Advances in Difference Equations
a SpringerOpen Journal

RESEARCH

Open Access

A Neumann problem for a diffusion equation with n-dimensional fractional Laplacian

Martin P. Arciga-Alejandre[1], Jorge Sanchez-Ortiz[1*], Francisco J. Ariza-Hernandez[1] and Eduard Garcia-Murcia[1]

*Correspondence: jsanchezmate@gmail.com
[1]Facultad de Matemáticas, Universidad Autónoma de Guerrero, Av. Lázaro Cárdenas S/N Cd. Universitaria, Chilpancingo, Guerrero, C.P. 39087, México

Abstract

We study an initial-boundary value problem for a n-dimensional stochastic diffusion equation with fractional Laplacian on $\mathbb{R}^n_+$. In order to prove existence and uniqueness, we generalize the Fokas method to construct the Green function for the associated linear problem and then we apply a fixed point argument. Also, we present an example where the explicit solutions are given.

MSC: 26A33; 31A10; 58J32

Keywords: Fractional Laplacian; Fokas method; Green function

1 Introduction

The classical diffusion phenomenon is governed by a second order linear partial differential equation, whose Green function is given by a Gaussian probability density function and which describes the movement of energy through a medium in response to a gradient of energy. On the other hand, the diffusion processes in various systems with complex structure, such as liquid crystals, glasses, polymers, biopolymers, and proteins, usually do not follow a Gaussian density, as a consequence the phenomenon is described by a fractional partial differential equation [7]. Dipierro et al., [4] have studied the asymptotic behavior of the solutions of the time-fractional diffusion equation.

There is some previous work for the initial-boundary value problem on the first quadrant $\mathbb{R}^2_+$ for fractional diffusion equations, where the Green function has been constructed and an integral representation of the solution was found [3, 6]. In this note, we consider the equation

$$u_t = \Delta^\alpha u, \tag{1}$$

where the operator Δ^α is defined via the Riesz fractional derivative, for each coordinate. Let us notice that the generalization of the Laplacian most commonly used [1, 9] is different from the one we use in this work.

However, Eq. (1) is an idealized version because many aspects are missing in the modeling; such as the inhomogeneity of the medium, external sources, and measurement errors.

Then a more realistic version is obtained by considering a stochastic version with additive noise. For example, Balanzario and Kaikina [2] studied the stochastic nonlinear Landau–Ginzburg equations on the half-line with Dirichlet white-noise boundary conditions, Shi and Wang [11] studied the solution for a stochastic fractional partial differential equation driven by an additive fractional space–time white noise. In Sanchez et al. [10], studied the stochastic version of (1) for the 2-dimensional case; however, the n-dimensional case on $\mathbb{R}^n_+ := \{\mathbf{x} = (x_1, \dots, x_n) : x_j \geq 0, j = 1, \dots n\}$ has not been studied. In the present work we tackle this problem via the main ideas of the Fokas method (unified transform) [5], this method is a technique for solving initial-boundary value problems for partial differential equations. Moreover, it generates integral representation formulas for solutions, where the integrals converge uniformly on the boundary.

2 Preliminaries

Let us give some known definitions and results.

Definition 1 The n-dimensional Fourier–Laplace transform is defined as follows:

$$\widehat{u}(\mathbf{k}, t) = \int_{\mathbb{R}^n_+} e^{-i\mathbf{k}\cdot\mathbf{x}} u(\mathbf{x}, t)\, d\mathbf{x},$$

where $\mathbf{x} \in \mathbb{R}^n_+$, $\mathbf{k} \in \mathbb{C}^n = \{\mathbf{k} = (k_1, \dots, k_n) : k_j \in \mathbb{C}, j = 1, \dots n\}$ and $\Im m(k_j) \leq 0$, $\mathbf{k} \cdot \mathbf{x}$ is the usual inner product, and its inverse is defined by

$$u(\mathbf{x}, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot\mathbf{x}} \widehat{u}(\mathbf{k}, t)\, d\mathbf{k}.$$

Definition 2 The Riesz fractional operator is defined by

$$\mathcal{D}^{\alpha}_{x_j} u(\mathbf{x}, t) = -\frac{1}{2\Gamma(3-\alpha)\cos(\frac{\pi}{2}\alpha)} \int_0^{\infty} \frac{\operatorname{sgn}(x_j - y_j)}{|x_j - y_j|^{\alpha-2}} \partial^3_{y_j} u(\mathbf{x}_j, t)\, dy_j.$$

Here, $\alpha \in (2, 3)$, $\mathbf{x}_j \in \mathbb{R}^n_+$ is the vector $\mathbf{x}$, where the jth coordinate is y_j, $j = 1, \dots n$.

Note that the operator, using integration by parts, $\mathcal{D}^{\alpha}_{x_j}$ can be represented in the following form [8]:

$$(-\Delta)^{\alpha}_j u(\mathbf{x}, t) = \frac{\alpha}{2\Gamma(1-\alpha)\cos(\frac{\pi}{2}\alpha)} \int_0^{\infty} \frac{u(\mathbf{x}_j, t) - u(\mathbf{x}, t)}{|x_j - y_j|^{1+\alpha}}\, dy_j.$$

Lemma 1 *If Δ^{α}, $\alpha \in (2, 3)$, is the fractional n-dimensional Laplace operator*

$$\Delta^{\alpha} = \mathcal{D}^{\alpha}_{x_1} + \mathcal{D}^{\alpha}_{x_2} + \cdots + \mathcal{D}^{\alpha}_{x_n},$$

then, for $\Im m(k_l) \leq 0$,

$$\widehat{\Delta^{\alpha} u}(\mathbf{k}) = |\mathbf{k}|^{\alpha} \widehat{u}(\mathbf{k}, t) - \sum_{l=1}^{n} \sum_{j=0}^{2} \frac{|k_l|^{\alpha}}{(ik_l)^{j+1}} \partial^j_{x_l} \widehat{u}(\mathbf{k}_{[-l]}, t).$$

Here, $|\mathbf{k}|^{\alpha} := \sum_{l=1}^{n} |k_l|^{\alpha}$ and $\mathbf{k}_{[-l]} \in \mathbb{C}^n$ is the $\mathbf{k}$ vector, where its lth coordinate is zero.

Proof The theorem follows from the linearity of the operator Δ^α and the well-known equation

$$\widehat{\mathcal{D}_x^\alpha u}(k) = |k|^\alpha \widehat{u}(k,t) - \sum_{j=0}^{2} \frac{|k|^\alpha}{(ik)^{j+1}} \partial_x^j \widehat{u}(0,t). \qquad \square$$

3 Green function

We consider a linear problem for an evolution equation with initial condition u_0 and boundary conditions h_j, $j = 1,\dots,n$,

$$\begin{cases} u_t = \Delta^\alpha u, \\ u(\mathbf{x},0) = u_0(\mathbf{x}), \\ u_{x_j}(\mathbf{x}_{[-j]},t) = h_j(\mathbf{x}_{[-j]},t), \end{cases} \tag{2}$$

where $\alpha \in (2,3)$, $t > 0$, $\mathbf{x}_{[-j]} \in \mathbb{R}_+^n$ means that the jth coordinate of $\mathbf{x}$ is zero, with the compatibility conditions $h_j(\mathbf{x}_{[-j,-l]},t) = h_l(\mathbf{x}_{[-j,-l]},t)$ where $\mathbf{x}_{[-j,-l]} \in \mathbb{R}_+^n$ is such that jth and lth coordinates, x_l and x_j, are equal to zero for $j \neq l$.

Theorem 1 *Let the initial data* $u_0(\mathbf{x}) \in \mathbf{L}^1(\mathbb{R}_+^n)$ *and the boundary data* $h_j(\mathbf{x}_{[-j]},t) \in \mathbf{C}(\mathbb{R}_+;\mathbf{L}^1(\mathbb{R}_+^n))$. *Suppose that there exists some function* $u(\mathbf{x},t)$, *which satisfies* (2). *Then* $u(\mathbf{x},t)$ *has the following integral representation*:

$$u(\mathbf{x},t) = \mathcal{G}^I(t)u_0 - \sum_{l=1}^{n} \int_0^t \mathcal{G}^{B_l}(t-s)h_l\,ds,$$

where the Green operators are given by

$$\begin{aligned} \mathcal{G}^I(t)u_0 &= \int_{\mathbb{R}_+^n} G^I(\mathbf{x},\mathbf{y},t)u_0(\mathbf{y})\,d\mathbf{y}, \\ \mathcal{G}^{B_l}(t)h_l &= \int_{\mathbb{R}_+^{n-1}} G^{B_l}(\mathbf{x},\mathbf{y}_{[-l]},t)h_l(\mathbf{y}_{[-l]},s)\,d\mathbf{y}_{[-l]}, \end{aligned} \tag{3}$$

and the Green functions are

$$\begin{aligned} G^I(\mathbf{x},\mathbf{y},\tau) &= \frac{2^n}{\pi^n} \int_{\mathbb{R}_+^n} e^{-\mathbf{k}^\alpha \tau} \prod_{l=1}^{n} \cos[k_l x_l]\cos[k_l y_l]\,d\mathbf{k}, \\ G^{B_l}(\mathbf{x},\mathbf{y}_l,\tau) &= \frac{2^n}{\pi^n} \int_{\mathbb{R}_+^n} e^{-\mathbf{k}^\alpha \tau} k_l^{\alpha-2} \cos[k_l x_l] \prod_{\substack{m=1 \\ m\neq l}}^{n} \cos[k_m x_m]\cos[k_m y_m]\,d\mathbf{k}. \end{aligned}$$

Here, $\mathbf{k}^\alpha = \sum_{l=1}^n k_l^\alpha$.

Proof Applying Theorem 1 to Eq. (2), we obtain

$$\widehat{u}_t(\mathbf{k},t) + |\mathbf{k}|^\alpha \widehat{u}(\mathbf{k},t) = \sum_{l=1}^{n}\sum_{j=0}^{2} \frac{|k_l|^\alpha}{(ik_l)^{j+1}} \partial_{x_l}^j \widehat{u}(\mathbf{k}_{[-l]},t).$$

Now, we multiply the above equation by $e^{|\mathbf{k}|^\alpha t}$ and integrate from 0 to t,

$$e^{|\mathbf{k}|^\alpha t}\widehat{u}(\mathbf{k},t) - \widehat{u}_0(\mathbf{k}) = \sum_{l=1}^{n}\sum_{j=0}^{2}\frac{|k_l|^\alpha}{(ik_l)^{j+1}}g_j^l\bigl(|\mathbf{k}|^\alpha,\mathbf{k}_{[-l]},t\bigr) \tag{4}$$

for $\Im m(k_l) \le 0$, where

$$g_j^l(\sigma,\mathbf{k}_{[-l]},t) = \int_0^t e^{\sigma s}\partial_{x_l}^j\widehat{u}(\mathbf{k}_{[-l]},s)\,ds.$$

Now, we initially consider 2-dimensional case. Thus, Eq. (4) is expressed as

$$\begin{aligned} e^{|\mathbf{k}|^\alpha t}\widehat{u}(\mathbf{k},t) - \widehat{u}_0(\mathbf{k}) &= \sum_{j=0}^{2}\frac{|k_1|^\alpha}{(ik_1)^{j+1}}g_j^1\bigl(|\mathbf{k}|^\alpha,\mathbf{k}_{[-1]},t\bigr) \\ &\quad + \sum_{j=0}^{2}\frac{|k_2|^\alpha}{(ik_2)^{j+1}}g_j^2\bigl(|\mathbf{k}|^\alpha,\mathbf{k}_{[-2]},t\bigr). \end{aligned} \tag{5}$$

Applying the inverse transform in (5) with respect to k_1 and moving the contour of integration for the terms with g_j^1 in the integrand, we obtain

$$\begin{aligned} \widehat{u}(x_1,k_2,t) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{ik_1x_1-|\mathbf{k}|^\alpha t}\left[\widehat{u}_0(\mathbf{k}) + \sum_{j=0}^{2}\frac{|k_2|^\alpha}{(ik_2)^{j+1}}g_j^2\bigl(|\mathbf{k}|^\alpha,\mathbf{k}_{[-2]},t\bigr)\right]dk_1 \\ &\quad + \frac{1}{2\pi}\int_{\partial D_1^+} e^{ik_1x_1-|\mathbf{k}|^\alpha t}\sum_{j=0}^{2}\frac{|k_1|^\alpha}{(ik_1)^{j+1}}g_j^1\bigl(|\mathbf{k}|^\alpha,\mathbf{k}_{[-1]},t\bigr)\,dk_1, \end{aligned} \tag{6}$$

where $D_1^+ = \{k_1 \in \mathbb{C} : 0 \le \Im m(k_1) \le \frac{\pi}{2\alpha}|\Re e(k_1)|\}$. Let us note the following: if we substitute k_1 by $-k_1$, the functions g_j^1 from Eq. (5) are invariant. Then, making this change of variables in (5), we get

$$\begin{aligned} e^{|\mathbf{k}|^\alpha t}\widehat{u}(-k_1,k_2,t) - \widehat{u}_0(-k_1,k_2) &= \sum_{j=0}^{2}\frac{|k_1|^\alpha}{(-ik_1)^{j+1}}g_j^1\bigl(|\mathbf{k}|^\alpha,\mathbf{k}_{[-1]},t\bigr) \\ &\quad + \sum_{j=0}^{2}\frac{|k_2|^\alpha}{(ik_2)^{j+1}}g_j^2\bigl(|\mathbf{k}|^\alpha,-\mathbf{k}_{[-2]},t\bigr), \end{aligned} \tag{7}$$

for $\Im m(-k_1), \Im m(k_2) \le 0$. Substituting g_2^1 from Eq. (7) in (6) and using the fact that

$$\int_{\partial D_1^+} e^{ik_1x_1}\widehat{u}(-k_1,k_2,t)\,dk_1 = 0,$$

by the Cauchy theorem, we obtain the following integral representation:

$$\begin{aligned} \widehat{u}(x_1,k_2,t) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{ik_1x_1-|\mathbf{k}|^\alpha t}\Biggl[\widehat{u}_0(\mathbf{k}) + \widehat{u}_0(-k_1,k_2) - \frac{2|k_1|^\alpha}{k_1^2}g_1^1\bigl(|\mathbf{k}|^\alpha,\mathbf{k}_{[-1]},t\bigr) \\ &\quad + \sum_{j=0}^{2}\frac{|k_2|^\alpha}{(ik_2)^{j+1}}\Bigl[g_j^2\bigl(|\mathbf{k}|^\alpha,\mathbf{k}_{[-2]},t\bigr) + g_j^2\bigl(|\mathbf{k}|^\alpha,-\mathbf{k}_{[-2]},t\bigr)\Bigr]\Biggr]dk_1. \end{aligned} \tag{8}$$

Applying the inverse transform in (8) with respect to k_2 and moving the contour of integration for the terms with g_j^2 in the integrand, we obtain

$$\begin{aligned} u(\mathbf{x},t) &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{i\mathbf{k}\cdot\mathbf{x}-|\mathbf{k}|^\alpha t}\big[\widehat{u}_0(\mathbf{k})+\widehat{u}_0(-k_1,k_2)\big] \\ &\quad - \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{i\mathbf{k}\cdot\mathbf{x}-|\mathbf{k}|^\alpha t}\frac{2|k_1|^\alpha}{k_1^2} g_1^1\big(|\mathbf{k}|^\alpha,\mathbf{k}_{[-1]},t\big)\,d\mathbf{k} \\ &\quad + \frac{1}{(2\pi)^2}\int_{\partial D_2^+}\int_{\mathbb{R}} e^{i\mathbf{k}\cdot\mathbf{x}-|\mathbf{k}|^\alpha t}\sum_{j=0}^{2}\frac{|k_2|^\alpha}{(ik_2)^{j+1}} \\ &\quad \times \big[g_j^2\big(|\mathbf{k}|^\alpha,\mathbf{k}_{[-2]},t\big)+g_j^2\big(|\mathbf{k}|^\alpha,-\mathbf{k}_{[-2]},t\big)\big]\,d\mathbf{k}, \end{aligned} \tag{9}$$

where $D_2^+ = \{k_2\in\mathbb{C} : 0\le \Im m(k_2)\le \frac{\pi}{2\alpha}|\Re e(k_2)|\}$. Let us note the following: if we substitute k_2 by $-k_2$, the functions g_j^2 from Eq. (8) are invariant. Then, making this change of variables in (7), we get

$$\begin{aligned} \widehat{u}(x_1,-k_2,t) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{ik_1x_1-|\mathbf{k}|^\alpha t}\big[\widehat{u}_0(k_1,-k_2)+\widehat{u}_0(-\mathbf{k})\big] \\ &\quad - \frac{1}{2\pi}\int_{\mathbb{R}} e^{ik_1x_1-|\mathbf{k}|^\alpha t}\Bigg[\frac{2|k_1|^\alpha}{k_1^2} g_1^1\big(|\mathbf{k}|^\alpha,-\mathbf{k}_{[-1]},t\big) \\ &\quad + \sum_{j=0}^{2}\frac{|k_2|^\alpha}{(-ik_2)^{j+1}}\big[g_j^2\big(|\mathbf{k}|^\alpha,\mathbf{k}_{[-2]},t\big)+g_j^2\big(|\mathbf{k}|^\alpha,-\mathbf{k}_{[-2]},t\big)\big]\Bigg]\,dk_1, \end{aligned} \tag{10}$$

for $\Im m(k_1), \Im m(k_2)\ge 0$. Substituting $g_2^2(|\mathbf{k}|^\alpha,\pm\mathbf{k}_{[-2]},t)$ from Eq. (10) in (9) and using the fact that

$$\int_{\partial D_2^+} e^{ik_2x_2}\widehat{u}(x_1,-k_2,t)\,dk_2 = 0,$$

by the Cauchy theorem, we obtain the following integral representation:

$$u = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{i\mathbf{k}\cdot\mathbf{x}-|\mathbf{k}|^\alpha t}\Bigg[\sum_{\mathbf{r}\in S_2}\widehat{u}_0(\mathbf{r}) - 2\sum_{l=1}^{2}\sum_{\mathbf{r}_{[-l]}\in S_2}\frac{|k_l|^\alpha}{k_l^2} g_1^l\big(|\mathbf{k}|^\alpha,\mathbf{r}_{[-l]},t\big)\Bigg]\,d\mathbf{k}, \tag{11}$$

where $\mathbf{r}\in S_2 = \{(\pm k_1,\pm k_2)\}$ and $\mathbf{r}_{[-l]}$ is such that the lth coordinate is equal to zero. In Eq. (11) we have, after interchanging the integration order, integrals of the form

$$\begin{aligned} &\int_{\mathbb{R}^2}\int_{\mathbb{R}_+^2} e^{i\mathbf{k}\cdot(x_1\pm y_1,x_2\pm y_2)-|\mathbf{k}|^\alpha t}u_0(\mathbf{y})\,d\mathbf{y}\,d\mathbf{k}, \\ &\int_{\mathbb{R}^2}\int_0^t\int_{\mathbb{R}_+} e^{i\mathbf{k}\cdot(x_1,x_2\pm y_2)-|\mathbf{k}|^\alpha(t-s)}\frac{|k_1|^\alpha}{k_1^2}u_{x_1}(0,\pm y_2,s)\,dy_2\,ds\,d\mathbf{k}, \end{aligned}$$

and

$$\int_{\mathbb{R}^2}\int_0^t\int_{\mathbb{R}_+} e^{i\mathbf{k}\cdot(x_1\pm y_1,x_2)-|\mathbf{k}|^\alpha(t-s)}\frac{|k_2|^\alpha}{k_2^2}u_{x_2}(\pm y_1,0,s)\,dy_1\,ds\,d\mathbf{k}.$$

We notice that all the integrals above are absolutely integrable, then using the Fubini theorem, after some simplifications, we arrive from Eq. (11) at the following equation:

$$u(\mathbf{x},t) = \mathcal{G}^I(t)u_0 - \sum_{l=1}^{2}\int_0^t \mathcal{G}^{B_l}(t-s)h_l\,ds,$$

where the Green operators are given by

$$\mathcal{G}^I(t)u_0 = \int_{\mathbb{R}_+^2} G^I(\mathbf{x},\mathbf{y},t)u_0(\mathbf{y})\,d\mathbf{y},$$
$$\mathcal{G}^{B_l}(t)h_l = \int_{\mathbb{R}_+} G^{B_l}(\mathbf{x},\mathbf{y}_{[-l]},t)h_l(\mathbf{y}_{[-l]},s)\,d\mathbf{y}_{[-l]},$$

and the Green functions are

$$G^I(\mathbf{x},\mathbf{y},\tau) = \left(\frac{2}{\pi}\right)^2\int_{\mathbb{R}_+^2} e^{-\mathbf{k}^\alpha\tau}\prod_{l=1}^{2}\cos[k_lx_l]\cos[k_ly_l]\,d\mathbf{k},$$
$$G^{B_l}(\mathbf{x},\mathbf{y}_{[-l]},\tau) = \left(\frac{2}{\pi}\right)^2\int_{\mathbb{R}_+^2} e^{-\mathbf{k}^\alpha\tau}\cos[k_lx_l]k_l^{\alpha-2}\prod_{\substack{m=1\\m\neq l}}^{2}\cos[k_mx_m]\cos[k_my_m]\,d\mathbf{k},$$

where $\mathbf{k}^\alpha = k_1^\alpha + k_2^\alpha$. Now, following the previous arguments we can tackle the n-dimensional case. This can be achieved, via mathematical induction over n, passing from Eq. (4) to Eq. (12), through the steps that we describe in the 2-dimensional case. Analogous to Eq. (11), we obtain an integral representation for u,

$$u(\mathbf{x},t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot\mathbf{x}-|\mathbf{k}|^\alpha t}\Bigg[\sum_{\mathbf{r}\in S_n}\widehat{u}_0(\mathbf{r}) - 2\sum_{l=1}^{n}\sum_{\mathbf{r}_{[-l]}\in S_n}\frac{|k_l|^\alpha}{k_l^2}g_1^l\big(|\mathbf{k}|^\alpha,\mathbf{r}_{[-l]},t\big)\Bigg]\,d\mathbf{k}, \tag{12}$$

where $\mathbf{r}\in S_n = \{(\pm k_1,\pm k_2,\dots,\pm k_n)\}$ and $\mathbf{r}_{[-l]}$ is such that the lth coordinate is equal to zero. Interchanging the integrals in the above equation, by Fubini's theorem, we obtain the desired result. □

4 Stochastic nonlinear problem

In order to state the problem, we define the Brownian sheet $\dot{B}$ on $\mathbb{R}_+^n\times[0,T]$ on a complete probability space $(\Omega,\mathcal{F},\mathcal{F}_t,P)$, here $\mathcal{F}$ is a σ-algebra, $\{\mathcal{F}_t\}_{t\geq 0}$ is a right-continuous filtration on $(\Omega,\mathcal{F})$ such that $\mathcal{F}_0$ contains all P-negligible subsets and P is a probability measure. We consider a center Gaussian field $B = \{B(\mathbf{x},t)|\mathbf{x}\geq 0, t\geq 0\}$ with covariance function given by

$$K\big((\mathbf{x},t),(\mathbf{y},s)\big) = \min\{t,s\}\operatorname{diag}\big(\min\{x_1,y_1\},\dots,\min\{x_n,y_n\}\big).$$

We suppose that B generates a $(\mathcal{F}_t, t\geq 0)$-martingale measure in the sense of Walsh [12]. Let the initial condition u_0 be $\mathcal{F}_0\times\mathcal{B}(\mathbb{R}_+^n)$ measurable, where $\mathcal{B}(\mathbb{R}_+^n)$ is the Borelian σ-algebra over $\mathbb{R}_+^n$.

Now, we consider the following initial-boundary value problem for a nonlinear equation:

$$\begin{cases} u_t - \Delta^\alpha u = \mathcal{N}u + \dot{B}, \\ u(\mathbf{x},0) = u_0(\mathbf{x}), \\ u_{x_j}(\mathbf{x}_{[-j]},t) = h_j(\mathbf{x}_{[-j]},t), \end{cases} \tag{13}$$

where $\mathbf{x} \in \mathbb{R}^n_+$, $t > 0$, $\alpha \in (2,3)$, $\mathcal{N}$ is a Lipschitzian operator; i.e., $|\mathcal{N}u - \mathcal{N}v| \leq C|u-v|$, $C > 0$, and the compatibility conditions $h_j(\mathbf{x}_{[-j,-l]},t) = h_l(\mathbf{x}_{[-j,-l]},t)$ are satisfied. We understand the solutions for the problem (13) in the following sense: u is a solution if, for all $\mathbf{x} \in \mathbb{R}^n_+$ and $t > 0$, the following equation is fulfilled:

$$\begin{aligned} u(\mathbf{x},t) = {}& \mathcal{G}^I(t)u_0 + \sum_{l=1}^n \int_0^t \mathcal{G}^{B_l}(t-s)h_l\,ds \\ &+ \int_0^t \int_{\mathbb{R}^n_+} G(\mathbf{x}-\mathbf{y},t-s)\mathcal{N}u(\mathbf{y},s)\,d\mathbf{y}\,ds, \\ &+ \int_0^t \int_{\mathbb{R}^n_+} G(\mathbf{x}-\mathbf{y},t-s)\,dB(\mathbf{y},s), \end{aligned} \tag{14}$$

where the Green operators $\mathcal{G}^I(t)$, $\mathcal{G}^{B_l}(t)$ are given in Eq. (3) and the Green function is

$$G(\mathbf{x},t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_+} e^{i\mathbf{k}\cdot\mathbf{x} - |\mathbf{k}|^\alpha t}\,d\mathbf{k}. \tag{15}$$

Theorem 2 *Let the initial data* $u_0(\mathbf{x}) \in \mathbf{L}^1(\mathbb{R}^n_+)$ *and the boundary data* $h_j(\mathbf{x}_{[-j]},t) \in \mathbf{C}(\mathbb{R}_+;\mathbf{L}^1(\mathbb{R}^n_+))$. *Suppose that, for each* $T > 0$, *there exists a constant* $C > 0$ *such that, for each* $\mathbf{x} \in \mathbb{R}^n_+$, $t \in [0,T]$ *and* $u, v \in \mathbb{R}^n$, $|\mathcal{N}u - \mathcal{N}v| \leq C|u-v|$, *and for some* $p \geq 1$,

$$\sup_{\mathbf{x}\geq 0} \mathbb{E}\big(|u_0(\mathbf{x})|^p\big) < \infty. \tag{16}$$

Then, there exists a unique solution $u(\mathbf{x},t)$ *to Eq.* (13). *Moreover, for all* $T > 0$ *and* $p \geq 1$,

$$\sup_{\substack{\mathbf{x}\geq 0 \\ t\in[0,T]}} \mathbb{E}\big(|u(\mathbf{x},t)|^p\big) < \infty.$$

Proof First, we define a Picard succession:

$$\begin{aligned} u^{n+1}(\mathbf{x},t) = {}& u^0(\mathbf{x},t) + \sum_{l=1}^n \int_0^t \int_{\mathbb{R}^{n-1}_+} G^{B_l}(\mathbf{x},\mathbf{y}_{[-l]},t-s)h_l(\mathbf{y}_{[-l]},s)\,d\mathbf{y}_{[-l]}\,ds \\ &+ \int_0^t \int_{\mathbb{R}^n_+} G(\mathbf{x}-\mathbf{y},t-s)\mathcal{N}u^n(\mathbf{y},s)\,d\mathbf{y}\,ds \\ &+ \int_0^t \int_{\mathbb{R}^n_+} G(\mathbf{x}-\mathbf{y},t-s)\,dB(\mathbf{y},s) \end{aligned} \tag{17}$$

where

$$u^0(\mathbf{x},t) = \int_{\mathbb{R}^n_+} G^I(\mathbf{x},\mathbf{y},t)u_0(\mathbf{y})\,d\mathbf{y}.$$

Now, let us prove that $\{u^n(\mathbf{x},t)\}_{n\geq 0}$ converges in $L^p(\Omega)$. Using the fact that, for all $t \geq 0$, $G(\mathbf{x},t)$ from Eq. (15) is a probability density function with respect to $\mathbf{x}$, we obtain, for $n \geq 2$,

$$\begin{aligned}
&\mathbb{E}\left(\left|u^{n+1}(\mathbf{x},t) - u^n(\mathbf{x},t)\right|^p\right) \\
&\quad = \mathbb{E}\left(\left|\int_0^t \int_{\mathbb{R}^n_+} G(\mathbf{x}-\mathbf{y},t-s)\left[\mathcal{N}u^n(\mathbf{y},s) - \mathcal{N}u^{n-1}(\mathbf{y},s)\right] d\mathbf{y}\,ds\right|^p\right) \\
&\quad \leq C(p) \int_0^t \int_{\mathbb{R}^n_+} G(\mathbf{x}-\mathbf{y},t-s)\mathbb{E}\left(\left|u^n(\mathbf{y},s) - u^{n-1}(\mathbf{y},s)\right|^p\right) d\mathbf{y}\,ds \\
&\quad \leq C(p) \int_0^t \sup_{\mathbf{x}\geq 0} \mathbb{E}\left(\left|u^n(\mathbf{y},s) - u^{n-1}(\mathbf{y},s)\right|^p\right) ds
\end{aligned}$$

and by (16) and Burkholder's inequality we have

$$\begin{aligned}
&\sup_{\mathbf{x}\geq 0} \mathbb{E}\left(|u^1(\mathbf{x},t) - u^0(\mathbf{x},t)|^p\right) \\
&\quad \leq C(p)\left(\sup_{\mathbf{x}\geq 0} \mathbb{E}\left(\left|u^1(\mathbf{x},t)\right|^p\right) + \sup_{\mathbf{x}\geq 0} \mathbb{E}\left(\left|u^0(\mathbf{x},t)\right|^p\right)\right) < \infty.
\end{aligned}$$

Then, by Gronwall's lemma we obtain

$$\sum_{n\geq 0} \sup_{\substack{\mathbf{x}\geq 0 \\ t\in[0,T]}} \mathbb{E}\left(\left|u^n(\mathbf{x},t) - u^{n-1}(\mathbf{x},t)\right|^p\right) < \infty.$$

Hence, $\{u^n(\mathbf{x},t)\}_{n\geq 0}$ is a Cauchy sequence in $L^p(\Omega)$. Let

$$u(\mathbf{x},t) = \lim_{n\to\infty} u^n(\mathbf{x},t).$$

Thus,

$$\sup_{\substack{\mathbf{x}\geq 0 \\ t\in[0,T]}} \mathbb{E}\left(\left|u(\mathbf{x},t)\right|^p\right) < \infty.$$

Taking $n \to \infty$ in $L^p(\Omega)$ at both sides of (17) shows that $u(\mathbf{x},t)$ satisfies the problem (2). Finally, we have to prove the uniqueness of the solution. Let u and v be the two solutions of problem (2), then

$$\begin{aligned}
&\mathbb{E}\left(\left|u(\mathbf{x},t) - v(\mathbf{x},t)\right|^p\right) \\
&\quad = \mathbb{E}\left(\left|\int_0^t \int_{\mathbb{R}^n_+} G(\mathbf{x}-\mathbf{y},t-s)\left[\mathcal{N}u(\mathbf{y},s) - \mathcal{N}v(\mathbf{y},s)\right] d\mathbf{y}\,ds\right|^p\right) \\
&\quad \leq C(p) \int_0^t \int_{\mathbb{R}^n_+} G(\mathbf{x}-\mathbf{y},t-s)\mathbb{E}\left(\left|u(\mathbf{y},s) - v(\mathbf{y},s)\right|^p\right) d\mathbf{y}\,ds \\
&\quad \leq C(p) \int_0^t \sup_{\mathbf{y}\geq 0} \mathbb{E}\left(\left|u(\mathbf{y},s) - v(\mathbf{y},s)\right|^p\right) ds.
\end{aligned}$$

Therefore, Gronwall's lemma yields

$$\mathbb{E}\left(\left|u(\mathbf{x},t) - v(\mathbf{x},t)\right|^p\right) = 0. \qquad \square$$

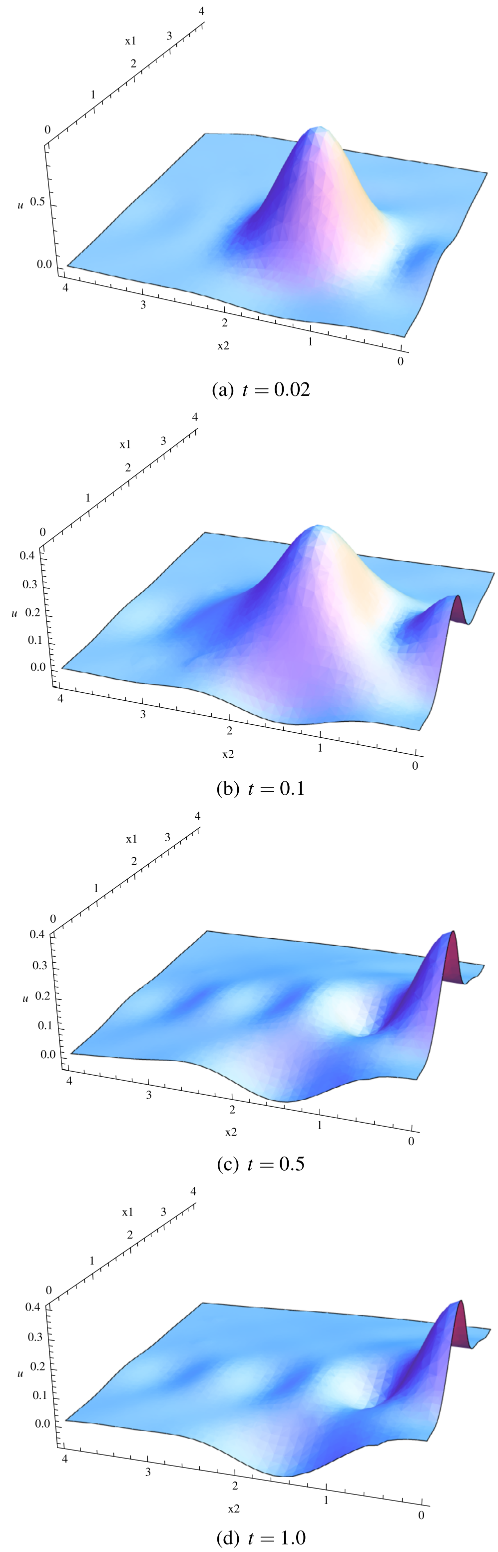

Figure 1 Anomalous diffusion for $\alpha = 2.5$

5 Example

In this section, we consider an example for the case $n = 2$, with the initial condition

$$u_0(x_1,x_2) = \begin{cases} 1, & 1 \le x_1, x_2 \le 2, \\ 0, & \text{in the other case,} \end{cases}$$

and the boundary conditions, for $l = 1, 2$,

$$h_l(\mathbf{x}_{[-l]}, t) = \begin{cases} (-1)^{l+1}, & 3/4 \le \mathbf{x}_{[-l]} \le 5/4, \\ 0, & \text{in the other case.} \end{cases}$$

In Fig. 1, we present the plot of the solution $u(\mathbf{x}, t)$ for $t = 0.02, 0.1, 0.5, 1$, and $\alpha = 2.5$.

Acknowledgements
Not applicable.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Publisher's Note

Received: 15 January 2021 Accepted: 5 May 2021 Published online: 17 May 2021

References

1. Abatangelo, N., Valdinoci, E.: Getting acquainted with the fractional Laplacian. In: Contemporary Research in Elliptic PDEs and Related Topics. Springer INdAM Ser., vol. 33, pp. 1–105. Springer, Cham (2019)
2. Balanzario, E.P., Kaikina, E.I.: Regularity analysis for stochastic complex Landau-Ginzburg equation with Dirichlet white-noise boundary conditions. SIAM J. Math. Anal. **52**(4), 3376–3396 (2020)
3. Bona, J.L., Luo, L.: Generalized Korteweg–de Vries equation in a quarter plane. Contemp. Math. **221**, 59–125 (1999)
4. Dipierro, S., Pellacci, B., Valdinoci, E., Verzini, G.: Time fractional equations with reaction terms: fundamental solutions and asymptotics. Discrete Contin. Dyn. Syst. **41**(1), 257–275 (2021)
5. Fokas, A.: A Unified Approach to Boundary Value Problems. SIAM, Philadelphia (2008)
6. Mantzavinos, D., Fokas, A.S.: The unified transform for the heat equation: II. Non-separable boundary conditions in two dimensions. Eur. J. Appl. Math. **26**, 887–916 (2015)
7. Metzler, R., Klafter, J.: The random walk's guide to anomalous diffusion: a fractional dynamics approach. Phys. Rep. **339**, 1–77 (2001)
8. Pozrikidis, C.: The Fractional Laplacian. CRC Press, Boca Raton (2016)
9. Ros-Oton, X., Valdinoci, E.: The Dirichlet problem for nonlocal operators with singular kernels: convex and nonconvex domains. Adv. Math. **288**, 732–790 (2016)
10. Sanchez-Ortiz, J., Ariza-Hernandez, F.J., Arciga-Alejandre, M.P., Garcia-Murcia, E.: Stochastic diffusion equation with fractional Laplacian on the first quadrant. Fract. Calc. Appl. Anal. **22**(3), 795–806 (2019)
11. Shi, K., Wang, Y.: On a stochastic fractional partial differential equation with a fractional noise. Stochastics **84**(1), 21–36 (2012)
12. Walsh, J.B.: An introduction to stochastic partial differential equations. Lect. Notes Math. **1180**, 265–439 (1986)