalities for convex functions in [11] and Kirmacı first established midpoint inequalities for convex functions in [21]. In [40], Sarikaya et al. generalized the inequalities (1.2) for fractional integrals and the authors also proved some corresponding trapezoidal type inequalities. Iqbal et al. presented some fractional midpoint type inequalities for convex functions in [18]. On the other hand, Dragomir proved Hermite–Hadamard inequalities for co-ordinated convex mappings in [9]. The midpoint and trapezoidal type inequalities for co-ordinated convex functions were established in [22] and

[39], respectively. Moreover, Sarikaya obtained fractional Hermite–Hadamard inequalities and fractional trapezoidal inequalities for functions with two variables in [32]. Tunç et al. presented some fractional midpoint type inequalities for co-ordinated convex functions in [46]. In [35], Sarikaya and Ertuğral first introduced new fractional integrals which are called generalized fractional integrals. In addition, they proved Hermite–Hadamard inequalities and several trapezoidal and midpoint type inequalities for generalized fractional integrals. Moreover, Budak et al. proved Midpoint type inequalities and extensions of Hermite–Hadamard inequalities in the papers [6] and [7], respectively. In [13], Ertuğral and Sarikaya presented some Simpson type inequalities for these fractional integral operators. For some of other papers on inequalities for generalized fractional integrals, we refer to [17, 48]. On the other hand, Turkay et al. described the generalized fractional integrals for functions with two variables. These authors presented Hermite–Hadamard and trapezoidal type inequalities for this kind of fractional integrals in [47]. For the other similar inequalities, we refer to [14, 15, 25, 30, 33, 45].

The aims of this paper is to establish some generalized inequalities for co-ordinated convex functions involving generalized fractional integrals. The general structure of the paper consists of five sections including an introduction. The remaining part of the paper proceeds as follows: In Sect. 2, we give the definitions of generalized fractional integrals and relations between generalized fractional integrals and other type of fractional integrals. In Sect. 3, an identity involving some parameters are proved for partially differentiable functions. Then we establish several generalized inequalities for mappings whose partially derivatives in absolute value are co-ordinated convex. With the special choice of the given parameters, we show that our results reproduce the results proved in the earlier work and we also give some new trapezoidal and Simpson type inequalities in Sect. 4. At the end of the paper, some conclusions and further directions of research are discussed in Sect. 5.

2 Generalized fractional Integrals

In this section, we summarize the generalized fractional integrals defined by Sarikaya and Ertuğral in [35].

Let us define a function $\varphi : [0,\infty) \to [0,\infty)$ satisfying the following condition:

$$\int_0^1 \frac{\varphi(t)}{t}\,dt < \infty.$$

We consider the following left-sided and right-sided generalized fractional integral operators:

$$_{a+}I_{\varphi}f(x) = \int_a^x \frac{\varphi(x-t)}{x-t} f(t)\,dt, \quad x > a, \tag{2.1}$$

and

$$_{b-}I_{\varphi}f(x) = \int_x^b \frac{\varphi(t-x)}{t-x} f(t)\,dt, \quad x < b, \tag{2.2}$$

respectively.

Some forms of fractional integrals, namely Riemann–Liouville fractional integrals, k-Riemann–Liouville fractional integrals, Katugampola fractional integrals, conformable fractional integrals, and Hadamard fractional integrals, are the most significant features of generalized fractional integrals. These important special cases of the integral operators (2.1) and (2.2) are mentioned below:

Remark 1 If we choose $\varphi(t)=t$, the operators (2.1) and (2.2) reduce to the Riemann integral.

Remark 2 Considering $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$ and $\alpha>0$, the operators (2.1) and (2.2) reduce to the Riemann–Liouville fractional integrals $J_{a+}^{\alpha}f(x)$ and $J_{b-}^{\alpha}f(x)$, respectively. Here, Γ is Gamma function.

Remark 3 For $\varphi(t)=\frac{1}{k\Gamma_k(\alpha)}t^{\frac{\alpha}{k}}$ and $\alpha,k>0$, the operators (2.1) and (2.2) reduce to the k-Riemann–Liouville fractional integrals $J_{a+,k}^{\alpha}f(x)$ and $J_{b-,k}^{\alpha}f(x)$, respectively. Here, Γ_k is the k-Gamma function.

Generalized double fractional integrals are given by Turkay et al. in [47], as follows.

Definition 1 The generalized double fractional integrals ${}_{a+,c+}I_{\varphi,\psi}$, ${}_{a+,d-}I_{\varphi,\psi}$, ${}_{b-,c+}I_{\varphi,\psi}$, ${}_{b-,d-}I_{\varphi,\psi}$ are defined by

$$ {}_{a+,c+}I_{\varphi,\psi}f(x,y)=\int_a^x\int_c^y\frac{\varphi(x-t)}{x-t}\frac{\psi(y-s)}{y-s}f(t,s)\,ds\,dt,\quad x>a,y>c, \tag{2.3}$$

$$ {}_{a+,d-}I_{\varphi,\psi}f(x,y)=\int_a^x\int_y^d\frac{\varphi(x-t)}{x-t}\frac{\psi(s-y)}{s-y}f(t,s)\,ds\,dt,\quad x>a,y<d, \tag{2.4}$$

$$ {}_{b-,c+}I_{\varphi,\psi}f(x,y)=\int_x^b\int_c^y\frac{\varphi(t-x)}{t-x}\frac{\psi(y-s)}{y-s}f(t,s)\,ds\,dt,\quad x<b,y>c, \tag{2.5}$$

and

$$ {}_{b-,d-}I_{\varphi,\psi}f(x,y)=\int_x^b\int_y^d\frac{\varphi(t-x)}{t-x}\frac{\psi(s-y)}{s-y}f(t,s)\,ds\,dt,\quad x<b,y<d. \tag{2.6}$$

Here, $f\in L_1([a,b]\times[c,d])$ and the functions $\varphi,\psi:[0,\infty)\to[0,\infty)$ satisfy, respectively, $\int_0^1\frac{\varphi(t)}{t}\,dt<\infty$ and $\int_0^1\frac{\psi(s)}{s}\,ds<\infty$.

By using Definition 1, well-known fractional integrals can be obtained by some special choices. For example;

(1) If we choose $\varphi(t)=t$ and $\psi(s)=s$, the operators (2.3), (2.4), (2.5) and (2.6) reduce to the double Riemann integral.
(2) Considering $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$, $\psi(s)=\frac{s^{\beta}}{\Gamma(\beta)}$, then, for $\alpha,\beta>0$, the operators (2.3), (2.4), (2.5) and (2.6) reduce to the Riemann–Liouville fractional integrals $J_{a+,c+}^{\alpha,\beta}f(x,y)$, $J_{a+,d-}^{\alpha,\beta}f(x,y)$ $J_{b-,c+}^{\alpha,\beta}f(x,y)$ and $J_{b-,d-}^{\alpha,\beta}f(x,y)$, respectively.
(3) For $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $\psi(s)=\frac{s^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$, for $\alpha,\beta,k>0$, the operators (2.3), (2.4), (2.5) and (2.6) reduce to the k-Riemann–Liouville fractional integrals $J_{a+,c+}^{\alpha,\beta,k}f(x,y)$, $J_{a+,d-}^{\alpha,\beta,k}f(x,y)$, $J_{b-,c+}^{\alpha,\beta,k}f(x,y)$ and $J_{b-,d-}^{\alpha,\beta,k}f(x,y)$, respectively.

For our work to be done in two-dimensional space, we will need the following descriptions.

3 An identity for Generalized double fractional integrals

Throughout this study, we assume that $\lambda_i, \mu_i \geq 0$, $i = 1,2$ and for brevity, we define

$$\Lambda_1(x,t) = \int_0^t \frac{\varphi((b-x)u)}{u}\,du, \qquad \Delta_1(x,t) = \int_0^t \frac{\varphi((x-a)u)}{u}\,du, \tag{3.1}$$

and

$$\Lambda_2(y,s) = \int_0^s \frac{\psi((d-y)u)}{u}\,du, \qquad \Delta_2(y,s) = \int_0^s \frac{\psi((y-c)u)}{u}\,du. \tag{3.2}$$

Lemma 1 *Let $f : \Delta := [a,b] \times [c,d] \to \mathbb{R}$ be an absolutely continuous function on Δ such that the partial derivative of order $\frac{\partial^2 f(t,s)}{\partial t \partial s}$ exist for all $(t,s) \in \Delta$. Then the following equality for generalized fractional integrals holds*:

$$\begin{aligned}
&\Omega(a,b,x;c,d,y) \\
&\quad= (b-x)(d-y)\int_0^1\int_0^1 \big(\lambda_1\Lambda_1(x,1) - \Lambda_1(x,t)\big)\big(\lambda_2\Lambda_2(y,1) - \Lambda_2(y,s)\big) \\
&\qquad\times \frac{\partial^2}{\partial t\partial s} f\big(tx + (1-t)b, sy + (1-s)d\big)\,ds\,dt \\
&\qquad- (b-x)(y-c)\int_0^1\int_0^1 \big(\lambda_1\Lambda_1(x,1) - \Lambda_1(x,t)\big)\big(\mu_2\Delta_2(y,1) - \Delta_2(y,s)\big) \\
&\qquad\times \frac{\partial^2}{\partial t\partial s} f\big(tx + (1-t)b, sy + (1-s)c\big)\,ds\,dt \\
&\qquad- (x-a)(d-y)\int_0^1\int_0^1 \big(\mu_1\Delta_1(x,1) - \Delta_1(x,t)\big)\big(\lambda_2\Lambda_2(y,1) - \Lambda_2(y,s)\big) \\
&\qquad\times \frac{\partial^2}{\partial t\partial s} f\big(tx + (1-t)a, sy + (1-s)d\big)\,ds\,dt \\
&\qquad+ (x-a)(y-c)\int_0^1\int_0^1 \big(\mu_1\Delta_1(x,1) - \Delta_1(x,t)\big)\big(\mu_2\Delta_2(y,1) - \Delta_2(y,s)\big) \\
&\qquad\times \frac{\partial^2}{\partial t\partial s} f\big(tx + (1-t)a, sy + (1-s)c\big)\,ds\,dt,
\end{aligned}$$

where

$$\begin{aligned}
&\Omega(a,b,x;c,d,y) \\
&\quad= \Lambda_1(x,1)\Lambda_2(y,1)\big[(1-\lambda_1)(1-\lambda_2)f(x,y) + (1-\lambda_1)\lambda_2 f(x,d) \\
&\qquad+ \lambda_1(1-\lambda_2)f(b,y) + \lambda_1\lambda_2 f(b,d)\big] \\
&\qquad+ \Lambda_1(x,1)\Delta_2(y,1)\big[(1-\lambda_1)(1-\mu_2)f(x,y) + (1-\lambda_1)\mu_2 f(x,c) \\
&\qquad+ \lambda_1(1-\mu_2)f(b,y) + \lambda_1\mu_2 f(b,c)\big] \\
&\qquad+ \Delta_1(x,1)\Lambda_2(y,1)\big[(1-\mu_1)(1-\lambda_2)f(x,y) + (1-\mu_1)\lambda_2 f(x,d) \\
&\qquad+ \mu_1(1-\lambda_2)f(a,y) + \mu_1\lambda_2 f(a,d)\big] \\
&\qquad+ \Delta_1(x,1)\Delta_2(y,1)\big[(1-\mu_1)(1-\mu_2)f(x,y) + (1-\mu_1)\mu_2 f(x,c) \\
&\qquad+ \mu_1(1-\mu_2)f(a,y) + \mu_1\mu_2 f(a,c)\big]
\end{aligned}$$

$$
\begin{aligned}
&-\big((1-\lambda_2)\Lambda_2(y,1)+(1-\mu_2)\Delta_2(y,1)\big)\big[{}_{x+}I_{\varphi}f(b,y)+{}_{x-}I_{\varphi}f(a,y)\big]\\
&-\big((1-\lambda_1)\Lambda_1(x,1)+(1-\mu_1)\Delta_1(x,1)\big)\big[{}_{y+}I_{\psi}f(x,d)+{}_{y-}I_{\psi}f(x,c)\big]\\
&-\lambda_2\Lambda_2(y,1)\big[{}_{x+}I_{\varphi}f(b,d)+{}_{x-}I_{\varphi}f(a,d)\big]-\mu_2\Delta_2(y,1)\big[{}_{x+}I_{\varphi}f(b,c)+{}_{x-}I_{\varphi}f(a,c)\big]\\
&-\lambda_1\Lambda_1(x,1)\big[{}_{y+}I_{\psi}f(b,d)+{}_{y-}I_{\psi}f(b,c)\big]-\mu_1\Delta_1(x,1)\big[{}_{y+}I_{\psi}f(a,d)+{}_{y-}I_{\psi}f(a,c)\big]\\
&+{}_{x+,y+}I_{\varphi,\psi}f(b,d)+{}_{x+,y-}I_{\varphi,\psi}f(b,c)+{}_{x-,y+}I_{\varphi,\psi}f(a,d)+{}_{x-,y-}I_{\varphi,\psi}f(a,c).
\end{aligned}
$$

Proof By using integration by parts, we have

$$
\begin{aligned}
H_1 &= \int_0^1\int_0^1 \big(\lambda_1\Lambda_1(x,1)-\Lambda_1(x,t)\big)\big(\lambda_2\Lambda_2(y,1)-\Lambda_2(y,s)\big)\frac{\partial^2}{\partial t\partial s}\\
&\quad\times f\big(tx+(1-t)b,sy+(1-s)d\big)\,ds\,dt\\
&=\frac{\Lambda_1(x,1)\Lambda_2(y,1)}{(b-x)(d-y)}\big[(1-\lambda_1)(1-\lambda_2)f(x,y)+(1-\lambda_1)\lambda_2 f(x,d)\\
&\quad+\lambda_1(1-\lambda_2)f(b,y)+\lambda_1\lambda_2 f(b,d)\big]\\
&\quad-\frac{(1-\lambda_2)\Lambda_2(y,1)}{(b-x)(d-y)}\,{}_{x+}I_{\varphi}f(b,y)-\frac{\lambda_2\Lambda_2(y,1)}{(b-x)(d-y)}\,{}_{x+}I_{\varphi}f(b,d)\\
&\quad-\frac{(1-\lambda_1)\Lambda_1(x,1)}{(b-x)(d-y)}\,{}_{y+}I_{\psi}f(x,d)-\frac{\lambda_1\Lambda_1(x,1)}{(b-x)(d-y)}\,{}_{y+}I_{\psi}f(b,d)\\
&\quad+\frac{1}{(b-x)(d-y)}\,{}_{x+,y+}I_{\varphi,\psi}f(b,d),
\end{aligned}
\tag{3.3}
$$

$$
\begin{aligned}
H_2 &= \int_0^1\int_0^1 \big(\lambda_1\Lambda_1(x,1)-\Lambda_1(x,t)\big)\big(\mu_2\Delta_2(y,1)-\Delta_2(y,s)\big)\frac{\partial^2}{\partial t\partial s}\\
&\quad\times f\big(tx+(1-t)b,sy+(1-s)c\big)\,ds\,dt\\
&=-\frac{\Lambda_1(x,1)\Lambda_2(y,1)}{(b-x)(y-c)}\big[(1-\lambda_1)(1-\mu_2)f(x,y)+(1-\lambda_1)\mu_2 f(x,c)\\
&\quad+\lambda_1(1-\mu_2)f(b,y)+\lambda_1\mu_2 f(b,c)\big]\\
&\quad+\frac{(1-\mu_2)\Delta_2(y,1)}{(b-x)(y-c)}\,{}_{x+}I_{\varphi}f(b,y)+\frac{\mu_2\Delta_2(y,1)}{(b-x)(y-c)}\,{}_{x+}I_{\varphi}f(b,c)\\
&\quad+\frac{(1-\lambda_1)\Lambda_1(x,1)}{(b-x)(y-c)}\,{}_{y-}I_{\psi}f(x,c)+\frac{\lambda_1\Lambda_1(x,1)}{(b-x)(y-c)}\,{}_{y-}I_{\psi}f(b,c)\\
&\quad-\frac{1}{(b-x)(y-c)}\,{}_{x+,y-}I_{\varphi,\psi}f(b,c),
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
H_3 &= \int_0^1\int_0^1 \big(\mu_1\Delta_1(x,1)-\Delta_1(x,t)\big)\big(\lambda_2\Lambda_2(y,1)-\Lambda_2(y,s)\big)\frac{\partial^2}{\partial t\partial s}\\
&\quad\times f\big(tx+(1-t)a,sy+(1-s)d\big)\,ds\,dt\\
&=-\frac{\Delta_1(x,1)\Lambda_2(y,1)}{(x-a)(d-y)}\big[(1-\mu_1)(1-\lambda_2)f(x,y)+(1-\mu_1)\lambda_2 f(x,d)\\
&\quad+\mu_1(1-\lambda_2)f(a,y)+\mu_1\lambda_2 f(a,d)\big]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(1-\lambda_2)\Lambda_2(y,1)}{(x-a)(d-y)}\,{}_{x-}I_{\varphi}f(a,y)+\frac{\lambda_2\Lambda_2(y,1)}{(x-a)(d-y)}\,{}_{x-}I_{\varphi}f(a,d) \\
&+\frac{(1-\mu_1)\Delta_1(x,1)}{(x-a)(d-y)}\,{}_{y+}I_{\psi}f(x,d)+\frac{\mu_1\Delta_1(x,1)}{(x-a)(d-y)}\,{}_{y+}I_{\psi}f(a,d) \\
&-\frac{1}{(x-a)(d-y)}\,{}_{x-,y+}I_{\varphi,\psi}f(a,d),
\end{aligned}
\tag{3.5}
$$

and

$$
\begin{aligned}
H_4 &= \int_0^1\int_0^1\bigl(\mu_1\Delta_1(x,1)-\Delta_1(x,t)\bigr)\bigl(\mu_2\Delta_2(y,1)-\Delta_2(y,s)\bigr)\frac{\partial^2}{\partial t\partial s} \\
&\quad\times f\bigl(tx+(1-t)a,sy+(1-s)c\bigr)\,ds\,dt \\
&= \frac{\Delta_1(x,1)\Delta_2(y,1)}{(x-a)(y-c)}\bigl[(1-\mu_1)(1-\mu_2)f(x,y)+(1-\mu_1)\mu_2 f(x,c) \\
&\quad+\mu_1(1-\mu_2)f(a,y)+\mu_1\mu_2 f(a,c)\bigr] \\
&\quad-\frac{(1-\mu_2)\Delta_2(y,1)}{(x-a)(y-c)}\,{}_{x-}I_{\varphi}f(a,y)-\frac{\mu_2\Delta_2(y,1)}{(x-a)(y-c)}\,{}_{x-}I_{\varphi}f(a,c) \\
&\quad-\frac{(1-\mu_1)\Delta_1(x,1)}{(x-a)(y-c)}\,{}_{y-}I_{\psi}f(x,c)-\frac{\mu_1\Delta_1(x,1)}{(x-a)(y-c)}\,{}_{y-}I_{\psi}f(a,c) \\
&\quad+\frac{1}{(x-a)(y-c)}\,{}_{x-,y-}I_{\varphi,\psi}f(a,c).
\end{aligned}
\tag{3.6}
$$

By using Eqs. (3.3)–(3.6), we have

$$
\begin{aligned}
&(b-x)(d-y)H_1-(b-x)(y-c)H_2-(x-a)(d-y)H_3+(x-a)(y-c)H_4 \\
&\quad=\Omega(a,b,x;c,d,y),
\end{aligned}
$$

which completes the proof of Lemma 1. □

Corollary 1 *In Lemma 1, if we choose $\varphi(t)=t$ and $\psi(s)=s$ for all $(t,s)\in\triangle$, then we obtain the equality for the Riemann integral*

$$
\begin{aligned}
&\aleph(a,b,x;c,d,y) \\
&\quad=(b-x)^2(d-y)^2\int_0^1\int_0^1(\lambda_1-t)(\lambda_2-s)\frac{\partial^2}{\partial t\partial s}f\bigl(tx+(1-t)b,sy+(1-s)d\bigr)\,ds\,dt \\
&\qquad-(b-x)^2(y-c)^2\int_0^1\int_0^1(\lambda_1-t)(\mu_2-s)\frac{\partial^2}{\partial t\partial s}f\bigl(tx+(1-t)b,sy+(1-s)c\bigr)\,ds\,dt \\
&\qquad-(x-a)^2(d-y)^2\int_0^1\int_0^1(\mu_1-t)(\lambda_2-s)\frac{\partial^2}{\partial t\partial s}f\bigl(tx+(1-t)a,sy+(1-s)d\bigr)\,ds\,dt \\
&\qquad+(x-a)^2(y-c)^2\int_0^1\int_0^1(\mu_1-t)(\mu_2-s)\frac{\partial^2}{\partial t\partial s}f\bigl(tx+(1-t)a,sy+(1-s)c\bigr)\,ds\,dt.
\end{aligned}
$$

Here,

$$\begin{aligned}
&\aleph(a,b,x;c,d,y)\\
&= (b-x)(d-y)\big[(1-\lambda_1)(1-\lambda_2)f(x,y) + (1-\lambda_1)\lambda_2 f(x,d)\\
&\quad + \lambda_1(1-\lambda_2)f(b,y) + \lambda_1\lambda_2 f(b,d)\big]\\
&\quad + (b-x)(y-c)\big[(1-\lambda_1)(1-\mu_2)f(x,y) + (1-\lambda_1)\mu_2 f(x,c)\\
&\quad + \lambda_1(1-\mu_2)f(b,y) + \lambda_1\mu_2 f(b,c)\big]\\
&\quad + (x-a)(d-y)\big[(1-\mu_1)(1-\lambda_2)f(x,y) + (1-\mu_1)\lambda_2 f(x,d)\\
&\quad + \mu_1(1-\lambda_2)f(a,y) + \mu_1\lambda_2 f(a,d)\big]\\
&\quad + (x-a)(y-c)\big[(1-\mu_1)(1-\mu_2)f(x,y) + (1-\mu_1)\mu_2 f(x,c)\\
&\quad + \mu_1(1-\mu_2)f(a,y) + \mu_1\mu_2 f(a,c)\big]\\
&\quad - \big((1-\lambda_2)(d-y) + (1-\mu_2)(y-c)\big)\int_a^b f(t,y)\,dt - \big((1-\lambda_1)(b-x)\\
&\quad + (1-\mu_1)(x-a)\big)\int_c^d f(x,s)\,ds\\
&\quad - \lambda_2(d-y)\int_a^b f(t,d)\,dt - \mu_2(y-c)\int_a^b f(t,c)\,dt\\
&\quad - \lambda_1(b-x)\int_c^d f(b,s)\,dt - \mu_1(x-a)\int_c^d f(a,s)\,ds + \int_a^b\int_c^d f(t,s)\,ds\,dt.
\end{aligned}$$

Corollary 2 *In Lemma 1, let us consider $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ and $\psi(s) = \frac{s^\beta}{\Gamma(\beta)}$ for all $(t,s) \in \triangle$. Then we obtain the equality for Riemann–Liouville fractional integrals*

$$\begin{aligned}
&\Phi(a,b,x;c,d,y)\\
&= (b-x)^{\alpha+1}(d-y)^{\beta+1}\int_0^1\int_0^1 \big(\lambda_1 - t^\alpha\big)\big(\lambda_2 - s^\beta\big)\frac{\partial^2}{\partial t\partial s}\\
&\quad \times f\big(tx+(1-t)b, sy+(1-s)d\big)\,ds\,dt\\
&\quad - (b-x)^{\alpha+1}(y-c)^{\beta+1}\int_0^1\int_0^1 \big(\lambda_1 - t^\alpha\big)\big(\mu_2 - s^\beta\big)\frac{\partial^2}{\partial t\partial s}\\
&\quad \times f\big(tx+(1-t)b, sy+(1-s)c\big)\,ds\,dt\\
&\quad - (x-a)^{\alpha+1}(d-y)^{\beta+1}\int_0^1\int_0^1 \big(\mu_1 - t^\alpha\big)\big(\lambda_2 - s^\beta\big)\frac{\partial^2}{\partial t\partial s}\\
&\quad \times f\big(tx+(1-t)a, sy+(1-s)d\big)\,ds\,dt\\
&\quad + (x-a)^{\alpha+1}(y-c)^{\beta+1}\int_0^1\int_0^1 \big(\mu_1 - t^\alpha\big)\big(\mu_2 - s^\beta\big)\frac{\partial^2}{\partial t\partial s}\\
&\quad \times f\big(tx+(1-t)a, sy+(1-s)c\big)\,ds\,dt,
\end{aligned}$$

where

$$
\begin{aligned}
&\Phi(a,b,x;c,d,y)\\
&\quad=(b-x)^{\alpha}(d-y)^{\beta}\bigl[(1-\lambda_1)(1-\lambda_2)f(x,y)+(1-\lambda_1)\lambda_2 f(x,d)\\
&\qquad+\lambda_1(1-\lambda_2)f(b,y)+\lambda_1\lambda_2 f(b,d)\bigr]\\
&\qquad+(b-x)^{\alpha}(y-c)^{\beta}\bigl[(1-\lambda_1)(1-\mu_2)f(x,y)+(1-\lambda_1)\mu_2 f(x,c)\\
&\qquad+\lambda_1(1-\mu_2)f(b,y)+\lambda_1\mu_2 f(b,c)\bigr]\\
&\qquad+(x-a)^{\alpha}(d-y)^{\beta}\bigl[(1-\mu_1)(1-\lambda_2)f(x,y)+(1-\mu_1)\lambda_2 f(x,d)\\
&\qquad+\mu_1(1-\lambda_2)f(a,y)+\mu_1\lambda_2 f(a,d)\bigr]\\
&\qquad+(x-a)^{\alpha}(y-c)^{\beta}\bigl[(1-\mu_1)(1-\mu_2)f(x,y)+(1-\mu_1)\mu_2 f(x,c)\\
&\qquad+\mu_1(1-\mu_2)f(a,y)+\mu_1\mu_2 f(a,c)\bigr]\\
&\qquad-\Gamma(\alpha+1)\bigl((1-\lambda_2)(d-y)^{\beta}+(1-\mu_2)(y-c)^{\beta}\bigr)\bigl[J_{x+}^{\alpha}f(b,y)+J_{x-}^{\alpha}f(a,y)\bigr]\\
&\qquad-\Gamma(\beta+1)\bigl((1-\lambda_1)(b-x)^{\alpha}+(1-\mu_1)(x-a)^{\alpha}\bigr)\bigl[J_{y+}^{\beta}f(x,d)+J_{y-}^{\beta}f(x,c)\bigr]\\
&\qquad-\lambda_2\Gamma(\alpha+1)(d-y)^{\beta}\bigl[J_{x+}^{\alpha}f(b,d)+J_{x-}^{\alpha}f(a,d)\bigr]\\
&\qquad-\mu_2\Gamma(\alpha+1)(y-c)^{\beta}\bigl[J_{x+}^{\alpha}f(b,c)+J_{x-}^{\alpha}f(a,c)\bigr]\\
&\qquad-\lambda_1\Gamma(\beta+1)(b-x)^{\alpha}\bigl[J_{y+}^{\beta}f(b,d)+J_{y-}^{\beta}f(b,c)\bigr]\\
&\qquad-\mu_1\Gamma(\beta+1)(x-a)^{\alpha}\bigl[J_{y+}^{\beta}f(a,d)+J_{y-}^{\beta}f(a,c)\bigr]\\
&\qquad+\Gamma(\alpha+1)\Gamma(\beta+1)\bigl[J_{x+,y+}^{\alpha,\beta}f(b,d)+J_{x+,y-}^{\alpha,\beta}f(b,c)+J_{x-,y+}^{\alpha,\beta}f(a,d)+J_{x-,y-}^{\alpha,\beta}f(a,c)\bigr].
\end{aligned}
$$

4 New inequalities for Generalized Fractional Integrals

Theorem 2 *Assume that the assumptions of Lemma 1 hold. Assume also that the mapping* $|\frac{\partial^2 f}{\partial t\partial s}|$ *is co-ordinated convex on* Δ. *Then we obtain the following inequality for generalized fractional integrals*:

$$
\begin{aligned}
&\bigl|\Omega(a,b,x;c,d,y)\bigr|\\
&\quad\le(b-x)(d-y)\left[A_1B_1\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|+A_1B_2\left|\frac{\partial^2}{\partial t\partial s}f(x,d)\right|\right.\\
&\qquad\left.+A_2B_1\left|\frac{\partial^2}{\partial t\partial s}f(b,y)\right|+A_2B_2\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|\right]\\
&\qquad+(b-x)(y-c)\left[A_1B_4\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|+A_1B_3\left|\frac{\partial^2}{\partial t\partial s}f(x,c)\right|\right.\\
&\qquad\left.+A_2B_4\left|\frac{\partial^2}{\partial t\partial s}f(b,y)\right|+A_2B_3\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|\right]\\
&\qquad+(x-a)(d-y)\left[A_4B_1\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|+A_4B_2\left|\frac{\partial^2}{\partial t\partial s}f(x,d)\right|\right.\\
&\qquad\left.+A_3B_1\left|\frac{\partial^2}{\partial t\partial s}f(a,y)\right|+A_3B_2\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|\right]\\
&\qquad+(x-a)(y-c)\left[A_4B_4\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|+A_4B_3\left|\frac{\partial^2}{\partial t\partial s}f(x,c)\right|\right.
\end{aligned}
$$

$$+ A_3B_4\left|\frac{\partial^2}{\partial t\partial s}f(a,y)\right| + A_3B_3\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|\Bigg].$$

Here,

$$\begin{cases} A_1 = \int_0^1 t|\lambda_1\Lambda_1(x,1) - \Lambda_1(x,t)|\,dt, \\ A_2 = \int_0^1 (1-t)|\lambda_1\Lambda_1(x,1) - \Lambda_1(x,t)|\,dt, \\ A_3 = \int_0^1 (1-t)|\mu_1\Delta_1(x,1) - \Delta_1(x,t)|\,dt, \\ A_4 = \int_0^1 t|\mu_1\Delta_1(x,1) - \Delta_1(x,t)|\,dt, \end{cases} \tag{4.1}$$

and

$$\begin{cases} B_1 = \int_0^1 s|\lambda_2\Lambda_2(y,1) - \Lambda_2(y,s)|\,ds, \\ B_2 = \int_0^1 (1-s)|\lambda_2\Lambda_2(y,1) - \Lambda_2(y,s)|\,ds, \\ B_3 = \int_0^1 (1-s)|\mu_2\Delta_2(y,1) - \Delta_2(y,s)|\,ds, \\ B_4 = \int_0^1 s|\mu_2\Delta_2(y,1) - \Delta_2(y,s)|\,ds. \end{cases} \tag{4.2}$$

Proof By taking the modulus in Lemma 1, we have

$$\begin{aligned} &\left|\Omega(a,b,x;c,d,y)\right| \\ &\quad\le (b-x)(d-y)\int_0^1\int_0^1 \left|\lambda_1\Lambda_1(x,1) - \Lambda_1(x,t)\right|\left|\lambda_2\Lambda_2(y,1) - \Lambda_2(y,s)\right| \\ &\qquad\times\left|\frac{\partial^2}{\partial t\partial s}f\big(tx+(1-t)b, sy+(1-s)d\big)\right|\,ds\,dt \\ &\qquad+ (b-x)(y-c)\int_0^1\int_0^1 \left|\lambda_1\Lambda_1(x,1) - \Lambda_1(x,t)\right|\left|\mu_2\Delta_2(y,1) - \Delta_2(y,s)\right| \\ &\qquad\times\left|\frac{\partial^2}{\partial t\partial s}f\big(tx+(1-t)b, sy+(1-s)c\big)\right|\,ds\,dt \\ &\qquad+ (x-a)(d-y)\int_0^1\int_0^1 \left|\mu_1\Delta_1(x,1) - \Delta_1(x,t)\right|\left|\lambda_2\Lambda_2(y,1) - \Lambda_2(y,s)\right| \\ &\qquad\times\left|\frac{\partial^2}{\partial t\partial s}f\big(tx+(1-t)a, sy+(1-s)d\big)\right|\,ds\,dt \\ &\qquad+ (x-a)(y-c)\int_0^1\int_0^1 \left|\mu_1\Delta_1(x,1) - \Delta_1(x,t)\right|\left|\mu_2\Delta_2(y,1) - \Delta_2(y,s)\right| \\ &\qquad\times\left|\frac{\partial^2}{\partial t\partial s}f\big(tx+(1-t)a, sy+(1-s)c\big)\right|\,ds\,dt. \end{aligned} \tag{4.3}$$

Since $|\frac{\partial^2 f}{\partial t\partial s}|$ is a co-ordinated convex, we get

$$\begin{aligned} &\int_0^1\int_0^1 \left|\lambda_1\Lambda_1(x,1) - \Lambda_1(x,t)\right|\left|\lambda_2\Lambda_2(y,1) - \Lambda_2(y,s)\right| \\ &\qquad\times\left|\frac{\partial^2}{\partial t\partial s}f\big(tx+(1-t)b, sy+(1-s)d\big)\right|\,ds\,dt \\ &\quad\le\int_0^1\int_0^1 \left|\lambda_1\Lambda_1(x,1) - \Lambda_1(x,t)\right|\left|\lambda_2\Lambda_2(y,1) - \Lambda_2(y,s)\right| \end{aligned} \tag{4.4}$$

$$\begin{aligned}&\times\Bigg(ts\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|+t(1-s)\left|\frac{\partial^2}{\partial t\partial s}f(x,d)\right|\\&+(1-t)s\left|\frac{\partial^2}{\partial t\partial s}f(b,y)\right|+(1-t)(1-s)\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|\Bigg)\,ds\,dt\\&=A_1B_1\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|+A_1B_2\left|\frac{\partial^2}{\partial t\partial s}f(x,d)\right|+A_2B_1\left|\frac{\partial^2}{\partial t\partial s}f(b,y)\right|+A_2B_2\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|.\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\int_0^1\int_0^1\left|\lambda_1\Lambda_1(x,1)-\Lambda_1(x,t)\right|\left|\mu_2\Delta_2(y,1)-\Delta_2(y,s)\right|\\&\quad\times\left|\frac{\partial^2}{\partial t\partial s}f\big(tx+(1-t)b,sy+(1-s)c\big)\right|\,ds\,dt\\&\le A_1B_4\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|+A_1B_3\left|\frac{\partial^2}{\partial t\partial s}f(x,c)\right|+A_2B_4\left|\frac{\partial^2}{\partial t\partial s}f(b,y)\right|+A_2B_3\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|,\end{aligned}\tag{4.5}$$

$$\begin{aligned}&\int_0^1\int_0^1\left|\mu_1\Delta_1(x,1)-\Delta_1(x,t)\right|\left|\lambda_2\Lambda_2(y,1)-\Lambda_2(y,s)\right|\\&\quad\times\left|\frac{\partial^2}{\partial t\partial s}f\big(tx+(1-t)a,sy+(1-s)d\big)\right|\,ds\,dt\\&\le A_4B_1\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|+A_4B_2\left|\frac{\partial^2}{\partial t\partial s}f(x,d)\right|+A_3B_1\left|\frac{\partial^2}{\partial t\partial s}f(a,y)\right|+A_3B_2\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|,\end{aligned}\tag{4.6}$$

and

$$\begin{aligned}&\int_0^1\int_0^1\left|\mu_1\Delta_1(x,1)-\Delta_1(x,t)\right|\left|\mu_2\Delta_2(y,1)-\Delta_2(y,s)\right|\\&\quad\times\left|\frac{\partial^2}{\partial t\partial s}f\big(tx+(1-t)a,sy+(1-s)c\big)\right|\,ds\,dt\\&\le A_4B_4\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|+A_4B_3\left|\frac{\partial^2}{\partial t\partial s}f(x,c)\right|+A_3B_4\left|\frac{\partial^2}{\partial t\partial s}f(a,y)\right|+A_3B_3\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|.\end{aligned}\tag{4.7}$$

If we substitute the inequalities (4.4)–(4.7) in (4.3), we obtain the desired result. This ends the proof of Theorem 2. □

Corollary 3 *In Theorem 2, if we assign $\varphi(t)=t$ and $\psi(s)=s$ for all $(t,s)\in\triangle$, then we obtain the following inequality for Riemann integrals*:

$$\begin{aligned}&\left|\aleph(a,b,x;c,d,y)\right|\\&\le\frac{(b-x)^2(d-y)^2}{36}\Bigg[\big(2\lambda_1^3-3\lambda_1+2\big)\big(2\lambda_2^3-3\lambda_2+2\big)\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|\\&\quad+\big(2\lambda_1^3-3\lambda_1+2\big)\big(-2\lambda_2^3+6\lambda_2^2-3\lambda_2+1\big)\left|\frac{\partial^2}{\partial t\partial s}f(x,d)\right|\\&\quad+\big(-2\lambda_1^3+6\lambda_1^2-3\lambda_1+1\big)\big(2\lambda_2^3-3\lambda_2+2\big)\left|\frac{\partial^2}{\partial t\partial s}f(b,y)\right|\\&\quad+\big(-2\lambda_1^3+6\lambda_1^2-3\lambda_1+1\big)\big(-2\lambda_2^3+6\lambda_2^2-3\lambda_2+1\big)\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|\Bigg]\end{aligned}$$

$$
\begin{aligned}
&+\frac{(b-x)^2(y-c)^2}{36}\Big[\big(2\lambda_1^3-3\lambda_1+2\big)\big(2\mu_2^3-3\mu_2+2\big)\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|\\
&+\big(2\lambda_1^3-3\lambda_1+2\big)\big(-2\mu_2^3+6\mu_2^2-3\mu_2+1\big)\left|\frac{\partial^2}{\partial t\partial s}f(x,c)\right|\\
&+\big(-2\lambda_1^3+6\lambda_1^2-3\lambda_1+1\big)\big(2\mu_2^3-3\mu_2+2\big)\left|\frac{\partial^2}{\partial t\partial s}f(b,y)\right|\\
&+\big(-2\lambda_1^3+6\lambda_1^2-3\lambda_1+1\big)\big(-2\mu_2^3+6\mu_2^2-3\mu_2+1\big)\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|\Big]\\
&+\frac{(x-a)^2(d-y)^2}{36}\Big[\big(2\mu_1^3-3\mu_1+2\big)\big(2\lambda_2^3-3\lambda_2+2\big)\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|\\
&+\big(2\mu_1^3-3\mu_1+2\big)\big(-2\lambda_2^3+6\lambda_2^2-3\lambda_2+1\big)\left|\frac{\partial^2}{\partial t\partial s}f(x,d)\right|\\
&+\big(-2\mu_1^3+6\mu_1^2-3\mu_1+1\big)\big(2\lambda_2^3-3\lambda_2+2\big)\left|\frac{\partial^2}{\partial t\partial s}f(a,y)\right|\\
&+\big(-2\mu_1^3+6\mu_1^2-3\mu_1+1\big)\big(-2\lambda_2^3+6\lambda_2^2-3\lambda_2+1\big)\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|\Big]\\
&+\frac{(x-a)^2(y-c)^2}{36}\Big[\big(2\mu_1^3-3\mu_1+2\big)\big(2\mu_2^3-3\mu_2+2\big)\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|\\
&+\big(2\mu_1^3-3\mu_1+2\big)\big(-2\mu_2^3+6\mu_2^2-3\mu_2+1\big)\left|\frac{\partial^2}{\partial t\partial s}f(x,c)\right|\\
&+\big(-2\mu_1^3+6\mu_1^2-3\mu_1+1\big)\big(2\mu_2^3-3\mu_2+2\big)\left|\frac{\partial^2}{\partial t\partial s}f(a,y)\right|\\
&+\big(-2\mu_1^3+6\mu_1^2-3\mu_1+1\big)\big(-2\mu_2^3+6\mu_2^2-3\mu_2+1\big)\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|\Big],
\end{aligned}
$$

where $\aleph(a,b,x;c,d,y)$ is defined as in Corollary 1.

Corollary 4 *In Theorem 2, let us now consider $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$ and $\psi(s)=\frac{s^\beta}{\Gamma(\beta)}$ for all $(t,s)\in\triangle$. Then we have the following inequality for Riemann–Liouville fractional integrals:*

$$
\begin{aligned}
&\big|\Phi(a,b,x;c,d,y)\big|\\
&\le(b-x)^{\alpha+1}(d-y)^{\beta+1}\Big[C_1(\alpha,\lambda_1)D_1(\beta,\lambda_2)\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|\\
&+C_1(\alpha,\lambda_1)D_2(\beta,\lambda_2)\left|\frac{\partial^2}{\partial t\partial s}f(x,d)\right|\\
&+C_2(\alpha,\lambda_1)D_1(\beta,\lambda_2)\left|\frac{\partial^2}{\partial t\partial s}f(b,y)\right|+C_2(\alpha,\lambda_1)D_2(\beta,\lambda_2)\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|\Big]\\
&+(b-x)^{\alpha+1}(y-c)^{\beta+1}\Big[C_1(\alpha,\lambda_1)D_1(\beta,\mu_2)\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|\\
&+C_1(\alpha,\lambda_1)D_2(\beta,\mu_2)\left|\frac{\partial^2}{\partial t\partial s}f(x,c)\right|\\
&+C_2(\alpha,\lambda_1)D_1(\beta,\mu_2)\left|\frac{\partial^2}{\partial t\partial s}f(b,y)\right|+C_2(\alpha,\lambda_1)D_2(\beta,\mu_2)\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|\Big]
\end{aligned}
$$

$$+ (x-a)^{\alpha+1}(d-y)^{\beta+1}\Bigg[C_1(\alpha,\mu_1)D_1(\beta,\lambda_2)\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|$$
$$+ C_1(\alpha,\mu_1)D_2(\beta,\lambda_2)\left|\frac{\partial^2}{\partial t\partial s}f(x,d)\right|$$
$$+ C_2(\alpha,\mu_1)D_1(\beta,\lambda_2)\left|\frac{\partial^2}{\partial t\partial s}f(a,y)\right| + C_2(\alpha,\mu_1)D_2(\beta,\lambda_2)\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|\Bigg]$$
$$+ (x-a)^{\alpha+1}(y-c)^{\beta+1}\Bigg[C_1(\alpha,\mu_1)D_1(\beta,\mu_2)\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|$$
$$+ C_1(\alpha,\mu_1)D_2(\beta,\mu_2)\left|\frac{\partial^2}{\partial t\partial s}f(x,c)\right|$$
$$+ C_2(\alpha,\mu_1)D_1(\beta,\mu_2)\left|\frac{\partial^2}{\partial t\partial s}f(a,y)\right| + C_2(\alpha,\mu_1)D_2(\beta,\mu_2)\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|\Bigg].$$

Here, $\Phi(a,b,x;c,d,y)$ *is defined as in Corollary 2 and*

$$\begin{cases} C_1(\alpha,\varsigma) = \frac{\alpha}{\alpha+2}\varsigma^{1+\frac{2}{\alpha}} + \frac{1}{\alpha+2} - \frac{\varsigma}{2}, \\ C_2(\alpha,\varsigma) = \frac{2\alpha}{\alpha+1}\varsigma^{1+\frac{1}{\alpha}} - \frac{\alpha}{\alpha+2}\varsigma^{1+\frac{2}{\alpha}} + \frac{1}{(\alpha+1)(\alpha+2)} - \frac{\varsigma}{2}, \\ D_1(\beta,\varsigma) = \frac{\beta}{\beta+2}\varsigma^{1+\frac{2}{\beta}} + \frac{1}{\beta+2} - \frac{\varsigma}{2}, \\ D_2(\beta,\varsigma) = \frac{2\beta}{\beta+1}\varsigma^{1+\frac{1}{\alpha}} - \frac{\beta}{\beta+2}\varsigma^{1+\frac{2}{\beta}} + \frac{1}{(\beta+1)(\beta+2)} - \frac{\varsigma}{2}. \end{cases} \tag{4.8}$$

Theorem 3 *Suppose that the assumptions of Lemma 1 hold. Suppose also that the mapping* $|\frac{\partial^2 f}{\partial t\partial s}|^q$, $q>1$ *is co-ordinated convex on* Δ. *Then we get the following inequality for generalized fractional integrals*:

$$\begin{aligned}
&\left|\Omega(a,b,x;c,d,y)\right| \\
&\le (b-x)(d-y)\left(\int_0^1\int_0^1\left|\lambda_1\Lambda_1(x,1)-\Lambda_1(x,t)\right|^p\left|\lambda_2\Lambda_2(y,1)-\Lambda_2(y,s)\right|^p\,ds\,dt\right)^{\frac{1}{p}} \\
&\quad\times\left(\frac{|\frac{\partial^2}{\partial t\partial s}f(x,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(x,d)|^q+|\frac{\partial^2}{\partial t\partial s}f(b,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(b,d)|^q}{4}\right)^{\frac{1}{q}} \\
&\quad+ (b-x)(y-c)\left(\int_0^1\int_0^1\left|\lambda_1\Lambda_1(x,1)-\Lambda_1(x,t)\right|^p\left|\mu_2\Delta_2(y,1)-\Delta_2(y,s)\right|^p\,ds\,dt\right)^{\frac{1}{p}} \\
&\quad\times\left(\frac{|\frac{\partial^2}{\partial t\partial s}f(x,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(x,c)|^q+|\frac{\partial^2}{\partial t\partial s}f(b,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(b,c)|^q}{4}\right)^{\frac{1}{q}} \\
&\quad+ (x-a)(d-y)\left(\int_0^1\int_0^1\left|\mu_1\Delta_1(x,1)-\Delta_1(x,t)\right|^p\left|\lambda_2\Lambda_2(y,1)-\Lambda_2(y,s)\right|^p\,ds\,dt\right)^{\frac{1}{p}} \\
&\quad\times\left(\frac{|\frac{\partial^2}{\partial t\partial s}f(x,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(x,d)|^q+|\frac{\partial^2}{\partial t\partial s}f(a,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(a,d)|^q}{4}\right)^{\frac{1}{q}} \\
&\quad+ (x-a)(y-c)\left(\int_0^1\int_0^1\left|\mu_1\Delta_1(x,1)-\Delta_1(x,t)\right|^p\left|\mu_2\Delta_2(y,1)-\Delta_2(y,s)\right|^p\,ds\,dt\right)^{\frac{1}{p}} \\
&\quad\times\left(\frac{|\frac{\partial^2}{\partial t\partial s}f(x,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(x,c)|^q+|\frac{\partial^2}{\partial t\partial s}f(a,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(a,c)|^q}{4}\right)^{\frac{1}{q}},
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and* $\Omega(a,b,x;c,d,y)$ *are defined as in Lemma 1.*

Proof With the help of the Hölder inequality and co-ordinated convexity of $|\frac{\partial^2 f}{\partial t\partial s}|^q$, we have

$$\begin{aligned}
&\int_0^1\int_0^1 \bigl|\lambda_1\Lambda_1(x,1)-\Lambda_1(x,t)\bigr|\bigl|\lambda_2\Lambda_2(y,1)-\Lambda_2(y,s)\bigr| \\
&\qquad\times\left|\frac{\partial^2}{\partial t\partial s}f\bigl(tx+(1-t)b,sy+(1-s)d\bigr)\right|\,ds\,dt \\
&\quad\le\left(\int_0^1\int_0^1\bigl|\lambda_1\Lambda_1(x,1)-\Lambda_1(x,t)\bigr|^p\bigl|\lambda_2\Lambda_2(y,1)-\Lambda_2(y,s)\bigr|^p\,ds\,dt\right)^{\frac{1}{p}} \\
&\qquad\times\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial t\partial s}f\bigl(tx+(1-t)b,sy+(1-s)d\bigr)\right|^q\,ds\,dt\right)^{\frac{1}{q}} \\
&\quad\le\left(\int_0^1\int_0^1\bigl|\lambda_1\Lambda_1(x,1)-\Lambda_1(x,t)\bigr|^p\bigl|\lambda_2\Lambda_2(y,1)-\Lambda_2(y,s)\bigr|^p\,ds\,dt\right)^{\frac{1}{p}} \\
&\qquad\times\left(\frac{|\frac{\partial^2}{\partial t\partial s}f(x,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(x,d)|^q+|\frac{\partial^2}{\partial t\partial s}f(b,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(b,d)|^q}{4}\right)^{\frac{1}{q}}.
\end{aligned}\tag{4.9}$$

Similarly, we get

$$\begin{aligned}
&\int_0^1\int_0^1 \bigl|\lambda_1\Lambda_1(x,1)-\Lambda_1(x,t)\bigr|\bigl|\mu_2\Delta_2(y,1)-\Delta_2(y,s)\bigr| \\
&\qquad\times\left|\frac{\partial^2}{\partial t\partial s}f\bigl(tx+(1-t)b,sy+(1-s)c\bigr)\right|\,ds\,dt \\
&\quad\le\left(\int_0^1\int_0^1\bigl|\lambda_1\Lambda_1(x,1)-\Lambda_1(x,t)\bigr|^p\bigl|\mu_2\Delta_2(y,1)-\Delta_2(y,s)\bigr|^p\,ds\,dt\right)^{\frac{1}{p}} \\
&\qquad\times\left(\frac{|\frac{\partial^2}{\partial t\partial s}f(x,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(x,c)|^q+|\frac{\partial^2}{\partial t\partial s}f(b,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(b,c)|^q}{4}\right)^{\frac{1}{q}},
\end{aligned}\tag{4.10}$$

$$\begin{aligned}
&\int_0^1\int_0^1 \bigl|\mu_1\Delta_1(x,1)-\Delta_1(x,t)\bigr|\bigl|\lambda_2\Lambda_2(y,1)-\Lambda_2(y,s)\bigr| \\
&\qquad\times\left|\frac{\partial^2}{\partial t\partial s}f\bigl(tx+(1-t)a,sy+(1-s)d\bigr)\right|\,ds\,dt \\
&\quad\le\left(\int_0^1\int_0^1\bigl|\mu_1\Delta_1(x,1)-\Delta_1(x,t)\bigr|^p\bigl|\lambda_2\Lambda_2(y,1)-\Lambda_2(y,s)\bigr|^p\,ds\,dt\right)^{\frac{1}{p}} \\
&\qquad\times\left(\frac{|\frac{\partial^2}{\partial t\partial s}f(x,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(x,d)|^q+|\frac{\partial^2}{\partial t\partial s}f(a,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(a,d)|^q}{4}\right)^{\frac{1}{q}},
\end{aligned}\tag{4.11}$$

and

$$\begin{aligned}
&\int_0^1\int_0^1 \bigl|\mu_1\Delta_1(x,1)-\Delta_1(x,t)\bigr|\bigl|\mu_2\Delta_2(y,1)-\Delta_2(y,s)\bigr| \\
&\qquad\times\left|\frac{\partial^2}{\partial t\partial s}f\bigl(tx+(1-t)a,sy+(1-s)c\bigr)\right|\,ds\,dt \\
&\quad\le\left(\int_0^1\int_0^1\bigl|\mu_1\Delta_1(x,1)-\Delta_1(x,t)\bigr|^p\bigl|\mu_2\Delta_2(y,1)-\Delta_2(y,s)\bigr|^p\,ds\,dt\right)^{\frac{1}{p}}
\end{aligned}\tag{4.12}$$

$$\times\left(\frac{|\frac{\partial^2}{\partial t\partial s}f(x,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(x,c)|^q+|\frac{\partial^2}{\partial t\partial s}f(a,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(a,c)|^q}{4}\right)^{\frac{1}{q}}.$$

By substituting the inequalities (4.9)–(4.12) in (4.3), we establish required result. □

Corollary 5 *In Theorem 3, if we select $\varphi(t)=t$ and $\psi(s)=s$ for all $(t,s)\in\triangle$, then we obtain the following inequality for the Riemann integral:*

$$\begin{aligned}
&\left|\aleph(a,b,x;c,d,y)\right|\\
&\le(b-x)^2(d-y)^2\left(\frac{(1-\lambda_1)^{p+1}+\lambda_1^{p+1}}{p+1}\right)^{\frac{1}{p}}\left(\frac{(1-\lambda_2)^{p+1}+\lambda_2^{p+1}}{p+1}\right)^{\frac{1}{p}}\\
&\quad\times\left(\frac{|\frac{\partial^2}{\partial t\partial s}f(x,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(x,d)|^q+|\frac{\partial^2}{\partial t\partial s}f(b,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(b,d)|^q}{4}\right)^{\frac{1}{q}}\\
&\quad+(b-x)^2(y-c)^2\left(\frac{(1-\lambda_1)^{p+1}+\lambda_1^{p+1}}{p+1}\right)^{\frac{1}{p}}\left(\frac{(1-\mu_2)^{p+1}+\mu_2^{p+1}}{p+1}\right)^{\frac{1}{p}}\\
&\quad\times\left(\frac{|\frac{\partial^2}{\partial t\partial s}f(x,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(x,c)|^q+|\frac{\partial^2}{\partial t\partial s}f(b,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(b,c)|^q}{4}\right)^{\frac{1}{q}}\\
&\quad+(x-a)^2(d-y)^2\left(\frac{(1-\mu_1)^{p+1}+\mu_1^{p+1}}{p+1}\right)^{\frac{1}{p}}\left(\frac{(1-\lambda_2)^{p+1}+\lambda_2^{p+1}}{p+1}\right)^{\frac{1}{p}}\\
&\quad\times\left(\frac{|\frac{\partial^2}{\partial t\partial s}f(x,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(x,d)|^q+|\frac{\partial^2}{\partial t\partial s}f(a,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(a,d)|^q}{4}\right)^{\frac{1}{q}}\\
&\quad+(x-a)^2(y-c)^2\left(\frac{(1-\mu_1)^{p+1}+\mu_1^{p+1}}{p+1}\right)^{\frac{1}{p}}\left(\frac{(1-\mu_2)^{p+1}+\mu_2^{p+1}}{p+1}\right)^{\frac{1}{p}}\\
&\quad\times\left(\frac{|\frac{\partial^2}{\partial t\partial s}f(x,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(x,c)|^q+|\frac{\partial^2}{\partial t\partial s}f(a,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(a,c)|^q}{4}\right)^{\frac{1}{q}}.
\end{aligned}$$

Here, $\aleph(a,b,x;c,d,y)$ is defined as in Corollary 1.

Corollary 6 *In Theorem 3, Let us consider $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$ and $\psi(s)=\frac{s^\beta}{\Gamma(\beta)}$ for all $(t,s)\in\triangle$. Then we have the inequality for Riemann–Liouville fractional integrals*

$$\begin{aligned}
&\left|\Phi(a,b,x;c,d,y)\right|\\
&\le(b-x)^{\alpha+1}(d-y)^{\beta+1}\left(\int_0^1\left|\lambda_1-t^\alpha\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\lambda_2-s^\beta\right|^p dt\right)^{\frac{1}{p}}\\
&\quad\times\left(\frac{|\frac{\partial^2}{\partial t\partial s}f(x,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(x,d)|^q+|\frac{\partial^2}{\partial t\partial s}f(b,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(b,d)|^q}{4}\right)^{\frac{1}{q}}\\
&\quad+(b-x)^{\alpha+1}(y-c)^{\beta+1}\left(\int_0^1\left|\lambda_1-t^\alpha\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\mu_2-s^\beta\right|^p dt\right)^{\frac{1}{p}}\\
&\quad\times\left(\frac{|\frac{\partial^2}{\partial t\partial s}f(x,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(x,c)|^q+|\frac{\partial^2}{\partial t\partial s}f(b,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(b,c)|^q}{4}\right)^{\frac{1}{q}}\\
&\quad+(x-a)^{\alpha+1}(d-y)^{\beta+1}\left(\int_0^1\left|\mu_1-t^\alpha\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\lambda_2-s^\beta\right|^p dt\right)^{\frac{1}{p}}
\end{aligned}$$

$$\times\left(\frac{|\frac{\partial^2}{\partial t\partial s}f(x,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(x,d)|^q+|\frac{\partial^2}{\partial t\partial s}f(a,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(a,d)|^q}{4}\right)^{\frac{1}{q}}$$

$$+(x-a)^{\alpha+1}(y-c)^{\beta+1}\left(\int_0^1\left|\mu_1-t^\alpha\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\mu_2-s^\beta\right|^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\frac{|\frac{\partial^2}{\partial t\partial s}f(x,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(x,c)|^q+|\frac{\partial^2}{\partial t\partial s}f(a,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(a,c)|^q}{4}\right)^{\frac{1}{q}},$$

where $\Phi(a,b,x;c,d,y)$ is defined as in Corollary 2.

Theorem 4 *Assume that the assumptions of Lemma 1 hold. If the mapping $|\frac{\partial^2 f}{\partial t\partial s}|^q$, $q\ge 1$ is co-ordinated convex on Δ, then we get the following inequality for generalized fractional integrals*:

$$\begin{aligned}
&\left|\Omega(a,b,x;c,d,y)\right|\\
&\le(b-x)(d-y)\left(\int_0^1\int_0^1\left|\lambda_1\Lambda_1(x,1)-\Lambda_1(x,t)\right|\left|\lambda_2\Lambda_2(y,1)-\Lambda_2(y,s)\right|ds\,dt\right)^{1-\frac{1}{q}}\\
&\quad\times\left(A_1B_1\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|^q+A_1B_2\left|\frac{\partial^2}{\partial t\partial s}f(x,d)\right|^q+A_2B_1\left|\frac{\partial^2}{\partial t\partial s}f(b,y)\right|^q\right.\\
&\quad\left.+A_2B_2\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|^q\right)^{\frac{1}{q}}\\
&\quad+(b-x)(y-c)\left(\int_0^1\int_0^1\left|\lambda_1\Lambda_1(x,1)-\Lambda_1(x,t)\right|\left|\mu_2\Delta_2(y,1)-\Delta_2(y,s)\right|ds\,dt\right)^{1-\frac{1}{q}}\\
&\quad\times\left(A_1B_4\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|^q+A_1B_3\left|\frac{\partial^2}{\partial t\partial s}f(x,c)\right|^q+A_2B_4\left|\frac{\partial^2}{\partial t\partial s}f(b,y)\right|^q\right.\\
&\quad\left.+A_2B_3\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|^q\right)^{\frac{1}{q}}\\
&\quad+(x-a)(d-y)\left(\int_0^1\int_0^1\left|\mu_1\Delta_1(x,1)-\Delta_1(x,t)\right|\left|\lambda_2\Lambda_2(y,1)-\Lambda_2(y,s)\right|ds\,dt\right)^{1-\frac{1}{q}}\\
&\quad\times\left(A_4B_1\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|^q+A_4B_2\left|\frac{\partial^2}{\partial t\partial s}f(x,d)\right|^q+A_3B_1\left|\frac{\partial^2}{\partial t\partial s}f(a,y)\right|^q\right.\\
&\quad\left.+A_3B_2\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|^q\right)^{\frac{1}{q}}\\
&\quad+(x-a)(y-c)\left(\int_0^1\int_0^1\left|\mu_1\Delta_1(x,1)-\Delta_1(x,t)\right|\left|\mu_2\Delta_2(y,1)-\Delta_2(y,s)\right|ds\,dt\right)^{1-\frac{1}{q}}\\
&\quad\times\left(A_4B_4\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|^q+A_4B_3\left|\frac{\partial^2}{\partial t\partial s}f(x,c)\right|^q+A_3B_4\left|\frac{\partial^2}{\partial t\partial s}f(a,y)\right|^q\right.\\
&\quad\left.+A_3B_3\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

Here, $\Omega(a,b,x;c,d,y)$ is defined as in Lemma 1, A_i, $i=1,2,3,4$ are defined as in (4.1) and B_i, $i=1,2,3,4$ are defined as in (4.2).

Proof Power mean inequality and co-ordinated convexity of $|\frac{\partial^2 f}{\partial t\partial s}|^q$ yield

$$\begin{aligned}
&\int_0^1\int_0^1 \bigl|\lambda_1\Lambda_1(x,1)-\Lambda_1(x,t)\bigr|\bigl|\lambda_2\Lambda_2(y,1)-\Lambda_2(y,s)\bigr| \\
&\qquad\times\left|\frac{\partial^2}{\partial t\partial s}f\bigl(tx+(1-t)b,sy+(1-s)\,d\bigr)\right|\,ds\,dt \\
&\quad\leq\left(\int_0^1\int_0^1 \bigl|\lambda_1\Lambda_1(x,1)-\Lambda_1(x,t)\bigr|\bigl|\lambda_2\Lambda_2(y,1)-\Lambda_2(y,s)\bigr|\,ds\,dt\right)^{1-\frac{1}{q}} \\
&\qquad\times\left(\int_0^1\int_0^1 \bigl|\lambda_1\Lambda_1(x,1)-\Lambda_1(x,t)\bigr|\bigl|\lambda_2\Lambda_2(y,1)-\Lambda_2(y,s)\bigr|\right. \\
&\qquad\left.\times\left|\frac{\partial^2}{\partial t\partial s}f\bigl(tx+(1-t)b,sy+(1-s)\,d\bigr)\right|^q\,ds\,dt\right)^{\frac{1}{q}} \\
&\quad\leq\left(\int_0^1\int_0^1 \bigl|\lambda_1\Lambda_1(x,1)-\Lambda_1(x,t)\bigr|\bigl|\lambda_2\Lambda_2(y,1)-\Lambda_2(y,s)\bigr|\,ds\,dt\right)^{1-\frac{1}{q}} \\
&\qquad\times\left[\int_0^1\int_0^1\left(\bigl|\lambda_1\Lambda_1(x,1)-\Lambda_1(x,t)\bigr|\bigl|\lambda_2\Lambda_2(y,1)-\Lambda_2(y,s)\bigr|ts\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|^q\right.\right. \\
&\qquad+\bigl|\lambda_1\Lambda_1(x,1)-\Lambda_1(x,t)\bigr|\bigl|\lambda_2\Lambda_2(y,1)-\Lambda_2(y,s)\bigr|t(1-s)\left|\frac{\partial^2}{\partial t\partial s}f(x,d)\right|^q \\
&\qquad+\bigl|\lambda_1\Lambda_1(x,1)-\Lambda_1(x,t)\bigr|\bigl|\lambda_2\Lambda_2(y,1)-\Lambda_2(y,s)\bigr|(1-t)s\left|\frac{\partial^2}{\partial t\partial s}f(b,y)\right|^q \\
&\qquad+\bigl|\lambda_1\Lambda_1(x,1)-\Lambda_1(x,t)\bigr|\bigl|\lambda_2\Lambda_2(y,1) \\
&\qquad\left.\left.-\Lambda_2(y,s)\bigr|(1-t)(1-s)\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|^q\right)\,ds\,dt\right]^{\frac{1}{q}} \\
&\quad=\left(\int_0^1\int_0^1 \bigl|\lambda_1\Lambda_1(x,1)-\Lambda_1(x,t)\bigr|\bigl|\lambda_2\Lambda_2(y,1)-\Lambda_2(y,s)\bigr|\,ds\,dt\right)^{1-\frac{1}{q}} \\
&\qquad\times\left(A_1B_1\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|^q+A_1B_2\left|\frac{\partial^2}{\partial t\partial s}f(x,d)\right|^q\right. \\
&\qquad\left.+A_2B_1\left|\frac{\partial^2}{\partial t\partial s}f(b,y)\right|^q+A_2B_2\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}\tag{4.13}$$

Similarly, we obtain

$$\begin{aligned}
&\int_0^1\int_0^1 \bigl|\lambda_1\Lambda_1(x,1)-\Lambda_1(x,t)\bigr|\bigl|\mu_2\Delta_2(y,1)-\Delta_2(y,s)\bigr| \\
&\qquad\times\left|\frac{\partial^2}{\partial t\partial s}f\bigl(tx+(1-t)b,sy+(1-s)c\bigr)\right|\,ds\,dt \\
&\quad\leq\left(\int_0^1\int_0^1 \bigl|\lambda_1\Lambda_1(x,1)-\Lambda_1(x,t)\bigr|\bigl|\mu_2\Delta_2(y,1)-\Delta_2(y,s)\bigr|\,ds\,dt\right)^{1-\frac{1}{q}} \\
&\qquad\times\left(A_1B_4\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|^q+A_1B_3\left|\frac{\partial^2}{\partial t\partial s}f(x,c)\right|^q+A_2B_4\left|\frac{\partial^2}{\partial t\partial s}f(b,y)\right|^q\right.
\end{aligned}\tag{4.14}$$

$$
\begin{aligned}
&\quad + A_2B_3\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|^q\Bigg)^{\frac{1}{q}}, \\
&\int_0^1\int_0^1 \left|\mu_1\Delta_1(x,1)-\Delta_1(x,t)\right|\left|\lambda_2\Lambda_2(y,1)-\Lambda_2(y,s)\right| \\
&\quad \times \left|\frac{\partial^2}{\partial t\partial s}f\bigl(tx+(1-t)a, sy+(1-s)\,d\bigr)\right|\,ds\,dt \\
&\le \left(\int_0^1\int_0^1 \left|\mu_1\Delta_1(x,1)-\Delta_1(x,t)\right|\left|\lambda_2\Lambda_2(y,1)-\Lambda_2(y,s)\right|\,ds\,dt\right)^{1-\frac{1}{q}} \\
&\quad \times \Bigg(A_4B_1\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|^q + A_4B_2\left|\frac{\partial^2}{\partial t\partial s}f(x,d)\right|^q + A_3B_1\left|\frac{\partial^2}{\partial t\partial s}f(a,y)\right|^q \\
&\quad + A_3B_2\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|^q\Bigg)^{\frac{1}{q}},
\end{aligned} \tag{4.15}
$$

and

$$
\begin{aligned}
&\int_0^1\int_0^1 \left|\mu_1\Delta_1(x,1)-\Delta_1(x,t)\right|\left|\mu_2\Delta_2(y,1)-\Delta_2(y,s)\right| \\
&\quad \times \left|\frac{\partial^2}{\partial t\partial s}f\bigl(tx+(1-t)a, sy+(1-s)c\bigr)\right|\,ds\,dt \\
&\le \left(\int_0^1\int_0^1 \left|\mu_1\Delta_1(x,1)-\Delta_1(x,t)\right|\left|\mu_2\Delta_2(y,1)-\Delta_2(y,s)\right|\,ds\,dt\right)^{1-\frac{1}{q}} \\
&\times \Bigg(A_4B_4\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|^q + A_4B_3\left|\frac{\partial^2}{\partial t\partial s}f(x,c)\right|^q \\
&\quad + A_3B_4\left|\frac{\partial^2}{\partial t\partial s}f(a,y)\right|^q + A_3B_3\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|^q\Bigg)^{\frac{1}{q}}.
\end{aligned} \tag{4.16}
$$

If we substitute the inequalities (4.13)–(4.16) in (4.3), then we establish the desired result. □

Corollary 7 *In Theorem 4, if we take $\varphi(t) = t$ and $\psi(s) = s$ for all $(t,s) \in \triangle$, then we have the following inequality for the Riemann integral*:

$$
\begin{aligned}
&\left|\aleph(a,b,x;c,d,y)\right| \\
&\le (b-x)^2(d-y)^2\frac{(\lambda_1^2-\lambda_1+\frac{1}{2})^{1-\frac{1}{q}}(\lambda_2^2-\lambda_2+\frac{1}{2})^{1-\frac{1}{q}}}{6^{\frac{2}{q}}} \\
&\times \Bigg[\bigl(2\lambda_1^3-3\lambda_1+2\bigr)\bigl(2\lambda_2^3-3\lambda_2+2\bigr)\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|^q \\
&\quad + \bigl(2\lambda_1^3-3\lambda_1+2\bigr)\bigl(-2\lambda_2^3+6\lambda_2^2-3\lambda_2+1\bigr)\left|\frac{\partial^2}{\partial t\partial s}f(x,d)\right|^q \\
&\quad + \bigl(-2\lambda_1^3+6\lambda_1^2-3\lambda_1+1\bigr)\bigl(2\lambda_2^3-3\lambda_2+2\bigr)\left|\frac{\partial^2}{\partial t\partial s}f(b,y)\right|^q \\
&\quad + \bigl(-2\lambda_1^3+6\lambda_1^2-3\lambda_1+1\bigr)\bigl(-2\lambda_2^3+6\lambda_2^2-3\lambda_2+1\bigr)\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|^q\Bigg]^{\frac{1}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&+(b-x)^2(y-c)^2\frac{(\lambda_1^2-\lambda_1+\frac{1}{2})^{1-\frac{1}{q}}(\mu_2^2-\mu_2+\frac{1}{2})^{1-\frac{1}{q}}}{6^{\frac{2}{q}}}\\
&\times\biggl[\bigl(2\lambda_1^3-3\lambda_1+2\bigr)\bigl(2\mu_2^3-3\mu_2+2\bigr)\biggl|\frac{\partial^2}{\partial t\partial s}f(x,y)\biggr|^q\\
&+\bigl(2\lambda_1^3-3\lambda_1+2\bigr)\bigl(-2\mu_2^3+6\mu_2^2-3\mu_2+1\bigr)\biggl|\frac{\partial^2}{\partial t\partial s}f(x,c)\biggr|^q\\
&+\bigl(-2\lambda_1^3+6\lambda_1^2-3\lambda_1+1\bigr)\bigl(2\mu_2^3-3\mu_2+2\bigr)\biggl|\frac{\partial^2}{\partial t\partial s}f(b,y)\biggr|^q\\
&+\bigl(-2\lambda_1^3+6\lambda_1^2-3\lambda_1+1\bigr)\bigl(-2\mu_2^3+6\mu_2^2-3\mu_2+1\bigr)\biggl|\frac{\partial^2}{\partial t\partial s}f(b,c)\biggr|^q\biggr]^{\frac{1}{q}}\\
&+(x-a)^2(d-y)^2\frac{(\mu_1^2-\mu_1+\frac{1}{2})^{1-\frac{1}{q}}(\lambda_2^2-\lambda_2+\frac{1}{2})^{1-\frac{1}{q}}}{6^{\frac{2}{q}}}\\
&\times\biggl[\bigl(2\mu_1^3-3\mu_1+2\bigr)\bigl(2\lambda_2^3-3\lambda_2+2\bigr)\biggl|\frac{\partial^2}{\partial t\partial s}f(x,y)\biggr|^q\\
&+\bigl(2\mu_1^3-3\mu_1+2\bigr)\bigl(-2\lambda_2^3+6\lambda_2^2-3\lambda_2+1\bigr)\biggl|\frac{\partial^2}{\partial t\partial s}f(x,d)\biggr|^q\\
&+\bigl(-2\mu_1^3+6\mu_1^2-3\mu_1+1\bigr)\bigl(2\lambda_2^3-3\lambda_2+2\bigr)\biggl|\frac{\partial^2}{\partial t\partial s}f(a,y)\biggr|^q\\
&+\bigl(-2\mu_1^3+6\mu_1^2-3\mu_1+1\bigr)\bigl(-2\lambda_2^3+6\lambda_2^2-3\lambda_2+1\bigr)\biggl|\frac{\partial^2}{\partial t\partial s}f(a,d)\biggr|^q\biggr]^{\frac{1}{q}}\\
&+(x-a)^2(y-c)^2\frac{(\mu_1^2-\mu_1+\frac{1}{2})^{1-\frac{1}{q}}(\mu_2^2-\mu_2+\frac{1}{2})^{1-\frac{1}{q}}}{6^{\frac{2}{q}}}\\
&\times\biggl[\bigl(2\mu_1^3-3\mu_1+2\bigr)\bigl(2\mu_2^3-3\mu_2+2\bigr)\biggl|\frac{\partial^2}{\partial t\partial s}f(x,y)\biggr|^q\\
&+\bigl(2\mu_1^3-3\mu_1+2\bigr)\bigl(-2\mu_2^3+6\mu_2^2-3\mu_2+1\bigr)\biggl|\frac{\partial^2}{\partial t\partial s}f(x,c)\biggr|^q\\
&+\bigl(-2\mu_1^3+6\mu_1^2-3\mu_1+1\bigr)\bigl(2\mu_2^3-3\mu_2+2\bigr)\biggl|\frac{\partial^2}{\partial t\partial s}f(a,y)\biggr|^q\\
&+\bigl(-2\mu_1^3+6\mu_1^2-3\mu_1+1\bigr)\bigl(-2\mu_2^3+6\mu_2^2-3\mu_2+1\bigr)\biggl|\frac{\partial^2}{\partial t\partial s}f(a,c)\biggr|^q\biggr]^{\frac{1}{q}},
\end{aligned}
$$

where $\aleph(a,b,x;c,d,y)$ is defined as in Corollary 1.

Corollary 8 *In Theorem 4, let us consider $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$ and $\psi(s)=\frac{s^\beta}{\Gamma(\beta)}$ for all $(t,s)\in\triangle$. Then we obtain the following inequality for Riemann–Liouville fractional integrals*:

$$
\begin{aligned}
&\bigl|\Phi(a,b,x;c,d,y)\bigr|\\
&\le(b-x)^{\alpha+1}(d-y)^{\beta+1}\bigl(C_1(\alpha,\lambda_1)+C_2(\alpha,\lambda_1)\bigr)^{1-\frac{1}{q}}\bigl(D_1(\beta,\lambda_2)+D_2(\beta,\lambda_2)\bigr)^{1-\frac{1}{q}}\\
&\quad\times\biggl[C_1(\alpha,\lambda_1)D_1(\beta,\lambda_2)\biggl|\frac{\partial^2}{\partial t\partial s}f(x,y)\biggr|^q+C_1(\alpha,\lambda_1)D_2(\beta,\lambda_2)\biggl|\frac{\partial^2}{\partial t\partial s}f(x,d)\biggr|^q
\end{aligned}
$$

$$
\begin{aligned}
&+ C_2(\alpha,\lambda_1)D_1(\beta,\lambda_2)\left|\frac{\partial^2}{\partial t\partial s}f(b,y)\right|^q + C_2(\alpha,\lambda_1)D_2(\beta,\lambda_2)\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|^q\Bigg]^{\frac{1}{q}} \\
&+ (b-x)^{\alpha+1}(y-c)^{\beta+1}\big(C_1(\alpha,\lambda_1)+C_2(\alpha,\lambda_1)\big)^{1-\frac{1}{q}}\big(D_1(\beta,\mu_2)+D_2(\beta,\mu_2)\big)^{1-\frac{1}{q}} \\
&\times\Bigg[C_1(\alpha,\lambda_1)D_1(\beta,\mu_2)\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|^q + C_1(\alpha,\lambda_1)D_2(\beta,\mu_2)\left|\frac{\partial^2}{\partial t\partial s}f(x,c)\right|^q \\
&+ C_2(\alpha,\lambda_1)D_1(\beta,\mu_2)\left|\frac{\partial^2}{\partial t\partial s}f(b,y)\right|^q + C_2(\alpha,\lambda_1)D_2(\beta,\mu_2)\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|^q\Bigg]^{\frac{1}{q}} \\
&+ (x-a)^{\alpha+1}(d-y)^{\beta+1}\big(C_1(\alpha,\mu_1)+C_2(\alpha,\mu_1)\big)^{1-\frac{1}{q}}\big(D_1(\beta,\lambda_2)+D_2(\beta,\lambda_2)\big)^{1-\frac{1}{q}} \\
&\times\Bigg[C_1(\alpha,\mu_1)D_1(\beta,\lambda_2)\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|^q + C_1(\alpha,\mu_1)D_2(\beta,\lambda_2)\left|\frac{\partial^2}{\partial t\partial s}f(x,d)\right|^q \\
&+ C_2(\alpha,\mu_1)D_1(\beta,\lambda_2)\left|\frac{\partial^2}{\partial t\partial s}f(a,y)\right|^q + C_2(\alpha,\mu_1)D_2(\beta,\lambda_2)\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|^q\Bigg]^{\frac{1}{q}} \\
&+ (x-a)^{\alpha+1}(y-c)^{\beta+1}\big(C_1(\alpha,\mu_1)+C_2(\alpha,\mu_1)\big)^{1-\frac{1}{q}}\big(D_1(\beta,\mu_2)+D_2(\beta,\mu_2)\big)^{1-\frac{1}{q}} \\
&\times\Bigg[C_1(\alpha,\mu_1)D_1(\beta,\mu_2)\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|^q + C_1(\alpha,\mu_1)D_2(\beta,\mu_2)\left|\frac{\partial^2}{\partial t\partial s}f(x,c)\right|^q \\
&+ C_2(\alpha,\mu_1)D_1(\beta,\mu_2)\left|\frac{\partial^2}{\partial t\partial s}f(a,y)\right|^q + C_2(\alpha,\mu_1)D_2(\beta,\mu_2)\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|^q\Bigg]^{\frac{1}{q}}.
\end{aligned}
$$

Here, $\Phi(a,b,x;c,d,y)$ is defined as in Corollary 2 and C_1, C_2 are defined as in (4.8).

5 Special Cases

In this section, some special cases of our results are presented and we show that our results reduce to inequalities obtained in earlier work.

Remark 4 In Corollary 3:

(1) If we assign $\lambda_1=\lambda_2=\mu_1=\mu_2=0$ and if $|\frac{\partial^2}{\partial t\partial s}f(t,s)|\le M$ for all $(t,s)\in\Delta$, then Corollary 3 reduces to [24, Theorem 3].

(2) Let us note that $\lambda_1=\lambda_2=\mu_1=\mu_2=1$, $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$. Then Corollary 3 reduces to [39, Theorem 2].

(3) For $\lambda_1=\lambda_2=\mu_1=\mu_2=\frac{1}{3}$, $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$, Corollary 3 is equal to [28, Theorem 3].

Remark 5 In Corollary 4;

(1) Assume $\lambda_1=\lambda_2=\mu_1=\mu_2=0$ and assume also $|\frac{\partial^2}{\partial t\partial s}f(t,s)|\le M$ for all $(t,s)\in\Delta$. Then Corollary 4 reduces to [23, Theorem 3].

(2) If we put $\lambda_1=\lambda_2=\mu_1=\mu_2=1$, then we have the following trapezoidal type inequalities for Riemann–Liouville fractional integrals:

$$
\begin{aligned}
&\big|\Im(a,b,x;c,d,y)\big| \\
&\quad\le (b-x)^{\alpha+1}(d-y)^{\beta+1}\Bigg[\frac{\alpha}{2(\alpha+2)}\frac{\beta}{2(\beta+2)}\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right| \\
&\qquad+\frac{\alpha}{2(\alpha+2)}\frac{\beta^2+3\beta}{2(\beta+1)(\beta+2)}\left|\frac{\partial^2}{\partial t\partial s}f(x,d)\right|
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\alpha^2+3\alpha}{2(\alpha+1)(\alpha+2)} \frac{\beta}{2(\beta+2)} \left| \frac{\partial^2}{\partial t \partial s} f(b,y) \right| \\
&+ \frac{\alpha^2+3\alpha}{2(\alpha+1)(\alpha+2)} \frac{\beta^2+3\beta}{2(\beta+1)(\beta+2)} \left| \frac{\partial^2}{\partial t \partial s} f(b,d) \right| \bigg] \\
&+ (b-x)^{\alpha+1}(y-c)^{\beta+1} \bigg[\frac{\alpha}{2(\alpha+2)} \frac{\beta}{2(\beta+2)} \left| \frac{\partial^2}{\partial t \partial s} f(x,y) \right| \\
&+ \frac{\alpha}{2(\alpha+2)} \frac{\beta^2+3\beta}{2(\beta+1)(\beta+2)} \left| \frac{\partial^2}{\partial t \partial s} f(x,c) \right| \\
&+ \frac{\alpha^2+3\alpha}{2(\alpha+1)(\alpha+2)} \frac{\beta}{2(\beta+2)} \left| \frac{\partial^2}{\partial t \partial s} f(b,y) \right| \\
&+ \frac{\alpha^2+3\alpha}{2(\alpha+1)(\alpha+2)} \frac{\beta^2+3\beta}{2(\beta+1)(\beta+2)} \left| \frac{\partial^2}{\partial t \partial s} f(b,c) \right| \bigg] \\
&+ (x-a)^{\alpha+1}(d-y)^{\beta+1} \bigg[\frac{\alpha}{2(\alpha+2)} \frac{\beta}{2(\beta+2)} \left| \frac{\partial^2}{\partial t \partial s} f(x,y) \right| \\
&+ \frac{\alpha}{2(\alpha+2)} \frac{\beta^2+3\beta}{2(\beta+1)(\beta+2)} \left| \frac{\partial^2}{\partial t \partial s} f(x,d) \right| \\
&+ \frac{\alpha^2+3\alpha}{2(\alpha+1)(\alpha+2)} \frac{\beta}{2(\beta+2)} \left| \frac{\partial^2}{\partial t \partial s} f(a,y) \right| \\
&+ \frac{\alpha^2+3\alpha}{2(\alpha+1)(\alpha+2)} \frac{\beta^2+3\beta}{2(\beta+1)(\beta+2)} \left| \frac{\partial^2}{\partial t \partial s} f(a,d) \right| \bigg] \\
&+ (x-a)^{\alpha+1}(y-c)^{\beta+1} \bigg[\frac{\alpha}{2(\alpha+2)} \frac{\beta}{2(\beta+2)} \left| \frac{\partial^2}{\partial t \partial s} f(x,y) \right| \\
&+ \frac{\alpha}{2(\alpha+2)} \frac{\beta^2+3\beta}{2(\beta+1)(\beta+2)} \left| \frac{\partial^2}{\partial t \partial s} f(x,c) \right| \\
&+ \frac{\alpha^2+3\alpha}{2(\alpha+1)(\alpha+2)} \frac{\beta}{2(\beta+2)} \left| \frac{\partial^2}{\partial t \partial s} f(a,y) \right| \\
&+ \frac{\alpha}{2(\alpha+2)} \frac{\beta^2+3\beta}{2(\beta+1)(\beta+2)} \left| \frac{\partial^2}{\partial t \partial s} f(a,c) \right| \bigg],
\end{aligned}
$$

where

$$
\begin{aligned}
&\Im(a,b,x;c,d,y) \\
&= (b-x)^{\alpha}(d-y)^{\beta}\big[f(b,d)\big] + (b-x)^{\alpha}(y-c)^{\beta}\big[f(b,c)\big] \\
&\quad + (x-a)^{\alpha}(d-y)^{\beta}\big[f(a,d)\big] + (x-a)^{\alpha}(y-c)^{\beta}\big[f(a,c)\big] \\
&\quad - \Gamma(\alpha+1)(d-y)^{\beta}\big[J_{x+}^{\alpha} f(b,d) + J_{x-}^{\alpha} f(a,d)\big] \\
&\quad - \Gamma(\alpha+1)(y-c)^{\beta}\big[J_{x+}^{\alpha} f(b,c) + J_{x-}^{\alpha} f(a,c)\big] \\
&\quad - \Gamma(\beta+1)(b-x)^{\alpha}\big[J_{y+}^{\beta} f(b,d) + J_{y-}^{\beta} f(b,c)\big] \\
&\quad - \Gamma(\beta+1)(x-a)^{\alpha}\big[J_{y+}^{\beta} f(a,d) + J_{y-}^{\beta} f(a,c)\big] \\
&\quad + \Gamma(\alpha+1)\Gamma(\beta+1)\big[J_{x+,y+}^{\alpha,\beta} f(b,d) + J_{x+,y-}^{\alpha,\beta} f(b,c) + J_{x-,y+}^{\alpha,\beta} f(a,d) + J_{x-,y-}^{\alpha,\beta} f(a,c)\big].
\end{aligned}
$$

(3) Let us consider $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = \frac{1}{3}$. Then we obtain the following Simpson type inequalities for Riemann–Liouville fractional integrals:

$$\begin{aligned}
&\left|\chi(a,b,x;c,d,y)\right| \\
&\quad\le (b-x)^{\alpha+1}(d-y)^{\beta+1}\left[C_1\left(\alpha,\frac{1}{3}\right)D_1\left(\beta,\frac{1}{3}\right)\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|\right. \\
&\qquad + C_1\left(\alpha,\frac{1}{3}\right)D_2\left(\beta,\frac{1}{3}\right)\left|\frac{\partial^2}{\partial t\partial s}f(x,d)\right| \\
&\qquad \left.+ C_2\left(\alpha,\frac{1}{3}\right)D_1\left(\beta,\frac{1}{3}\right)\left|\frac{\partial^2}{\partial t\partial s}f(b,y)\right| + C_2\left(\alpha,\frac{1}{3}\right)D_2\left(\beta,\frac{1}{3}\right)\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|\right] \\
&\qquad + (b-x)^{\alpha+1}(y-c)^{\beta+1}\left[C_1\left(\alpha,\frac{1}{3}\right)D_1\left(\beta,\frac{1}{3}\right)\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|\right. \\
&\qquad + C_1\left(\alpha,\frac{1}{3}\right)D_2\left(\beta,\frac{1}{3}\right)\left|\frac{\partial^2}{\partial t\partial s}f(x,c)\right| \\
&\qquad \left.+ C_2\left(\alpha,\frac{1}{3}\right)D_1\left(\beta,\frac{1}{3}\right)\left|\frac{\partial^2}{\partial t\partial s}f(b,y)\right| + C_2\left(\alpha,\frac{1}{3}\right)D_2\left(\beta,\frac{1}{3}\right)\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|\right] \\
&\qquad + (x-a)^{\alpha+1}(d-y)^{\beta+1}\left[C_1\left(\alpha,\frac{1}{3}\right)D_1\left(\beta,\frac{1}{3}\right)\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|\right. \\
&\qquad + C_1\left(\alpha,\frac{1}{3}\right)D_2\left(\beta,\frac{1}{3}\right)\left|\frac{\partial^2}{\partial t\partial s}f(x,d)\right| \\
&\qquad \left.+ C_2\left(\alpha,\frac{1}{3}\right)D_1\left(\beta,\frac{1}{3}\right)\left|\frac{\partial^2}{\partial t\partial s}f(a,y)\right| + C_2\left(\alpha,\frac{1}{3}\right)D_2\left(\beta,\frac{1}{3}\right)\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|\right] \\
&\qquad + (x-a)^{\alpha+1}(y-c)^{\beta+1}\left[C_1\left(\alpha,\frac{1}{3}\right)D_1\left(\beta,\frac{1}{3}\right)\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|\right. \\
&\qquad + C_1\left(\alpha,\frac{1}{3}\right)D_2\left(\beta,\frac{1}{3}\right)\left|\frac{\partial^2}{\partial t\partial s}f(x,c)\right| \\
&\qquad \left.+ C_2\left(\alpha,\frac{1}{3}\right)D_1\left(\beta,\frac{1}{3}\right)\left|\frac{\partial^2}{\partial t\partial s}f(a,y)\right| + C_2\left(\alpha,\frac{1}{3}\right)D_2\left(\beta,\frac{1}{3}\right)\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|\right].
\end{aligned}$$

Here,

$$\begin{aligned}
&\chi(a,b,x;c,d,y) \\
&\quad= \frac{(b-x)^\alpha(d-y)^\beta}{9}\left[4f(x,y) + 2f(x,d) + 2f(b,y) + f(b,d)\right] \\
&\qquad + \frac{(b-x)^\alpha(y-c)^\beta}{9}\left[4f(x,y) + 2f(x,c) + 2f(b,y) + f(b,c)\right] \\
&\qquad + \frac{(x-a)^\alpha(d-y)^\beta}{9}\left[4f(x,y) + 2f(x,d) + 2f(a,y) + f(a,d)\right] \\
&\qquad + \frac{(x-a)^\alpha(y-c)^\beta}{9}\left[4f(x,y) + 2f(x,c) + 2f(a,y) + f(a,c)\right] \\
&\qquad - \frac{2}{3}\Gamma(\alpha+1)\left((d-y)^\beta + (y-c)^\beta\right)\left[J_{x+}^{\alpha}f(b,y) + J_{x-}^{\alpha}f(a,y)\right] \\
&\qquad - \frac{2}{3}\Gamma(\beta+1)\left((b-x)^\alpha + (x-a)^\alpha\right)\left[J_{y+}^{\beta}f(x,d) + J_{y-}^{\beta}f(x,c)\right]
\end{aligned}$$

$$
\begin{aligned}
&-\Gamma(\alpha+1)\frac{(d-y)^{\beta}}{3}\left[J_{x+}^{\alpha}f(b,d)+J_{x-}^{\alpha}f(a,d)\right] \\
&-\Gamma(\alpha+1)\frac{(y-c)^{\beta}}{3}\left[J_{x+}^{\alpha}f(b,c)+J_{x-}^{\alpha}f(a,c)\right] \\
&-\Gamma(\beta+1)\frac{(b-x)^{\alpha}}{3}\left[J_{y+}^{\beta}f(b,d)+J_{y-}^{\beta}f(b,c)\right] \\
&-\Gamma(\beta+1)\frac{(x-a)^{\alpha}}{3}\left[J_{y+}^{\beta}f(a,d)+J_{y-}^{\beta}f(a,c)\right] \\
&+\Gamma(\alpha+1)\Gamma(\beta+1)\left[J_{x+,y+}^{\alpha,\beta}f(b,d)+J_{x+,y-}^{\alpha,\beta}f(b,c)+J_{x-,y+}^{\alpha,\beta}f(a,d)+J_{x-,y-}^{\alpha,\beta}f(a,c)\right].
\end{aligned}
$$

Remark 6 In Corollary 5;

(1) Suppose $\lambda_1=\lambda_2=\mu_1=\mu_2=0$ and suppose also $|\frac{\partial^2}{\partial t\partial s}f(t,s)|\leq M$ for all $(t,s)\in\Delta$. Then Corollary 5 equals [24, Theorem 4].

(2) If we take $\lambda_1=\lambda_2=\mu_1=\mu_2=1$, then we get the following trapezoidal type inequalities for the Riemann integral:

$$
\begin{aligned}
&\bigg|(b-x)(d-y)f(b,d)+(b-x)(y-c)f(b,c)+(x-a)(d-y)f(a,d) \\
&\quad +(x-a)(y-c)f(a,c) \\
&\quad -(d-y)\int_a^b f(t,d)\,dt-(y-c)\int_a^b f(t,c)\,dt \\
&\quad -(b-x)\int_c^d f(b,s)\,dt-(x-a)\int_c^d f(a,s)\,ds+\int_a^b\int_c^d f(t,s)\,ds\,dt\bigg| \\
&\leq (b-x)^2(d-y)^2\left(\frac{1}{p+1}\right)^{\frac{2}{p}} \\
&\quad \times\left(\frac{|\frac{\partial^2}{\partial t\partial s}f(x,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(x,d)|^q+|\frac{\partial^2}{\partial t\partial s}f(b,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(b,d)|^q}{4}\right)^{\frac{1}{q}} \\
&\quad +(b-x)^2(y-c)^2\left(\frac{1}{p+1}\right)^{\frac{2}{p}} \\
&\quad \times\left(\frac{|\frac{\partial^2}{\partial t\partial s}f(x,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(x,c)|^q+|\frac{\partial^2}{\partial t\partial s}f(b,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(b,c)|^q}{4}\right)^{\frac{1}{q}} \\
&\quad +(x-a)^2(d-y)^2\left(\frac{1}{p+1}\right)^{\frac{2}{p}} \\
&\quad \times\left(\frac{|\frac{\partial^2}{\partial t\partial s}f(x,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(x,d)|^q+|\frac{\partial^2}{\partial t\partial s}f(a,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(a,d)|^q}{4}\right)^{\frac{1}{q}} \\
&\quad +(x-a)^2(y-c)^2\left(\frac{1}{p+1}\right)^{\frac{2}{p}} \\
&\quad \times\left(\frac{|\frac{\partial^2}{\partial t\partial s}f(x,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(x,c)|^q+|\frac{\partial^2}{\partial t\partial s}f(a,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(a,c)|^q}{4}\right)^{\frac{1}{q}}.
\end{aligned}
$$

(3) For $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = \frac{1}{3}$, we have the following Simpson type inequalities for the Riemann integral:

$$\begin{aligned}
&\bigg| \frac{(b-x)(d-y)}{9}\big[4f(x,y) + 2_2 f(x,d) + 2f(b,y) + f(b,d)\big] \\
&\quad + \frac{(b-x)(y-c)}{9}\big[4f(x,y) + 2f(x,c) + 2f(b,y) + f(b,c)\big] \\
&\quad + \frac{(x-a)(d-y)}{9}\big[4f(x,y) + 2f(x,d) + 2f(a,y) + f(a,d)\big] \\
&\quad + \frac{(x-a)(y-c)}{9}\big[4f(x,y) + 2f(x,c) + 2f(a,y) + f(a,c)\big] \\
&\quad - \frac{2}{3}\big((d-y) + (y-c)\big) \int_a^b f(t,y)\,dt - \frac{2}{3}\big((b-x) + (x-a)\big) \int_c^d f(x,s)\,ds \\
&\quad - \frac{(d-y)}{3} \int_a^b f(t,d)\,dt - \frac{(y-c)}{3} \int_a^b f(t,c)\,dt \\
&\quad - \frac{(b-x)}{3} \int_c^d f(b,s)\,dt - \frac{(x-a)}{3} \int_c^d f(a,s)\,ds + \int_a^b \int_c^d f(t,s)\,ds\,dt \bigg| \\
&\leq (b-x)^2(d-y)^2 \left(\frac{(\frac{2}{3})^{p+1} + (\frac{1}{3})^{p+1}}{p+1} \right)^{\frac{2}{p}} \\
&\quad \times \left(\frac{|\frac{\partial^2}{\partial t \partial s} f(x,y)|^q + |\frac{\partial^2}{\partial t \partial s} f(x,d)|^q + |\frac{\partial^2}{\partial t \partial s} f(b,y)|^q + |\frac{\partial^2}{\partial t \partial s} f(b,d)|^q}{4} \right)^{\frac{1}{q}} \\
&\quad + (b-x)^2(y-c)^2 \left(\frac{(\frac{2}{3})^{p+1} + (\frac{1}{3})^{p+1}}{p+1} \right)^{\frac{2}{p}} \\
&\quad \times \left(\frac{|\frac{\partial^2}{\partial t \partial s} f(x,y)|^q + |\frac{\partial^2}{\partial t \partial s} f(x,c)|^q + |\frac{\partial^2}{\partial t \partial s} f(b,y)|^q + |\frac{\partial^2}{\partial t \partial s} f(b,c)|^q}{4} \right)^{\frac{1}{q}} \\
&\quad + (x-a)^2(d-y)^2 \left(\frac{(\frac{2}{3})^{p+1} + (\frac{1}{3})^{p+1}}{p+1} \right)^{\frac{2}{p}} \\
&\quad \times \left(\frac{|\frac{\partial^2}{\partial t \partial s} f(x,y)|^q + |\frac{\partial^2}{\partial t \partial s} f(x,d)|^q + |\frac{\partial^2}{\partial t \partial s} f(a,y)|^q + |\frac{\partial^2}{\partial t \partial s} f(a,d)|^q}{4} \right)^{\frac{1}{q}} \\
&\quad + (x-a)^2(y-c)^2 \left(\frac{(\frac{2}{3})^{p+1} + (\frac{1}{3})^{p+1}}{p+1} \right)^{\frac{2}{p}} \\
&\quad \times \left(\frac{|\frac{\partial^2}{\partial t \partial s} f(x,y)|^q + |\frac{\partial^2}{\partial t \partial s} f(x,c)|^q + |\frac{\partial^2}{\partial t \partial s} f(a,y)|^q + |\frac{\partial^2}{\partial t \partial s} f(a,c)|^q}{4} \right)^{\frac{1}{q}}.
\end{aligned}$$

Remark 7 In Corollary 6;

(1) If we assign $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 0$ and if $|\frac{\partial^2}{\partial t \partial s} f(t,s)| \leq M$ for all $(t,s) \in \Delta$, then Corollary 6 reduces to [23, Theorem 4].

(2) For $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1$, the following trapezoidal type inequality for Riemann–Liouville fractional integrals holds:

$$\begin{aligned}
&\big|\Im(a,b,x;c,d,y)\big| \\
&\leq (b-x)^{\alpha+1}(d-y)^{\beta+1} \left(\int_0^1 \big|1 - t^\alpha\big|^p\,dt \right)^{\frac{1}{p}} \left(\int_0^1 \big|1 - s^\beta\big|^p\,dt \right)^{\frac{1}{p}}
\end{aligned}$$

$$\times\left(\frac{|\frac{\partial^2}{\partial t\partial s}f(x,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(x,d)|^q+|\frac{\partial^2}{\partial t\partial s}f(b,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(b,d)|^q}{4}\right)^{\frac{1}{q}}$$

$$+(b-x)^{\alpha+1}(y-c)^{\beta+1}\left(\int_0^1\left|1-t^{\alpha}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|1-s^{\beta}\right|^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\frac{|\frac{\partial^2}{\partial t\partial s}f(x,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(x,c)|^q+|\frac{\partial^2}{\partial t\partial s}f(b,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(b,c)|^q}{4}\right)^{\frac{1}{q}}$$

$$+(x-a)^{\alpha+1}(d-y)^{\beta+1}\left(\int_0^1\left|1-t^{\alpha}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|1-s^{\beta}\right|^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\frac{|\frac{\partial^2}{\partial t\partial s}f(x,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(x,d)|^q+|\frac{\partial^2}{\partial t\partial s}f(a,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(a,d)|^q}{4}\right)^{\frac{1}{q}}$$

$$+(x-a)^{\alpha+1}(y-c)^{\beta+1}\left(\int_0^1\left|1-t^{\alpha}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|1-s^{\beta}\right|^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\frac{|\frac{\partial^2}{\partial t\partial s}f(x,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(x,c)|^q+|\frac{\partial^2}{\partial t\partial s}f(a,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(a,c)|^q}{4}\right)^{\frac{1}{q}},$$

where $\Im(a,b,x;c,d,y)$ is defined as in Remark 5.

(3) Considering $\lambda_1=\lambda_2=\mu_1=\mu_2=\frac{1}{3}$, we obtain the following Simpson type inequality for Riemann–Liouville fractional integrals:

$$\begin{aligned}
&\left|\chi(a,b,x;c,d,y)\right| \\
&\leq (b-x)^{\alpha+1}(d-y)^{\beta+1}\left(\int_0^1\left|\frac{1}{3}-t^{\alpha}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\frac{1}{3}-s^{\beta}\right|^p dt\right)^{\frac{1}{p}} \\
&\quad\times\left(\frac{|\frac{\partial^2}{\partial t\partial s}f(x,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(x,d)|^q+|\frac{\partial^2}{\partial t\partial s}f(b,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(b,d)|^q}{4}\right)^{\frac{1}{q}} \\
&\quad+(b-x)^{\alpha+1}(y-c)^{\beta+1}\left(\int_0^1\left|\frac{1}{3}-t^{\alpha}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\frac{1}{3}-s^{\beta}\right|^p dt\right)^{\frac{1}{p}} \\
&\quad\times\left(\frac{|\frac{\partial^2}{\partial t\partial s}f(x,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(x,c)|^q+|\frac{\partial^2}{\partial t\partial s}f(b,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(b,c)|^q}{4}\right)^{\frac{1}{q}} \\
&\quad+(x-a)^{\alpha+1}(d-y)^{\beta+1}\left(\int_0^1\left|\frac{1}{3}-t^{\alpha}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\frac{1}{3}-s^{\beta}\right|^p dt\right)^{\frac{1}{p}} \\
&\quad\times\left(\frac{|\frac{\partial^2}{\partial t\partial s}f(x,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(x,d)|^q+|\frac{\partial^2}{\partial t\partial s}f(a,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(a,d)|^q}{4}\right)^{\frac{1}{q}} \\
&\quad+(x-a)^{\alpha+1}(y-c)^{\beta+1}\left(\int_0^1\left|\frac{1}{3}-t^{\alpha}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\frac{1}{3}-s^{\beta}\right|^p dt\right)^{\frac{1}{p}} \\
&\quad\times\left(\frac{|\frac{\partial^2}{\partial t\partial s}f(x,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(x,c)|^q+|\frac{\partial^2}{\partial t\partial s}f(a,y)|^q+|\frac{\partial^2}{\partial t\partial s}f(a,c)|^q}{4}\right)^{\frac{1}{q}}.
\end{aligned}$$

Here, $\chi(a,b,x;c,d,y)$ is defined as in Remark 5.

Remark 8 In Corollary 7;

(1) Let us consider $\lambda_1=\lambda_2=\mu_1=\mu_2=0$ and $|\frac{\partial^2}{\partial t\partial s}f(t,s)|\leq M$ for all $(t,s)\in\Delta$. Then Corollary 7 equals [24, Theorem 5].

(2) For $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1$, we have the following trapezoidal type inequalities for the Riemann integral:

$$\begin{aligned}
&\Big|(b-x)(d-y)f(b,d) + (b-x)(y-c)f(b,c) + (x-a)(d-y)f(a,d) \\
&\quad + (x-a)(y-c)f(a,c) \\
&\quad - (d-y)\int_a^b f(t,d)\,dt - (y-c)\int_a^b f(t,c)\,dt \\
&\quad - (b-x)\int_c^d f(b,s)\,dt - (x-a)\int_c^d f(a,s)\,ds + \int_a^b\int_c^d f(t,s)\,ds\,dt\Big| \\
&\leq (b-x)^2(d-y)^2\frac{1}{2^2\cdot 3^{\frac{2}{q}}}\Big[\Big|\frac{\partial^2}{\partial t\partial s}f(x,y)\Big|^q + 2\Big|\frac{\partial^2}{\partial t\partial s}f(x,d)\Big|^q \\
&\quad + 2\Big|\frac{\partial^2}{\partial t\partial s}f(b,y)\Big|^q + 4\Big|\frac{\partial^2}{\partial t\partial s}f(b,d)\Big|^q\Big]^{\frac{1}{q}} \\
&\quad + (b-x)^2(y-c)^2\frac{1}{2^2\cdot 3^{\frac{2}{q}}}\Big[\Big|\frac{\partial^2}{\partial t\partial s}f(x,y)\Big|^q + 2\Big|\frac{\partial^2}{\partial t\partial s}f(x,c)\Big|^q \\
&\quad + 2\Big|\frac{\partial^2}{\partial t\partial s}f(b,y)\Big|^q + 4\Big|\frac{\partial^2}{\partial t\partial s}f(b,c)\Big|^q\Big]^{\frac{1}{q}} \\
&\quad + (x-a)^2(d-y)^2\frac{1}{2^2\cdot 3^{\frac{2}{q}}}\Big[\Big|\frac{\partial^2}{\partial t\partial s}f(x,y)\Big|^q + 2\Big|\frac{\partial^2}{\partial t\partial s}f(x,d)\Big|^q \\
&\quad + 2\Big|\frac{\partial^2}{\partial t\partial s}f(a,y)\Big|^q + 4\Big|\frac{\partial^2}{\partial t\partial s}f(a,d)\Big|^q\Big]^{\frac{1}{q}} \\
&\quad + (x-a)^2(y-c)^2\frac{1}{2^2\cdot 3^{\frac{2}{q}}}\Big[\Big|\frac{\partial^2}{\partial t\partial s}f(x,y)\Big|^q + 2\Big|\frac{\partial^2}{\partial t\partial s}f(x,c)\Big|^q \\
&\quad + 2\Big|\frac{\partial^2}{\partial t\partial s}f(a,y)\Big|^q + 4\Big|\frac{\partial^2}{\partial t\partial s}f(a,c)\Big|^q\Big]^{\frac{1}{q}}.
\end{aligned}$$

(3) If we elect $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = \frac{1}{3}$, then we have he following Simpson type inequalities for the Riemann integral:

$$\begin{aligned}
&\Big|\frac{(b-x)(d-y)}{9}\big[4f(x,y) + 2_2f(x,d) + 2f(b,y) + f(b,d)\big] \\
&\quad + \frac{(b-x)(y-c)}{9}\big[4f(x,y) + 2f(x,c) + 2f(b,y) + f(b,c)\big] \\
&\quad + \frac{(x-a)(d-y)}{9}\big[4f(x,y) + 2f(x,d) + 2f(a,y) + f(a,d)\big] \\
&\quad + \frac{(x-a)(y-c)}{9}\big[4f(x,y) + 2f(x,c) + 2f(a,y) + f(a,c)\big] \\
&\quad - \frac{2}{3}\big((d-y) + (y-c)\big)\int_a^b f(t,y)\,dt - \frac{2}{3}\big((b-x) + (x-a)\big)\int_c^d f(x,s)\,ds \\
&\quad - \frac{(d-y)}{3}\int_a^b f(t,d)\,dt - \frac{(y-c)}{3}\int_a^b f(t,c)\,dt
\end{aligned}$$

$$
\begin{aligned}
&-\frac{(b-x)}{3}\int_c^d f(b,s)\,dt-\frac{(x-a)}{3}\int_c^d f(a,s)\,ds+\int_a^b\int_c^d f(t,s)\,ds\,dt\Bigg|\\
&\leq (b-x)^2(d-y)^2\frac{5^2}{18^2(45)^{\frac{2}{q}}}\\
&\quad\times\Bigg[29\cdot 29\bigg|\frac{\partial^2}{\partial t\partial s}f(x,y)\bigg|^q+29\cdot 16\bigg|\frac{\partial^2}{\partial t\partial s}f(x,d)\bigg|^q+29\cdot 16\bigg|\frac{\partial^2}{\partial t\partial s}f(b,y)\bigg|^q\\
&\quad+16\cdot 16\bigg|\frac{\partial^2}{\partial t\partial s}f(b,d)\bigg|^q\Bigg]^{\frac{1}{q}}\\
&\quad+(b-x)^2(y-c)^2\frac{5^2}{18^2(45)^{\frac{2}{q}}}\\
&\quad\times\Bigg[29\cdot 29\bigg|\frac{\partial^2}{\partial t\partial s}f(x,y)\bigg|^q+29\cdot 16\bigg|\frac{\partial^2}{\partial t\partial s}f(x,c)\bigg|^q+29\cdot 16\bigg|\frac{\partial^2}{\partial t\partial s}f(b,y)\bigg|^q\\
&\quad+16\cdot 16\bigg|\frac{\partial^2}{\partial t\partial s}f(b,c)\bigg|^q\Bigg]^{\frac{1}{q}}\\
&\quad+(x-a)^2(d-y)^2\frac{5^2}{18^2(45)^{\frac{2}{q}}}\\
&\quad\times\Bigg[29\cdot 29\bigg|\frac{\partial^2}{\partial t\partial s}f(x,y)\bigg|^q+29\cdot 16\bigg|\frac{\partial^2}{\partial t\partial s}f(x,d)\bigg|^q+29\cdot 16\bigg|\frac{\partial^2}{\partial t\partial s}f(a,y)\bigg|^q\\
&\quad+16\cdot 16\bigg|\frac{\partial^2}{\partial t\partial s}f(a,d)\bigg|^q\Bigg]^{\frac{1}{q}}\\
&\quad+(x-a)^2(y-c)^2\frac{5^2}{18^2(45)^{\frac{2}{q}}}\\
&\quad\times\Bigg[29\cdot 29\bigg|\frac{\partial^2}{\partial t\partial s}f(x,y)\bigg|^q+29\cdot 16\bigg|\frac{\partial^2}{\partial t\partial s}f(x,c)\bigg|^q+29\cdot 16\bigg|\frac{\partial^2}{\partial t\partial s}f(a,y)\bigg|^q\\
&\quad+16\cdot 16\bigg|\frac{\partial^2}{\partial t\partial s}f(a,c)\bigg|^q\Bigg]^{\frac{1}{q}}.
\end{aligned}
$$

Remark 9 In Corollary 8;

(1) Assume $\lambda_1=\lambda_2=\mu_1=\mu_2=0$ and assume also $|\frac{\partial^2}{\partial t\partial s}f(t,s)|\leq M$ for all $(t,s)\in\Delta$. Then Corollary 8 reduces to [23, Theorem 5].

(2) If $\lambda_1=\lambda_2=\mu_1=\mu_2=1$ is chosen, then we get the following trapezoidal type inequality for Riemann–Liouville fractional integrals:

$$
\begin{aligned}
&\big|\Im(a,b,x;c,d,y)\big|\\
&\leq (b-x)^{\alpha+1}(d-y)^{\beta+1}\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{\beta}{\beta+1}\right)^{1-\frac{1}{q}}\\
&\quad\times\Bigg[\frac{\alpha}{2(\alpha+2)}\frac{\beta}{2(\beta+2)}\bigg|\frac{\partial^2}{\partial t\partial s}f(x,y)\bigg|^q\\
&\quad+\frac{\alpha}{2(\alpha+2)}\frac{\beta^2+3\beta}{2(\beta+1)(\beta+2)}\bigg|\frac{\partial^2}{\partial t\partial s}f(x,d)\bigg|^q\\
&\quad+\frac{\alpha^2+3\alpha}{2(\alpha+1)(\alpha+2)}\frac{\beta}{2(\beta+2)}\bigg|\frac{\partial^2}{\partial t\partial s}f(b,y)\bigg|^q
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\alpha^2+3\alpha}{2(\alpha+1)(\alpha+2)} \frac{\beta^2+3\beta}{2(\beta+1)(\beta+2)} \left| \frac{\partial^2}{\partial t \partial s} f(b,d) \right|^q \Bigg]^{\frac{1}{q}} \\
&+ (b-x)^{\alpha+1}(y-c)^{\beta+1} \left(\frac{\alpha}{\alpha+1} \right)^{1-\frac{1}{q}} \left(\frac{\beta}{\beta+1} \right)^{1-\frac{1}{q}} \\
&\times \Bigg[\frac{\alpha}{2(\alpha+2)} \frac{\beta}{2(\beta+2)} \left| \frac{\partial^2}{\partial t \partial s} f(x,y) \right|^q \\
&+ \frac{\alpha}{2(\alpha+2)} \frac{\beta^2+3\beta}{2(\beta+1)(\beta+2)} \left| \frac{\partial^2}{\partial t \partial s} f(x,c) \right|^q \\
&+ \frac{\alpha^2+3\alpha}{2(\alpha+1)(\alpha+2)} \frac{\beta}{2(\beta+2)} \left| \frac{\partial^2}{\partial t \partial s} f(b,y) \right|^q \\
&+ \frac{\alpha^2+3\alpha}{2(\alpha+1)(\alpha+2)} \frac{\beta^2+3\beta}{2(\beta+1)(\beta+2)} \left| \frac{\partial^2}{\partial t \partial s} f(b,c) \right|^q \Bigg]^{\frac{1}{q}} \\
&+ (x-a)^{\alpha+1}(d-y)^{\beta+1} \left(\frac{\alpha}{\alpha+1} \right)^{1-\frac{1}{q}} \left(\frac{\beta}{\beta+1} \right)^{1-\frac{1}{q}} \\
&\times \Bigg[\frac{\alpha}{2(\alpha+2)} \frac{\beta}{2(\beta+2)} \left| \frac{\partial^2}{\partial t \partial s} f(x,y) \right|^q \\
&+ \frac{\alpha}{2(\alpha+2)} \frac{\beta^2+3\beta}{2(\beta+1)(\beta+2)} \left| \frac{\partial^2}{\partial t \partial s} f(x,d) \right|^q \\
&+ \frac{\alpha^2+3\alpha}{2(\alpha+1)(\alpha+2)} \frac{\beta}{2(\beta+2)} \left| \frac{\partial^2}{\partial t \partial s} f(a,y) \right|^q \\
&+ \frac{\alpha^2+3\alpha}{2(\alpha+1)(\alpha+2)} \frac{\beta^2+3\beta}{2(\beta+1)(\beta+2)} \left| \frac{\partial^2}{\partial t \partial s} f(a,d) \right|^q \Bigg]^{\frac{1}{q}} \\
&+ (x-a)^{\alpha+1}(y-c)^{\beta+1} \left(\frac{\alpha}{\alpha+1} \right)^{1-\frac{1}{q}} \left(\frac{\beta}{\beta+1} \right)^{1-\frac{1}{q}} \\
&\times \Bigg[\frac{\alpha}{2(\alpha+2)} \frac{\beta}{2(\beta+2)} \left| \frac{\partial^2}{\partial t \partial s} f(x,y) \right|^q \\
&+ \frac{\alpha}{2(\alpha+2)} \frac{\beta^2+3\beta}{2(\beta+1)(\beta+2)} \left| \frac{\partial^2}{\partial t \partial s} f(x,c) \right|^q \\
&+ \frac{\alpha^2+3\alpha}{2(\alpha+1)(\alpha+2)} \frac{\beta}{2(\beta+2)} \left| \frac{\partial^2}{\partial t \partial s} f(a,y) \right|^q \\
&+ \frac{\alpha^2+3\alpha}{2(\alpha+1)(\alpha+2)} \frac{\beta^2+3\beta}{2(\beta+1)(\beta+2)} \left| \frac{\partial^2}{\partial t \partial s} f(a,c) \right|^q \Bigg]^{\frac{1}{q}},
\end{aligned}
$$

where $\Im(a,b,x;c,d,y)$ is defined as in Remark 5.

(3) For $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = \frac{1}{3}$, the following Simpson type inequality for Riemann–Liouville fractional integrals holds:

$$
\begin{aligned}
&\left| \chi(a,b,x;c,d,y) \right| \\
&\leq (b-x)^{\alpha+1}(d-y)^{\beta+1} \left(C_1\left(\alpha, \frac{1}{3}\right) + C_2\left(\alpha, \frac{1}{3}\right) \right)^{1-\frac{1}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(D_1\left(\beta,\frac{1}{3}\right)+D_2\left(\beta,\frac{1}{3}\right)\right)^{1-\frac{1}{q}}\\
&\times\left[C_1\left(\alpha,\frac{1}{3}\right)D_1\left(\beta,\frac{1}{3}\right)\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|^q\right.\\
&+C_1\left(\alpha,\frac{1}{3}\right)D_2\left(\beta,\frac{1}{3}\right)\left|\frac{\partial^2}{\partial t\partial s}f(x,d)\right|^q\\
&+C_2\left(\alpha,\frac{1}{3}\right)D_1(\beta,\lambda_2)\left|\frac{\partial^2}{\partial t\partial s}f(b,y)\right|^q\\
&\left.+C_2\left(\alpha,\frac{1}{3}\right)D_2\left(\beta,\frac{1}{3}\right)\left|\frac{\partial^2}{\partial t\partial s}f(b,d)\right|^q\right]^{\frac{1}{q}}\\
&+(b-x)^{\alpha+1}(y-c)^{\beta+1}\left(C_1\left(\alpha,\frac{1}{3}\right)+C_2\left(\alpha,\frac{1}{3}\right)\right)^{1-\frac{1}{q}}\\
&\times\left(D_1\left(\beta,\frac{1}{3}\right)+D_2\left(\beta,\frac{1}{3}\right)\right)^{1-\frac{1}{q}}\\
&\times\left[C_1\left(\alpha,\frac{1}{3}\right)D_1\left(\beta,\frac{1}{3}\right)\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|^q\right.\\
&+C_1\left(\alpha,\frac{1}{3}\right)D_2\left(\beta,\frac{1}{3}\right)\left|\frac{\partial^2}{\partial t\partial s}f(x,c)\right|^q\\
&+C_2\left(\alpha,\frac{1}{3}\right)D_1\left(\beta,\frac{1}{3}\right)\left|\frac{\partial^2}{\partial t\partial s}f(b,y)\right|^q\\
&\left.+C_2\left(\alpha,\frac{1}{3}\right)D_2\left(\beta,\frac{1}{3}\right)\left|\frac{\partial^2}{\partial t\partial s}f(b,c)\right|^q\right]^{\frac{1}{q}}\\
&+(x-a)^{\alpha+1}(d-y)^{\beta+1}\left(C_1\left(\alpha,\frac{1}{3}\right)+C_2\left(\alpha,\frac{1}{3}\right)\right)^{1-\frac{1}{q}}\\
&\times\left(D_1\left(\beta,\frac{1}{3}\right)+D_2\left(\beta,\frac{1}{3}\right)\right)^{1-\frac{1}{q}}\\
&\times\left[C_1\left(\alpha,\frac{1}{3}\right)D_1\left(\beta,\frac{1}{3}\right)\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|^q\right.\\
&+C_1\left(\alpha,\frac{1}{3}\right)D_2\left(\beta,\frac{1}{3}\right)\left|\frac{\partial^2}{\partial t\partial s}f(x,d)\right|^q\\
&+C_2\left(\alpha,\frac{1}{3}\right)D_1\left(\beta,\frac{1}{3}\right)\left|\frac{\partial^2}{\partial t\partial s}f(a,y)\right|^q\\
&\left.+C_2\left(\alpha,\frac{1}{3}\right)D_2\left(\beta,\frac{1}{3}\right)\left|\frac{\partial^2}{\partial t\partial s}f(a,d)\right|^q\right]^{\frac{1}{q}}\\
&+(x-a)^{\alpha+1}(y-c)^{\beta+1}\left(C_1\left(\alpha,\frac{1}{3}\right)+C_2\left(\alpha,\frac{1}{3}\right)\right)^{1-\frac{1}{q}}\\
&\times\left(D_1\left(\beta,\frac{1}{3}\right)+D_2\left(\beta,\frac{1}{3}\right)\right)^{1-\frac{1}{q}}\\
&\times\left[C_1\left(\alpha,\frac{1}{3}\right)D_1\left(\beta,\frac{1}{3}\right)\left|\frac{\partial^2}{\partial t\partial s}f(x,y)\right|^q\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ C_1\left(\alpha,\frac{1}{3}\right)D_2\left(\beta,\frac{1}{3}\right)\left|\frac{\partial^2}{\partial t\partial s}f(x,c)\right|^q \\
&+ C_2\left(\alpha,\frac{1}{3}\right)D_1\left(\beta,\frac{1}{3}\right)\left|\frac{\partial^2}{\partial t\partial s}f(a,y)\right|^q \\
&+ C_2\left(\alpha,\frac{1}{3}\right)D_2\left(\beta,\frac{1}{3}\right)\left|\frac{\partial^2}{\partial t\partial s}f(a,c)\right|^q\Bigg]^{\frac{1}{q}}.
\end{aligned}
$$

Here, $\chi(a,b,x;c,d,y)$ is defined as in Remark 5.

6 Conclusion

In this paper, we used the concepts of generalized fractional integrals and proved some new generalized inequalities for partially differentiable co-ordinated convex mappings. In addition, we discussed the special cases of the main results. Furthermore, several new inequalities of trapezoidal and Simpson type for partially differentiable co-ordinated convex functions via Riemann and Riemann–Liouville fractional integrals. It is an interesting and new problem that the upcoming researchers can address similar inequalities for other type co-ordinated convex functions in their future research. By using the Jensen inequality and a coordinated concave function in our main lemma, one can obtain new inequalities for generalized fractional integrals

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