

RESEARCH

Open Access



Exponential stability in the Lagrange sense for Clifford-valued recurrent neural networks with time delays

G. Rajchakit¹, R. Sriraman², N. Boonsatit^{3*} , P. Hammachukiattikul⁴, C.P. Lim⁵ and P. Agarwal⁶

*Correspondence:

nattakan.b@rmutsb.ac.th

³Department of Mathematics,
Rajamangala University of
Technology Suvarnabhumi,
Nonthaburi 11000, Thailand
Full list of author information is
available at the end of the article

Abstract

This paper considers the Clifford-valued recurrent neural network (RNN) models, as an augmentation of real-valued, complex-valued, and quaternion-valued neural network models, and investigates their global exponential stability in the Lagrange sense. In order to address the issue of non-commutative multiplication with respect to Clifford numbers, we divide the original n -dimensional Clifford-valued RNN model into $2^m n$ real-valued models. On the basis of Lyapunov stability theory and some analytical techniques, several sufficient conditions are obtained for the considered Clifford-valued RNN models to achieve global exponential stability according to the Lagrange sense. Two examples are presented to illustrate the applicability of the main results, along with a discussion on the implications.

Keywords: Clifford-valued neural network; Exponential stability; Lyapunov functional; Lagrange stability

1 Introduction

Recurrent neural network (RNN) models have been successfully employed for solving problems of optimization, control, associative memory, and signal and image processing. Recently, the dynamic analysis of RNN models has attracted deep interest from various researchers, and many scientific papers with respect to the stability theory of RNNs have been published [1–6]. Due to the limited speed of signal propagation, time delays (either constant or time-varying) are often encountered in neural network (NN) models employed for solving real-world applications. Time delays are the main source of various dynamics such as chaos, divergence, poor functionality, and instability [4–9]. Therefore, studies on NN dynamics that incorporate either constant or time-varying delays are necessary. On the other hand, complex-valued and quaternion-valued NN models are important in light of their potential applications in many fields including color night vision, radar imaging, polarized signal classification, 3D wind forecasting, and others [9–13]. Recently, many important results have been published concerning different dynamics of the complex-valued and quaternion-valued NN models [14–20]. Particularly, stability analysis [14, 18–20], synchronization analysis [15], stabilizability and instabilizability analysis [16], controllability and observability [21], optimization [22, 23], and so on.

© The Author(s) 2021. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Clifford algebra provides a powerful theory for solving problems in geometry problems. In addition, Clifford algebra has been applied to numerous areas of science and engineering such as neural computation, computer and robot vision, and control problems [24–28]. In dealing with high-dimensional data and spatial geometric transformation, Clifford-valued NN models are superior to real-valued, complex-valued, and quaternion-valued NN models [26–29]. Recently, theoretical and applied studies on Clifford-valued NN models have become a new research subject. However, the dynamic properties of Clifford-valued NN models are usually more complex than those of real-valued, complex-valued and quaternion-valued NN models. Due to the issue of non-commutativity of multiplication with respect to Clifford numbers, studies on Clifford-valued NN dynamics are still limited [30–39].

By using the linear matrix inequality approach, the authors in [30] derived the global exponential stability criteria for delayed Clifford recurrent NN models. For Clifford-valued NN models with time delays, their global asymptotic stability problems were investigated in [31] by a decomposition method. Using a direct approach, the presence and global exponential stability conditions pertaining to almost periodic solutions were extracted for Clifford-valued neutral high-order Hopfield NN models with leakage delays in [34]. Using the Banach fixed point theorem and the Lyapunov functional, the global asymptotic almost periodic synchronization problems for the Clifford-valued NN cellular models were examined in [35]. The weighted pseudo-almost automorphic solutions for neutral type fuzzy cellular Clifford NN models were discussed in [37]. In [39], the authors explored the presence of an anti-periodic solution to the problem of Clifford-valued inertial Cohen-Grossberg NN models by using suitable Lyapunov functional. Recently, the effects of neutral delay and discrete delays have been considered in a class of Clifford-valued NNs [40], and the associated existence, uniqueness and global stability criteria have been obtained.

It is worth noting that many previous NNs studies focus mostly on the global stability in the Lyapunov sense. In many real physical systems, however, the equilibrium point may not exist. Besides, computationally restrictive and multi-stability dynamics have been found to be necessary for dealing with critical neural computation in applications [41, 42]. As an example, when employing NN models for associative memory or pattern recognition, multiple equilibrium points are often required. In these cases, NN models are no longer globally stable in the Lyapunov sense. In dealing with multi-stable models, more realistic concepts of stability are necessary [43–46]. It should be noticed that in Lyapunov stability of NN models, the existence and uniqueness of the equilibrium points are required. In contrast, the Lagrange stability, which is concerned with stability of the total model, does not require the information of equilibrium points. Therefore, the Lagrange stability or the attractive sets of NN models is useful, and many studies have been published [45–48].

Motivated by the above discussions, we investigate the exponential stability in the Lagrange sense for Clifford-valued RNNs with time delays in this paper. To the best of our knowledge, there is no result on the topic of Lagrange stability of Clifford-valued NNs. Undoubtedly, this interesting topic is still an open challenge. Therefore, our main objective of this paper is to derive new sufficient conditions to ensure the Lagrange exponential stability for Clifford-valued RNNs. The main contributions of this work are as follows: (1) This is the first paper to study such a problem for global exponential stability

of Clifford-valued RNN models in the Lagrange sense, which encompasses real-valued, complex-valued, and quaternion-valued NN models as special cases. (2) By using the Lyapunov functional and some analytical methods, new sufficient conditions for ascertaining the exponential stability of the considered Clifford-valued RNN models in the Lagrange sense are derived. This is achieved by decomposing the n -dimensional Clifford-valued RNN model into $2^m n$ -dimensional real-valued RNN models. (3) The results obtained can be used for the study of monostable, multi-stable, and other complex NN models. In addition, the obtained results are new and different compared with those in the existing literature. The usefulness of the obtained results is validated with two numerical examples.

We organize this paper as follows. The proposed Clifford-valued RNN model is formally defined in Sect. 2. The new global Lagrange exponential stability criterion is presented in Sect. 3, while two numerical examples are given in Sect. 4. The conclusions of the results are given in Sect. 5.

2 Mathematical fundamentals and problem formulation

2.1 Notations

\mathbb{A} is defined as the Clifford algebra and has m generators over the real number \mathbb{R} . Let \mathbb{R}^n and \mathbb{A}^n be an n -dimensional real vector space and an n -dimensional real Clifford vector space, respectively; while $\mathbb{R}^{n \times n}$ and $\mathbb{A}^{n \times n}$ denote the set of all $n \times n$ real matrices and the set of all $n \times n$ real Clifford matrices, respectively. We define the norm of \mathbb{R}^n as $\|r\| = \sum_{i=1}^n |r_i|$, and for $A = (a_{ij})_{n \times n} \in \mathbb{R}^{n \times n}$, denote $\|A\| = \max_{1 \leq i \leq n} \{\sum_{j=1}^n |a_{ij}|\}$. While $r = \sum_A r^A e_A \in \mathbb{A}$, denote $|r|_{\mathbb{A}} = \sum_A |r^A|$, and for $A = (a_{ij})_{n \times n} \in \mathbb{A}^{n \times n}$, denote $\|A\|_{\mathbb{A}} = \max_{1 \leq i \leq n} \{\sum_{j=1}^n |a_{ij}|_{\mathbb{A}}\}$. Superscripts T and $*$, respectively, denote the matrix transposition and matrix involution transposition. For any matrix, $\mathcal{P} > 0$ (< 0) denotes a positive (negative) definite matrix. Let $\mathcal{C}\mathcal{C}\mathcal{F}^+$ be the set of all nonnegative continuous functionals $\mathcal{L} : \mathcal{C} \rightarrow [0, \infty)$. For $\varphi \in \mathcal{C}([-\tau, 0], \mathbb{A}^n)$, $\|\varphi\|_{\tau} \leq \sup_{-\tau \leq s \leq 0} \|\varphi(t+s)\|$.

2.2 Clifford algebra

The Clifford real algebra over \mathbb{R}^m is defined as

$$\mathbb{A} = \left\{ \sum_{A \subseteq \{1, 2, \dots, m\}} a^A e_A, a^A \in \mathbb{R} \right\},$$

where $e_A = e_{l_1} e_{l_2} \dots e_{l_v}$ with $A = \{l_1, l_2, \dots, l_v\}$, $1 \leq l_1 < l_2 < \dots < l_v \leq m$.

Moreover, $e_{\emptyset} = e_0 = 1$ and $e_l = e_{\{l\}}$, $l = 1, 2, \dots, m$, are denoted as the Clifford generators, and they fulfill the following relations:

$$\begin{cases} e_i e_j + e_j e_i = 0, & i \neq j, i, j = 1, 2, \dots, m, \\ e_i^2 = -1, & i = 1, 2, \dots, m, \\ e_0^2 = 1. \end{cases} \tag{1}$$

For simplicity, when an element is the product of multiple Clifford generators, its subscripts are incorporated together, e.g., $e_4 e_5 e_6 e_7 = e_{4567}$.

Let $\Lambda = \{\emptyset, 1, 2, \dots, A, \dots, 12 \dots m\}$, and we have

$$\mathbb{A} = \left\{ \sum_A a^A e_A, a^A \in \mathbb{R} \right\},$$

where \sum_A denotes $\sum_{A \in \Lambda}$ and \mathbb{A} is isomorphic to \mathbb{R}^{2^m} .

For any Clifford number $r = \sum_A r^A e_A$, the involution of r is defined by

$$\bar{r} = \sum_A r^A \bar{e}_A,$$

where $\bar{e}_A = (-1)^{\frac{\sigma[A](\sigma[A]+1)}{2}} e_A$, and

$$\sigma[A] = \begin{cases} 0, & \text{if } A = \emptyset, \\ \nu, & \text{if } A = l_1 l_2 \dots l_\nu. \end{cases}$$

From the definition, we can directly deduce that $e_A \bar{e}_A = \bar{e}_A e_A = 1$. For a Clifford-valued function $r = \sum_A r^A e_A : \mathbb{R} \rightarrow \mathbb{A}$, where $r^A : \mathbb{R} \rightarrow \mathbb{R}, A \in \Lambda$, its derivative is represented by $\frac{dr(t)}{dt} = \sum_A \frac{dr^A(t)}{dt} e_A$.

Since $e_B \bar{e}_A = (-1)^{\frac{\sigma[A](\sigma[A]+1)}{2}} e_B e_A$, we can write $e_B \bar{e}_A = e_C$ or $e_B \bar{e}_A = -e_C$, where e_C is a basis of Clifford algebra \mathbb{A} . As an example, $e_{l_1 l_2} \bar{e}_{l_2 l_3} = -e_{l_1 l_2} e_{l_2 l_3} = -e_{l_1} e_{l_2} e_{l_2} e_{l_3} = -e_{l_1} (-1) e_{l_3} = e_{l_1} e_{l_3} = e_{l_1 l_3}$. Therefore, it is possible to identify a unique corresponding basis e_C for a given $e_B \bar{e}_A$. Define

$$\sigma[B, \bar{A}] = \begin{cases} 0, & \text{if } e_B \bar{e}_A = e_C, \\ 1, & \text{if } e_B \bar{e}_A = -e_C, \end{cases}$$

and then, $e_B \bar{e}_A = (-1)^{\sigma[B, \bar{A}]} e_C$.

Moreover, for $\mathcal{G} = \sum_C \mathcal{G}^C e_C \in \mathbb{A}$, we define $\mathcal{G}^{B, \bar{A}} = (-1)^{\sigma[B, \bar{A}]} \mathcal{G}^C$ for $e_B \bar{e}_A = (-1)^{\sigma[B, \bar{A}]} e_C$. Therefore

$$\mathcal{G}^{B, \bar{A}} e_B \bar{e}_A = \mathcal{G}^{B, \bar{A}} (-1)^{\sigma[B, \bar{A}]} e_C = (-1)^{\sigma[B, \bar{A}]} \mathcal{G}^C (-1)^{\sigma[B, \bar{A}]} e_C = \mathcal{G}^C e_C.$$

2.3 Problem definition

The following Clifford-valued RNN model with discrete time-varying delays is considered:

$$\begin{cases} \dot{r}_i(t) = -d_i r_i(t) + \sum_{j=1}^n a_{ij} h_j(r_j(t)) + \sum_{j=1}^n b_{ij} h_j(r_j(t - \tau_j(t))) + u_i, & t \geq 0, \\ r_i(s) = \varphi_i(s), & s \in [-\tau, 0], \end{cases} \tag{2}$$

where $i, j = 1, 2, \dots, n$, and n corresponds to the number of neurons; $r_i(t) \in \mathbb{A}$ represents the state vector of the i th unit; $d_i \in \mathbb{R}^+$ indicates the rate with which the i th unit will reset its potential to the resting state in isolation when it is disconnected from the network and external inputs; $a_{ij}, b_{ij} \in \mathbb{A}$ indicate the strengths of connection weights without and with time-varying delays between cells i and j , respectively; $u_i \in \mathbb{A}$ is an external input on the i th unit; $h_j(\cdot) : \mathbb{A}^n \rightarrow \mathbb{A}^n$ is the activation function of signal transmission; $\tau_j(t)$ corresponds

to the transmission delay which satisfies $0 \leq \tau_j(t) \leq \tau$, where τ is a positive constant and $\tau = \max_{1 \leq j \leq n} \{\tau_j(t)\}$. Furthermore, $\varphi_i \in \mathcal{C}([-\tau, 0], \mathbb{A}^n)$ is the initial condition for NN model (1).

For convenience of discussion, we rewrite (1) in the vector form

$$\begin{cases} \dot{r}(t) = -\mathcal{D}r(t) + \mathcal{A}h(r(t)) + \mathcal{B}h(r(t - \tau(t))) + \mathcal{U}, & t \geq 0, t \neq t_k, \\ r(s) = \varphi(s), & s \in [-\tau, 0], \end{cases} \tag{3}$$

where $r(t) = (r_1(t), r_2(t), \dots, r_n(t))^T \in \mathbb{A}^n$; $\mathcal{D} = \text{diag}\{d_1, d_2, \dots, d_n\} \in \mathbb{R}^n$ with $d_i > 0$, $i = 1, 2, \dots, n$; and $\mathcal{A} = (a_{ij})_{n \times n} \in \mathbb{A}^{n \times n}$; $\mathcal{B} = (b_{ij})_{n \times n} \in \mathbb{A}^{n \times n}$; $\mathcal{U} = (u_1, u_2, \dots, u_n)^T \in \mathbb{A}^n$; $h(r(t)) = (h_1(r_1(t)), h_2(r_2(t)), \dots, h_n(r_n(t)))^T \in \mathbb{A}^n$; $h(r(t - \tau(t))) = (h_1(r_1(t - \tau(t))), h_2(r_2(t - \tau(t))), \dots, h_n(r_n(t - \tau(t))))^T \in \mathbb{A}^n$.

(A1) Function $h_j(\cdot)$ fulfills the Lipschitz continuity condition with respect to the n -dimensional Clifford vector. For each $j = 1, 2, \dots, n$, there exists a positive constant k_j such that, for any $x, y \in \mathbb{A}$,

$$|h_j(x) - h_j(y)|_{\mathbb{A}} \leq k_j |x - y|_{\mathbb{A}}, \quad j = 1, 2, \dots, n, \tag{4}$$

where k_j ($j = 1, 2, \dots, n$) is known as the Lipschitz constant and $h_j(0) = 0$. In addition, there exists a positive constant k_j such that $|h(x)|_{\mathbb{A}} \leq k_j$ for any $x \in \mathbb{A}$.

According to Assumption (A1), it is clear that

$$(h(x) - h(y))^* (h(x) - h(y)) \leq (x - y)^* \mathcal{K}^T \mathcal{K} (x - y), \tag{5}$$

where $\mathcal{K} = \text{diag}\{k_1, k_2, \dots, k_n\}$.

Remark 2.1 There exists a constant $k_j > 0$ ($j = 1, 2, \dots, n$) such that $\forall x = (x_1, x_2, \dots, x_{2^m})^T \in \mathbb{R}^{2^m}$ and $\forall y = (y_1, y_2, \dots, y_{2^m})^T \in \mathbb{R}^{2^m}$

$$|h_j^A(x) - h_j^A(y)| \leq k_j \sum_{\ell=1}^{2^m} |x_\ell - y_\ell|, \quad A \in \Lambda, j = 1, 2, \dots, n.$$

Remark 2.2 The proposed system model in this paper is more general than the system model proposed in previous works [4, 6, 19]. For example, when we consider $m = 0$ in NN model (3), then the model can be reduced to the real-valued NN model proposed in [4]. If we take $m = 1$ in NN model (3), then the model can be reduced to the complex-valued NN model proposed in [6]. If we choose $m = 2$ in NN model (3), then the model can be reduced to the quaternion-valued NN model proposed in [19].

3 Main results

In order to overcome the issue of non-commutativity of multiplication of Clifford numbers, we transform the original Clifford-valued model (3) into multidimensional real-valued model. This can be achieved with the help of $e_A \bar{e}_A = \bar{e}_A e_A = 1$ and $e_B \bar{e}_A e_A = e_B$. Given any $\mathcal{G} \in \mathbb{A}$, unique \mathcal{G}^C that is able to satisfy $\mathcal{G}^C e_C h^A e_A = (-1)^{\sigma[B, \bar{A}]} \mathcal{G}^C h^A e_B = \mathcal{G}^{B, \bar{A}} h^A e_B$ can

be identified. By decomposing (3) into $\dot{r} = \sum_A \dot{r}^A e_A$, we have

$$\begin{cases} \dot{r}^A(t) = -\mathcal{D}r^A(t) + \sum_B \mathcal{A}^{A,\bar{B}} h^B(r(t)) + \sum_B \mathcal{B}^{A,\bar{B}} h^B(r(t-\tau(t))) + \mathcal{U}^A, & t \geq 0, \\ r^A(s) = \varphi^A(s), & s \in [-\tau, 0], A, B \in \Lambda, \end{cases} \tag{6}$$

where

$$\begin{aligned} r(t) &= \sum_{A \in \Lambda} r^A(t) e_A, \\ \mathcal{U} &= \sum_{A \in \Lambda} \mathcal{U}^A e_A, \\ h^B(r(t)) &= (h_1^B(r_1^{C_1}(t), r_1^{C_2}(t), \dots, r_1^{C_{2^m}}(t)), h_2^B(r_2^{C_1}(t), r_2^{C_2}(t), \dots, r_2^{C_{2^m}}(t)), \\ &\quad \dots, h_n^B(r_n^{C_1}(t), r_n^{C_2}(t), \dots, r_n^{C_{2^m}}(t)))^T, \\ h^B(r(t-\tau(t))) &= (h_1^B(r_1^{C_1}(t-\tau(t)), r_1^{C_2}(t-\tau(t)), \dots, r_1^{C_{2^m}}(t-\tau(t))), \\ &\quad h_2^B(r_2^{C_1}(t-\tau(t)), r_2^{C_2}(t-\tau(t)), \dots, r_2^{C_{2^m}}(t-\tau(t))), \\ &\quad \dots, h_n^B(r_n^{C_1}(t-\tau(t)), r_n^{C_2}(t-\tau(t)), \dots, r_n^{C_{2^m}}(t-\tau(t))))^T, \\ \mathcal{A} &= \sum_{C \in \Lambda} \mathcal{A}^C e_C, \quad \mathcal{A}^{A,\bar{B}} = (-1)^{\sigma[A,\bar{B}]} \mathcal{A}^C, \\ \mathcal{B} &= \sum_{C \in \Lambda} \mathcal{B}^C e_C, \quad \mathcal{B}^{A,\bar{B}} = (-1)^{\sigma[A,\bar{B}]} \mathcal{B}^C, \\ \mathcal{A}^{A,\bar{B}} &= (a_{ij}^{A,\bar{B}})_{n \times n}, \quad \mathcal{B}^{A,\bar{B}} = (b_{ij}^{A,\bar{B}})_{n \times n}, \\ e_A \bar{e}_B &= (-1)^{\sigma[A,\bar{B}]} e_C. \end{aligned}$$

Corresponding to the basis of Clifford algebra, the Clifford-valued NN model can be reformulated into novel real-valued ones. Define

$$\begin{aligned} \tilde{r}(t) &= ((r^0(t))^T, (r^1(t))^T, \dots, (r^A(t))^T, \dots, (r^{12\dots m}(t))^T)^T \in \mathbb{R}^{2^m n}, \\ \tilde{h}(\tilde{r}(t)) &= ((h^0(r(t)))^T, (h^1(r(t)))^T, \dots, (h^A(r(t)))^T, \dots, (h^{12\dots m}(r(t)))^T)^T \in \mathbb{R}^{2^m n}, \\ \tilde{h}(\tilde{r}(t-\tau(t))) &= ((h^0(r(t-\tau(t))))^T, (h^1(r(t-\tau(t))))^T, \dots, (h^A(r(t-\tau(t))))^T, \dots, \\ &\quad (h^{12\dots m}(r(t-\tau(t))))^T)^T \in \mathbb{R}^{2^m n}, \\ \tilde{\mathcal{U}} &= ((\mathcal{U}^0)^T, (\mathcal{U}^1)^T, \dots, (\mathcal{U}^A)^T, \dots, (\mathcal{U}^{12\dots m})^T)^T \in \mathbb{R}^{2^m n}, \\ \tilde{\mathcal{D}} &= \begin{pmatrix} \mathcal{D} & 0 & \dots & 0 \\ 0 & \mathcal{D} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathcal{D} \end{pmatrix}_{2^m n \times 2^m n}, \\ \tilde{\mathcal{A}} &= \begin{pmatrix} \mathcal{A}^0 & \mathcal{A}^{\bar{1}} & \dots & \mathcal{A}^{\bar{A}} & \dots & \mathcal{A}^{\overline{12\dots m}} \\ \mathcal{A}^1 & \mathcal{A}^{1\bar{1}} & \dots & \mathcal{A}^{1\bar{A}} & \dots & \mathcal{A}^{1\bar{1}2\dots m} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ \mathcal{A}^{12\dots m} & \mathcal{A}^{12\dots m\bar{1}} & \dots & \mathcal{A}^{12\dots m\bar{A}} & \dots & \mathcal{A}^{12\dots m\bar{1}2\dots m} \end{pmatrix}_{2^m n \times 2^m n}, \end{aligned}$$

$$\tilde{\mathcal{B}} = \begin{pmatrix} \mathcal{B}^0 & \mathcal{B}^{\bar{1}} & \dots & \mathcal{B}^{\bar{A}} & \dots & \mathcal{B}^{\overline{12\dots m}} \\ \mathcal{B}^1 & \mathcal{B}^{1\bar{1}} & \dots & \mathcal{B}^{1\bar{A}} & \dots & \mathcal{B}^{1\overline{12\dots m}} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ \mathcal{B}^{12\dots m} & \mathcal{B}^{12\dots m\bar{1}} & \dots & \mathcal{B}^{12\dots m\bar{A}} & \dots & \mathcal{B}^{12\dots m\overline{12\dots m}} \end{pmatrix}_{2^m n \times 2^m n},$$

and then (6) can be written as

$$\dot{\tilde{r}}(t) = -\tilde{\mathcal{D}}\tilde{r}(t) + \tilde{\mathcal{A}}\tilde{h}(\tilde{r}(t)) + \tilde{\mathcal{B}}\tilde{h}(\tilde{r}(t - \tau(t))) + \tilde{\mathcal{U}}, \quad t \geq 0, \tag{7}$$

with the initial value

$$\tilde{r}(s) = \tilde{\varphi}(s), \quad s \in [-\tau, 0], \tag{8}$$

where $\tilde{\varphi}(s) = ((\varphi^0(s))^T, (\varphi^1(s))^T, \dots, (\varphi^A(s))^T, \dots, (\varphi^{12\dots m}(s))^T)^T \in \mathbb{R}^{2^m n}$.

In addition, notice that (4) can be expressed as the following inequality:

$$(\tilde{h}(\tilde{r}_1) - \tilde{h}(\tilde{r}_2))^T (\tilde{h}(\tilde{r}_1) - \tilde{h}(\tilde{r}_2)) \leq (\tilde{r}_1 - \tilde{r}_2)^T \tilde{\mathcal{K}}^T (\tilde{r}_1 - \tilde{r}_2), \tag{9}$$

where

$$\tilde{\mathcal{K}} = \begin{pmatrix} \mathcal{K}^T \mathcal{K} & 0 & \dots & 0 \\ 0 & \mathcal{K}^T \mathcal{K} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathcal{K}^T \mathcal{K} \end{pmatrix}_{2^m n \times 2^m n}.$$

Definition 3.1 ([47]) NN model (7) is said to be uniformly stable in the Lagrange sense if, for any $\beta > 0$, there exists a constant $\mathcal{L} = \mathcal{L}(\beta) > 0$ such that $\|\tilde{r}(t, \tilde{\varphi})\| < \mathcal{L}$ for any $\tilde{\varphi} \in \mathcal{C}_\beta = \{\tilde{\varphi} \in \mathcal{C}([-\tau, 0], \mathbb{R}^{2^m n}) \mid \|\tilde{\varphi}\| \leq \beta\}$ and $t \geq 0$.

Definition 3.2 ([47]) Given a radially unbounded and positive definite function $\mathcal{V}(\cdot)$, a nonnegative continuous function $\mathcal{L}(\cdot)$, and two constants $\rho > 0$ and $\nu > 0$ such that, for any solution $\tilde{r}(t)$ of NN model (7), $\mathcal{V}(t) > \rho$ implies

$$\mathcal{V}(t) - \rho \leq \mathcal{L}(\tilde{\varphi})e^{-\nu t}$$

for any $t \geq 0$ and $\tilde{\varphi} \in \mathcal{C}_\beta$. Then NN (7) is said to be globally exponentially attractive (GEA) with respect to $\mathcal{V}(t)$, and the compact set $\Omega = \{\tilde{r}(t) \in \mathbb{R}^{2^m n} \mid \mathcal{V}(t) \leq \rho\}$ is said to be a GEA set of NN model (7).

Definition 3.3 ([47]) NN model (7) is globally exponentially stable in the Lagrange sense if it is both uniformly stable in the Lagrange sense and is globally exponentially attractive. If there is a need to emphasize the Lyapunov-like function, NN model (7) becomes globally exponentially stable in the Lagrange sense with respect to \mathcal{V} .

Lemma 3.4 ([47]) For positive definite matrix $\mathcal{P} \in \mathbb{R}^{2^m n \times 2^m n}$, positive real constant ϵ , and $x, y \in \mathbb{R}^{2^m n}$, it holds that $x^T y + y^T x \leq \epsilon x^T \mathcal{P} x + \epsilon^{-1} y^T \mathcal{P} y$.

Lemma 3.5 ([47]) *Given $\mathcal{S} = \begin{pmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} \\ \mathcal{S}_{21} & \mathcal{S}_{22} \end{pmatrix} \in \mathbb{R}^{2^m n}$ with $\mathcal{S}_{11}^T = \mathcal{S}_{11}$, $\mathcal{S}_{22}^T = \mathcal{S}_{22}$, $\mathcal{S}_{12}^T = \mathcal{S}_{21}$, this is equivalent to one of the following conditions: (i) $\mathcal{S}_{22} < 0$, $\mathcal{S}_{11} - \mathcal{S}_{12}\mathcal{S}_{22}^{-1}\mathcal{S}_{12}^T < 0$, (ii) $\mathcal{S}_{11} < 0$, $\mathcal{S}_{22} - \mathcal{S}_{12}^T\mathcal{S}_{11}^{-1}\mathcal{S}_{12} < 0$.*

Lemma 3.6 ([47]) *Suppose that there exists $\varsigma_1 > \varsigma_2 > 0$, $\tilde{r}(t)$ which satisfies $D^+\tilde{r}(t) \leq -\varsigma_1\tilde{r}(t) + \varsigma_2\tilde{\tilde{r}}(t)$, a nonnegative continuous quantity function for all $t \in [t_0 - \tau, t_0]$, then $\tilde{r}(t) \leq \tilde{\tilde{r}}(t_0)e^{(-\lambda(t-t_0))}$ holds for any $t \geq t_0$, where $\tilde{\tilde{r}}(t) = \sup_{t-\tau \leq s \leq t} \tilde{r}(s)$, $\tau \geq 0$, and λ is the unique positive root of $\lambda = \varsigma_1 - \varsigma_2e^{\lambda\tau}$.*

Lemma 3.7 ([47]) *Given matrices $\mathcal{E} \in \mathbb{R}^{2^m n \times 2^m n}$, $\mathcal{F} \in \mathbb{R}^{2^m n \times 2^m n}$, $\mathcal{G} \in \mathbb{R}^{2^m n \times 2^m n}$, $\mathcal{H} \in \mathbb{R}^{2^m n \times 2^m n}$, and appropriate reversible matrices \mathcal{X} , \mathcal{Y} , then*

$$\begin{pmatrix} \mathcal{E} \\ \mathcal{F} \end{pmatrix} \mathcal{X}^{-1} \begin{pmatrix} \mathcal{E} \\ \mathcal{F} \end{pmatrix}^T + \begin{pmatrix} \mathcal{G} \\ \mathcal{H} \end{pmatrix} \mathcal{Y}^{-1} \begin{pmatrix} \mathcal{G} \\ \mathcal{H} \end{pmatrix}^T = \begin{pmatrix} \mathcal{E} & \mathcal{G} \\ \mathcal{F} & \mathcal{H} \end{pmatrix} \begin{pmatrix} \mathcal{X}^{-1} & 0 \\ 0 & \mathcal{Y}^{-1} \end{pmatrix} \begin{pmatrix} \mathcal{E} & \mathcal{G} \\ \mathcal{F} & \mathcal{H} \end{pmatrix}^T.$$

Lemma 3.8 ([48]) *Given positive constants β and γ and suppose $\mathcal{V}(t) \in \mathcal{C}([0, +\infty), \mathbb{R})$, in which*

$$D^+\mathcal{V}(t) \leq -\beta\mathcal{V}(t) + \gamma, \quad t \geq 0,$$

then

$$\mathcal{V}(t) - \frac{\gamma}{\beta} \leq \left(\mathcal{V}(0) - \frac{\gamma}{\beta} \right) e^{-\beta t}, \quad t \geq 0.$$

If $\mathcal{V}(t) \geq \frac{\gamma}{\beta}$, $t \geq 0$, then $\mathcal{V}(t)$ exponentially approaches $\frac{\gamma}{\beta}$ as t increases.

3.1 Exponential stability

Theorem 3.9 *Under Assumption (A1), if there exist positive definite matrices $\mathcal{P}, \mathcal{R}, \mathcal{S} \in \mathbb{R}^{2^m n \times 2^m n}$, positive diagonal matrices $\mathcal{C}, \mathcal{Q} \in \mathbb{R}^{2^m n \times 2^m n}$ such that the following LMI hold:*

$$\begin{pmatrix} \Xi & \mathcal{P}\tilde{\mathcal{A}} - \tilde{\mathcal{D}}\mathcal{C} & \mathcal{P}\tilde{\mathcal{B}} & \mathcal{P} \\ \tilde{\mathcal{A}}^T\mathcal{P} - \mathcal{C}^T\tilde{\mathcal{D}} & \mathcal{C}\tilde{\mathcal{A}} + \tilde{\mathcal{A}}^T\mathcal{C} - \mathcal{Q} & \mathcal{C}\tilde{\mathcal{B}} & \mathcal{C} \\ \tilde{\mathcal{B}}^T\mathcal{P} & \tilde{\mathcal{B}}^T\mathcal{C} & -\mathcal{R} & 0 \\ \mathcal{P} & \mathcal{C}^T & 0 & -\mathcal{S} \end{pmatrix} < 0, \tag{10}$$

$$\tilde{\mathcal{K}}\mathcal{R}\tilde{\mathcal{K}} \leq \mathcal{P}, \tag{11}$$

where $\Xi = \mathcal{P} + \mathcal{C}\tilde{\mathcal{K}} - \mathcal{P}\tilde{\mathcal{D}} - \tilde{\mathcal{D}}^T\mathcal{P} + \tilde{\mathcal{K}}\mathcal{Q}\tilde{\mathcal{K}}$, then NN model (7) is globally exponentially stable in the Lagrange sense. Moreover, the set

$$\Omega = \left\{ \tilde{r} \in \mathbb{R}^{2^m n} \mid \tilde{r}^T(t)\mathcal{P}\tilde{r}(t) \leq \frac{\tilde{u}^T\mathcal{S}\tilde{u}}{\epsilon} \right\} \tag{12}$$

is a globally exponentially attractive set of NN model (7), where ϵ is a proper positive constant.

Proof Consider the following Lyapunov function which is positive definite and radially unbounded

$$\mathcal{V}(t) = \tilde{r}^T(t)\mathcal{P}\tilde{r}(t) + 2 \sum_{i=1}^n c_i \int_0^{\tilde{r}_i(t)} \tilde{h}_i(s) ds. \tag{13}$$

Then, the Dini derivative $\mathcal{D}^+\mathcal{V}(t)$ can be computed along the solutions of NN model (7), we get

$$\begin{aligned} \mathcal{D}^+\mathcal{V}(t) &= 2\tilde{r}^T(t)\mathcal{P}\dot{\tilde{r}}(t) + 2\tilde{h}^T(\tilde{r}(t))\mathcal{C}\dot{\tilde{r}}(t) \\ &= 2\tilde{r}^T(t)\mathcal{P}(-\tilde{\mathcal{D}}\tilde{r}(t) + \tilde{\mathcal{A}}\tilde{h}(\tilde{r}(t)) + \tilde{\mathcal{B}}\tilde{h}(\tilde{r}(t-\tau(t))) + \tilde{\mathcal{U}}) \\ &\quad + 2\tilde{h}^T(\tilde{r}(t))\mathcal{C}(-\tilde{\mathcal{D}}\tilde{r}(t) + \tilde{\mathcal{A}}\tilde{h}(\tilde{r}(t)) + \tilde{\mathcal{B}}\tilde{h}(\tilde{r}(t-\tau(t))) + \tilde{\mathcal{U}}) \\ &= 2(\tilde{r}^T(t)\mathcal{P} + \tilde{h}^T(\tilde{r}(t))\mathcal{C})(-\tilde{\mathcal{D}}\tilde{r}(t) + \tilde{\mathcal{A}}\tilde{h}(\tilde{r}(t))) + 2(\tilde{r}^T(t)\mathcal{P}\tilde{\mathcal{B}} \\ &\quad + \tilde{h}^T(\tilde{r}(t))\mathcal{C}\tilde{\mathcal{B}})\tilde{h}(\tilde{r}(t-\tau(t))) + 2(\tilde{r}^T(t)\mathcal{P} + \tilde{h}^T(\tilde{r}(t))\mathcal{C})\tilde{\mathcal{U}}. \end{aligned} \tag{14}$$

There exists a positive diagonal matrix \mathcal{Q} , and through Assumption (A1) we have

$$\begin{aligned} &2(\tilde{r}^T(t)\mathcal{P} + \tilde{h}^T(\tilde{r}(t))\mathcal{C})(-\tilde{\mathcal{D}}\tilde{r}(t) + \tilde{\mathcal{A}}\tilde{h}(\tilde{r}(t))) \\ &\leq -2\tilde{r}^T(t)(\mathcal{P}\tilde{\mathcal{D}})\tilde{r}(t) + 2\tilde{r}^T(t)(\mathcal{P}\tilde{\mathcal{A}} - \tilde{\mathcal{D}}\mathcal{C})\tilde{h}(\tilde{r}(t)) \\ &\quad + 2\tilde{h}^T(\tilde{r}(t))(\mathcal{C}\tilde{\mathcal{A}})\tilde{h}(\tilde{r}(t)) + \tilde{r}^T(t)\mathcal{K}\mathcal{Q}\mathcal{K}\tilde{r}(t) - \tilde{h}^T(\tilde{r}(t))\mathcal{Q}\tilde{h}(\tilde{r}(t)) \\ &= \begin{pmatrix} \tilde{r}(t) \\ \tilde{h}(\tilde{r}(t)) \end{pmatrix}^T \begin{pmatrix} -\mathcal{P}\tilde{\mathcal{D}} - \tilde{\mathcal{D}}^T\mathcal{P} + \mathcal{K}\mathcal{Q}\mathcal{K} & \mathcal{P}\tilde{\mathcal{A}} - \tilde{\mathcal{D}}\mathcal{C} \\ \tilde{\mathcal{A}}^T\mathcal{P} - \mathcal{C}^T\tilde{\mathcal{D}} & \mathcal{C}\tilde{\mathcal{A}} + \tilde{\mathcal{A}}^T\mathcal{C} - \mathcal{Q} \end{pmatrix} \begin{pmatrix} \tilde{r}(t) \\ \tilde{h}(\tilde{r}(t)) \end{pmatrix}. \end{aligned} \tag{15}$$

Using Assumption (A1) and Lemma 3.4, there exist positive defined matrices \mathcal{R} and \mathcal{S} such that the following inequalities are true:

$$\begin{aligned} &2(\tilde{r}^T(t)\mathcal{P}\tilde{\mathcal{B}} + \tilde{h}^T(\tilde{r}(t))\mathcal{C}\tilde{\mathcal{B}})\tilde{h}(\tilde{r}(t-\tau(t))) \\ &\leq (\tilde{r}^T(t)\mathcal{P}\tilde{\mathcal{B}} + \tilde{h}^T(\tilde{r}(t))\mathcal{C}\tilde{\mathcal{B}})\mathcal{R}^{-1}(\tilde{r}^T(t)\mathcal{P}\tilde{\mathcal{B}} + \tilde{h}^T(\tilde{r}(t))\mathcal{C}\tilde{\mathcal{B}})^T \\ &\quad + \tilde{h}^T(\tilde{r}(t-\tau(t)))\mathcal{R}\tilde{h}(\tilde{r}(t-\tau(t))) \\ &\leq \begin{pmatrix} \tilde{r}(t) \\ \tilde{h}(\tilde{r}(t)) \end{pmatrix}^T \begin{pmatrix} \mathcal{P}\tilde{\mathcal{B}} \\ \mathcal{C}\tilde{\mathcal{B}} \end{pmatrix} \mathcal{R}^{-1} \begin{pmatrix} \mathcal{P}\tilde{\mathcal{B}} \\ \mathcal{C}\tilde{\mathcal{B}} \end{pmatrix}^T \begin{pmatrix} \tilde{r}(t) \\ \tilde{h}(\tilde{r}(t)) \end{pmatrix} \\ &\quad + \tilde{r}^T(t-\tau(t))\tilde{\mathcal{K}}\mathcal{R}\tilde{\mathcal{K}}\tilde{r}(t-\tau(t)), \end{aligned} \tag{16}$$

and

$$\begin{aligned} &2(\tilde{r}^T(t)\mathcal{P} + \tilde{h}^T(\tilde{r}(t))\mathcal{C})\tilde{\mathcal{U}} \\ &\leq (\tilde{r}^T(t)\mathcal{P} + \tilde{h}^T(\tilde{r}(t))\mathcal{C})\mathcal{S}^{-1}(\tilde{r}^T(t)\mathcal{P} + \tilde{h}^T(\tilde{r}(t))\mathcal{C})^T + \tilde{\mathcal{U}}^T\mathcal{S}\tilde{\mathcal{U}} \\ &\leq \begin{pmatrix} \tilde{r}(t) \\ \tilde{h}(\tilde{r}(t)) \end{pmatrix}^T \begin{pmatrix} \mathcal{P} \\ \mathcal{C} \end{pmatrix} \mathcal{S}^{-1} \begin{pmatrix} \mathcal{P} \\ \mathcal{C} \end{pmatrix}^T \begin{pmatrix} \tilde{r}(t) \\ \tilde{h}(\tilde{r}(t)) \end{pmatrix} + \tilde{\mathcal{U}}^T\mathcal{S}\tilde{\mathcal{U}}. \end{aligned} \tag{17}$$

Through Lemma 3.7 and (14)–(17), we have

$$\begin{aligned}
 \mathcal{D}^+ \mathcal{V}(t) &\leq \begin{pmatrix} \tilde{r}(t) \\ \tilde{h}(\tilde{r}(t)) \end{pmatrix}^T \begin{pmatrix} -\mathcal{P}\tilde{\mathcal{D}} - \tilde{\mathcal{D}}^T \mathcal{P} + \tilde{\mathcal{K}}\mathcal{Q}\tilde{\mathcal{K}} & \mathcal{P}\tilde{\mathcal{A}} - \tilde{\mathcal{D}}\mathcal{C} \\ \tilde{\mathcal{A}}^T \mathcal{P} - \mathcal{C}^T \tilde{\mathcal{D}} & \mathcal{C}\tilde{\mathcal{A}} + \tilde{\mathcal{A}}^T \mathcal{C} - \mathcal{Q} \end{pmatrix} \\
 &\quad \times \begin{pmatrix} \tilde{r}(t) \\ \tilde{h}(\tilde{r}(t)) \end{pmatrix} + \begin{pmatrix} \tilde{r}(t) \\ \tilde{h}(\tilde{r}(t)) \end{pmatrix}^T \begin{pmatrix} \mathcal{P}\tilde{\mathcal{B}} \\ \mathcal{C}\tilde{\mathcal{B}} \end{pmatrix} \mathcal{R}^{-1} \begin{pmatrix} \mathcal{P}\tilde{\mathcal{B}} \\ \mathcal{C}\tilde{\mathcal{B}} \end{pmatrix}^T \begin{pmatrix} \tilde{r}(t) \\ \tilde{h}(\tilde{r}(t)) \end{pmatrix} \\
 &\quad + r^T(t - \tau(t)) \tilde{\mathcal{K}}\mathcal{R}\tilde{\mathcal{K}}r(t - \tau(t)) + \begin{pmatrix} \tilde{r}(t) \\ \tilde{h}(\tilde{r}(t)) \end{pmatrix}^T \begin{pmatrix} \mathcal{P} \\ \mathcal{C} \end{pmatrix} \mathcal{S}^{-1} \\
 &\quad \times \begin{pmatrix} \mathcal{P} \\ \mathcal{C} \end{pmatrix}^T \begin{pmatrix} \tilde{r}(t) \\ \tilde{h}(\tilde{r}(t)) \end{pmatrix} + \tilde{\mathcal{U}}^T \mathcal{S}\tilde{\mathcal{U}} \\
 &= \begin{pmatrix} \tilde{r}(t) \\ \tilde{h}(\tilde{r}(t)) \end{pmatrix}^T \Theta_1 \begin{pmatrix} \tilde{r}(t) \\ \tilde{h}(\tilde{r}(t)) \end{pmatrix} + \begin{pmatrix} \tilde{r}(t) \\ \tilde{h}(\tilde{r}(t)) \end{pmatrix}^T \Theta_2 \begin{pmatrix} \tilde{r}(t) \\ \tilde{h}(\tilde{r}(t)) \end{pmatrix} \\
 &\quad + r^T(t - \tau(t)) \tilde{\mathcal{K}}\mathcal{R}\tilde{\mathcal{K}}r(t - \tau(t)) + \tilde{\mathcal{U}}^T \mathcal{S}\tilde{\mathcal{U}}, \tag{18}
 \end{aligned}$$

where

$$\Theta_1 = \begin{pmatrix} -\mathcal{P}\tilde{\mathcal{D}} - \tilde{\mathcal{D}}^T \mathcal{P} + \tilde{\mathcal{K}}\mathcal{Q}\tilde{\mathcal{K}} & \mathcal{P}\tilde{\mathcal{A}} - \tilde{\mathcal{D}}\mathcal{C} \\ \tilde{\mathcal{A}}^T \mathcal{P} - \mathcal{C}^T \tilde{\mathcal{D}} & \mathcal{C}\tilde{\mathcal{A}} + \tilde{\mathcal{A}}^T \mathcal{C} - \mathcal{Q} \end{pmatrix}$$

and

$$\Theta_2 = \begin{pmatrix} \mathcal{P}\tilde{\mathcal{B}} & \mathcal{P} \\ \mathcal{C}\tilde{\mathcal{B}} & \mathcal{C} \end{pmatrix} \begin{pmatrix} \mathcal{R}^{-1} & 0 \\ 0 & \mathcal{S}^{-1} \end{pmatrix} \begin{pmatrix} \mathcal{P}\tilde{\mathcal{B}} & \mathcal{P} \\ \mathcal{C}\tilde{\mathcal{B}} & \mathcal{C} \end{pmatrix}^T.$$

There exists $\epsilon > 0$, and from (10) we have

$$\begin{pmatrix} \tilde{\mathcal{E}} & \mathcal{P}\tilde{\mathcal{A}} - \tilde{\mathcal{D}}\mathcal{C} & \mathcal{P}\tilde{\mathcal{B}} & \mathcal{P} \\ \tilde{\mathcal{A}}^T \mathcal{P} - \mathcal{C}^T \tilde{\mathcal{D}} & \mathcal{C}\tilde{\mathcal{A}} + \tilde{\mathcal{A}}^T \mathcal{C} - \mathcal{Q} & \mathcal{C}\tilde{\mathcal{B}} & \mathcal{C} \\ \tilde{\mathcal{B}}^T \mathcal{P} & \tilde{\mathcal{B}}^T \mathcal{C} & -\mathcal{R} & 0 \\ \mathcal{P} & \mathcal{C}^T & 0 & -\mathcal{S} \end{pmatrix} < 0, \tag{19}$$

where $\tilde{\mathcal{E}} = (1 + \epsilon)(\mathcal{P} + \mathcal{C}\mathcal{K}) - \mathcal{P}\tilde{\mathcal{D}} - \tilde{\mathcal{D}}^T \mathcal{P} + \tilde{\mathcal{K}}\mathcal{Q}\tilde{\mathcal{K}}$.

According to Lemma 3.5, we have

$$\begin{aligned}
 &\begin{pmatrix} (1 + \epsilon)(\mathcal{P} + \mathcal{C}\mathcal{K}) - \mathcal{P}\tilde{\mathcal{D}} - \tilde{\mathcal{D}}^T \mathcal{P} + \tilde{\mathcal{K}}\mathcal{Q}\tilde{\mathcal{K}} & \mathcal{P}\tilde{\mathcal{A}} - \tilde{\mathcal{D}}\mathcal{C} \\ \tilde{\mathcal{A}}^T \mathcal{P} - \mathcal{C}^T \tilde{\mathcal{D}} & \mathcal{C}\tilde{\mathcal{A}} + \tilde{\mathcal{A}}^T \mathcal{C} - \mathcal{Q} \end{pmatrix} \\
 &\quad + \begin{pmatrix} \mathcal{P}\tilde{\mathcal{B}} & \mathcal{P} \\ \mathcal{C}\tilde{\mathcal{B}} & \mathcal{C} \end{pmatrix} \begin{pmatrix} \mathcal{R}^{-1} & 0 \\ 0 & \mathcal{S}^{-1} \end{pmatrix} \begin{pmatrix} \mathcal{P}\tilde{\mathcal{B}} & \mathcal{P} \\ \mathcal{C}\tilde{\mathcal{B}} & \mathcal{C} \end{pmatrix}^T < 0.
 \end{aligned}$$

This indicates that

$$\Theta_1 + \Theta_2 < \begin{pmatrix} -(1 + \epsilon)(\mathcal{P} + \mathcal{C}\mathcal{K}) & 0 \\ 0 & 0 \end{pmatrix}. \tag{20}$$

By combining (11), (18), and (20), we have

$$\begin{aligned} \mathcal{D}^+ \mathcal{V}(t) &\leq -(1 + \epsilon) \tilde{r}^T(t) (\mathcal{P} + \mathcal{C}\mathcal{K}) \tilde{r}(t) \\ &\quad + r^T(t - \tau(t)) \tilde{\mathcal{K}} \mathcal{R} \tilde{\mathcal{K}} r(t - \tau(t)) + \tilde{\mathcal{U}}^T \mathcal{S} \tilde{\mathcal{U}}. \end{aligned} \tag{21}$$

Through (A1) and (13), we have

$$\mathcal{V}(t) \leq \tilde{r}^T(t) (\mathcal{P} + \mathcal{C}\mathcal{K}) \tilde{r}(t), \quad t \geq 0. \tag{22}$$

From (21) and (22), we have

$$\mathcal{D}^+ \mathcal{V}(t) \leq -(1 + \epsilon) \mathcal{V}(t) + \tilde{\mathcal{V}}(t) + \tilde{\mathcal{U}}^T \mathcal{S} \tilde{\mathcal{U}}, \quad t \geq 0, \tag{23}$$

where $\tilde{\mathcal{V}}(t) = \sup_{t-\tau \leq s \leq t} \mathcal{V}(s)$.

According to (23), we have

$$\mathcal{D}^+ (\mathcal{V}(t) - \vartheta) \leq -(1 + \epsilon) (\mathcal{V}(t) - \vartheta) + \tilde{\mathcal{V}}(t) - \vartheta, \quad t \geq 0, \tag{24}$$

where $\vartheta = \frac{\tilde{\mathcal{U}}^T \mathcal{S} \tilde{\mathcal{U}}}{\epsilon}$.

We can derive $\mathcal{V}(t) - \vartheta \leq (\tilde{\mathcal{V}}(t) - \vartheta) e^{-\lambda t}$ from Lemma 3.6, where λ is the unique positive root of $\lambda = (1 + \epsilon) - e^{\lambda t}$.

As such, NN model (7) is globally exponentially stable in the Lagrange sense, and the set

$$\Omega = \left\{ \tilde{r} \in \mathbb{R}^{2m_n} \mid \tilde{r}^T(t) \mathcal{P} \tilde{r}(t) \leq \frac{\tilde{\mathcal{U}}^T \mathcal{S} \tilde{\mathcal{U}}}{\epsilon} \right\}$$

is a globally exponentially attractive set of NN model (7). The proof of Theorem 3.9 is completed. □

Now we have the following Corollary 3.10.

Corollary 3.10 *Under (A1), if there exist positive definite matrices $\mathcal{P}, \mathcal{R}, \mathcal{S} \in \mathbb{R}^{2m_n \times 2m_n}$, positive diagonal matrix $\mathcal{Q} \in \mathbb{R}^{2m_n \times 2m_n}$ such that the following LMI hold:*

$$\begin{pmatrix} \hat{\Theta} & \mathcal{P} \tilde{\mathcal{A}} & \mathcal{P} \tilde{\mathcal{B}} & \mathcal{P} \\ \tilde{\mathcal{A}}^T \mathcal{P} & -\mathcal{Q} & 0 & 0 \\ \tilde{\mathcal{B}}^T \mathcal{P} & 0 & -\mathcal{R} & 0 \\ \mathcal{P} & 0 & 0 & -\mathcal{S} \end{pmatrix} < 0, \tag{25}$$

$$\tilde{\mathcal{K}} \mathcal{R} \tilde{\mathcal{K}} \leq \mathcal{P}, \tag{26}$$

where $\hat{\Theta} = \mathcal{P} - \mathcal{P} \tilde{\mathcal{D}} - \tilde{\mathcal{D}}^T \mathcal{P} + \tilde{\mathcal{K}} \mathcal{Q} \tilde{\mathcal{K}}$. As such, NN model (7) is globally exponentially stable in the Lagrange sense. Moreover, the set

$$\Omega = \left\{ \tilde{r} \in \mathbb{R}^{2m_n} \mid \tilde{r}^T(t) \mathcal{P} \tilde{r}(t) \leq \frac{\tilde{\mathcal{U}}^T \mathcal{S} \tilde{\mathcal{U}}}{\epsilon} \right\}$$

is a globally exponentially attractive set of NN model (7).

Proof Consider the following Lyapunov function which is positive definite and radially unbounded:

$$\mathcal{V}(t) = \tilde{r}^T(t) \mathcal{P} \tilde{r}(t). \tag{27}$$

Corollary 3.10 can be proven by applying a similar approach pertaining to Theorem 3.9. So, the proof is omitted here. \square

Corollary 3.11 *Suppose that the activation function $h_i(\cdot)$ is bounded, i.e., $|h_i(\cdot)|_A \leq k_i$, where k_i ($i = 1, 2, \dots, n$) is a positive constant, then NN (6) is globally exponentially stable in the Lagrange sense. Moreover, the compact set*

$$\tilde{\Omega}_1 = \left\{ r^A \mid \sum_{i=1}^n \sum_A \frac{1}{2} (r_i^A(t))^2 \leq \frac{\sum_{i=1}^n \mathcal{N}_i^2 / \mu_i}{2 \min_{1 \leq i \leq n} (d_i - \mu_i)}, \text{ where } 0 < \mu_i < d_i \right\} \tag{28}$$

is the globally exponentially attractive set of (6), where

$$\mathcal{N}_i = \frac{1}{2} \left(\sum_{i=1}^n \sum_{j=1}^n \sum_A \sum_B (|a_{ij}^{A,\bar{B}}| + |b_{ij}^{A,\bar{B}}|) k_j + |u_i^A| \right).$$

Proof First of all, we prove that NN model (6) is uniformly stable in the Lagrange sense. Consider the following Lyapunov function which is positive definite and radially unbounded:

$$\mathcal{V}(t) = \frac{1}{2} \sum_{i=1}^n \sum_A (r_i^A(t))^2. \tag{29}$$

Let $0 < \mu_i < d_i$ ($i = 1, 2, \dots, n$). Then the Dini derivative $\mathcal{D}^+ \mathcal{V}(t)$ can be computed along the positive half trajectory of NN model (6), and we have

$$\begin{aligned} \mathcal{D}^+ \mathcal{V}(t)|_{(6)} &= \sum_{i=1}^n \sum_A (r_i^A(t)) (\dot{r}_i^A(t)) \\ &= \sum_{i=1}^n \sum_A (r_i^A(t)) \left(-d_i(r_i^A(t)) + \sum_{j=1}^n \sum_B a_{ij}^{A,\bar{B}} h_j^A(r_j(t)) \right. \\ &\quad \left. + \sum_{j=1}^n \sum_B b_{ij}^{A,\bar{B}} h_j^A(r_j(t - \tau_j(t))) + u_i^A \right) \\ &\leq - \sum_{i=1}^n \sum_A d_i (r_i^A(t))^2 + \sum_{i=1}^n \sum_{j=1}^n \sum_A \sum_B |a_{ij}^{A,\bar{B}}| |h_j^A(r_j(t))| |r_i^A(t)| \\ &\quad + \sum_{i=1}^n \sum_{j=1}^n \sum_A \sum_B |b_{ij}^{A,\bar{B}}| |h_j^A(r_j(t - \tau_j(t)))| |r_i^A(t)| + \sum_{i=1}^n \sum_A |u_i^A| |r_i^A(t)| \\ &\leq - \sum_{i=1}^n \sum_A d_i (r_i^A(t))^2 + \sum_{i=1}^n \sum_{j=1}^n \sum_A \sum_B |a_{ij}^{A,\bar{B}}| k_j |r_i^A(t)| \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i=1}^n \sum_{j=1}^n \sum_A \sum_B |b_{ij}^{A,\bar{B}}| k_j |r_i^A(t)| + \sum_{i=1}^n \sum_A |u_i^A| |r_i^A(t)| \\
 & = - \sum_{i=1}^n \sum_A d_i (r_i^A(t))^2 + \sum_{i=1}^n \sum_{j=1}^n \sum_A \sum_B (|\alpha_{ij}^{A,\bar{B}}| k_j + |b_{ij}^{A,\bar{B}}| k_j + |u_i^A|) |r_i^A(t)| \\
 & = - \sum_{i=1}^n \sum_A d_i (r_i^A(t))^2 + \sum_{i=1}^n \sum_A 2\mathcal{N}_i |r_i^A(t)| \\
 & \leq - \sum_{i=1}^n \sum_A d_i (r_i^A(t))^2 + \sum_{i=1}^n \sum_A \mu_i (r_i^A(t))^2 + \sum_{i=1}^n \frac{\mathcal{N}_i^2}{\mu_i} \\
 & \leq - \min_{1 \leq i \leq n} \{d_i - \mu_i\} \sum_{i=1}^n \sum_A (r_i^A(t))^2 + \sum_{i=1}^n \frac{\mathcal{N}_i^2}{\mu_i} \\
 & = -2 \min_{1 \leq i \leq n} \{d_i - \mu_i\} \mathcal{V}(t) + \sum_{i=1}^n \frac{\mathcal{N}_i^2}{\mu_i}. \tag{30}
 \end{aligned}$$

Through Lemma 3.8, for any $t \geq 0$, we have

$$\mathcal{V}(t) - \rho \leq (\mathcal{V}(0) - \rho) e^{-vt}, \tag{31}$$

where $\rho = \frac{\sum_{i=1}^n \mathcal{N}_i^2 / \mu_i}{2 \min_{1 \leq i \leq n} (d_i - \mu_i)}$ and $v = 2 \min_{1 \leq i \leq n} \{d_i - \mu_i\}$.

This ensures that the solution of NN model (6) is uniformly bounded. Hence, NN model (6) is uniformly stable in the Lagrange sense. Observe that

$$\begin{aligned}
 \mathcal{V}(0) - \rho & \leq \mathcal{V}(0) = \frac{1}{2} \sum_{i=1}^n \sum_A (r_i^A(0))^2, \\
 & = \frac{1}{2} \sum_{i=1}^n \sum_A (\varphi_i^A(0))^2 \triangleq \mathcal{L}(\varphi^A).
 \end{aligned}$$

Then $\mathcal{L} \in \mathcal{CCF}^+$, and (31) implies that, for any $t \geq 0$,

$$\mathcal{V}(t) - \rho \leq \mathcal{L}(\varphi^A) e^{-vt}. \tag{32}$$

Through Definition 3.2, NN model (6) is globally exponentially stable and $\tilde{\Omega}_1$ is a globally exponentially attractive set. This proves the global exponential stability in the Lagrange sense of NN model (6). \square

Corollary 3.12 *If the activation function $h_i(\cdot)$ is bounded, i.e., $|h_i(\cdot)|_A \leq k_i$, where k_i ($i = 1, 2, \dots, n$) is a positive constant, then NN (6) is globally exponentially stable in the Lagrange sense. Moreover, the compact set*

$$\tilde{\Omega}_2 = \left\{ r^A \mid \sum_{i=1}^n \sum_A |r_i^A(t)| \leq \sum_{i=1}^n \frac{\mathcal{M}_i}{\min_{1 \leq i \leq n} d_i} \right\} \tag{33}$$

is a globally exponentially attractive set of NN model (6), where

$$\mathcal{M}_i = \sum_{i=1}^n \sum_{j=1}^n \sum_A \sum_B ((|a_{ij}^{A,\bar{B}}| + |b_{ij}^{A,\bar{B}}|) k_j + |u_i^A|).$$

Proof We first prove that $\tilde{\Omega}_2$ is a globally exponentially attractive set. We employ the positive definite and radially unbounded Lyapunov function

$$\mathcal{V}(t) = \sum_{i=1}^n \sum_A |r_i^A(t)|. \tag{34}$$

Then the Dini derivative $\mathcal{D}^+ \mathcal{V}(t)$ can be computed along the positive half trajectory of NN model (6), and we have

$$\begin{aligned} \mathcal{D}^+ \mathcal{V}(t)|_{(6)} &= \sum_{i=1}^n \sum_A r_i^A(t) \operatorname{sign}\{r_i^A(t)\} \\ &= \sum_{i=1}^n \sum_A \left[-d_i r_i^A(t) + \sum_{j=1}^n \sum_B a_{ij}^{A,\bar{B}} h_j^B(r_j(t)) + \sum_{j=1}^n \sum_B b_{ij}^{A,\bar{B}} h_j^B(r_j(t - \tau_j(t))) \right. \\ &\quad \left. + u_i^A \right] \operatorname{sign}\{r_i^A(t)\} \\ &\leq \sum_{i=1}^n \sum_A \left[-d_i |r_i^A(t)| + \sum_{j=1}^n \sum_B |a_{ij}^{A,\bar{B}}| |h_j^A(r_j(t))| \right. \\ &\quad \left. + \sum_{j=1}^n \sum_B |b_{ij}^{A,\bar{B}}| |h_j^A(r_j(t - \tau_j(t)))| + |u_i^A| \right] \\ &\leq - \sum_{i=1}^n \sum_A d_i |r_i^A(t)| + \sum_{i=1}^n \sum_{j=1}^n \sum_A \sum_B (|a_{ij}^{A,\bar{B}}| k_j + |b_{ij}^{A,\bar{B}}| k_j + |u_i^A|) \\ &\leq - \min_{1 \leq i \leq n} d_i \sum_{i=1}^n \sum_A |r_i^A(t)| + \sum_{i=1}^n \mathcal{M}_i \\ &= - \min_{1 \leq i \leq n} d_i \mathcal{V}(t) + \sum_{i=1}^n \mathcal{M}_i. \tag{35} \end{aligned}$$

Similarly, we infer that $\tilde{\Omega}_2$ is a globally exponentially attractive set by Lemma 3.8 and Definition 3.2. Thus, the proof procedure is omitted. \square

Remark 3.13 It is worth mentioning that the Clifford number multiplication does not satisfy commutativity, which complicates the investigation of Clifford-valued NNs’ dynamics. On the other hand, the method of decomposition is very successful in overcoming the difficulty of non-commutative Clifford numbers multiplication. Recent authors have obtained sufficient conditions for global stability and periodic solutions of Clifford-valued NN models by using the decomposition method [31, 33, 35, 39, 40].

Remark 3.14 In [30–39], the authors studied the global asymptotic stability or global exponential stability and periodic solutions of Clifford-valued NN models in the Lyapunov

sense. Compared with some previous studies of Clifford-valued NN models [30–40], in this paper, for the first time we have derived new sufficient conditions with respect to the global exponential stability in the Lagrange sense for a class of Clifford-valued NNs. Therefore, the proposed results in this paper are different and new compared with those in the existing literature [30–40].

4 Numerical examples

Two numerical examples are presented to demonstrate the feasibility and effectiveness of the results established in Sect. 3.

Example 1 For $m = 2$ and $n = 2$, a two-neuron Clifford-valued NN model can be described by

$$\dot{r}(t) = -\mathcal{D}r(t) + \mathcal{A}h(r(t)) + \mathcal{B}h(r(t - \tau(t))) + \mathcal{U}, \quad t \geq 0. \tag{36}$$

The multiplication generators are: $e_1^2 = e_2^2 = e_{12}^2 = e_1e_2e_{12} = -1$, $e_1e_2 = -e_2e_1 = e_{12}$, $e_1e_{12} = -e_{12}e_1 = -e_2$, $e_2e_{12} = -e_{12}e_2 = e_1$, $\dot{r}_1 = \dot{r}_1^0e_0 + \dot{r}_1^1e_1 + \dot{r}_1^2e_2 + \dot{r}_1^{12}e_{12}$, $\dot{r}_2 = \dot{r}_2^0e_0 + \dot{r}_2^1e_1 + \dot{r}_2^2e_2 + \dot{r}_2^{12}e_{12}$. Furthermore, we take

$$\begin{aligned} \mathcal{D} &= \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}, & \mathcal{K} &= \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}, \\ \mathcal{A} &= \begin{pmatrix} 0.2e_0 + e_1 & 0.1e_0 + 0.3e_2 + 0.6e_{12} \\ 0.05e_0 - 0.2e_2 + 0.4e_{12} & 0.1e_0 + 0.2e_1 + 0.05e_{12} \end{pmatrix}, \\ \mathcal{B} &= \begin{pmatrix} 0.3e_0 + 0.01e_1 & 0.1e_0 + 0.02e_2 + 0.3e_{12} \\ 0.05e_0 - 0.2e_2 + 0.05e_{12} & 0.2e_0 + 0.2e_1 + 0.05e_{12} \end{pmatrix}, \\ \mathcal{U} &= \begin{pmatrix} 0.2e_0 - 0.4e_1 + 0.1e_2 + 0.6e_{12} \\ 0.3e_0 + 0.8e_1 - 0.5e_2 - 0.2e_{12} \end{pmatrix}. \end{aligned}$$

According to their definitions, we have

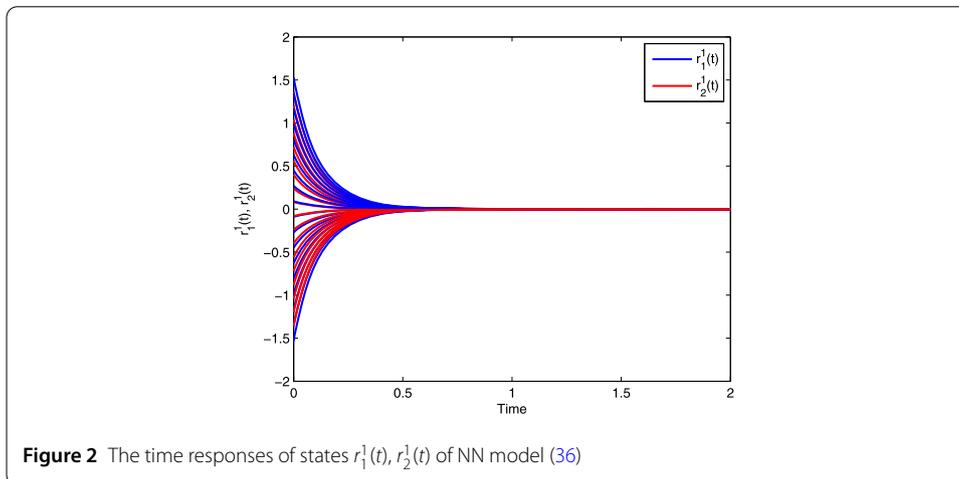
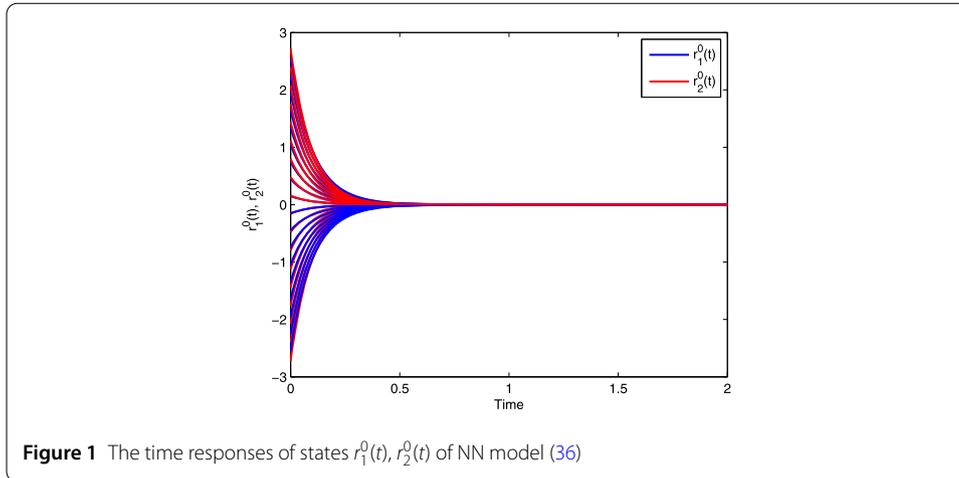
$$\begin{aligned} \mathcal{A}^0 &= \begin{pmatrix} 0.2 & 0.1 \\ 0.05 & 0.1 \end{pmatrix}, & \mathcal{A}^1 &= \begin{pmatrix} 1 & 0 \\ 0 & 0.2 \end{pmatrix}, \\ \mathcal{A}^2 &= \begin{pmatrix} 0 & 0.3 \\ -0.2 & 0 \end{pmatrix}, & \mathcal{A}^{12} &= \begin{pmatrix} 0 & 0.6 \\ 0.4 & 0.05 \end{pmatrix}, \\ \mathcal{B}^0 &= \begin{pmatrix} 0.3 & 0.1 \\ 0.05 & 0.2 \end{pmatrix}, & \mathcal{B}^1 &= \begin{pmatrix} 0.01 & 0 \\ 0 & 0.2 \end{pmatrix}, \\ \mathcal{B}^2 &= \begin{pmatrix} 0 & 0.02 \\ -0.2 & 0 \end{pmatrix}, & \mathcal{B}^{12} &= \begin{pmatrix} 0 & 0.3 \\ 0.05 & 0.05 \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} \tilde{\mathcal{A}} &= \begin{pmatrix} \mathcal{A}^0 & \mathcal{A}^{\bar{1}} & \mathcal{A}^{\bar{2}} & \mathcal{A}^{\bar{1}\bar{2}} \\ \mathcal{A}^1 & \mathcal{A}^{1.\bar{1}} & \mathcal{A}^{1.\bar{2}} & \mathcal{A}^{1.\bar{1}\bar{2}} \\ \mathcal{A}^2 & \mathcal{A}^{2.\bar{1}} & \mathcal{A}^{2.\bar{2}} & \mathcal{A}^{2.\bar{1}\bar{2}} \\ \mathcal{A}^{12} & \mathcal{A}^{12.\bar{1}} & \mathcal{A}^{12.\bar{2}} & \mathcal{A}^{12.\bar{1}\bar{2}} \end{pmatrix}, \\ &= \begin{pmatrix} \mathcal{A}^0 & -\mathcal{A}^1 & -\mathcal{A}^2 & -\mathcal{A}^{12} \\ \mathcal{A}^1 & \mathcal{A}^0 & -\mathcal{A}^{12} & \mathcal{A}^2 \\ \mathcal{A}^2 & \mathcal{A}^{12} & \mathcal{A}^0 & -\mathcal{A}^1 \\ \mathcal{A}^{12} & -\mathcal{A}^2 & \mathcal{A}^1 & \mathcal{A}^0 \end{pmatrix}, \\ \tilde{\mathcal{B}} &= \begin{pmatrix} \mathcal{B}^0 & \mathcal{B}^{\bar{1}} & \mathcal{B}^{\bar{2}} & \mathcal{B}^{\bar{1}\bar{2}} \\ \mathcal{B}^1 & \mathcal{B}^{1.\bar{1}} & \mathcal{B}^{1.\bar{2}} & \mathcal{B}^{1.\bar{1}\bar{2}} \\ \mathcal{B}^2 & \mathcal{B}^{2.\bar{1}} & \mathcal{B}^{2.\bar{2}} & \mathcal{B}^{2.\bar{1}\bar{2}} \\ \mathcal{B}^{12} & \mathcal{B}^{12.\bar{1}} & \mathcal{B}^{12.\bar{2}} & \mathcal{B}^{12.\bar{1}\bar{2}} \end{pmatrix}, \\ &= \begin{pmatrix} \mathcal{B}^0 & -\mathcal{B}^1 & -\mathcal{B}^2 & -\mathcal{B}^{12} \\ \mathcal{B}^1 & \mathcal{B}^0 & -\mathcal{B}^{12} & \mathcal{B}^2 \\ \mathcal{B}^2 & \mathcal{B}^{12} & \mathcal{B}^0 & -\mathcal{B}^1 \\ \mathcal{B}^{12} & -\mathcal{B}^2 & \mathcal{B}^1 & \mathcal{B}^0 \end{pmatrix}. \end{aligned}$$

Then we choose the activation function for NN model (36) as follows: $h_1(r) = h_2(r) = \frac{1}{4}(|r + 1| - |r - 1|)$ for all $r = r^0e_0 + r^1e_1 + r^2e_2 + r^{12}e_{12}$. It is obvious that the function satisfies (A1) with $k_1 = k_2 = 0.25$. Let the time-varying delay $\tau(t) = 0.4|\cos(t)| + 0.2$ with $\tau = 0.6$ and $\epsilon = 0.6$. By using MATLAB, the LMI conditions of (10), (11) in Theorem 3.9 are true with $t_{\min} = -0.1027$. The feasible solutions of the existing positive definite matrices are as follows:

$$\begin{aligned} \mathcal{P} &= \begin{pmatrix} 5.0730 & -0.0021 & -0.0050 & -0.0028 & -0.0004 & 0.0167 & 0 & -0.0054 \\ -0.0021 & 5.0974 & 0.0023 & 0.0015 & -0.0197 & 0 & 0.0053 & 0 \\ -0.0050 & 0.0023 & 5.0754 & -0.0022 & 0 & -0.0111 & -0.0004 & -0.0195 \\ -0.0028 & 0.0015 & -0.0022 & 5.0986 & 0.0108 & -0.0000 & 0.0167 & 0 \\ -0.0004 & -0.0197 & 0 & 0.0108 & 5.0757 & -0.0022 & 0.0049 & -0.0024 \\ 0.0167 & 0 & -0.0111 & 0 & -0.0022 & 5.0987 & 0.0028 & -0.0013 \\ 0 & 0.0053 & -0.0004 & 0.0167 & 0.0049 & 0.0028 & 5.0731 & -0.0021 \\ -0.0054 & 0 & -0.0195 & 0 & -0.0024 & -0.0013 & -0.0021 & 5.0981 \end{pmatrix}, \\ \mathcal{R} &= \begin{pmatrix} 22.4397 & 0.0464 & -0.0134 & -0.0123 & -0.0043 & -0.0028 & 0 & -0.1101 \\ 0.0464 & 22.3761 & 0.0106 & -0.0175 & 0.1232 & -0.0007 & 0.1090 & 0 \\ -0.0134 & 0.0106 & 22.4221 & 0.0463 & 0 & -0.0920 & -0.0054 & 0.1209 \\ -0.0123 & -0.0175 & 0.0463 & 22.3900 & 0.0880 & -0.0001 & -0.0028 & -0.0007 \\ -0.0043 & 0.1232 & 0 & 0.0880 & 22.4274 & 0.0463 & 0.0120 & -0.0116 \\ -0.0028 & -0.0007 & -0.0920 & -0.0001 & 0.0463 & 22.3901 & 0.0122 & 0.0185 \\ 0 & 0.1090 & -0.0054 & -0.0028 & 0.0120 & 0.0122 & 22.4397 & 0.0464 \\ -0.1101 & 0 & 0.1209 & -0.0007 & -0.0116 & 0.0185 & 0.0464 & 22.3758 \end{pmatrix}, \\ \mathcal{S} &= \begin{pmatrix} 35.9175 & 0.0030 & -0.0014 & -0.0005 & 0 & 0.0009 & 0 & -0.0057 \\ 0.0030 & 35.9200 & 0.0004 & -0.0001 & 0.0068 & 0 & 0.0056 & 0 \\ -0.0014 & 0.0004 & 35.9192 & 0.0030 & 0 & -0.0059 & 0 & 0.0066 \\ -0.0005 & -0.0001 & 0.0030 & 35.9181 & 0.0059 & 0 & 0.0009 & 0 \\ 0 & 0.0068 & 0 & 0.0059 & 35.9192 & 0.0030 & 0.0014 & -0.0004 \\ 0.0009 & 0 & -0.0059 & 0 & 0.0030 & 35.9181 & 0.0005 & 0.0001 \\ 0 & 0.0056 & 0 & 0.0009 & 0.0014 & 0.0005 & 35.9175 & 0.0031 \\ -0.0057 & 0 & 0.0066 & 0 & -0.0004 & 0.0001 & 0.0031 & 35.9197 \end{pmatrix}, \end{aligned}$$



and the positive diagonal matrices $\mathbf{C} = \text{diag}\{2.5673, 2.5579, 2.5666, 2.5626, 2.5665, 2.5627, 2.5674, 2.5580\}$, and $\mathbf{Q} = \text{diag}\{49.1714, 49.0520, 49.0625, 49.1578, 49.0658, 49.1574, 49.1719, 49.0450\}$. Calculating the eigenvalues of \mathcal{P} , we have 5.0600, 5.0600, 5.0648, 5.0649, 5.1072, 5.1074, 5.1127, and 5.1130. Therefore, we establish that NN model (7) is globally exponentially stable in the Lagrange sense, and the set

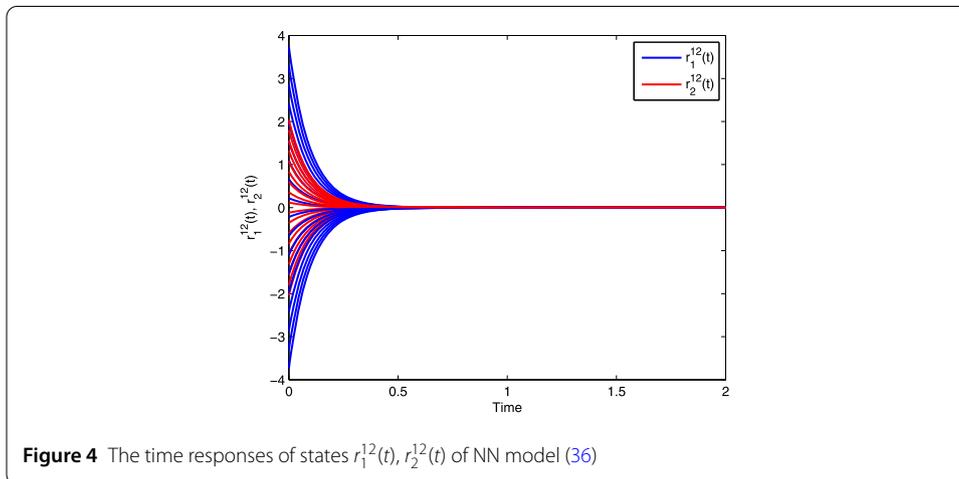
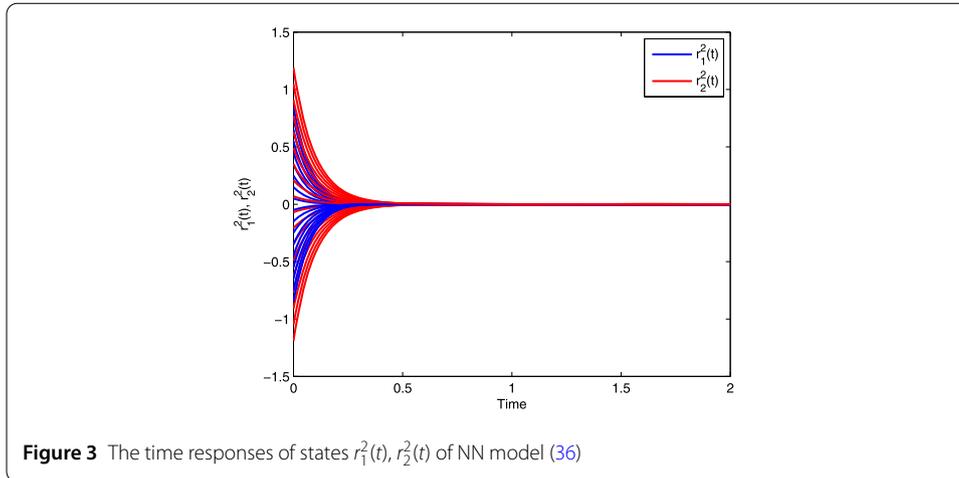
$$\Omega = \left\{ \tilde{r} \in \mathbb{R}^{2m_n} \mid \tilde{r}^T(t) \mathcal{P} \tilde{r}(t) \leq \frac{58.9174}{\epsilon} \right\}$$

is a globally exponentially attractive set of model (7) corresponding to Theorem 3.9, where $0 < \epsilon \leq 0.2491$.

It is straightforward to verify that all conditions of Theorem 3.9 are fulfilled. Under 20 randomly selected initial values, the time responses of the states $r_1^0(t), r_2^0(t), r_1^1(t), r_2^1(t), r_1^2(t), r_2^2(t), r_1^{12}(t)$, and $r_2^{12}(t)$ of model (36) are presented in Figures 1 to 4, respectively.

Example 2 Consider the Clifford-valued NN model (36) with the following parameters:

$$\mathcal{D} = \begin{pmatrix} 2.2 & 0 \\ 0 & 2.4 \end{pmatrix},$$



$$\mathcal{A} = \begin{pmatrix} 0.5e_0 + 0.1e_1 & 0.1e_0 + 0.3e_2 + 0.6e_{12} \\ 0.5e_0 - 0.1e_1 + 0.3e_{12} & 0.3e_0 + 0.1e_1 + 0.5e_{12} \end{pmatrix},$$

$$\mathcal{B} = \begin{pmatrix} 0.1e_0 + 0.2e_1 + 0.5e_2 & 0.3e_0 + 0.1e_2 + 0.4e_{12} \\ 0.6e_0 - 0.2e_1 + 0.3e_{12} & 0.4e_0 + 0.1e_{12} \end{pmatrix},$$

$$\mathcal{U} = \begin{pmatrix} 0.2e_0 - 0.1e_1 + 0.2e_2 + 0.1e_{12} \\ 0.1e_0 + 0.3e_1 + 0.1e_2 - 0.2e_{12} \end{pmatrix}.$$

We choose the activation function of NN model (36) as follows: $h_1(r) = h_2(r) = \frac{1}{2}(|r + 1| - |r - 1|)$ for all $r = r^0e_0 + r^1e_1 + r^2e_2 + r^{12}e_{12}$. It is obvious that the function satisfies (A1) with $k_1 = k_2 = 0.5$. The time-varying delay is considered as $\tau(t) = 0.2|\cos(t)| + 0.03$ with $\tau = 0.23$. Besides, it is easy to obtain $d_1 = 2.2, d_2 = 2.4, a_{11}^{A,\bar{B}} = 0.6, a_{12}^{A,\bar{B}} = 0.9, a_{21}^{A,\bar{B}} = 0.7, a_{22}^{A,\bar{B}} = 0.9, b_{11}^{A,\bar{B}} = 0.8, b_{12}^{A,\bar{B}} = 0.8, b_{21}^{A,\bar{B}} = 0.7, b_{22}^{A,\bar{B}} = 0.5, u_1 = 0.4, u_2 = 0.3$.

Select $\mu_1 = \mu_2 = 0.6$ to satisfy $0 < \mu_i < d_i$ ($i = 1, 2$). To obtain our global exponential attractive set, we firstly calculate $\mathcal{N}_i = \frac{1}{2}(\sum_{i=1}^n \sum_{j=1}^n \sum_A \sum_B (|a_{ij}^{A,\bar{B}}| + |b_{ij}^{A,\bar{B}}|)k_j + |u_i^A|), i = 1, 2$. Furthermore, we calculate $\mathcal{N}_1 = \frac{1}{2}(\sum_{j=1}^2 \sum_A \sum_B (|a_{1j}^{A,\bar{B}}| + |b_{1j}^{A,\bar{B}}|)k_j + |u_1^A|) = 0.975$ and $\mathcal{N}_2 = \frac{1}{2}(\sum_{j=1}^2 \sum_A \sum_B (|a_{2j}^{A,\bar{B}}| + |b_{2j}^{A,\bar{B}}|)k_j + |u_2^A|) = 0.85$. Then we obtain the set from Corollary 3.11: $\tilde{\Omega}_2 = \{r^A \mid \sum_A \frac{(r_1^A(t))^2 + (r_2^A(t))^2}{2} \leq 0.8714\}$.

5 Conclusions

In this paper, the global exponential stability problem for Clifford-valued RNN models with time-varying delays in the Lagrange sense has been examined with the use of Lyapunov functional and LMI methods. The sufficient conditions have been obtained by decomposing the n -dimensional Clifford-valued RNN model into a $2^m n$ -dimensional real-valued RNN model to ensure the global exponential stability of the considered NN model in the Lagrange sense. Furthermore, the estimated global exponentially attractive set has been obtained. In addition, the validity and feasibility of the results obtained have been demonstrated with two numerical examples. The results obtained in this paper can be further extended to other complex systems. We will be exploring the stabilizability and instabilizability analysis of Clifford-valued NN models with the help of various control systems. The corresponding results will be carried out in the near future.

Acknowledgements

This research is made possible through financial support from the Rajamangala University of Technology Suvarnabhumi, Thailand. The authors are grateful to the Rajamangala University of Technology Suvarnabhumi, Thailand for supporting this research.

Funding

The research is supported by the Rajamangala University of Technology Suvarnabhumi, Thailand.

Availability of data and materials

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Funding acquisition, NB; Conceptualization, GR; Software, GR and NB; Formal analysis, GR and NB; Methodology, GR and NB; Supervision, GR, PA, RS, PH, and CPL; Writing—original draft, GR; Validation, GR and NB; Writing—review and editing, GR and NB. All authors have read and agreed to the published version of the manuscript.

Author details

¹Department of Mathematics, Faculty of Science, Maejo University, Chiang Mai 52290, Thailand. ²Department of Mathematics, Thiruvalluvar University, Vellore, Tamil Nadu 632115, India. ³Department of Mathematics, Rajamangala University of Technology Suvarnabhumi, Nonthaburi 11000, Thailand. ⁴Department of Mathematics, Phuket Rajabhat University, Phuket 83000, Thailand. ⁵Institute for Intelligent Systems Research and Innovation, Deakin University, Waurn Ponds, VIC 3216, Australia. ⁶Department of Mathematics, Anand International College of Engineering, Jaipur 303012, India.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 15 January 2021 Accepted: 9 May 2021 Published online: 17 May 2021

References

1. Cao, J., Wang, J.: Global asymptotic stability of a general class of recurrent neural networks with time-varying delays. *IEEE Trans. Circuits Syst. I* **50**, 34–44 (2003)
2. Cao, J., Wang, J.: Global asymptotic and robust stability of recurrent neural networks with time delays. *IEEE Trans. Circuits Syst. I* **52**, 417–426 (2005)
3. Cao, J., Yuan, K., Li, H.: Global asymptotical stability of recurrent neural networks with multiple discrete delays and distributed delays. *IEEE Trans. Neural Netw.* **17**, 1646–1651 (2006)
4. Yang, B., Hao, M., Cao, J., Zhao, X.: Delay-dependent global exponential stability for neural networks with time-varying delay. *Neurocomputing* **338**, 172–180 (2019)
5. Zhang, Z., Liu, X., Chen, J., Guo, R., Zhou, S.: Further stability analysis for delayed complex-valued recurrent neural networks. *Neurocomputing* **251**, 81–89 (2017)
6. Pan, J., Liu, X., Xie, W.: Exponential stability of a class of complex-valued neural networks with time-varying delays. *Neurocomputing* **164**, 293–299 (2015)
7. Kaviarasan, B., Kwon, O.M., Park, M.J., Sakthivel, R.: Stochastic faulty estimator-based non-fragile tracking controller for multi-agent systems with communication delay. *Appl. Math. Comput.* **392**, 125704 (2021)
8. Sakthivel, R., Sakthivel, R., Kaviarasan, B., Alzahrani, F.: Leader-following exponential consensus of input saturated stochastic multi-agent systems with Markov jump parameters. *Neurocomputing* **287**, 84–92 (2018)
9. Hirose, A.: *Complex-Valued Neural Networks: Theories and Applications*. World Scientific, Singapore (2003)

10. Nitta, T.: Solving the XOR problem and the detection of symmetry using a single complex-valued neuron. *Neural Netw.* **16**, 1101–1105 (2003)
11. Isokawa, T., Nishimura, H., Kamiura, N., Matsui, N.: Associative memory in quaternionic Hopfield neural network. *Int. J. Neural Syst.* **18**, 135–145 (2008)
12. Matsui, N., Isokawa, T., Kusamichi, H., Peper, F., Nishimura, H.: Quaternion neural network with geometrical operators. *J. Intell. Fuzzy Syst.* **15**, 149–164 (2004)
13. Mandic, D.P., Jahanchahi, C., Took, C.C.: A quaternion gradient operator and its applications. *IEEE Signal Process. Lett.* **18**, 47–50 (2011)
14. Samidurai, R., Sriraman, R., Zhu, S.: Leakage delay-dependent stability analysis for complex-valued neural networks with discrete and distributed time-varying delays. *Neurocomputing* **338**, 262–273 (2019)
15. Aouiti, C., Bessifi, M., Li, X.: Finite-time and fixed-time synchronization of complex-valued recurrent neural networks with discontinuous activations and time-varying delays. *Circuits Syst. Signal Process.* **39**, 5406–5428 (2020)
16. Zhang, Z., Liu, X., Zhou, D., Lin, C., Chen, J., Wang, H.: Finite-time stabilizability and instabilizability for complex-valued memristive neural networks with time delays. *IEEE Trans. Syst. Man Cybern. Syst.* **48**, 2371–2382 (2018)
17. Li, Y., Meng, X.: Almost automorphic solutions for quaternion-valued Hopfield neural networks with mixed time-varying delays and leakage delays. *J. Syst. Sci. Complex.* **33**, 100–121 (2020)
18. Tu, Z., Zhao, Y., Ding, N., Feng, Y., Zhang, W.: Stability analysis of quaternion-valued neural networks with both discrete and distributed delays. *Appl. Math. Comput.* **343**, 342–353 (2019)
19. Shu, H., Song, Q., Liu, Y., Zhao, Z., Alsaadi, F.E.: Global μ -stability of quaternion-valued neural networks with non-differentiable time-varying delays. *Neurocomputing* **247**, 202–212 (2017)
20. Tan, M., Liu, Y., Xu, D.: Multistability analysis of delayed quaternion-valued neural networks with nonmonotonic piecewise nonlinear activation functions. *Appl. Math. Comput.* **341**, 229–255 (2019)
21. Jiang, B.X., Liu, Y., Kou, K.I., Wang, Z.: Controllability and observability of linear quaternion-valued systems. *Acta Math. Sin. Engl. Ser.* **36**, 1299–1314 (2020)
22. Liu, Y., Zheng, Y., Lu, J., Cao, J., Rutkowski, L.: Constrained quaternion-variable convex optimization: a quaternion-valued recurrent neural network approach. *IEEE Trans. Neural Netw. Learn. Syst.* **31**, 1022–1035 (2020)
23. Xia, Z., Liu, Y., Lu, J., Cao, J., Rutkowski, L.: Penalty method for constrained distributed quaternion-variable optimization. *IEEE Trans. Cybern.* (2020). <https://doi.org/10.1109/TCYB.2020.3031687>
24. Pearson, J.K., Bisset, D.L.: Neural networks in the Clifford domain. In: *Proc. IEEE ICNN*, Orlando, FL, USA (1994)
25. Pearson, J.K., Bisset, D.L.: Back Propagation in a Clifford Algebra. *ICANN*, Brighton (1992)
26. Buchholz, S., Sommer, G.: On Clifford neurons and Clifford multi-layer perceptrons. *Neural Netw.* **21**, 925–935 (2008)
27. Kuroe, Y.: Models of Clifford recurrent neural networks and their dynamics. In: *IJCNN-2011*, San Jose, CA, USA (2011).
28. Hitzer, E., Nitta, T., Kuroe, Y.: Applications of Clifford's geometric algebra. *Adv. Appl. Clifford Algebras* **23**, 377–404 (2013)
29. Buchholz, S.: A theory of neural computation with Clifford algebras. PhD thesis, University of Kiel (2005)
30. Zhu, J., Sun, J.: Global exponential stability of Clifford-valued recurrent neural networks. *Neurocomputing* **173**, 685–689 (2016)
31. Liu, Y., Xu, P., Lu, J., Liang, J.: Global stability of Clifford-valued recurrent neural networks with time delays. *Nonlinear Dyn.* **332**, 259–269 (2019)
32. Shen, S., Li, Y.: S^p -Almost periodic solutions of Clifford-valued fuzzy cellular neural networks with time-varying delays. *Neural Process. Lett.* **51**, 1749–1769 (2020)
33. Li, Y., Xiang, J., Li, B.: Globally asymptotic almost automorphic synchronization of Clifford-valued recurrent neural networks with delays. *IEEE Access* **7**, 54946–54957 (2019)
34. Li, B., Li, Y.: Existence and global exponential stability of pseudo almost periodic solution for Clifford-valued neutral high-order Hopfield neural networks with leakage delays. *IEEE Access* **7**, 150213–150225 (2019)
35. Li, Y., Xiang, J.: Global asymptotic almost periodic synchronization of Clifford-valued CNNs with discrete delays. *Complexity* **2019**, Article ID 6982109 (2019)
36. Li, B., Li, Y.: Existence and global exponential stability of almost automorphic solution for Clifford-valued high-order Hopfield neural networks with leakage delays. *Complexity* **2019**, Article ID 6751806 (2019)
37. Aouiti, C., Dridi, F.: Weighted pseudo almost automorphic solutions for neutral type fuzzy cellular neural networks with mixed delays and D operator in Clifford algebra. *Int. J. Syst. Sci.* **51**, 1759–1781 (2020)
38. Aouiti, C., Gharbia, I.B.: Dynamics behavior for second-order neutral Clifford differential equations: inertial neural networks with mixed delays. *Comput. Appl. Math.* **39**, 120 (2020)
39. Li, Y., Xiang, J.: Existence and global exponential stability of anti-periodic solution for Clifford-valued inertial Cohen-Grossberg neural networks with delays. *Neurocomputing* **332**, 259–269 (2019)
40. Rajchakit, G., Sriraman, R., Lim, C.P., Unyong, B.: Existence, uniqueness and global stability of Clifford-valued neutral-type neural networks with time delays. *Math. Comput. Simul.* (2021). <https://doi.org/10.1016/j.matcom.2021.02.023>
41. Liao, X.X., Luo, Q., Zeng, Z.G., Guo, Y.: Global exponential stability in Lagrange sense for recurrent neural networks with time delays. *Nonlinear Anal., Real World Appl.* **9**, 1535–1557 (2008)
42. Liao, X.X., Luo, Q., Zeng, Z.G.: Positive invariant and global exponential attractive sets of neural networks with time-varying delays. *Neurocomputing* **71**, 513–518 (2008)
43. Wang, X., Jiang, M., Fang, S.: Stability analysis in Lagrange sense for a non-autonomous Cohen-Grossberg neural network with mixed delays. *Nonlinear Anal.* **70**, 4294–4306 (2009)
44. Luo, Q., Zeng, Z., Liao, X.: Global exponential stability in Lagrange sense for neutral type recurrent neural networks. *Neurocomputing* **74**, 638–645 (2011)
45. Tu, Z., Wang, L.: Global Lagrange stability for neutral type neural networks with mixed time-varying delays. *Int. J. Mach. Learn. Cybern.* **9**, 599–609 (2018)
46. Wang, B., Jian, J., Jiang, M.: Stability in Lagrange sense for Cohen-Grossberg neural networks with time-varying delays and finite distributed delays. *Nonlinear Anal. Hybrid Syst.* **4**, 65–78 (2010)
47. Tu, Z., Jian, J., Wang, K.: Global exponential stability in Lagrange sense for recurrent neural networks with both time-varying delays and general activation functions via LMI approach. *Nonlinear Anal., Real World Appl.* **12**, 2174–2182 (2011)

48. Song, Q., Shu, H., Zhao, Z., Liu, Y., Alsaadi, F.E.: Lagrange stability analysis for complex-valued neural networks with leakage delay and mixed time-varying delays. *Neurocomputing* **244**, 33–41 (2017)

Submit your manuscript to a SpringerOpen[®] journal and benefit from:

- ▶ Convenient online submission
- ▶ Rigorous peer review
- ▶ Open access: articles freely available online
- ▶ High visibility within the field
- ▶ Retaining the copyright to your article

Submit your next manuscript at ▶ [springeropen.com](https://www.springeropen.com)
