Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Dynamics of a lake-eutrophication model with nontransient/transient impulsive dredging and pulse inputting

Jianjun Jiao[1*] and Wenyan Tang[1]

*Correspondence:
jiaojianjun05@126.com;
jiaojianjun@mail.gufe.edu.cn
[1] School of Mathematics and Statistics, Guizhou University of Finance and Economics, Guiyang 550004, People's Republic of China

Abstract

In this work, we present a lake-eutrophication model with nontransient/transient impulsive dredging and pulse inputting. We obtain globally asymptotically stable conditions for the phytoplankton-extinction periodic solution of system (2.1). Furthermore, we gain the permanent conditions for system (2.1). Finally, we employ computer simulations to illustrate the results. Our results indicate the effective controlling strategy for water resource management.

Keywords: Lake-eutrophication model; Nontransient/transient impulsive dredging; Pulse inputting; Phytoplankton-extinction

1 Introduction

Lakes are very important water resources; many lakes have water supply, shipping, flood control, irrigation, aquaculture, tourism, and other functions [1]. Lake eutrophication has become a worldwide environmental problem. According to statistics, the proportion of eutrophic water bodies in Asia, Europe, North America, and Africa reached 54%, 53%, 46%, and 28%, respectively [2]. Bennett et al. [3] investigated human impact on erodable phosphorus and eutrophication. The main characteristic of lake pollution is eutrophication of water body. Because of human interference of activities, eutrophication process is very rapid. Deposing the sediment is an important reservoir of nutrients in lakes. After the nutrient load of the lake is reduced or completely cut off, the nutrient salt in the sediment will gradually released to become the dominant factor of lake eutrophication endogenous [4]. So the preventing and controlling phytoplankton in eutrophication lake ecosystem have also become an important subject of water environmental protection. Partly and periodically dredging sediments can protect lake ecosystem and water resource. At present, physical, chemical, and biological methods are the common methods of controlling phytoplankton (cyanobacteria) in eutrophication lake ecosystem [5]. The physical methods are relatively safe ways to remove algae. Impulsive differential equations are found in almost every domain of applied science and have been studied in many investigations [6–13]. However, the authors did not applied impulsive differential equations to describe the physical methods for water resource management. In this paper, we present a lake-eutrophication model for water resource management, which considers effects of nontransient/transient impulsive dredging and pulse inputting.

2 The model

For the diagram in Fig. 1, in this paper, we consider a like-eutrophication model with nontransient/transient impulsive dredging and pulse inputting on nutrients

$$\begin{cases} \left.\begin{aligned} \frac{ds(t)}{dt} &= \lambda_1 - d_1 s(t) \\ &\quad - \frac{\beta_{11}}{\delta_{11}} s(t)x_1(t) - \frac{\beta_{12}}{\delta_{12}} s(t)x_2(t), \\ \frac{dx_1(t)}{dt} &= \beta_{11} s(t)x_1(t) - d_{11}x_1(t), \\ \frac{dx_2(t)}{dt} &= \beta_{12} s(t)x_2(t) - d_{12}x_2(t), \end{aligned}\right\} & t \in (n\tau, (n+l)\tau], \\ \left.\begin{aligned} \triangle s(t) &= -\mu_s s(t), \\ \triangle x_1(t) &= -\mu_1 x_1(t), \\ \triangle x_2(t) &= -\mu_2 x_2(t), \end{aligned}\right\} & t = (n+l)\tau, n \in Z^+, \\ \left.\begin{aligned} \frac{ds(t)}{dt} &= \lambda_2 - (d_2 + E_s)s(t) \\ &\quad - \frac{\beta_{21}}{\delta_{21}} s(t)x_1(t) - \frac{\beta_{22}}{\delta_{22}} s(t)x_2(t), \\ \frac{dx_1(t)}{dt} &= \beta_{21} s(t)x_1(t) - (d_{21} + E_1)x_1(t), \\ \frac{dx_2(t)}{dt} &= \beta_{22} s(t)x_2(t) - (d_{22} + E_2)x_2(t), \end{aligned}\right\} & t \in ((n+l)\tau, (n+1)\tau], \\ \left.\begin{aligned} \triangle s(t) &= \mu, \\ \triangle x_1(t) &= 0, \\ \triangle x_2(t) &= 0, \end{aligned}\right\} & t = (n+1)\tau, n \in Z^+. \end{cases} \tag{2.1}$$

Here $s(t)$ represents the concentrations of the nutrients at time t, $x_i(t)$ $(i = 1,2)$ represent the concentrations of phytoplankton in lake at time t, $\lambda_1 > 0$ represents the in-

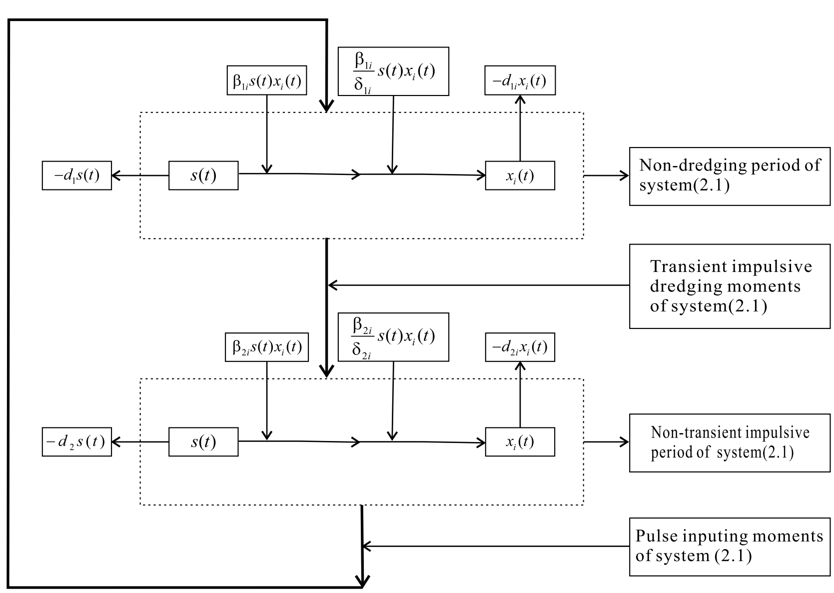

Figure 1 Diagram for the dynamics of a lake-eutrophication model with nontransient/transient impulsive dredging and pulse inputting

put concentration of the nutrients from ravine streams around the eutrophication-lake in the interval $(n\tau, (n + l)\tau)]$, $d_1 > 0$ represents washout and loss rate of the nutrient in eutrophication-lake in the interval $(n\tau, (n + l)\tau)]$, $\beta_{11} > 0$ represents the maximum growth rate of phytoplankton x_1 in eutrophication-lake in the interval $(n\tau, (n + l)\tau)]$, $0 < \delta_{11} < 1$ represents the yield of the nutrients for phytoplankton x_1 in eutrophication-lake in the interval $(n\tau, (n + l)\tau)]$, $\beta_{12} > 0$ represents the maximum growth rate of phytoplankton x_2 in eutrophication-lake in the interval $(n\tau, (n + l)\tau)]$, $0 < \delta_{12} < 1$ represents the yield of the nutrients for phytoplankton x_2 in eutrophication-lake in the interval $(n\tau, (n + l)\tau)]$, $d_{12} > 0$ represents the death and loss rate of the phytoplankton in the eutrophication-lake in the interval $(n\tau, (n + l)\tau)]$, $0 < \mu_s < 1$ represents the impulsive dredging effect on the nutrients in the eutrophication-lake at moment $t = (n + l)\tau$, $0 < \mu_1 < 1$ represents the impulsive dredging effect on phytoplankton x_1 in the eutrophication-lake at moment $t = (n + l)\tau$, $0 < \mu_2 < 1$ represents the impulsive dredging effect on phytoplankton x_2 in the eutrophication-lake at moment $t = (n + l)\tau$, $\lambda_2 > 0$ represents the input concentration of the nutrients from ravine streams around the eutrophication-lake in the interval $((n + l)\tau, (n + 1)\tau)]$, $d_2 > 0$ represents washout and loss rate of the nutrient in eutrophication-lake on interval $((n + l)\tau, (n + 1)\tau)]$, $E_s > 0$ represents the nontransient impulsive dredging effect on the nutrients in the eutrophication-lake in the interval $((n + l)\tau, (n + 1)\tau)]$, $\beta_{21} > 0$ represents the maximum growth rate of phytoplankton x_1 in eutrophication-lake in the interval $((n + l)\tau, (n + 1)\tau)]$, $0 < \delta_{21} < 1$ represents the yield of the nutrients for phytoplankton x_1 in eutrophication-lake in the interval $((n + l)\tau, (n + 1)\tau)]$, $\beta_{22} > 0$ represents the maximum growth rate of phytoplankton x_2 in eutrophication-lake in the interval $((n + l)\tau, (n + 1)\tau)]$, $0 < \delta_{22} < 1$ represents the yield of the nutrients for phytoplankton x_2 in eutrophication-lake in the interval $((n + l)\tau, (n + 1)\tau)]$, $d_{22} > 0$ represents the death and loss rate of the phytoplankton in the eutrophication-lake in the interval $((n + l)\tau, (n + 1)\tau)]$, $E_1 > 0$ represents the nontransient impulsive dredging effect on phytoplankton x_1 in the eutrophication-lake in the interval $((n + l)\tau, (n + 1)\tau)]$, $E_2 > 0$ represents the nontransient impulsive dredging effect on phytoplankton x_2 in the eutrophication-lake in the interval $((n + l)\tau, (n + 1)\tau)]$, $\mu > 0$ represents the pulse inputting amount of the nutrients with seasonally rainstorm washing from soil around the lake at moment $t = (n + 1)\tau$. The time interval $(n\tau, (n + l)\tau]$ represents the nondredging period, the time interval $((n + l)\tau, (n + 1)\tau]$ represents the dredging period, and $0 < l < 1$ represents the interval length of the nondredging.

3 Some lemmas

The solution $X(t) = (s(t), x_1(t), x_2(t))^T$ of system (2.1) is a nonsmooth function $X: R_+ \to R^3_+$. It is continuous on $(n\tau, (n + l)\tau]$ and $((n + l)\tau, (n + 1)\tau]$, $n \in Z_+$, and the limits $X(n\tau^+) = \lim_{t\to n\tau^+} X(t)$ and $X((n + l)\tau^+) = \lim_{t\to(n+l)\tau^+} X(t)$ exist. Obviously, the global existence and uniqueness of solutions of system (2.1) are guaranteed by the smoothness properties of f defined by right-side of system (2.1) [6].

Lemma 3.1 *For solution $(s(t), x_1(t), x_2(t))$ of system (2.1), there exists a constant $M > 0$ such that $s(t) \le M$, $x_1(t) \le M$, and $x_2(t) \le M$ for all t large enough.*

Proof Defining $V(t) = \delta s(t) + x_1(t) + x_2(t)$ and taking $\delta = \max\{\delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}\}$ and $d = \min\{d_1, d_{11}, d_{12}, d_2, d_{21}, d_{22}\}$, we have $D^+V(t) + dV(t) \le \delta\lambda_1$ for $t \in (n\tau, (n + l)\tau]$. We also have $D^+V(t) + dV(t) \le \delta\lambda_2$ for $t \in ((n + l)\tau, (n + 1)\tau]$. Denoting $\xi = \max\{\delta\lambda_1, \delta\lambda_2\}$, we have the following inequality for $t \neq n\tau$, $t \neq (n+l)\tau$:

$$D^+V(t) + dV(t) \leq \xi.$$

We haver $V(n\tau^+) = V(n\tau) + \mu$ for $t = n\tau$ and $V((n+l)\tau^+) \leq V((n+l)\tau)$ for $t = (n+l)\tau$. By the lemma of [6] we have

$$\begin{aligned} V(t) &\leq V(0)\exp(-dt) + \frac{\xi}{d}\big(1 - \exp(-dt)\big) + \frac{\mu e^{-d(t-\tau)}}{1 - e^{d\tau}} + \frac{\mu e^{d\tau}}{e^{d\tau} - 1} \\ &\to \frac{\xi}{d} + \frac{\mu e^{d\tau}}{e^{d\tau} - 1} \quad \text{as } t \to \infty. \end{aligned}$$

So $V(t)$ is uniformly ultimately bounded. By the definition of $V(t)$ we have that there exists a constant $M > 0$ such that $s(t) \leq M$, $x_1(t) \leq M$, and $x_2(t) \leq M$ for t large enough.

If $x_i(t) = 0$ $(i = 1, 2)$, then a subsystem of system (2.1) is

$$\begin{cases} \frac{ds(t)}{dt} = \lambda_1 - d_1 s(t), & t \in (n\tau, (n+l)\tau], \\ \triangle s(t) = -\mu_s s(t), & t = (n+l)\tau, n \in Z^+, \\ \frac{ds(t)}{dt} == \lambda_2 - d_2 s(t), & t \in ((n+l)\tau, (n+1)\tau], \\ \triangle s(t) = \mu, & t = (n+1)\tau, n \in Z^+. \end{cases} \tag{3.1}$$

Between the impulsive points, system (3.1) has the analytic solution

$$s(t) = \begin{cases} \frac{1}{d_1}[\lambda_1 - (\lambda_1 - d_1 s(n\tau^+))e^{-d_1(t-n\tau)}], & t \in (n\tau, (n+l)\tau], \\ \frac{1}{d_2}[\lambda_2 - (\lambda_2 - d_2 s((n+l)\tau^+))e^{-d_2(t-n\tau)}], & t \in ((n+l)\tau, (n+1)\tau]. \end{cases} \tag{3.2}$$

Considering the second and fourth equations of system (3.1), the stroboscopic map of system (3.1) is presented by

$$\begin{aligned} s\big((n+1)\tau^+\big) = {} & e^{-d_1 l\tau} s\big(n\tau^+\big) \\ & + \mu + \frac{(1-\mu_s)\lambda_1}{d_1} e^{-d_2(1-l)\tau}\big(1 - e^{-d_1 l\tau}\big) + \frac{\lambda_2}{d_2}\big(1 - e^{-d_2(1-l)\tau}\big). \end{aligned} \tag{3.3}$$

The unique fixed point s^* of (3.3) is

$$s^* = \frac{\mu + \frac{(1-\mu_s)\lambda_1}{d_1} e^{-d_2(1-l)\tau}(1 - e^{-d_1 l\tau}) + \frac{\lambda_2}{d_2}(1 - e^{-d_2(1-l)\tau})}{1 - e^{-d_1 l\tau}}. \tag{3.4}$$

□

Similarly to [12], we can easily obtain the following two lemmas.

Lemma 3.2 *The fixed point s^* of (3.3) defined in (3.4) is globally asymptotically stable.*

Lemma 3.3 *The periodic solution $\widetilde{s(t)}$ of system (3.1) is globally asymptotically stable, where $\widetilde{s(t)}$ is defined as*

$$\widetilde{s(t)} = \begin{cases} \frac{1}{d_1}[\lambda_1 - (\lambda_1 - d_1 s^*)e^{-d_1(t-n\tau)}], & t \in (n\tau, (n+l)\tau], \\ \frac{1}{d_2}[\lambda_2 - (\lambda_2 - d_2 s^{**})e^{-d_2(t-n\tau)}], & t \in ((n+l)\tau, (n+1)\tau], \end{cases} \tag{3.5}$$

where s^* is defined in (3.4), and s^{**} is defined as

$$s^{**} = \frac{(1-\mu_s)}{d_1}\left[\lambda_1 - \left(\lambda_1 - d_1 s^*\right)e^{-d_1 l\tau}\right]. \tag{3.6}$$

4 The dynamics

Theorem 4.1 *If*

$$\begin{aligned}\ln\frac{1}{1-\mu_1} > {} & \left(\frac{\beta_{11}}{d_1}\lambda_1 - d_{11}\right)l\tau + \left[\frac{\beta_{21}}{d_2}\lambda_2 - (d_{21}+E_1)\right](1-l)\tau \\ & - \frac{\beta_{11}}{d_1^2}\left(\lambda_1 - d_1 s^*\right)\left(1-e^{-d_1 l\tau}\right) \\ & - \frac{\beta_{21}}{d_2^2}\left(\lambda_2 - d_2 s^{**}\right)e^{-d_2 l\tau}\left(1-e^{-d_2(1-l)\tau}\right)\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}\ln\frac{1}{1-\mu_2} > {} & \left(\frac{\beta_{12}}{d_1}\lambda_1 - d_{12}\right)l\tau + \left[\frac{\beta_{22}}{d_2}\lambda_2 - (d_{22}+E_1)\right](1-l)\tau \\ & - \frac{\beta_{12}}{d_1^2}\left(\lambda_1 - d_1 s^*\right)\left(1-e^{-d_1 l\tau}\right) \\ & - \frac{\beta_{22}}{d_2^2}\left(\lambda_2 - d_2 s^{**}\right)e^{-d_2 l\tau}\left(1-e^{-d_2(1-l)\tau}\right),\end{aligned} \tag{4.2}$$

then the phytoplankton-extinction periodic solution $(\widetilde{s(t)},0,0)$ of system (2.1) is globally asymptotically stable, where s^* is defined in (3.4), and s^{**} is defined in (3.6).

Proof We first prove that the phytoplankton-extinction solution $(\widetilde{s(t)},0,0)$ of (2.1) is locally stable. Defining $s_1(t) = s(t) - \widetilde{s(t)}$, $x_1(t) = x_1(t)$, and $x_2(t) = x_2(t)$, we have the following linearly similar system for system (2.1), which is concerning one periodic solution $(\widetilde{s(t)},0,0)$ of system (2.1):

$$\begin{aligned}\begin{pmatrix}\frac{ds_1(t)}{dt}\\ \frac{dx_1(t)}{dt}\\ \frac{dx_2(t)}{dt}\end{pmatrix} = {} & \begin{pmatrix}-d_1 & \frac{\beta_{11}}{\delta_{11}}\widetilde{s(t)} & \frac{\beta_{12}}{\delta_{12}}\widetilde{s(t)}\\ 0 & \beta_{11}\widetilde{s(t)} - d_{11} & 0\\ 0 & 0 & \beta_{12}\widetilde{s(t)} - d_{12}\end{pmatrix} \\ & \times \begin{pmatrix}s_1(t)\\ x_1(t)\\ x_2(t)\end{pmatrix}, \quad t \in (n\tau,(n+l)\tau],\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}\begin{pmatrix}\frac{ds_1(t)}{dt}\\ \frac{dx_1(t)}{dt}\\ \frac{dx_2(t)}{dt}\end{pmatrix} = {} & \begin{pmatrix}-(d_2+E_s) & \frac{\beta_{21}}{\delta_{21}}\widetilde{s(t)} & \frac{\beta_{22}}{\delta_{22}}\widetilde{s(t)}\\ 0 & \beta_{21}\widetilde{s(t)} - (d_{21}+E_1) & 0\\ 0 & 0 & \beta_{22}\widetilde{s(t)} - (d_{22}+E_2)\end{pmatrix} \\ & \times \begin{pmatrix}s_1(t)\\ x_1(t)\\ x_2(t)\end{pmatrix}, \quad t \in (n\tau,(n+l)\tau].\end{aligned} \tag{4.4}$$

It is easy to obtain the fundamental solution matrix on interval $(n\tau, (n+l)\tau]$

$$\Phi_1(t) = \begin{pmatrix} e^{-d_1(t-n\tau)} & *_{11} & *_{12} \\ 0 & \exp(\int_{n\tau}^{t}(\beta_{11}\widetilde{s(t)} - d_{11})\,ds) & *_{13} \\ 0 & 0 & \exp(\int_{n\tau}^{t}(\beta_{12}\widetilde{s(t)} - d_{12})\,ds) \end{pmatrix}. \tag{4.5}$$

There is no need to calculate the exact form of $*_{1j}$ $(j = 1, 2, 3)$ as they are not required in the analysis that follows, and the fundamental solution matrix on the interval $((n+l)\tau, (n+1)\tau]$ is

$$\Phi_2(t) = \begin{pmatrix} e^{-(d_2+E_s)(t-n\tau)} & *_{21} & *_{22} \\ 0 & A & *_{23} \\ 0 & 0 & \exp(\int_{(n+l)\tau}^{t}(\beta_{22}\widetilde{s(t)} - (d_{22} + E_2)\,ds) \end{pmatrix}, \tag{4.6}$$

where $A = \exp(\int_{(n+l)\tau}^{t}(\beta_{21}\widetilde{s(t)} - (d_{21} + E_1)\,ds)$. There is no need to calculate the exact form of $\star_{2j}$ $(j = 1, 2, 3)$ as they are not required in the analysis that follows.

For $t = (n + l)\tau$, the linearization of the fourth, fifth, and sixth equations of (2.1) is

$$\begin{pmatrix} s_1((n+l)\tau^+) \\ x_1((n+l)\tau^+) \\ x_2((n+l)\tau^+) \end{pmatrix} = \begin{pmatrix} 1-\mu_s & 0 & 0 \\ 0 & 1-\mu_1 & 0 \\ 0 & 0 & 1-\mu_2 \end{pmatrix} \begin{pmatrix} s_1((n+l)\tau) \\ x_1((n+l)\tau) \\ x_2((n+l)\tau) \end{pmatrix}. \tag{4.7}$$

For $t = (n + 1)\tau$, the linearization of the tenth, eleventh, and twelfth equations of (2.1) is

$$\begin{pmatrix} s_1((n+1)\tau^+) \\ x_1((n+1)\tau^+) \\ x_2((n+1)\tau^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} s_1((n+1)\tau) \\ x_1((n+1)\tau) \\ x_2((n+1)\tau) \end{pmatrix}. \tag{4.8}$$

The stability of the periodic solution $(\widetilde{s(t)}, 0, 0)$ is determined by the eigenvalues of

$$M = \begin{pmatrix} 1-\mu_s & 0 & 0 \\ 0 & 1-\mu_1 & 0 \\ 0 & 0 & 1-\mu_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Phi_1(l\tau)\Phi_2(\tau). \tag{4.9}$$

The eigenvalues of (4.9) are represented as

$$\lambda_1 = (1-\mu_s)e^{-[d_1 l + d_2(1-l)]\tau} < 1,$$
$$\lambda_2 = (1-\mu_1)e^{\int_0^{l\tau}(\beta_{11}\widetilde{s(t)} - d_{11})\,ds + \int_{l\tau}^{\tau}(\beta_{21}\widetilde{s(t)} - (d_{21}+E_1)\,ds},$$

and

$$\lambda_3 = (1-\mu_2)e^{\int_0^{l\tau}(\beta_{12}\widetilde{s(t)} - d_{12})\,ds + \int_{(n+l)\tau}^{t}(\beta_{22}\widetilde{s(t)} - (d_{22}+E_2))\,ds}.$$

From (4.1) and (4.2) we have $|\lambda_2| < 1$ and $|\lambda_3| < 1$. Then, according to the Floquet theory [6], we can obtain that the phytoplankton-extinction solution $(\widetilde{s(t)}, 0, 0)$ of system (2.1) is locally stable.

In the next step, we prove that the phytoplankton-extinction solution $(\widetilde{s(t)}, 0, 0)$ of system (2.1) is globally attractive. Choosing $\varepsilon > 0$ such that

$$\rho_1 = (1-\mu_1)e^{\int_0^{l\tau}(\beta_{11}(\widetilde{s(t)}+\varepsilon)-d_{11})\,ds+\int_{l\tau}^{\tau}(\beta_{21}(\widetilde{s(t)}+\varepsilon)-(d_{21}+E_1))\,ds} < 1$$

and

$$\rho_2 = (1-\mu_2)e^{\int_0^{l\tau}(\beta_{12}(\widetilde{s(t)}+\varepsilon)-d_{12})\,ds+\int_{(n+l)\tau}^{t}(\beta_{22}(\widetilde{s(t)}+\varepsilon)-(d_{22}+E_2))\,ds} < 1,$$

we have the following two inequalities by the first and seventh equations of (2.1):

$$\frac{ds(t)}{dt} \le \lambda_1 - d_1 s(t) \tag{4.10}$$

and

$$\frac{ds(t)}{dt} \le \lambda_2 - (d_2 + E_s)s(t). \tag{4.11}$$

Therefore we find the comparatively impulsive differential equation

$$\begin{cases} \frac{ds_1(t)}{dt} = \lambda_1 - d_1 s_1(t), & t \in (n\tau, (n+l)\tau], \\ \triangle s_1(t) = -\mu_s s_1(t), & t = (n+l)\tau, n \in Z^+, \\ \frac{ds_1(t)}{dt} = \lambda_2 - (d_2 + E_s)s_1(t), & t \in ((n+l)\tau, (n+1)\tau], \\ \triangle s_1(t) = \mu, & t = (n+1)\tau, n \in Z^+. \end{cases} \tag{4.12}$$

From Lemma 3.3. and the comparison theorem of impulsive equation [6] we have $s(t) \le s_1(t)$ and $s_1(t) \to \widetilde{s_1(t)}$ as $t \to \infty$. Then there exists $\varepsilon > 0$ small enough such that

$$s(t) \le s_1(t) \le \widetilde{s_1(t)} + \varepsilon = \widetilde{s(t)} + \varepsilon \tag{4.13}$$

for all t large enough. For convenience, we assume that (4.13) holds for all $t \ge 0$. From (2.1) and (4.13) we have

$$\begin{cases} \left.\begin{aligned} \frac{dx_1(t)}{dt} &\le [\beta_{11}(\widetilde{s(t)}+\varepsilon) - d_{11}]x_1(t), \\ \frac{dx_2(t)}{dt} &\le [\beta_{12}(\widetilde{s(t)}+\varepsilon) - d_{12}]x_2(t), \end{aligned}\right\} & t \in (n\tau, (n+l)\tau], \\ \left.\begin{aligned} \triangle x_1(t) &= -\mu_1 x_1(t), \\ \triangle x_2(t) &= -\mu_2 x_2(t), \end{aligned}\right\} & t = (n+l)\tau, n \in Z^+, \\ \left.\begin{aligned} \frac{dx_1(t)}{dt} &\le [\beta_{21}(\widetilde{s(t)}+\varepsilon) - (d_{21}+E_1)]x_1(t), \\ \frac{dx_2(t)}{dt} &\le [\beta_{22}(\widetilde{s(t)}+\varepsilon) - (d_{21}+E_2)]x_2(t), \end{aligned}\right\} & t \in ((n+l)\tau, (n+1)\tau], \\ \left.\begin{aligned} \triangle x_1(t) &= 0, \\ \triangle x_2(t) &= 0, \end{aligned}\right\} & t = (n+1)\tau, n \in Z^+. \end{cases} \tag{4.14}$$

Therefore

$$x_1\big((n+1)\tau\big) \le x_1\big(n\tau^+\big)(1-\mu_1)e^{\int_{n\tau}^{(n+l)\tau}(\beta_{11}(\widetilde{s(t)}+\varepsilon)-d_{11})\,ds+\int_{(n+l)\tau}^{(n+1)\tau}(\beta_{21}(\widetilde{s(t)}+\varepsilon)-(d_{21}+E_1))\,ds}$$

and

$$\begin{aligned}&x_2\big((n+1)\tau\big)\\&\quad\le x_2\big(n\tau^+\big)(1-\mu_2)e^{\int_{n\tau}^{(n+l)\tau}(\beta_{12}(\widetilde{s(t)}+\varepsilon)-d_{12})\,ds+\int_{(n+l)\tau}^{(n+1)\tau}(\beta_{22}(\widetilde{s(t)}+\varepsilon)-(d_{22}+E_2))\,ds}.\end{aligned}$$

Hence $x_i(n\tau) \le x_i(0^+)\rho_i^n$ $(i=1,2)$. So $x_i(n\tau) \to 0$ $(i=1,2)$ as $n \to \infty$. Therefore $x_i(t) \to 0$ $(i=1,2)$ as $t \to \infty$.

In the third step, we prove that $s(t) \to \widetilde{s(t)}$ as $t \to \infty$. For $\varepsilon_1 > 0$ small enough, there exists $t_0 > 0$ such that $0 < x_1(t) < \varepsilon_1$ and $0 < x_2(t) < \varepsilon_1$ for all $t \ge t_0$. Without loss of generality, we assume that $0 < x_1(t) < \varepsilon_1$ and $0 < x_2(t) < \varepsilon_1$ for all $t \ge 0$. Then we have

$$\lambda_1 - \left[d_1 + \left(\frac{\beta_{11}}{d_{11}} + \frac{\beta_{12}}{d_{12}}\right)\varepsilon_1\right]s(t) \le \frac{ds(t)}{dt} \le \lambda_1 - d_1 s(t) \tag{4.15}$$

and

$$\lambda_2 - \left[d_2 + \left(\frac{\beta_{12}}{d_{12}} + \frac{\beta_{22}}{d_{22}}\right)\varepsilon_1\right]s(t) \le \frac{ds(t)}{dt} \le \lambda_2 - d_2 s(t), \tag{4.16}$$

and $z_2(t) \le s(t) \le z_1(t)$ and $z_1(t) \to \widetilde{z_1(t)}$, $z_2(t) \to \widetilde{z_2(t)}$ as $t \to \infty$, where $z_1(t)$ and $z_2(t)$ are the solutions of

$$\begin{cases}\frac{dz_1(t)}{dt} = \lambda_1 - [d_1 + (\frac{\beta_{11}}{d_{11}} + \frac{\beta_{12}}{d_{12}})\varepsilon_1]z_1(t), & t \in (n\tau, (n+l)\tau],\\ \triangle z_1(t) = -\mu_s z_1(t), & t = (n+l)\tau,\\ \frac{dz_1(t)}{dt} = \lambda_2 - [d_2 + (\frac{\beta_{12}}{d_{12}} + \frac{\beta_{22}}{d_{22}})\varepsilon_1]z_1(t), & t \in ((n+l)\tau, (n+1)\tau],\\ \triangle z_1(t) = \mu, & t = (n+1)\tau, n \in Z^+,\end{cases} \tag{4.17}$$

and

$$\begin{cases}\frac{dz_2(t)}{dt} = \lambda_1 - d_1 z_2(t), & t \in (n\tau, (n+l)\tau],\\ \triangle z_2(t) = -\mu_s z_2(t), & t = (n+l)\tau,\\ \frac{dz_2(t)}{dt} = \lambda_2 - d_2 z_2(t), & t \in ((n+l)\tau, (n+1)\tau],\\ \triangle z_2(t) = \mu, & t = (n+1)\tau, n \in Z^+,\end{cases} \tag{4.18}$$

respectively. Similarly to Lemma 3.3, the periodic solution of (4.17), which is globally asymptotically stable, is

$$\widetilde{z_1(t)} = \begin{cases}\frac{1}{d_1}[\lambda_1 - (\lambda_1 - [d_1 + (\frac{\beta_{11}}{d_{11}} + \frac{\beta_{12}}{d_{12}})\varepsilon_1]z_1^*)\\ \quad \times e^{-[d_1+(\frac{\beta_{11}}{d_{11}}+\frac{\beta_{12}}{d_{12}})\varepsilon_1](t-n\tau)}], \quad t \in (n\tau, (n+l)\tau],\\ \frac{1}{[d_2+(\frac{\beta_{12}}{d_{12}}+\frac{\beta_{22}}{d_{22}})\varepsilon_1]}[\lambda_2 - (\lambda_2 - (d_2 + (\frac{\beta_{12}}{d_{12}} + \frac{\beta_{22}}{d_{22}})\varepsilon_1)z_1^{**})\\ \quad \times e^{-[d_2+(\frac{\beta_{12}}{d_{12}}+\frac{\beta_{22}}{d_{22}})\varepsilon_1](t-n\tau)}], \quad t \in ((n+l)\tau, (n+1)\tau],\end{cases} \tag{4.19}$$

where

$$z_1^* = \frac{\mu + \frac{(1-\mu_s)\lambda_1}{[d_1+(\frac{\beta_{11}}{d_{11}}+\frac{\beta_{12}}{d_{12}})\varepsilon_1]} e^{-[d_2+(\frac{\beta_{12}}{d_{12}}+\frac{\beta_{22}}{d_{22}})\varepsilon_1](1-l)\tau}(1-e^{-[d_1+(\frac{\beta_{11}}{d_{11}}+\frac{\beta_{12}}{d_{12}})\varepsilon_1]l\tau})}{1-e^{-[d_1+(\frac{\beta_{11}}{d_{11}}+\frac{\beta_{12}}{d_{12}})\varepsilon_1]l\tau}} + \frac{\frac{\lambda_2}{[d_2+(\frac{\beta_{12}}{d_{12}}+\frac{\beta_{22}}{d_{22}})\varepsilon_1]}(1-e^{-[d_2+(\frac{\beta_{12}}{d_{12}}+\frac{\beta_{22}}{d_{22}})\varepsilon_1](1-l)\tau})}{1-e^{-[d_1+(\frac{\beta_{11}}{d_{11}}+\frac{\beta_{12}}{d_{12}})\varepsilon_1]l\tau}}, \tag{4.20}$$

and

$$z_1^{**} = \frac{(1-\mu_s)}{[d_1+(\frac{\beta_{11}}{d_{11}}+\frac{\beta_{12}}{d_{12}})\varepsilon_1]}\left[\lambda_1 - \left(\lambda_1 - \left[d_1 + \left(\frac{\beta_{11}}{d_{11}}+\frac{\beta_{12}}{d_{12}}\right)\varepsilon_1\right]s^*\right) \times e^{-[d_1+(\frac{\beta_{11}}{d_{11}}+\frac{\beta_{12}}{d_{12}})\varepsilon_1]l\tau}\right]. \tag{4.21}$$

Therefore, for any $\varepsilon > 0$, there exists $t > t_1$ such that

$$\widetilde{z_1(t)} - \varepsilon < s(t) < \widetilde{z_2(t)} + \varepsilon.$$

Letting $\varepsilon_1 \to 0$, we have

$$\widetilde{s(t)} - \varepsilon < s(t) < \widetilde{s(t)} + \varepsilon$$

for t large enough, which implies $s(t) \to \widetilde{s(t)}$ as $t \to \infty$. This completes the proof. □

Theorem 4.2 *If*

$$\begin{aligned} \ln\frac{1}{1-\mu_1} &< \left(\frac{\beta_{11}}{d_1}\lambda_1 - d_{11}\right)l\tau + \left[\frac{\beta_{21}}{d_2}\lambda_2 - (d_{21}+E_1)\right](1-l)\tau \\ &\quad - \frac{\beta_{11}}{d_1^2}\left(\lambda_1 - d_1 s^*\right)\left(1-e^{-d_1 l\tau}\right) \\ &\quad - \frac{\beta_{21}}{d_2^2}\left(\lambda_2 - d_2 s^{**}\right)e^{-d_2 l\tau}\left(1-e^{-d_2(1-l)\tau}\right) \end{aligned} \tag{4.22}$$

and

$$\begin{aligned} \ln\frac{1}{1-\mu_2} &< \left(\frac{\beta_{12}}{d_1}\lambda_1 - d_{12}\right)l\tau + \left[\frac{\beta_{22}}{d_2}\lambda_2 - (d_{22}+E_1)\right](1-l)\tau \\ &\quad - \frac{\beta_{12}}{d_1^2}\left(\lambda_1 - d_1 s^*\right)\left(1-e^{-d_1 l\tau}\right) \\ &\quad - \frac{\beta_{22}}{d_2^2}\left(\lambda_2 - d_2 s^{**}\right)e^{-d_2 l\tau}\left(1-e^{-d_2(1-l)\tau}\right), \end{aligned} \tag{4.23}$$

then system (2.1) *is permanent, where* s^* *is defined in* (3.4) *and* s^{**} *is defined in* (3.6).

Proof By Lemma 3.1, $s(t) \le M$, $x_1(t) \le M$, and $x_2(t) \le M$ for t large enough. We may assum that $s(t) \le M$, $x_1(t) \le M$, and $x_2(t) \le M$ for $t \ge 0$. Therefore we have

$$\frac{ds(t)}{dt} \ge \lambda_1 - \left[d_1 + \left(\frac{\beta_{11}}{d_{11}} + \frac{\beta_{12}}{d_{12}}\right)M\right]s(t), \tag{4.24}$$

$$\frac{ds(t)}{dt} \ge \lambda_2 - \left[d_2 + \left(\frac{\beta_{12}}{d_{12}} + \frac{\beta_{22}}{d_{22}}\right)\varepsilon_1\right]s(t), \tag{4.25}$$

and $s(t) \ge z_3(t)$ and $z_3(t) \to \widetilde{z_3(t)}$ as $t \to \infty$, where $z_3(t)$ is the globally asymptotically stable solution of the comparatively impulsive differential equation

$$\begin{cases} \frac{dz_3(t)}{dt} = \lambda_1 - [d_1 + (\frac{\beta_{11}}{d_{11}} + \frac{\beta_{12}}{d_{12}})M]z_3(t), & t \in (n\tau, (n+l)\tau], \\ \triangle z_3(t) = -\mu_s z_3(t), & t = (n+l)\tau, n \in Z^+, \\ \frac{dz_3(t)}{dt} = \lambda_2 - [d_2 + (\frac{\beta_{12}}{d_{12}} + \frac{\beta_{22}}{d_{22}})M]z_3(t), & t \in ((n+l)\tau, (n+1)\tau], \\ \triangle z_3(t) = \mu, & t = (n+1)\tau, n \in Z^+, \end{cases} \tag{4.26}$$

with

$$\widetilde{z_3(t)} = \begin{cases} \frac{1}{d_1}[\lambda_1 - (\lambda_1 - [d_1 + (\frac{\beta_{11}}{d_{11}} + \frac{\beta_{12}}{d_{12}})M]z_3^*) \\ \quad \times e^{-[d_1 + (\frac{\beta_{11}}{d_{11}} + \frac{\beta_{12}}{d_{12}})M](t-n\tau)}], & t \in (n\tau, (n+l)\tau], \\ \frac{1}{[d_2 + (\frac{\beta_{12}}{d_{12}} + \frac{\beta_{22}}{d_{22}})M]}[\lambda_2 - (\lambda_2 - (d_2 + (\frac{\beta_{12}}{d_{12}} + \frac{\beta_{22}}{d_{22}})M)z_3^{**}) \\ \quad \times e^{-[d_2 + (\frac{\beta_{12}}{d_{12}} + \frac{\beta_{22}}{d_{22}})M](t-n\tau)}], & t \in ((n+l)\tau, (n+1)\tau], \end{cases} \tag{4.27}$$

where

$$\begin{aligned} z_3^* = {} & \frac{\mu + \frac{(1-\mu_s)\lambda_1}{[d_1 + (\frac{\beta_{11}}{d_{11}} + \frac{\beta_{12}}{d_{12}})M]} e^{-[d_2 + (\frac{\beta_{12}}{d_{12}} + \frac{\beta_{22}}{d_{22}})M](1-l)\tau} (1 - e^{-[d_1 + (\frac{\beta_{11}}{d_{11}} + \frac{\beta_{12}}{d_{12}})M]l\tau})}{1 - e^{-[d_1 + (\frac{\beta_{11}}{d_{11}} + \frac{\beta_{12}}{d_{12}})M]l\tau}} \\ & + \frac{\frac{\lambda_2}{[d_2 + (\frac{\beta_{12}}{d_{12}} + \frac{\beta_{22}}{d_{22}})M]} (1 - e^{-[d_2 + (\frac{\beta_{12}}{d_{12}} + \frac{\beta_{22}}{d_{22}})M](1-l)\tau})}{1 - e^{-[d_1 + (\frac{\beta_{11}}{d_{11}} + \frac{\beta_{12}}{d_{12}})M]l\tau}} \end{aligned} \tag{4.28}$$

and

$$\begin{aligned} z_3^{**} = {} & \frac{(1-\mu_s)}{[d_1 + (\frac{\beta_{11}}{d_{11}} + \frac{\beta_{12}}{d_{12}})M]} \\ & \times \left[\lambda_1 - \left(\lambda_1 - \left[d_1 + \left(\frac{\beta_{11}}{d_{11}} + \frac{\beta_{12}}{d_{12}}\right)M\right]z_3^*\right) e^{-[d_1 + (\frac{\beta_{11}}{d_{11}} + \frac{\beta_{12}}{d_{12}})M]l\tau}\right]. \end{aligned} \tag{4.29}$$

Therefore, for any $\varepsilon_2 > 0$,

$$s(t) > \widetilde{z_3(t)} - \varepsilon_2 \tag{4.30}$$

for t large enough, which implies that

$$
\begin{aligned}
s(t) \geq{}& \frac{1}{d_1}\left[\lambda_1 - \left(\lambda_1 - \left[d_1 + \left(\frac{\beta_{11}}{d_{11}} + \frac{\beta_{12}}{d_{12}}\right)M\right]z_1^*\right) \times e^{-[d_1+(\frac{\beta_{11}}{d_{11}}+\frac{\beta_{12}}{d_{12}})M]\tau}\right] \\
&+ \frac{1}{[d_2 + (\frac{\beta_{12}}{d_{12}} + \frac{\beta_{22}}{d_{22}})M]}\left[\lambda_2 - \left(\lambda_2 - \left(d_2 + \left(\frac{\beta_{12}}{d_{12}} + \frac{\beta_{22}}{d_{22}}\right)M\right)z_1^{**}\right)\right. \\
&\left.\times e^{-[d_2+(\frac{\beta_{12}}{d_{12}}+\frac{\beta_{22}}{d_{22}})M]\tau}\right] - \varepsilon_2 = m_2.
\end{aligned}
$$

Thus we only need to find $m_1 > 0$ such that $x_1(t) \geq m_1$ and $x_2(t) \geq m_1$ for t large enough.

By the conditions of this theorem we can select $m_3 > 0$ and $\varepsilon_1 > 0$ small enough such that

$$
\begin{aligned}
\sigma_1 ={}& \left(\frac{\beta_{11}}{[d_1 + (\frac{\beta_{11}}{d_{11}} + \frac{\beta_{12}}{d_{12}})m_3]}\lambda_1 - d_{11} - \varepsilon_1\right)l\tau \\
&+ \left[\frac{\beta_{21}}{[d_2 + (\frac{\beta_{12}}{d_{12}} + \frac{\beta_{22}}{d_{22}})m_3]}\lambda_2 - (d_{21} + E_1 + \varepsilon_1)\right](1-l)\tau \\
&- \frac{\beta_{11}}{[d_1 + (\frac{\beta_{11}}{d_{11}} + \frac{\beta_{12}}{d_{12}})m_3]^2}\left(\lambda_1 - \left[d_1 + \left(\frac{\beta_{11}}{d_{11}} + \frac{\beta_{12}}{d_{12}}\right)m_3\right]s^*\right) \\
&\times \left(1 - e^{-[d_1+(\frac{\beta_{11}}{d_{11}}+\frac{\beta_{12}}{d_{12}})m_3]l\tau}\right) \\
&- \frac{\beta_{21}}{[d_2 + (\frac{\beta_{12}}{d_{12}} + \frac{\beta_{22}}{d_{22}})m_3]^2}\left(\lambda_2 - d_2 s^{**}\right)e^{-[d_2+(\frac{\beta_{12}}{d_{12}}+\frac{\beta_{22}}{d_{22}})m_3]l\tau}\left(1 - e^{-d_2(1-l)\tau}\right) > 1
\end{aligned}
$$

and

$$
\begin{aligned}
\sigma_2 ={}& \left(\frac{\beta_{12}}{[d_1 + (\frac{\beta_{11}}{d_{11}} + \frac{\beta_{12}}{d_{12}})m_3]}\lambda_1 - d_{12} - \varepsilon_1\right)l\tau \\
&+ \left[\frac{\beta_{22}}{[d_2 + (\frac{\beta_{12}}{d_{12}} + \frac{\beta_{22}}{d_{22}})m_3]}\lambda_2 - (d_{22} + E_1 + \varepsilon_1)\right](1-l)\tau \\
&- \frac{\beta_{12}}{[d_1 + (\frac{\beta_{11}}{d_{11}} + \frac{\beta_{12}}{d_{12}})m_3]^2}\left(\lambda_1 - \left[d_1 + \left(\frac{\beta_{11}}{d_{11}} + \frac{\beta_{12}}{d_{12}}\right)m_3\right]s^*\right) \\
&\times \left(1 - e^{-[d_1+(\frac{\beta_{11}}{d_{11}}+\frac{\beta_{12}}{d_{12}})m_3]l\tau}\right) \\
&- \frac{\beta_{22}}{[d_2 + (\frac{\beta_{12}}{d_{12}} + \frac{\beta_{22}}{d_{22}})m_3]^2}\left(\lambda_2 - \left[d_2 + \left(\frac{\beta_{12}}{d_{12}} + \frac{\beta_{22}}{d_{22}}\right)m_3\right]s^{**}\right) \\
&\times e^{-d_2 l\tau}\left(1 - e^{-[d_2+(\frac{\beta_{12}}{d_{12}}+\frac{\beta_{22}}{d_{22}})m_3](1-l)\tau}\right) > 1.
\end{aligned}
$$

We prove that $x_1(t) < m_3$ and $x_2(t) < m_3$ cannot hold for $t \geq 0$. Otherwise,

$$
\begin{cases}
\frac{ds(t)}{dt} \geq \lambda_1 - [d_1 + (\frac{\beta_{11}}{d_{11}} + \frac{\beta_{12}}{d_{12}})m_3]s(t), & t \in (n\tau, (n+l)\tau], \\
\triangle s(t) = -\mu_s s(t), & t = (n+l)\tau, n \in Z^+, \\
\frac{ds(t)}{dt} \geq \lambda_2 - [d_2 + (\frac{\beta_{12}}{d_{12}} + \frac{\beta_{22}}{d_{22}})m_3]s(t), & t \in ((n+l)\tau, (n+1)\tau], \\
\triangle s(t) = \mu, & t = (n+1)\tau, n \in Z^+.
\end{cases}
\tag{4.31}
$$

By Lemma 3.3 we have $s(t) \geq z(t)$ and $z(t) \to \overline{z(t)}$, $t \to \infty$, where $z(t)$ is the globally asymptotically stable solution of

$$\begin{cases} \frac{dz(t)}{dt} = \lambda_1 - [d_1 + (\frac{\beta_{11}}{d_{11}} + \frac{\beta_{12}}{d_{12}})m_3]z(t), \quad t \in (n\tau, (n+l)\tau], \\ \triangle z(t) = -\mu_s z(t), \quad t = (n+l)\tau, n \in Z^+, \\ \frac{dz(t)}{dt} = \lambda_2 - [d_2 + (\frac{\beta_{12}}{d_{12}} + \frac{\beta_{22}}{d_{22}})m_3]z(t), \quad t \in ((n+l)\tau, (n+1)\tau], \\ \triangle z(t) = \mu, \quad t = (n+1)\tau, n \in Z^+, \end{cases} \tag{4.32}$$

with

$$\overline{z(t)} = \begin{cases} \frac{1}{d_1}[\lambda_1 - (\lambda_1 - [d_1 + (\frac{\beta_{11}}{d_{11}} + \frac{\beta_{12}}{d_{12}})m_3]z^*) \\ \quad \times e^{-[d_1+(\frac{\beta_{11}}{d_{11}}+\frac{\beta_{12}}{d_{12}})m_3](t-n\tau)}], \quad t \in (n\tau, (n+l)\tau], \\ \frac{1}{[d_2+(\frac{\beta_{12}}{d_{12}}+\frac{\beta_{22}}{d_{22}})m_3]}[\lambda_2 - (\lambda_2 - (d_2 + (\frac{\beta_{12}}{d_{12}} + \frac{\beta_{22}}{d_{22}})m_3)z^{**}) \\ \quad \times e^{-[d_2+(\frac{\beta_{12}}{d_{12}}+\frac{\beta_{22}}{d_{22}})m_3](t-n\tau)}], \quad t \in ((n+l)\tau, (n+1)\tau], \end{cases} \tag{4.33}$$

where

$$\begin{aligned} z^* = {} & \frac{\mu + \frac{(1-\mu_s)\lambda_1}{[d_1+(\frac{\beta_{11}}{d_{11}}+\frac{\beta_{12}}{d_{12}})m_3]} e^{-[d_2+(\frac{\beta_{12}}{d_{12}}+\frac{\beta_{22}}{d_{22}})m_3](1-l)\tau}(1 - e^{-[d_1+(\frac{\beta_{11}}{d_{11}}+\frac{\beta_{12}}{d_{12}})m_3]l\tau})}{1 - e^{-[d_1+(\frac{\beta_{11}}{d_{11}}+\frac{\beta_{12}}{d_{12}})m_3]l\tau}} \\ & + \frac{\frac{\lambda_2}{[d_2+(\frac{\beta_{12}}{d_{12}}+\frac{\beta_{22}}{d_{22}})m_3]}(1 - e^{-[d_2+(\frac{\beta_{12}}{d_{12}}+\frac{\beta_{22}}{d_{22}})m_3](1-l)\tau})}{1 - e^{-[d_1+(\frac{\beta_{11}}{d_{11}}+\frac{\beta_{12}}{d_{12}})m_3]l\tau}}, \end{aligned} \tag{4.34}$$

and

$$\begin{aligned} z^{**} = {} & \frac{(1-\mu_s)}{[d_1 + (\frac{\beta_{11}}{d_{11}} + \frac{\beta_{12}}{d_{12}})m_3]} \\ & \times \left[\lambda_1 - \left(\lambda_1 - \left[d_1 + \left(\frac{\beta_{11}}{d_{11}} + \frac{\beta_{12}}{d_{12}}\right)m_3\right]z^*\right)e^{-[d_1+(\frac{\beta_{11}}{d_{11}}+\frac{\beta_{12}}{d_{12}})m_3]l\tau}\right]. \end{aligned} \tag{4.35}$$

Therefore there $T_1 > 0$ such that, for $t \geq T_1$,

$$s(t) \geq z(t) \geq \overline{z(t)} - \varepsilon_1$$

and

$$\begin{cases} \left.\begin{aligned} \frac{dx_1(t)}{dt} \geq [\beta_{11}(\overline{z(t)} - \varepsilon) - d_{11}]x_1(t), \\ \frac{dx_2(t)}{dt} \geq [\beta_{12}(\overline{z(t)} - \varepsilon) - d_{12}]x_2(t), \end{aligned}\right\} \quad t \in (n\tau, (n+l)\tau], \\ \left.\begin{aligned} \triangle x_1(t) = -\mu_1 x_1(t), \\ \triangle x_2(t) = -\mu_2 x_2(t), \end{aligned}\right\} \quad t = (n+l)\tau, n \in Z^+, \\ \left.\begin{aligned} \frac{dx_1(t)}{dt} \geq [\beta_{21}(\overline{z(t)} - \varepsilon) - (d_{21} + E_1)]x_1(t), \\ \frac{dx_2(t)}{dt} \geq [\beta_{22}(\overline{z(t)} - \varepsilon) - (d_{21} + E_2)]x_2(t), \end{aligned}\right\} \quad t \in ((n+l)\tau, (n+1)\tau], \\ \left.\begin{aligned} \triangle x_1(t) = 0, \\ \triangle x_2(t) = 0, \end{aligned}\right\} \quad t = (n+1)\tau, n \in Z^+. \end{cases} \tag{4.36}$$

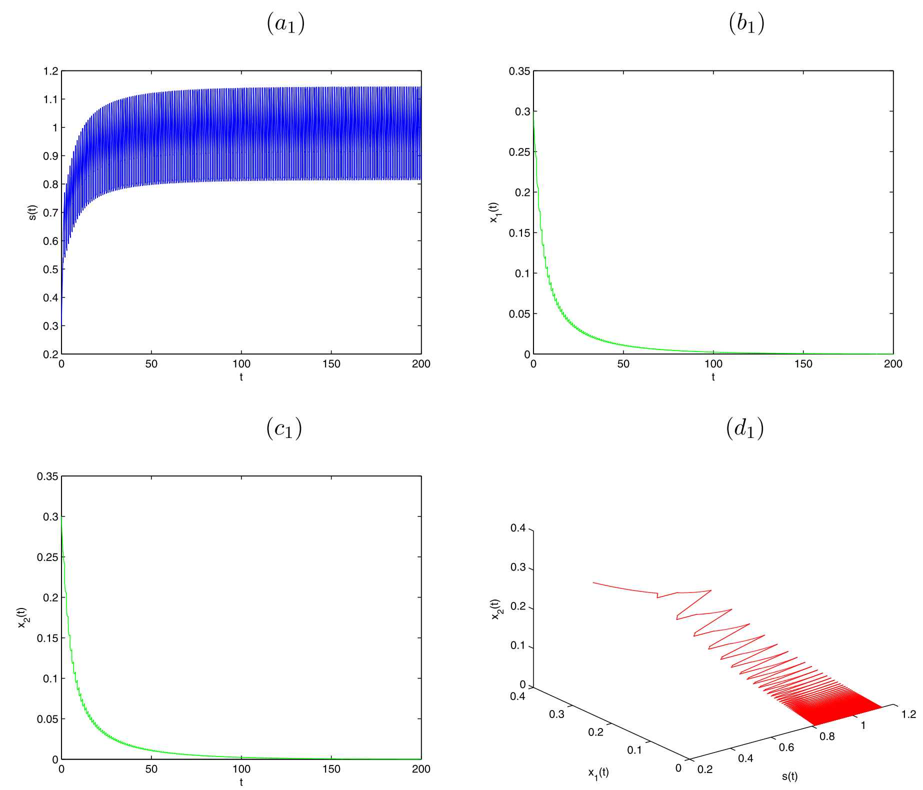

Figure 2 Globally asymptotically stable phytoplankton-extinction periodic solution of system (2.1) with $s(0) = 0.3$, $x_1(0) = 0.3$, $x_2(0) = 0.3$, $\lambda_1 = 0.5$, $d_1 = 0.2$, $\beta_{11} = 0.5$, $\delta_{11} = 1$, $\beta_{12} = 0.5$, $\delta_{12} = 1$, $d_{11} = 0.4$, $d_{12} = 0.4$, $\lambda_2 = 0.1$, $d_2 = 0.2$, $\beta_{21} = 0.3$, $\delta_{21} = 1$, $\beta_{22} = 0.3$, $\delta_{22} = 1$, $d_{21} = 0.3$, $d_{22} = 0.3$, $E_s = 0.3$, $E_1 = 0.2$, $E_2 = 0.2$, $\mu_s = 0.28$, $\mu_1 = 0.1$, $\mu_2 = 0.1$, $\mu = 0.1$, $l = 0.8$, $\tau = 1$; (**a**) time-series of $s(t)$; (**b**) time-series of $x_1(t)$; (**c**) time-series of $x_1(t)$; (**d**) the phase portrait of the globally asymptotically stable phytoplankton-extinction periodic solution of system (2.1)

Let $N_1 \in N$ and $N_1\tau > T_1$, Integrating (4.36) on $(n\tau, (n+1)\tau)$, $n \geq N_1$, we have

$$\begin{aligned} x_1\bigl((n+1)\tau\bigr) &\geq x_1\bigl(n\tau^+\bigr)(1-\mu_1) \\ &\quad \times e^{\int_{n\tau}^{(n+l)\tau}(\beta_{11}(\overline{z(t)}-\varepsilon)-d_{11})\,ds+\int_{(n+l)\tau}^{(n+1)\tau}(\beta_{21}(\overline{z(t)}-\varepsilon)-(d_{21}+E_1))\,ds} \\ &= (1-\mu_1)x_1\bigl(n\tau^+\bigr)e^{\sigma_1} \end{aligned} \tag{4.37}$$

and

$$\begin{aligned} x_2\bigl((n+1)\tau\bigr) &\geq x_2\bigl(n\tau^+\bigr)(1-\mu_2) \\ &\quad \times e^{\int_{n\tau}^{(n+l)\tau}(\beta_{12}(\overline{z(t)}-\varepsilon)-d_{12})\,ds+\int_{(n+l)\tau}^{(n+1)\tau}(\beta_{22}(\overline{z(t)}-\varepsilon)-(d_{22}+E_2))\,ds} \\ &= (1-\mu_2)x_2\bigl(n\tau^+\bigr)e^{\sigma_2}. \end{aligned} \tag{4.38}$$

Then $x_1((N_1+k)\tau) \geq (1-\mu_1)^k x_1(N_1\tau^+)e^{k\sigma_1} \to \infty$ and $x_2((N_1+k)\tau) \geq (1-\mu_2)^k x_2(N_1\tau^+) \times e^{k\sigma_2} \to \infty$ as $k \to \infty$, which is a contradiction to the boundedness of $x_1(t)$ and $x_1(t)$. Hence there exists $t_1 > 0$ such that $x_1(t) \geq m_1$ and $x_2(t) \geq m_1$. The proof is complete. □

5 Discussion

According to the fact of water management, we propose a periodic lake-eutrophication model with nontransient/transient impulsive dredging and pulse inputting on nutrients. We proved that the phytoplankton-extinction boundary periodic solution of system (2.1) is globally asymptotically stable and obtained the conditions for the permanence of system (2.1).

If we suppose that $s(0) = 0.3$, $x_1(0) = 0.3$, $x_2(0) = 0.3$, $\lambda_1 = 0.5$, $d_1 = 0.2$, $\beta_{11} = 0.5$, $\delta_{11} = 1$, $\beta_{12} = 0.5$, $\delta_{12} = 1$, $d_{11} = 0.4$, $d_{12} = 0.4$, $\lambda_2 = 0.1$, $d_2 = 0.2$, $\beta_{21} = 0.3$, $\delta_{21} = 1$, $\beta_{22} = 0.3$, $\delta_{22} = 1$, $d_{21} = 0.3$, $d_{22} = 0.3$, $E_s = 0.3$, $E_1 = 0.2$, $E_2 = 0.2$, $\mu_s = 0.28$, $\mu_1 = 0.1$, $\mu_2 = 0.1$, $\mu = 0.1$, $l = 0.8$, $\tau = 1$, then these parameter values satisfy Theorem 4.1. Then the phytoplankton-extinction periodic solution $(\widetilde{s(t)}, 0, 0)$ of system (2.1) is globally asymptotically stable (see Fig. 2). If we assume that $s(0) = 0.3$, $x_1(0) = 0.3$, $x_2(0) = 0.3$, $\lambda_1 = 0.5$, $d_1 = 0.2$, $\beta_{11} = 0.8$, $\delta_{11} = 1$, $\beta_{12} = 0.8$, $\delta_{12} = 1$, $d_{11} = 0.4$, $d_{12} = 0.4$, $\lambda_2 = 0.2$, $d_2 = 0.2$, $\beta_{21} = 0.5$, $\delta_{21} = 1$, $\beta_{22} = 0.5$, $\delta_{22} = 1$, $d_{21} = 0.3$, $d_{22} = 0.3$, $E_s = 0.3$, $E_1 = 0.2$, $E_2 = 0.2$, $\mu_s = 0.2$, $\mu_1 = 0.1$, $\mu_2 = 0.1$, $\mu = 0.1$, $l = 0.8$, $\tau = 1$, then these parameter values satisfy Theorem 4.2. Then system (2.1) is permanent (see Fig. 3). From Theorems 4.1 and 4.2, and Figs. 2 and 3 we can deduce that the parameter λ_2 has a controlling threshold λ_2^*. When $\lambda_2 < \lambda_2^*$, the phytoplankton-extinction periodic solution $(\widetilde{s(t)}, 0, 0)$ of system (2.1) is globally asymptotically stable. When $\lambda_2 > \lambda_2^*$, system (2.1) is permanent. That is to say, we should re-

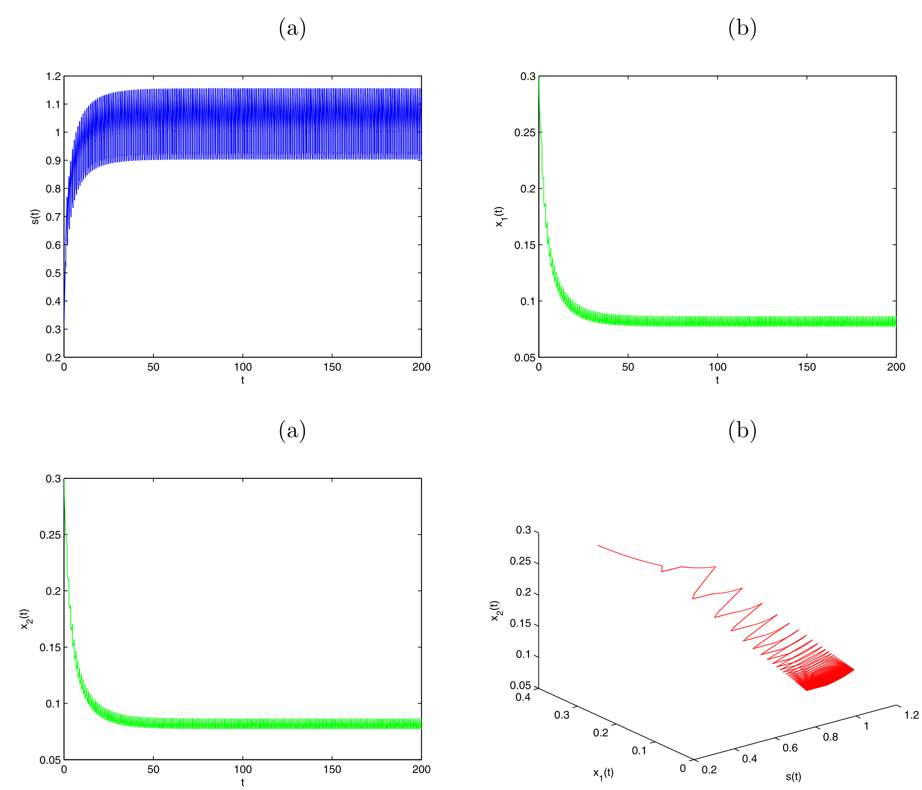

Figure 3 The permanence of system (2.1) with $s(0) = 0.3$, $x_1(0) = 0.3$, $x_2(0) = 0.3$, $\lambda_1 = 0.5$, $d_1 = 0.2$, $\beta_{11} = 0.5$, $\delta_{11} = 1$, $\beta_{12} = 0.5$, $\delta_{12} = 1$, $d_{11} = 0.4$, $d_{12} = 0.4$, $\lambda_2 = 0.1$, $d_2 = 0.2$, $\beta_{21} = 0.3$, $\delta_{21} = 1$, $\beta_{22} = 0.3$, $\delta_{22} = 1$, $d_{21} = 0.3$, $d_{22} = 0.3$, $E_s = 0.3$, $E_1 = 0.2$, $E_2 = 0.2$, $\mu_s = 0.2$, $\mu_1 = 0.1$, $\mu_2 = 0.1$, $\mu = 0.1$, $l = 0.8$, $\tau = 1$; (**a**) time-series of $s(t)$; (**b**) time-series of $x_1(t)$; (**c**) time-series of $x_2(t)$; (**d**) the phase portrait of the permanence of system (2.1)

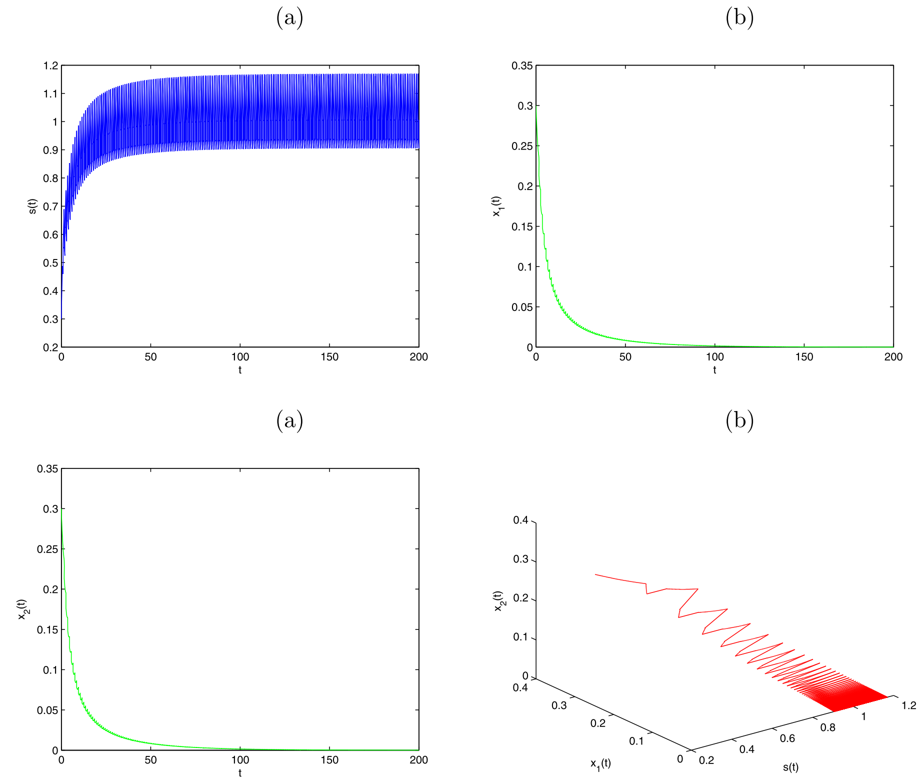

Figure 4 Globally asymptotically stable phytoplankton-extinction periodic solution of system (2.1) with $s(0) = 0.3$, $x_1(0) = 0.3$, $x_2(0) = 0.3$, $\lambda_1 = 0.5$, $d_1 = 0.2$, $\beta_{11} = 0.5$, $\delta_{11} = 1$, $\beta_{12} = 0.5$, $\delta_{12} = 1$, $d_{11} = 0.4$, $d_{12} = 0.4$, $\lambda_2 = 0.1$, $d_2 = 0.2$, $\beta_{21} = 0.3$, $\delta_{21} = 1$, $\beta_{22} = 0.3$, $\delta_{22} = 1$, $d_{21} = 0.3$, $d_{22} = 0.3$, $E_s = 0.3$, $E_1 = 0.2$, $E_2 = 0.2$, $\mu_s = 0.28$, $\mu_1 = 0.1$, $\mu_2 = 0.1$, $\mu = 0.1$, $l = 0.6$, $\tau = 1$, (**a**) time-series of $s(t)$; (**b**) time-series of $x_1(t)$; (**c**) time-series of $x_1(t)$; (**d**) the phase portrait of the globally asymptotically stable phytoplankton-extinction periodic solution of system (2.1)

duce the nutrients indraughting lake-ecosystem during nontransient impulsive dredging.

The parameter values $s(0) = 0.3$, $x_1(0) = 0.3$, $x_2(0) = 0.3$, $\lambda_1 = 0.5$, $d_1 = 0.2$, $\beta_{11} = 0.5$, $\delta_{11} = 1$, $\beta_{12} = 0.5$, $\delta_{12} = 1$, $d_{11} = 0.4$, $d_{12} = 0.4$, $\lambda_2 = 0.1$, $d_2 = 0.2$, $\beta_{21} = 0.3$, $\delta_{21} = 1$, $\beta_{22} = 0.3$, $\delta_{22} = 1$, $d_{21} = 0.3$, $d_{22} = 0.3$, $E_s = 0.3$, $E_1 = 0.2$, $E_2 = 0.2$, $\mu_s = 0.28$, $\mu_1 = 0.1$, $\mu_2 = 0.1$, $\mu = 0.1$, $l = 0.6$, $\tau = 1$ satisfy Theorem 4.1. Then the phytoplankton-extinction periodic solution $(\widetilde{s(t)}, 0, 0)$ of system (2.1) is globally asymptotically stable (see Fig. 4). From Theorems 4.1 and 4.2 and from the simulation experiments of Figs. 3 and 4 we can easily deduce that there exists a threshold l^*. If $l > l^*$, then system (2.1) is permanent. If $l < l^*$, then the phytoplankton-extinction periodic solution $(\widetilde{s(t)}, 0, 0)$ of system (2.1) is globally asymptotically stable. That is to say, a too long nontransient impulsive period will confuse the lake-ecosystem. Then appropriate extending the nontransient impulsive period will be beneficial to water resource management. A similar discussion may do with thresholds of the parameters λ_1, μ_s, μ_1, μ_2, and so on. Therefore the method of dredging sediment engineering should be combined with implementing ecological engineering to restore and rebuild healthy and stable aquatic ecosystem, which should be an effective way to control eutrophic lakes. Our results also provide reliable tactic basis for the practical water resource management.

Acknowledgements
Not applicable.

Funding
This paper is supported by National Natural Science Foundation of China (11761019, 11361014), the Science Technology Foundation of Guizhou Education Department (20175736-001, 2008038), the Science Technology Foundation of Guizhou (2010J2130), the Project of High Level Creative Talents in Guizhou Province (20164035), and the Joint Project of Department of Commerce and Guizhou University of Finance and Economics (2016SWBZD18).

Availability of data and materials
Data sharing not applicable to this paper as no datasets were generated or analyzed during the current study.

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Consent for publication
Not applicable.

Authors' contributions
Both authors contributed equally to the writing of this paper. Both authors read and approved the final manuscript.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 14 February 2021 Accepted: 25 May 2021 Published online: 07 June 2021

References
1. Qin, B.Q.: Lake eutrophication: control countermeasures and recycling exploitation. Ecol. Eng. **35**, 1569–1573 (2009)
2. Nyenje, P.M., Foppen, J.W., Uhlenbrook, S.: Eutrophication and nutrient release in urban areas of sub-Saharan Africa – a review. Sci. Total Environ. **408**, 447–455 (2010)
3. Bennett, E.M., Carpenter, S.R., Caraco, N.F.: Human impact on erodable phosphorus and eutrophication: a global perspective: increasing accumulation of phosphorus in soil threatens rivers, lakes, and coastal oceans with eutrophication. AIBS Bull. **51**, 227–234 (2001)
4. Wang, X., Feng, J., Hu, M.: Sediment dredging engineering for eutrophication control in lake. Environ. Prot. **2**, 22–23 (2003) (In Chinese)
5. Reynolds, C.S.: The development of perceptions of eutrophication and its control. Ecohydrol. Hydrobiol. **3**(2), 149–163 (2003)
6. Lakshmikantham, V.: Theory of Impulsive Differential Equations. World Scientific, Singapore (1989)
7. Jiao, J., et al.: Dynamics of a stage-structured predator-prey model with prey impulsively diffusing between two patches. Nonlinear Anal., Real World Appl. **11**, 2748–2756 (2010)
8. Jiao, J., et al.: Dynamical analysis of a five-dimensioned chemostat model with impulsive diffusion and pulse input environmental toxicant. Chaos Solitons Fractals **44**, 17–27 (2011)
9. Jiao, J.J., et al.: An appropriate pest management SI model with biological and chemical control concern. Appl. Math. Comput. **196**, 285–293 (2008)
10. Liu, X.N., Chen, L.S.: Complex dynamics of Holling II Lotka–Volterra predator–prey system with impulsive perturbations on the predator. Chaos Solitons Fractals **16**, 311–320 (2003)
11. Jiao, J., Chen, L.: Nonlinear incidence rate of a pest management SI model with biological and chemical control concern. Appl. Math. Mech. **28**(4), 541–551 (2007)
12. Jiao, J., Cai, S., Chen, L.: Dynamical analysis of a three-dimensional predator-prey model with impulsive harvesting and diffusion. Int. J. Bifurc. Chaos **21**(2), 453–465 (2011)
13. Tang, W., Jiao, J.: Dynamics of a periodic switched predator–prey system with hibernation, birth pulse and impulsive harvesting. J. Appl. Math. Comput. **51**, 161–175 (2016)