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Generalizations of some contractions in b-metric-like spaces and applications to boundary value problems

Kastriot Zoto[1], Hassen Aydi[2,3,4] and Habes Alsamir[5*]

*Correspondence:
habes@dau.edu.sa
[5]College of Business Administration-Finance Department, Dar Al Uloom University Saudi Arabia, Riyadh, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

This paper provides two wide classes of contractions, which are obtained by using notions of α_{s^p}-admissibility and the rich set of C-class functions in the setting of a complete b-metric-like space under more general contractive conditions. An application is provided and many known results in the literature can be derived.

MSC: 47H10; 54H25

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1 Introduction and preliminaries

Fixed point theory is still a central topic with a broad focus on applications of fixed point models not only in mathematical analysis, but also in other branches of natural sciences. The Banach contraction theorem continues to be generalized in other metric settings. For more related results, see [1–23].

Following the generalizations made by Matthews [24], Hitzler and Seda [25], and Amini-Harandi [26], Alghamdi *et al.* [27] introduced the concept of b-metric-like spaces. Many authors have obtained interesting results in these areas associated with many more applications in the field of nonlinear analysis and main areas of interdisciplinary research.

In our work, we use the notions of α-admissible functions, (α,ψ,φ)-contractive mappings, F-contractions, and Kannan type contractions. In this paper, we introduce $\alpha_{s^p}-F$ contractive mappings by means of α_{s^p}-admissible functions and auxiliary functions, named as C-class functions. We also provide two wide classes of contractions selected among b-metric and b-metric-like settings, giving knew extensions of $\alpha_{s^p}-F$ contractions and Kannan type contractions. These new generalized classes not only generalize the known ones, but also include and unify a huge number of existing ones selected in the corresponding literature, and the corresponding results are supported by an application on boundary value problems.

Let T be a nonempty set and $s\geq 1$ be a given real number. Let $\sigma_b : T\times T\to[0,\infty)$ be a mapping satisfying the following conditions for each $h,k,z\in T$:

I. $\sigma_b(h,k) = 0$ *if and only if* $h = k$;
II. $\sigma_b(h,k) = 0$ *implies* $h = k$;
III. $\sigma_b(h,k) = \sigma_b(k,h)$;
IV. $\sigma_b(h,k) \leq \sigma_b(h,z) + \sigma_b(z,k)$;
V. $\sigma_b(h,k) \leq s[\sigma_b(h,z) + \sigma_b(z,k)]$.

Definition 1.1 ([28]) A pair (T,σ_b) satisfying axioms I, III, and V is called a b-metric space with parameter s.

Definition 1.2 ([26]) A pair (T,σ_b) satisfying axioms II, III, and IV is called a metric-like space.

Definition 1.3 ([27]) A pair (T,σ_b) satisfying axioms II, III, and V is called a b-metric-like space with parameter s.

It is true that if $h,k \in T$ and $\sigma_b(h,k) = 0$, then $h = k$; however, the converse need not be true, and $\sigma_b(h,h)$ may be positive for $h \in T$.

Example 1.4 Let $T = \mathbb{R}$ and $\sigma_b : T^2 \to [0,\infty)$ be a given function as $\sigma_b(h,k) = (|h| + |k|)^2$ for all $h,k \in T$. Then (T,σ_b) is a b-metric-like space with parameter $s = 2$.

Definition 1.5 ([27]) Let (T,σ_b) be a b-metric-like space.
(a) A sequence $\{h_n\}$ in T is called convergent to a point $h \in T$ if $\lim_{n\to\infty} \sigma_b(h_n,h) = \sigma_b(h,h)$;
(b) A sequence $\{h_n\}$ in T is called Cauchy if $\lim_{n,m\to\infty} \sigma_b(h_n,h_m)$ exists and is finite;
(c) The b-metric-like space (T,σ_b) is called complete if, for every Cauchy sequence $\{h_n\}$ in T, there exists $h \in T$ such that $\lim_{n,m\to\infty} \sigma_b(h_n,h_m) = \lim_{n\to\infty} \sigma_b(h_n,h) = \sigma_b(h,h)$.

In 2012, the introduction of α-admissible functions by Samet et al. in [29] leads to an extensive development of many notions and properties related to fixed point theory and its applications.

Definition 1.6 Let T be a nonempty set. Let $f : T \to T$ and $\alpha : T \times T \to \mathbb{R}^+$ be given functions. We say that f is an α-admissible mapping if $\alpha(h,k) \geq 1$ implies that $\alpha(fh,fk) \geq 1$ for all $h,k \in T$.

Further, Aydi [30] extended this definition to a pair of mappings.

Definition 1.7 For a nonempty set T, let $f,g : T \to T$ and $\alpha : T \times T \to \mathbb{R}^+$ be mappings. We say that (f,g) is an α-admissible pair if, for all $h,k \in T$ we have

$$\alpha(h,k) \geq 1 \Rightarrow \alpha(fh,gk) \geq 1 \quad \text{and} \quad \alpha(gk,fh) \geq 1.$$

We here summarize the most important lemmas and results very useful in the main section of the paper.

Lemma 1.8 ([21]) *Let (T,σ_b) be a b-metric-like space with parameter $s \geq 1$. If a given mapping $f : T \to T$ is continuous at $h^* \in T$, then we have*

$$\sigma_b(fh_n, fh^*) \to \sigma_b(fh^*, fh^*) \quad \text{whenever } \sigma_b(h_n, h^*) \to \sigma_b(h^*, h^*) \text{ for each } \{h_n\} \textit{in } T.$$

The following is a short revised version of the lemma in [22].

Lemma 1.9 *Let (T,σ_b) be a b-metric-like space with parameter $s \geq 1$, and suppose that $\{h_n\}$ is σ_b-convergent to h with $\sigma_b(h,h) = 0$. Then, for each $j \in T$, we have*

$$s^{-1}\sigma_b(h,j) \leq \liminf_{n\to\infty} \sigma_b(h_n,j) \leq \limsup_{n\to\infty} \sigma_b(h_n,j) \leq s\sigma_b(h,j).$$

Lemma 1.10 ([21]) *In a b-metric-like space (T,σ_b) with parameter $s \geq 1$, for $h,k \in T$ and $\{h_n\} \subset T$, we have*:

(a) $\sigma_b(h,k) = 0 \Rightarrow \sigma_b(h,h) = \sigma_b(k,k) = 0$;

(b) *If* $\lim_{n\to\infty} \sigma_b(h_n,h_{n+1}) = 0$, *then* $\lim_{n\to\infty} \sigma_b(h_n,h_n) = \lim_{n\to\infty} \sigma_b(h_{n+1},h_{n+1}) = 0$;

(c) $h \neq k \Rightarrow \sigma_b(h,k) > 0$.

Lemma 1.11 ([22]) *Let (T,σ_b) be a complete b-metric-like space and $\{h_n\}$ be a sequence such that*

$$\lim_{n\to\infty} \sigma_b(h_n,h_{n+1}) = 0.$$

If, for such a sequence $\{h_n\}$, $\lim_{n,m\to\infty} \sigma_b(h_n,h_m) \neq 0$, then there are $\varepsilon > 0$ and subsequences of positive integers $\{m(i)\}$; $\{n(i)\}$ with $n_i > m_i > i$ such that

$$\varepsilon \leq \limsup_{i\to\infty} \sigma_b(h_{2n_i},h_{2m_i}) \leq \varepsilon s, \qquad \varepsilon/s \leq \limsup_{i\to\infty} \sigma_b(h_{2m_i},h_{2n_i-1}) \leq \varepsilon s,$$

$$\varepsilon/s^2 \leq \limsup_{i\to\infty} \sigma_b(h_{2n_i-1},h_{2m_i+1}) \leq \varepsilon s^2,$$

$$\varepsilon/s \leq \limsup_{i\to\infty} \sigma_b(h_{2m_i+1},h_{2n_i}) \leq \varepsilon s^2.$$

Lemma 1.12 ([22]) *Let $\{h_n\}$ be a sequence in a b-metric-like space (T,σ_b) with parameter $s \geq 1$ such that $\sigma_b(h_n,h_{n+1}) \leq \lambda\sigma_b(h_{n-1},h_n)$ for all $n > 0$, for some λ, where $0 \leq \lambda < 1/s$. Then*:

1. $\lim_{n\to\infty} \sigma_b(h_n,h_{n+1}) = 0$,
2. *$\{h_n\}$ is a Cauchy sequence in (T,σ_b) and* $\lim_{n,m\to\infty} \sigma_b(h_n,h_m) = 0$.

Definition 1.13 ([22]) Let (T,σ_b) be a b-metric-like space, $f,g : T \to T$ and $\alpha : T \times T \to \mathbb{R}^+$ be given mappings, and let $p \geq 1$ be an arbitrary constant. We say that (f,g) is an α_{s^p}-admissible pair if $\alpha(h,k) \geq s^p$ implies $\min\{\alpha(fh,gk),\alpha(gk,fh)\} \geq s^p$ for all $h,k \in T$.

Examples 2 and 3 in [22] illustrate Definition 1.13.

Definition 1.14 ([22]) Let (T,σ_b) be a b-metric-like space, $f : T \to T$ and $\alpha : T \times T \to \mathbb{R}^+$ be given mappings, and let $p \geq 1$ be an arbitrary constant. We say that f is an α_{s^p}-admissible mapping if $\alpha(h,k) \geq s^p$ implies $\min\{\alpha(fh,fk),\alpha(fk,fh)\} \geq s^p$ for all $h,k \in T$.

Also, in the sequel, we recall additional properties given in [22].

(H_{s^p}): If $\{h_n\}$ is a sequence in T such that $h_n \to h \in T$ as $n \to \infty$ and $\alpha(h_n, h_{n+1}) \geq s^p$ and $\alpha(h_{n+1}, h_n) \geq s^p$, then there exists a subsequence $\{h_{n_i}\}$ of $\{h_n\}$ with $\alpha(h_{n_i}, h) \geq s^p$ and $\alpha(h, h_{n_i}) \geq s^p$ for all $i \in \mathbb{N}$.

(U_{s^p}): For all $h, k \in CF(f, g)$, we have $\alpha(h, k) \geq s^p$, where $CF(f, g)$ denotes the set of common fixed points of f and g (also $\mathrm{Fix}(f)$ is the set of fixed points of f).

Definition 1.15 ([31]) A mapping $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is called a C-class function if

1) $F(m, n) \leq m$ for all $m, n \geq 0$;
2) $F(m, n) = m$ implies that either $m = 0$ or $n = 0$ for all $m, n \geq 0$;
3) $F(m, n)$ is continuous on its variables $m, n \geq 0$.

2 Main results

In this section we present two main theorems. The first is a general result in a larger ambient of spaces that extends and unifies a number of well-known corresponding results related to fixed point theory. The second is an extension of the outstanding classical result of Kannan contraction to the setting of b-metric-like spaces.

Let $f : T \to T$ be a mapping. We denote

$$N(h,k) = \max \left\{ \begin{array}{l} \sigma_b(h,k), \sigma_b(h,fh), \sigma_b(k,fk), \frac{\sigma_b(h,fk)+\sigma_b(k,fh)}{4s}, \frac{\sigma_b(h,fh)\sigma_b(h,fk)}{1+s[\sigma_b(h,fh)+\sigma_b(k,fk)]}, \\ \frac{\sigma_b(k,fk)[1+\sigma_b(h,fh)]}{1+\sigma_b(h,k)}, \frac{\sigma_b(h,fh)[1+\sigma_b(k,fk)]}{1+\sigma_b(fh,fk)} \end{array} \right\} \tag{1}$$

and the following sets of functions:

$$\begin{aligned} \mathrm{I} &:= \bigl\{\psi : [0,+\infty) \to [0,+\infty) \text{ is strictly increasing, continuous such that } \psi(m) = 0 \\ &\qquad \text{iff } m = 0\bigr\}; \\ \Theta &:= \bigl\{\theta : [0,+\infty) \to [0,+\infty) \text{ is continuous with } \theta(m) < \psi(m) \text{ for all } m > 0\bigr\}; \\ \Gamma &:= \left\{ \begin{array}{l} (\beta,\gamma,\delta)/\beta,\gamma,\delta : R^+ \to [0,1) \text{ satisfying} \\ \limsup_{n\to m} \beta(n) + \limsup_{n\to m} \gamma(n) + \limsup_{n\to m} \delta(n) < 1, \text{for all } m > 0 \end{array} \right\}. \end{aligned}$$

Definition 2.1 Let (T, σ_b) be a b-metric-like space with parameter $s \geq 1$. A mapping $f : T \to T$ is said to be an $\alpha_{s^p} - (C, N, \Theta)$ contraction if f is an α_{s^p}-admissible mapping ($p > 1$) and it satisfies

$$\alpha(h,k)\sigma_b(fh,fk) \leq F\bigl(N(h,k), \theta\bigl(N(h,k)\bigr)\bigr) \tag{2}$$

for all $h, k \in T$, where $F \in C$, $\theta \in \Theta$ and $N(h,k)$ is defined by (1).

We now state the following general result.

Theorem 2.2 *Let (T, σ_b) be a complete b-metric-like space with parameter $s > 1$, and $f : T \to T$ be an $\alpha_{s^p} - (C, N, \Theta)$ contraction. Suppose that the following conditions hold*:

(i) *there exists $h_0 \in T$ such that* $\min\{\alpha(h_0, fh_0), \alpha(fh_0, h_0)\} \geq s^p$;
(ii) *the properties H_{s^p} and U_{s^p} are satisfied.*

Then f has a unique fixed point $h \in T$.

Proof From assumption (i) there exists $h_0 \in T$ such that $\alpha(h_0, fh_0) \geq s^p$. We construct a sequence $\{h_n\}$ in T by $h_n = f^n h_0 = f(h_{n-1})$ for all $n \in \mathbb{N}$. If we suppose that $\sigma_b(h_n, h_{n+1}) = 0$ for some n, then $h_{n+1} = h_n$, and so f has a fixed point. Consequently, throughout the proof, we assume that

$$\sigma_b(h_n, h_{n+1}) > 0 \quad \text{for all } n \in \mathbb{N}. \tag{3}$$

By the α_{s^p}-type admissibility of f, we observe that

$$\begin{aligned} &\alpha(h_0, h_1) = \alpha(h_0, fh_0) \geq s^p, \\ &\alpha(fh_0, fh_1) = \alpha(h_1, h_2) \geq s^p \quad \text{and} \quad \alpha(fh_1, fh_2) = \alpha(h_2, h_3) \geq s^p. \end{aligned}$$

Then, inductively,

$$\alpha(h_n, h_{n+1}) \geq s^p \quad \text{for all } n \in \mathbb{N}. \tag{4}$$

By (1) and condition (2), we have

$$\begin{aligned} s^p \sigma_b(h_n, h_{n+1}) &= s^p \sigma_b(fh_{n-1}, fh_n) \\ &\leq \alpha(h_{n-1}, h_n)\sigma_b(fh_{n-1}, fh_n) \\ &\leq F\big(N(h_{n-1}, h_n), \theta\big(N(h_{n-1}, h_n)\big)\big) \\ &\leq N(h_{n-1}, h_n), \end{aligned} \tag{5}$$

where

$$\begin{aligned} N(h_{n-1}, h_n) &= \max \left\{ \begin{array}{l} \sigma_b(h_{n-1}, h_n), \sigma_b(h_{n-1}, fh_{n-1}), \sigma_b(h_n, fh_n), \frac{\sigma_b(h_{n-1}, fh_n) + \sigma_b(h_n, fh_{n-1})}{4s}, \\ \frac{\sigma_b(h_{n-1}, h_n)\sigma_b(h_{n-1}, fh_n)}{1 + s[\sigma_b(h_{n-1}, fh_{n-1}) + \sigma_b(h_n, fh_n)]} \frac{\sigma_b(h_n, fh_n)[1 + \sigma_b(h_{n-1}, fh_{n-1})]}{1 + \sigma_b(h_{n-1}, h_n)}, \\ \frac{\sigma_b(h_{n-1}, fh_{n-1})[1 + \sigma_b(h_n, fh_n)]}{1 + \sigma_b(fh_{n-1}, fh_n)} \end{array} \right\} \\ &= \max \left\{ \begin{array}{l} \sigma_b(h_{n-1}, h_n), \sigma_b(h_{n-1}, h_n), \sigma_b(h_n, h_{n+1}), \frac{\sigma_b(h_{n-1}, h_{n+1}) + \sigma_b(h_n, h_n)}{4s}, \\ \frac{\sigma_b(h_{n-1}, h_n)\sigma_b(h_{n-1}, h_{n+1})}{1 + s[\sigma_b(h_{n-1}, h_n) + \sigma_b(h_n, h_{n+1})]} \frac{\sigma_b(h_n, h_{n+1})[1 + \sigma_b(h_{n-1}, h_n)]}{1 + \sigma_b(h_{n-1}, h_n)}, \\ \frac{\sigma_b(h_{n-1}, h_n)[1 + \sigma_b(h_n, h_{n+1})]}{1 + \sigma_b(h_n, h_{n+1})} \end{array} \right\} \\ &\leq \max \left\{ \begin{array}{l} \sigma_b(h_{n-1}, h_n), \sigma_b(h_{n-1}, h_n), \sigma_b(h_n, h_{n+1}), \\ \frac{s[\sigma_b(h_{n-1}, h_n) + \sigma_b(h_n, h_{n+1})] + 2s\sigma_b(h_{n-1}, h_n)}{4s}, \\ \frac{\sigma_b(h_{n-1}, h_n)s[\sigma_b(h_{n-1}, h_n) + \sigma_b(h_n, h_{n+1})]}{1 + s[\sigma_b(h_{n-1}, h_n) + \sigma_b(h_n, h_{n+1})]}, \sigma_b(h_n, h_{n+1}), \sigma_b(h_{n-1}, h_n) \end{array} \right\} \\ &= \max\big\{\sigma_b(h_{n-1}, h_n), \sigma_b(h_n, h_{n+1})\big\}. \end{aligned} \tag{6}$$

If we have

$$\sigma_b(h_{n-1}, h_n) < \sigma_b(h_n, h_{n+1}) \quad \text{for some } n \in \mathbb{N},$$

then, from inequality (6), we get

$$N(h_{n-1}, h_n) \leq \sigma_b(h_n, h_{n+1}). \tag{7}$$

Using (5), we obtain

$$s^p\sigma_b(h_n,h_{n+1}) \le \sigma_b(h_n,h_{n+1}).$$

Hence $\sigma_b(h_n,h_{n+1}) = 0$, that is a contradiction.

Thus, for all $n \in \mathbb{N}$, $\sigma_b(h_n,h_{n+1}) \le \sigma_b(h_{n-1},h_n)$ and by (5), we can establish that

$$s^p\sigma_b(h_n,h_{n+1}) \le \sigma_b(h_{n-1},h_n).$$

As a result, the above inequality can be written as

$$\sigma_b(h_n,h_{n+1}) \le \lambda\sigma_b(h_{n-1},h_n), \tag{8}$$

where $\lambda = 1/s^p \in [0,1/s)$. By Lemma 1.12 and using (8), we claim

$$\lim_{n\to\infty}\sigma_b(h_n,h_{n+1}) = 0, \tag{9}$$

and the sequence $\{h_n\}$ is Cauchy. (T,σ_b) is complete, so there is some $h \in T$ such that $\{h_n\}$ converges to h. That is,

$$\lim_{n\to\infty}\sigma_b(h_n,h) = \sigma_b(h,h) = \lim_{n,m\to\infty}\sigma_b(h_n,h_m) = 0. \tag{10}$$

The self-map f is not continuous, then from (4) and property H_{s^p}, there exists a subsequence $\{h_{n_i}\}$ of $\{h_n\}$ such that $\alpha(h_{n_i},h) \ge s^p$ for all $i \in \mathbb{N}$. Applying contractive condition (2) to h_{n_i} and h, we obtain

$$\begin{aligned} s^p\sigma_b(h_{n_i+1},fh) &= s^p\sigma_b(fh_{n_i},fh) \\ &\le \alpha(h_{n_i},h)\sigma_b(fh_{n_i},fh) \\ &\le F\bigl(N(h_{n_i},h),\theta\bigl(N(h_{n_i},h)\bigr)\bigr) \\ &\le N(h_{n_i},h), \end{aligned} \tag{11}$$

where

$$\begin{aligned} N(h_{n_i},h) &= \max\left\{\begin{matrix} \sigma_b(h_{n_i},h),\sigma_b(h_{n_i},fh_{n_i}),\sigma_b(h,fh), \\ \frac{\sigma_b(h_{n_i},fh)+\sigma_b(h,fh_{n_i})}{4s}, \frac{\sigma_b(h_{n_i},h)\sigma_b(h_{n_i},fh)}{1+s[\sigma_b(h_{n_i},fh_{n_i})+\sigma_b(h,fh)]} \\ \frac{\sigma_b(h,fh)[1+\sigma_b(h_{n_i},fh_{n_i})]}{1+\sigma_b(h_{n_i},h)}, \frac{\sigma_b(h_{n_i},fh_{n_i})[1+\sigma_b(h,fh)]}{1+\sigma_b(fh_{n_i},fh)} \end{matrix}\right\} \\ &= \max\left\{\begin{matrix} \sigma_b(h_{n_i},h),\sigma_b(h_{n_i},h_{n_i+1}),\sigma_b(h,fh), \\ \frac{\sigma_b(h_{n_i},fh)+\sigma_b(h,h_{n_i+1})}{4s}, \frac{\sigma_b(h_{n_i},h)\sigma_b(h_{n_i},fh)}{1+s[\sigma_b(h_{n_i},h_{n_i+1})+\sigma_b(h,fh)]} \\ \frac{\sigma_b(h,fh)[1+\sigma_b(h_{n_i},h_{n_i+1})]}{1+\sigma_b(h_{n_i},h)}, \frac{\sigma_b(h_{n_i},h_{n_i+1})[1+\sigma_b(h,fh)]}{1+\sigma_b(h_{n_i+1},fh)} \end{matrix}\right\}. \end{aligned} \tag{12}$$

By the upper limit in (12) and due to Lemma 1.9, Lemma 1.10, and Eq. (10), we derive

$$\limsup_{i\to\infty} N(h_{n_i},h) \le \max\left\{0,0,\sigma_b(h,fh),\frac{s\sigma_b(h,fh)}{4s},0,\sigma_b(h,fh),0\right\} = \sigma_b(h,fh). \tag{13}$$

Letting $i \to \infty$ in (11), and in view of (13) and Lemma 1.9, it follows that

$$\begin{aligned} s^{p-1}\sigma_b(h,fh) &= s^p \frac{1}{s}\sigma_b(h,fh) \le s^p \limsup_{i\to\infty} \sigma_b(h_{n_i},fh) \\ &\le \limsup_{i\to\infty} N(h_{n_i},h) \le \sigma_b(h,fh). \end{aligned} \tag{14}$$

From (14) we get $\sigma_b(h,fh) = 0$, which implies that $fh = h$. Hence h is a fixed point of f.

If $h,z \in \mathrm{Fix}(f)$, by the hypothesis U_{s^p}, $\alpha(h,z) \ge s^p$, and applying (2), we have

$$\begin{aligned} s^p\sigma_b(h,h) &= s^p\sigma_b(fh,fh) \le \alpha(h,h)\sigma_b(fh,fh) \\ &\le F\bigl(N(h,h),\theta\bigl(N(h,h)\bigr)\bigr) \\ &\le N(h,h) = \sigma_b(h,h), \end{aligned} \tag{15}$$

where

$$\begin{aligned} N(h,h) &= \max\left\{ \begin{array}{l} \sigma_b(h,h), \sigma_b(h,fh), \sigma_b(h,fh), \frac{\sigma_b(h,fh)+\sigma_b(h,fh)}{4s}, \frac{\sigma_b(h,fh)\sigma_b(h,fh)}{1+s[\sigma_b(h,fh)+\sigma_b(h,fh)]}, \\ \frac{\sigma_b(h,fh)[1+\sigma_b(h,fh)]}{1+\sigma_b(h,h)}, \frac{\sigma_b(h,fh)[1+\sigma_b(h,fh)]}{1+\sigma_b(fh,fh)} \end{array} \right\} \\ &= \max\left\{ \begin{array}{l} \sigma_b(h,h), \sigma_b(h,h), \sigma_b(h,h), \frac{\sigma_b(h,h)+\sigma_b(h,h)}{4s}, \frac{\sigma_b(h,h)\sigma_b(h,h)}{1+s[\sigma_b(h,h)+\sigma_b(h,h)]}, \\ \frac{\sigma_b(h,h)[1+\sigma_b(h,h)]}{1+\sigma_b(h,h)}, \frac{\sigma_b(h,h)[1+\sigma_b(h,h)]}{1+\sigma_b(h,h)} \end{array} \right\} \\ &= \sigma_b(h,h). \end{aligned} \tag{16}$$

$$\text{By (15) it follows } s^p\sigma_b(h,h) \le \sigma_b(h,h). \tag{17}$$

Since $s > 1$, the inequality above implies $\sigma_b(h,h) = 0$ (similarly, $\sigma_b(z,z) = 0$).

Again by condition (2), we have

$$\begin{aligned} s^p\sigma_b(h,z) &= s^p\sigma_b(fh,fz) \le \alpha(h,z)\sigma_b(fh,fz) \\ &\le F\bigl(N(h,z),\theta\bigl(N(h,z)\bigr)\bigr) \\ &\le N(h,z) \\ &= \sigma_b(h,z), \end{aligned} \tag{18}$$

where

$$\begin{aligned} N(h,z) &= \max\left\{ \begin{array}{l} \sigma_b(h,z), \sigma_b(h,fh), \sigma_b(z,fz), \frac{\sigma_b(h,fz)+\sigma_b(z,fh)}{4s}, \frac{\sigma_b(h,fh)\sigma_b(h,fz)}{1+s[\sigma_b(h,fh)+\sigma_b(z,fz)]}, \\ \frac{\sigma_b(z,fz)[1+\sigma_b(h,fh)]}{1+\sigma_b(h,z)}, \frac{\sigma_b(h,fh)[1+\sigma_b(z,fz)]}{1+\sigma_b(fh,fz)} \end{array} \right\} \\ &= \max\left\{ \begin{array}{l} \sigma_b(h,z), \sigma_b(h,h), \sigma_b(z,z), \frac{\sigma_b(h,z)+\sigma_b(z,h)}{4s}, \frac{\sigma_b(h,h)\sigma_b(h,z)}{1+s[\sigma_b(h,h)+\sigma_b(z,z)]}, \\ \frac{\sigma_b(z,z)[1+\sigma_b(h,h)]}{1+\sigma_b(h,z)}, \frac{\sigma_b(h,h)[1+\sigma_b(z,z)]}{1+\sigma_b(h,z)} \end{array} \right\} \\ &= \max\left\{\sigma_b(h,z), 0, 0, \frac{\sigma_b(h,z)}{2s}, 0, 0, 0\right\} \\ &= \sigma_b(h,z). \end{aligned}$$

Inequality (18) implies that $\sigma_b(h,z) = 0$. Therefore, $h = z$ and the fixed point is unique. □

Remark 2.3

(i) The proof of Theorem 2.2 is simply constructive somewhat shorter, and avoid the use of Lemma 1.11.

(ii) The above result reduces to other settings of spaces for the choice of parameters s and p.

(iii) Many applications of Theorem 2.2 are attributed to the variety of class C that makes it to contain many known theorems as special cases.

In the sequel, we provide an illustrative example of Theorem 2.2.

Example 2.4 In $T = [0,+\infty)$, we take $\sigma_b(h,k) = (h+k)^2$ for all $h,k \in T$. Clearly, (T,σ_b) is a b-metric-like space with coefficient $s = 2$. Let us define the mappings $f : T \to T$ and $\alpha : T \times T \to [0,+\infty[$ by

$$fh = \begin{cases} \frac{1}{5}h & \text{if } h \in [0,1), \\ \frac{1}{10}h & \text{if } h \in [1,2), \\ 2h & \text{if } h \geq 2 \end{cases} \quad \text{and} \quad \alpha(h,k) = \begin{cases} h+k+4 & \text{if } h,k \in [0,2], \\ 0 & \text{otherwise.} \end{cases}$$

Let $h,k \in T$, if $\alpha(h,k) \geq 4 = s^2$, then $h,k \in [0,2)$, and also we have $fh,fk \in [0,1/5)$ and $\alpha(fh,fk) \geq s^2$. Thus we have shown that f is an α_{s^p}-admissible mapping. Choosing $F \in C$ as $\mathrm{F}(m,n) = m - n$, we discuss the following cases:

Let $h,k \in [0,1)$, then we get

$$\begin{aligned} \alpha(h,k)\sigma_b(fh,fk) &= s^2\sigma_b(fh,fk) = 4\sigma_b\left(\frac{1}{5}h, \frac{1}{5}k\right) = 4\left(\frac{1}{5}h + \frac{1}{5}k\right)^2 \\ &= \frac{4}{25}(h+k)^2 < \frac{1}{6}\sigma_b(h,k) \leq \frac{1}{6}N(h,k) = N(h,k) - \frac{5}{6}N(h,k) \\ &= N(h,k) - \theta\big(N(h,k)\big) \\ &= F\big(N(h,k), \theta\big(N(h,k)\big)\big). \end{aligned}$$

Let $h,k \in [1,2)$, then we get

$$\begin{aligned} \alpha(h,k)\sigma_b(fh,fk) &= s^2\sigma_b(fh,fk) = 4\sigma_b\left(\frac{1}{10}h, \frac{1}{10}k\right) = 4\left(\frac{1}{10}h + \frac{1}{10}k\right)^2 \\ &= \frac{4}{100}(h+k)^2 < \frac{1}{6}\sigma_b(h,k) \leq \frac{1}{6}N(h,k) = N(h,k) - \frac{5}{6}N(h,k) \\ &= N(h,k) - \theta\big(N(h,k)\big) \\ &= F\big(N(h,k), \theta\big(N(h,k)\big)\big). \end{aligned}$$

Let $h \in [0,1)$, $k \in [1,2)$, then we get

$$\begin{aligned} \alpha(h,k)\sigma_b(fh,fk) &= s^2\sigma_b(fh,fk) = 4\sigma_b\left(\frac{1}{5}h, \frac{1}{10}k\right) = 4\left(\frac{1}{5}h + \frac{1}{10}k\right)^2 \leq 4\left(\frac{1}{5}h + \frac{1}{5}k\right)^2 \\ &= \frac{4}{25}(h+k)^2 < \frac{1}{6}\sigma_b(h,k) \leq \frac{1}{6}N(h,k) = N(h,k) - \frac{5}{6}N(h,k) \end{aligned}$$

$$= N(h,k) - \theta\big(N(h,k)\big)$$
$$= F\big(N(h,k), \theta\big(N(h,k)\big)\big).$$

The other case $k \in [0,1)$, $h \in [1,2)$ is the same as the previous case.

Obviously, the other assumptions of Theorem 2.2 can be verified and f has $h = 0$ as a unique fixed point.

On the other hand, if we refer to the metric space with the standard metric $d(h,k) = |h - k|$ for points $h = 0$, $k = 2$ in case $N(0,2) = d(0,2)$, we see that

$$4 = d(f0, f2) \leq F\big(d(0,2), \theta\big(d(0,2)\big)\big) = F\big(2, \theta(2)\big),$$

that is, there exists no function $F \in C$ that satisfies the inequality (and also the Banach contraction principle).

Theorem 2.5 *Let (T,σ_b) be a complete b-metric-like space with parameter $s > 1$ and $f : T \to T$ be a mapping satisfying*

$$s^p \sigma_b(fh, fk) \leq F\big(N(h,k), \theta\big(N(h,k)\big)\big)$$

for all $h,k \in T$, where $\theta \in \Theta$, $F \in C$, $p > 1$ and $N(h,k)$ is defined by (1). Then f has a fixed point in T.

Proof It is obtained from Theorem 2.2 by setting $\alpha(h,k) = s^p$ $(p > 1)$. □

Some applications of Theorem 2.2 are the following results by choosing the function $F \in C$, based on Example 2.13 (see [31]).

Corollary 2.6 *Let $f : T \to T$ be an α_{s^p}-admissible mapping on a complete b-metric-like space (T,σ_b) with parameter $s > 1$. Suppose that the following assertions hold:*

(i) *There exists a function $\beta : [0,\infty) \to [0,1)$ satisfying the condition: $\beta(h_n) \to 1$ as $n \to \infty$ implies that $h_n \to 0$ as $n \to \infty$ such that*

$$\alpha(h,k)\sigma_b(fh, fk) \leq \beta\big(N(h,k)\big)\big(N(h,k)\big)$$

for all $h,k \in T$;where $N(h,k)$ is defined by (1);

(ii) *There exists $h_0 \in T$ with $\min\{\alpha(h_0, fh_0), \alpha(fh_0, h_0)\} \geq s^p$;*

(iii) *Properties H_{s^p}; U_{s^p} are satisfied.*

Then f has a unique fixed point $h \in T$.

Proof It follows from Theorem 2.2 by setting the function $F \in C$ as $F(m,n) = \beta(m)m$. □

Corollary 2.7 *Let $f : T \to T$ be an α_{s^p}-admissible mapping on a complete b-metric-like space (T,σ_b) with parameter $s > 1$. Suppose that the following conditions are satisfied:*

(i) *There exists a continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(t) < t$ for all $t > 0$, satisfying*

$$\alpha(h,k)\sigma_b(fh, fk) \leq \varphi\big(N(h,k)\big)$$

for all $h,k \in T$, where $N(h,k)$ is defined by (1);

(ii) *There exists $h_0 \in T$ with* $\min\{\alpha(h_0,fh_0),\alpha(fh_0,h_0)\} \geq s^p$;

(iii) *Properties H_{s^p}; U_{s^p} are satisfied.*

Then f has a unique fixed point $h \in T$.

Proof It is derived from Theorem 2.2 by setting $F(m,n) = \varphi(m)$ where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous such that $\varphi(m) < m$ for all $m > 0$. □

The next theorem is a new extension of Kannan type contraction and is concerned with common fixed points for a pair of self-mappings. It uses the following definitions.

Definition 2.8 Let (T,σ_b) be a complete b-metric-like space with parameter $s \geq 1$, and $f,g : T \to T$, $\alpha : T \times T \to \mathbb{R}^+$ be given mappings. The pair (f,g) is called a generalized $\alpha_{s^p} - (I,\Theta,\Gamma)$-Kannan contraction pair if there exist $\psi \in I$, $\theta \in \Theta$, $\beta,\gamma,\delta \in \Gamma$ satisfying

$$\begin{aligned}\psi\big(\alpha(h,k)\sigma_b(fh,gk)\big) \leq{}& \beta\big[\sigma_b(h,k)\big]\theta\big[\sigma_b(h,k)\big]\\&+\gamma\big[\sigma_b(h,k)\big]\theta\big[\sigma_b(h,fh)\big]\\&+\delta\big[\sigma_b(h,k)\big]\theta\big[\sigma_b(k,gk)\big]\end{aligned}\tag{19}$$

for all $h,k \in T$ with $\alpha(h,k) \geq s^p (p > 1)$ and $\theta(m) < \psi(m)$ for all $m > 0$.

Remark 2.9 If we put $g = f$, then Definition 2.8 can be stated as generalized $\alpha_{s^p} - (I,\Theta,\Gamma)$ Kannan contraction for one mapping.

Theorem 2.10 *Let (f,g) be a pair of self-mappings on a complete b-metric-like space (T,σ_b) with coefficient $s \geq 1$. If (f,g) is a generalized $\alpha_{s^p} - (I,\Theta,\Gamma)$ Kannan contraction pair, and the following conditions hold*:

(i) *There exists $h_0 \in T$ with* $\min\{\alpha(h_0,fh_0),\alpha(fh_0,h_0)\} \geq s^p$;

(ii) *(f,g) is an α_{s^p}-admissible pair*;

(iii) *Properties H_{s^p}; U_{s^p} are satisfied.*

Then f and g have a unique common fixed point $h \in T$.

Proof Since condition (i) holds, there exists $h_0 \in T$ with $\alpha(h_0,fh_0) \geq s^p$ and $\alpha(fh_0,h_0) \geq s^p$. Take $h_1 = fh_0$ and $h_2 = gh_1$. By induction, we construct an iterative sequence $\{h_n\}$ in T such that $h_{2n+1} = fh_{2n}$ and $h_{2n+2} = gh_{2n+1}$ for all $n \geq 0$. Then $\alpha(h_0,h_1) \geq s^p$ and $\alpha(h_1,h_0) \geq s^p$, by condition (ii) (f,g) is an α_{s^p}-admissible pair, so we obtain that

$$\alpha(h_1,h_2) = \alpha(fh_0,gh_1) \geq s^p \quad \text{and} \quad \alpha(h_2,h_1) = \alpha(gh_1,fh_0) \geq s^p.$$

Also, we have

$$\alpha(h_3,h_2) = \alpha(fh_2,gh_1) \geq s^p \quad \text{and} \quad \alpha(h_2,h_3) = \alpha(gh_1,fh_2) \geq s^p.$$

Proceeding inductively, we obtain

$$\alpha(h_n,h_{n+1}) \geq s^p \quad \text{and} \quad \alpha(h_{n+1},h_n) \geq s^p \quad \text{for all } n \geq 0.\tag{20}$$

If, for some $n \in N$, $\sigma_b(h_{2n+1}, h_{2n}) = 0$, then by (19) we have

$$\begin{aligned}
\psi\left(\sigma_b(h_{2n+1}, h_{2n+2})\right) &\le \psi\left(s^p\sigma_b(h_{2n+1}, h_{2n+2})\right) = \psi\left(s^p\sigma_b(fh_{2n}, gh_{2n+1})\right) \\
&\le \psi\left(\alpha(h_{2n}, h_{2n+1})\sigma_b(fh_{2n}, gh_{2n+1})\right) \\
&\le \beta\left[\sigma_b(h_{2n}, h_{2n+1})\right]\theta\left[\sigma_b(h_{2n}, h_{2n+1})\right] \\
&\quad + \gamma\left[\sigma_b(h_{2n}, h_{2n+1})\right]\theta\left[\sigma_b(h_{2n}, fh_{2n})\right] \\
&\quad + \delta\left[\sigma_b(h_{2n}, h_{2n+1})\right]\theta\left[\sigma_b(h_{2n+1}, gh_{2n+1})\right] \\
&= \beta\left[\sigma_b(h_{2n}, h_{2n+1})\right]\theta\left[\sigma_b(h_{2n}, h_{2n+1})\right] \\
&\quad + \gamma\left[\sigma_b(h_{2n}, h_{2n+1})\right]\theta\left[\sigma_b(h_{2n}, h_{2n+1})\right] \\
&\quad + \delta\left[\sigma_b(h_{2n}, h_{2n+1})\right]\theta\left[\sigma_b(h_{2n+1}, h_{2n+2})\right] \\
&= \delta\left[\sigma_b(h_{2n}, h_{2n+1})\right]\theta\left[\sigma_b(h_{2n+1}, h_{2n+2})\right] < \theta\left[\sigma_b(h_{2n+1}, h_{2n+2})\right].
\end{aligned}$$

By properties of ψ, θ, we get $\sigma_b(h_{2n+1}, h_{2n+2}) = 0$, that is, $h_{2n+1} = h_{2n+2}$. Furthermore, that is $h_{2n} = h_{2n+1} = fh_{2n}$ and $h_{2n} = h_{2n+2} = gh_{2n+1} = gfh_{2n} = gh_{2n}$. Hence, the proof is concluded. Now, we assume that $\sigma_b(h_n, h_{n+1}) > 0$ for all $n \ge 0$. By (20), applying condition (19), we have

$$\begin{aligned}
\psi\left(\sigma_b(h_{2n+1}, h_{2n})\right) &\le \psi\left(s^p\sigma_b(h_{2n+1}, h_{2n})\right) = \psi\left(s^p\sigma_b(fh_{2n}, gh_{2n-1})\right) \\
&\le \psi\left(\alpha(h_{2n}, h_{2n-1})\sigma_b(fh_{2n}, gh_{2n-1})\right) \\
&\le \beta\left[\sigma_b(h_{2n}, h_{2n-1})\right]\theta\left[\sigma_b(h_{2n}, h_{2n-1})\right] \\
&\quad + \gamma\left[\sigma_b(h_{2n}, h_{2n-1})\right]\theta\left[\sigma_b(h_{2n}, fh_{2n})\right] \\
&\quad + \delta\left[\sigma_b(h_{2n}, h_{2n-1})\right]\theta\left[\sigma_b(h_{2n-1}, gh_{2n-1})\right] \\
&= \beta\left[\sigma_b(h_{2n}, h_{2n-1})\right]\theta\left[\sigma_b(h_{2n}, h_{2n-1})\right] \\
&\quad + \gamma\left[\sigma_b(h_{2n}, h_{2n-1})\right]\theta\left[\sigma_b(h_{2n}, h_{2n+1})\right] \\
&\quad + \delta\left[\sigma_b(h_{2n}, h_{2n-1})\right]\theta\left[\sigma_b(h_{2n}, h_{2n-1})\right].
\end{aligned} \tag{21}$$

If we suppose that $\theta[\sigma_b(h_{2n-1}, h_{2n})] \le \theta[\sigma_b(h_{2n}, h_{2n+1})]$ for some $n \in \mathbb{N}$, then inequality (21) takes the form

$$\begin{aligned}
&\psi\left(\sigma_b(h_{2n+1}, h_{2n})\right) \\
&\quad \le \psi\left(s^p\sigma_b(h_{2n+1}, h_{2n})\right) \\
&\quad = \psi\left(s^p\sigma_b(fh_{2n}, gh_{2n-1})\right) \\
&\quad \le \left(\beta\left[\sigma_b(h_{2n}, h_{2n-1})\right] + \gamma\left[\sigma_b(h_{2n}, h_{2n-1})\right] + \delta\left[\sigma_b(h_{2n}, h_{2n-1})\right]\right)\theta\left[\sigma_b(h_{2n}, h_{2n+1})\right] \\
&\quad < \theta\left[\sigma_b(h_{2n}, h_{2n+1})\right],
\end{aligned}$$

that is, a contradiction. Hence

$$\theta\left[\sigma_b(h_{2n}, h_{2n+1})\right] < \theta\left[\sigma_b(h_{2n-1}, h_{2n})\right]. \tag{22}$$

By (22) and the properties of ψ, θ, we get

$$\psi\left[\sigma_b(h_{2n+1},h_{2n})\right]\le\theta\left[\sigma_b(h_{2n},h_{2n+1})\right]<\theta\left[\sigma_b(h_{2n-1},h_{2n})\right]<\psi\left[\sigma_b(h_{2n-1},h_{2n})\right]. \tag{23}$$

Inequality (23) implies $\sigma_b(h_{2n},h_{2n+1})\le\sigma_b(h_{2n-1},h_{2n})$ for all $n\in\mathbb{N}$.

That is, the sequence $\{\sigma_b(h_{2n+1},h_{2n})\}$ is decreasing. Thus, it is convergent to $\inf\{\sigma_b(h_{2n+1},h_{2n})\}=r\ge 0$. That is, $\lim_{n\to\infty}\sigma_b(h_n,h_{n+1})=r$, and also $\lim_{n\to\infty}\sigma_b(h_{2n},h_{2n+1})=\lim_{n\to\infty}\sigma_b(h_{2n-1},h_{2n})=r$.

If we suppose $r>0$, then we consider

$$\begin{aligned}\psi\left(\sigma_b(h_{2n+1},h_{2n})\right)&\le\psi\left(s^p\sigma_b(h_{2n+1},h_{2n})\right)=\psi\left(s^p\sigma_b(fh_{2n},gh_{2n-1})\right)\\&\le\psi\left(\alpha(h_{2n},h_{2n-1})\sigma_b(fh_{2n},gh_{2n-1})\right)\\&\le\beta\left[\sigma_b(h_{2n},h_{2n-1})\right]\theta\left[\sigma_b(h_{2n},h_{2n-1})\right]\\&\quad+\gamma\left[\sigma_b(h_{2n},h_{2n-1})\right]\theta\left[\sigma_b(h_{2n},h_{2n+1})\right]\\&\quad+\delta\left[\sigma_b(h_{2n},h_{2n-1})\right]\theta\left[\sigma_b(h_{2n-1},h_{2n})\right]\end{aligned}\tag{24}$$

and, letting $n\to\infty$ in (24), we obtain $\psi(r)\le\theta(r)$, which implies that $r=0$, that is,

$$\lim_{n\to\infty}\sigma_b(h_n,h_{n+1})=\lim_{n\to\infty}\sigma_b(h_{n-1},h_n)=0. \tag{25}$$

Now, we prove that $\lim_{n,m\to\infty}\sigma_b(h_n,h_m)=0$. It is sufficient to show that $\lim_{n,m\to\infty}\sigma_b(h_{2n},h_{2m})=0$. If we assume $\lim_{n,m\to\infty}\sigma_b(h_{2n},h_{2m})\neq 0$ then, using Lemma 1.11, there exists $\varepsilon>0$, and we can find subsequences $\{m_i\}$ and $\{n_i\}$ of positive integers, with $n_i>m_i>i$, such that

$$\begin{aligned}&\varepsilon\le\limsup_{i\to\infty}\sigma_b(h_{2n_i},h_{2m_i})\le\varepsilon s, &&\frac{\varepsilon}{s}\le\limsup_{i\to\infty}\sigma_b(h_{2m_i},h_{2n_i-1})\le\varepsilon s,\\&\frac{\varepsilon}{s^2}\le\limsup_{i\to\infty}\sigma_b(h_{2n_i-1},h_{2m_i+1})\le\varepsilon s^2, &&\frac{\varepsilon}{s}\le\limsup_{i\to\infty}\sigma_b(h_{2m_i+1},h_{2n_i})\le\varepsilon s^2.\end{aligned}\tag{26}$$

Since $\alpha(h_{2m_i},h_{2n_i-1})\ge s^p$ from (19), we have

$$\begin{aligned}\psi\left(s^p\sigma_b(h_{2m_i+1},h_{2n_i})\right)&\le\psi\left(s^p\sigma_b(fh_{2m_i},gh_{2n_i-1})\right)\\&\le\psi\left(\alpha(h_{2m_i},h_{2n_i-1})\sigma_b(fh_{2m_i},gh_{2n_i-1})\right)\\&\le\beta\left[\sigma_b(h_{2m_i},h_{2n_i-1})\right]\theta\left(\sigma_b(h_{2m_i},h_{2n_i-1})\right)\\&\quad+\gamma\left[\sigma_b(h_{2m_i},h_{2n_i-1})\right]\theta\left(\sigma_b(h_{2m_i},fh_{2m_i})\right)\\&\quad+\delta\left[\sigma_b(h_{2m_i},h_{2n_i-1})\right]\theta\left(\sigma_b(h_{2n_i-1},gh_{2n_i-1})\right)\\&=\beta\left[\sigma_b(h_{2m_i},h_{2n_i-1})\right]\theta\left(\sigma_b(h_{2m_i},h_{2n_i-1})\right)\\&\quad+\gamma\left[\sigma_b(h_{2m_i},h_{2n_i-1})\right]\theta\left(\sigma_b(h_{2m_i},h_{2m_i+1})\right)\\&\quad+\delta\left[\sigma_b(h_{2m_i},h_{2n_i-1})\right]\theta\left(\sigma_b(h_{2n_i-1},h_{2n_i})\right).\end{aligned}\tag{27}$$

Hence, by (26), (27), and (25), we obtain

$$
\begin{aligned}
\psi(\varepsilon s) &\le \psi\left(\varepsilon s^{p-1}\right) = \psi\left(s^p \frac{\varepsilon}{s}\right) \le \psi\left(\limsup_{i\to\infty} \sigma_b(h_{m_i}, h_{n_i})\right) \\
&\le \limsup_{i\to\infty} \beta\left[\sigma_b(h_{2m_i}, h_{2n_i-1})\right]\theta\left(\limsup_{i\to\infty} \sigma_b(h_{2m_i}, h_{2n_i-1})\right) \\
&\quad + \limsup_{i\to\infty} \gamma\left[\sigma_b(h_{2m_i}, h_{2n_i-1})\right]\theta\left(\limsup_{i\to\infty} \sigma_b(h_{2m_i}, h_{2m_i+1})\right) \\
&\quad + \limsup_{i\to\infty} \delta\left[\sigma_b(h_{2m_i}, h_{2n_i-1})\right]\theta\left(\limsup_{i\to\infty} \sigma_b(h_{2n_i-1}, h_{2n_i})\right) \\
&= \limsup_{i\to\infty} \beta\left[\sigma_b(h_{2m_i}, h_{2n_i-1})\right]\theta(\varepsilon s) + \limsup_{i\to\infty} \gamma\left[\sigma_b(h_{2m_i}, h_{2n_i-1})\right]\theta(0) \\
&\quad + \limsup_{i\to\infty} \delta\left[\sigma_b(h_{2m_i}, h_{2n_i-1})\right]\theta(0) \\
&\le \theta(\varepsilon s),
\end{aligned}
$$

which implies $\varepsilon = 0$, a contradiction. Thus, $\lim_{n,m\to\infty} \sigma_b(h_n, h_m) = 0$, and the sequence $\{h_n\}$ is Cauchy. (T, σ_b) is complete, so there exists $h \in T$ such that $\{h_n\}$ is convergent to h, that is,

$$
\lim_{n\to\infty} \sigma_b(h_n, h) = \lim_{n\to\infty} \sigma_b(h_n, h_m) = \sigma_b(h, h) = 0. \tag{28}
$$

By property H_{s^p}, there exists a subsequence $\{h_{n_i}\}$ of $\{h_n\}$ with $\alpha(h_{n_i}, h) \ge s^p$ and $\alpha(h, h_{n_i}) \ge s^p$ for all $i \in \mathbb{N}$. Then, from condition (19), we have

$$
\begin{aligned}
\psi\left(s^p \sigma_b(h_{2n_i+1}, gh)\right) = \psi\left(s^p \sigma_b(fh_{2n(i)}, gh)\right) &\le \psi\left(\alpha(h_{2n(i)}, h)\sigma_b(fh_{2n(i)}, gh)\right) \\
&\le \beta\left[\sigma_b(h_{2n(i)}, h)\right]\theta\left(\sigma_b(h_{2n(i)}, h)\right) \\
&\quad + \gamma\left[\sigma_b(h_{2n(i)}, h)\right]\theta\left(\sigma_b(h_{2n(i)}, fh_{2n(i)})\right) \\
&\quad + \delta\left[\sigma_b(h_{2n(i)}, h)\right]\theta\left(\sigma_b(h, gh)\right) \\
&= \beta\left[\sigma_b(h_{2n_i}, h)\right]\theta\left(\sigma_b(h_{2n_i}, h)\right) \\
&\quad + \gamma\left[\sigma_b(h_{2n_i}, h)\right]\theta\left(\sigma_b(h_{2n_i}, h_{2n_i+1})\right) \\
&\quad + \delta\left[\sigma_b(h_{2n_i}, h)\right]\theta\left(\sigma_b(h, gh)\right).
\end{aligned} \tag{29}
$$

Considering limit superior as $i \to \infty$ in (29), and due to (25), (28), and Lemma 1.9, we obtain

$$
\begin{aligned}
\psi\left(s^{p-1}\sigma_b(h, gh)\right) = \psi\left(s^p s^{-1}\sigma_b(h, gh)\right) &\le \psi\left(s^p \limsup_{i\to\infty} \sigma_b(h_{2n_i+1}, gh)\right) \\
&\le \limsup_{i\to\infty} \beta\left[\sigma_b(h_{2n_i}, h)\right]\theta\left(\limsup_{i\to\infty} \sigma_b(h_{2n_i}, h)\right) \\
&\quad + \limsup_{i\to\infty} \gamma\left[\sigma_b(h_{2n_i}, h)\right]\theta\left(\limsup_{i\to\infty} \sigma_b(h_{2n_i}, h_{2n_i+1})\right) \\
&\quad + \limsup_{i\to\infty} \delta\left[\sigma_b(h_{2n_i}, h)\right]\theta\left(\limsup_{i\to\infty} \sigma_b(h, gh)\right) \\
&\le \theta\left(\sigma_b(h, gh)\right).
\end{aligned} \tag{30}
$$

Inequality (30) yields that $\sigma_b(h,gh) = 0$, so $gh = h$. Similarly, $fh = h$.

If $h,j \in C(f,g)$ with $h \neq j$, then, by hypothesis U_{s^p} and applying (19), we obtain

$$\begin{aligned}
&\psi\big(s^p\sigma_b(h,h)\big)\\
&\quad\le \psi\big(\alpha(h,h)\sigma_b(fh,gh)\big)\\
&\quad\le \beta\big[\sigma_b(h,h)\big]\theta\big(\sigma_b(h,h)\big) + \gamma\big[\sigma_b(h,h)\big]\theta\big(\sigma_b(h,fh)\big) + \delta\big[\sigma_b(h,h)\big]\theta\big(\sigma_b(h,gh)\big)\\
&\quad= \beta\big[\sigma_b(h,h)\big]\theta\big(\sigma_b(h,h)\big) + \gamma\big[\sigma_b(h,h)\big]\theta\big(\sigma_b(h,h)\big) + \delta\big[\sigma_b(h,h)\big]\theta\big(\sigma_b(h,h)\big)\\
&\quad= \big(\beta\big[\sigma_b(h,h)\big] + \gamma\big[\sigma_b(h,h)\big] + \delta\big[\sigma_b(h,h)\big]\big)\theta\big(\sigma_b(h,h)\big)\\
&\quad< \theta\big(\sigma_b(h,h)\big),
\end{aligned}$$

that implies $\sigma_b(h,h) = 0$ (also $\sigma_b(j,j) = 0$).

Again from (19), we have

$$\begin{aligned}
\psi\big(s^p\sigma_b(h,j)\big) &\le \psi\big(\alpha(h,j)\sigma_b(fh,gj)\big)\\
&\le \beta\big[\sigma_b(h,j)\big]\theta\big(\sigma_b(h,j)\big) + \gamma\big[\sigma_b(h,j)\big]\theta\big(\sigma_b(h,fh)\big)\\
&\quad + \delta\big[\sigma_b(h,j)\big]\theta\big(\sigma_b(j,gj)\big)\\
&\le \theta\big(\sigma_b(h,j)\big),
\end{aligned}$$

a contradiction. Hence, $h = j$. □

Corollary 2.11 *Let (f,g) be an α_{s^p}-admissible pair of self-mappings on a complete b-metric-like space (T,σ_b) with coefficient $s \ge 1$. If there exist $\psi \in I$, $\theta \in \Theta$ and $c_1, c_2, c_3 \in \mathbb{R}^+$ with $c_1 + c_2 + c_3 < 1$ such that*

$$\psi\big(\alpha(h,k)\sigma_b(fh,gk)\big) \le c_1\theta\big[\sigma_b(h,k)\big] + c_2\theta\big[\sigma_b(h,fh)\big] + c_3\theta\big[\sigma_b(k,gk)\big]$$

for all $h,k \in T$; furthermore, the following conditions hold:

(i) *there exists $h_0 \in T$ such that* $\min\{\alpha(h_0,fh_0),\alpha(fh_0,h_0)\} \ge s^p$;

(ii) *properties H_{s^p}; U_{s^p} are satisfied,*

then f and g have a unique common fixed point $h \in T$.

Proof Take in Theorem 2.10, $\beta(m) = c_1$, $\gamma(m) = c_2$, $\delta(m) = c_3$, $m \ge 0$. □

Corollary 2.12 *Let f be an α_{s^p}-admissible self-mapping on a b-metric-like space (T,σ_b) with coefficient $s \ge 1$. If f is a generalized $\alpha_{s^p} - (I \times \Theta \times \Gamma)$-Kannan contraction, and the following assertions hold:*

(i) *there exists $h_0 \in T$ such that* $\min\{\alpha(h_0,fh_0),\alpha(fh_0,h_0)\} \ge s^p$;

(ii) *conditions H_{s^p}; U_{s^p} are satisfied,*

then f has a unique fixed point $h \in T$.

Proof The proof follows from Theorem 2.10 if we take $g = f$. □

Corollary 2.13 *Let (f,g) be an α_{s^p}-admissible pair of self-mappings on a complete b-metric-like space (T,σ_b) with coefficient $s \ge 1$. If there exist $\psi \in I$, $\theta \in \Theta$, and $\beta \in \Gamma$ such* *that*

$$\psi\left(\alpha(h,k)\sigma_b(fh,gk)\right) \le \beta\left[\sigma_b(h,k)\right]\left(\theta\left[\sigma_b(h,k)\right] + \theta\left[\sigma_b(h,fh)\right] + \theta\left[\sigma_b(k,gk)\right]\right)$$

for all $h,k \in T$ *and* $\theta(m) < \psi(m)$ *for all* $m > 0$; *and the following assertions hold*:

(i) *there exists* $h_0 \in T$ *such that* $\min\{\alpha(h_0,fh_0),\alpha(fh_0,h_0)\} \ge s^p$;

(ii) *conditions* H_{s^p}; U_{s^p} *are satisfied*,

then f *and* g *have a unique common fixed point* $h \in T$.

Proof By taking $\gamma(m) = \delta(m) = \beta(m)$. □

Remark 2.14 It is evident that we can generate a variety of other corollaries as special cases by putting $\alpha(h,k) = s^p$ $(p > 1)$, or $g = f$ or $\psi(m) = m$, or defining $\beta,\gamma,\delta \in \Gamma$ as constant functions.

3 Applications

In this section, we discuss an application that attributes the solvability of boundary value problems of second order ordinary differential equations:

$$\begin{cases} h''(u) = M_1(u,h(u)), & u \in [0,1], \\ h''(u) = M_2(u,h(u)), & u \in [0,1], \\ h(0) = h(1) = 0 \end{cases} \tag{31}$$

for given continuous functions $M_1; M_2 : [0,1] \times \mathbb{R} \to \mathbb{R}$.

Let $T = \mathbb{C}([0,1],\mathbb{R})$ be the set of real continuous functions defined on $[0,1]$, endowed with the b-metric-like

$$\sigma_b(h,k) = \max_{u\in[0,1]}\left(\left|h(u)\right| + \left|k(u)\right|\right)^n \quad \text{for all } h,k \in T.$$

It is evident that (T,σ_b) is a complete b-metric-like space with parameter $s = 2^{n-1}$ where $n > 1$.

The equivalent system of integral equations corresponding to boundary value problems (31) is the following:

$$\begin{cases} h(u) = \int_0^1 G(u,\rho)M_1(\rho,h(\rho))\,d\rho, \\ h(u) = \int_0^1 G(u,\rho)M_2(\rho,h(\rho))\,d\rho, \\ \quad \text{for } u \in [0,1] \end{cases} \tag{32}$$

and $G(u,\rho)$ is the Green function given as

$$G(u,\rho) = \begin{cases} \rho(u-\rho) & 0 \le \rho \le u \le 1, \\ u(\rho-u) & 0 \le u \le \rho \le 1. \end{cases}$$

Consider the mappings $f,g : T \to T$ by

$$fh(u) = \int_0^1 G(u,\rho)M_1\left(\rho,h(\rho)\right)d\rho \quad \text{for } u \in [0,1],$$

$$gh(u) = \int_0^1 G(u,\rho)M_2\big(\rho,h(\rho)\big)\,d\rho \quad \text{for } u \in [0,1],$$

and let $\zeta : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a given function.

Theorem 3.1 *Consider the system of integral Eqs.* (32) *and suppose that the following assertions hold:*

(i) *There exists $h_0 \in T$ such that $\zeta(h_0(u), fh_0(u)) \geq 0$ for all $u \in [0,1]$;*

(ii) *For all $u \in [0,1]$ and $h,k \in T$,*

$$\zeta\big(h(u),k(u)\big) \geq 0 \quad \text{implies that } \zeta\big(fh(u),gk(u)\big) \geq 0;$$

(iii) *Properties H_{s^p} and U_{s^p} are satisfied;*

(iv) *There exist $n > 1$, $p > 1$, $\lambda \in (0,1)$ and a continuous function $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\big(\big|M_1\big(\rho,h(\rho)\big)\big| + \big|M_2\big(\rho,h(\rho)\big)\big|\big) \leq \sqrt[n]{L2^p\theta\big[\big(\big|h(\rho)\big| + \big|k(\rho)\big|\big)^n\big]}$$

for all $\rho \in [0,1]$, $h,k \in T$;

(v) *For all $\rho \in [0,1]$, $\sup_{u\in[0,1]}\int_0^1 G(u,\rho)\,d\rho \leq \frac{1}{2}$.*

Then the system of integral Eqs. (32) (*or equivalently,* (31)) *has a unique solution in T.*

Proof We define a function $\alpha : T\times T \to [0,\infty)$ by

$$\alpha(h,k) = \begin{cases} s^p & \text{if } \zeta(h(u),k(u)) \geq 0, \text{ for all } u \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that (f,g) is an α_{s^p}-admissible pair.

Let $h,k \in T = \mathbb{C}([0,1],\mathbb{R})$ be such that $\alpha(h(u),k(u)) \geq s^p$, i.e., $\zeta(fh(u),gk(u)) \geq 0$, then from the assertions above, for all $u \in [0,1]$, we might observe that

$$\begin{aligned}
\sigma_b\big(fh(u),gk(u)\big) &= \max_{u\in[0,1]}\big(\big|fh(u)\big| + \big|gk(u)\big|\big)^n \\
&= \left(\left|\int_0^1 G(u,\rho)M_1\big(\rho,h(\rho)\big)\,d\rho\right| + \left|\int_0^1 G(u,\rho)M_2\big(\rho,h(\rho)\big)\,d\rho\right|\right)^n \\
&\leq \left(\int_0^1 G(u,\rho)\big|M_1\big(\rho,h(\rho)\big)\big|\,d\rho + \int_0^1 G(u,\rho)\big|M_2\big(\rho,h(\rho)\big)\big|\,d\rho\right)^n \\
&= \left(\int_0^1 G(u,\rho)\big(\big|M_1\big(\rho,h(\rho)\big)\big| + \big|M_2\big(\rho,h(\rho)\big)\big|\big)\,d\rho\right)^n \\
&\leq \left(\int_0^1 G(u,\rho)\sqrt[n]{\lambda 2^p\theta\big[\big(\big|h(\rho)\big| + \big|k(\rho)\big|\big)^n\big]}\,d\rho\right)^n \\
&= \left(\int_0^1 G(u,\rho)\sqrt[n]{\lambda 2^p\theta\big[\sigma_b\big(h(\rho),k(\rho)\big)\big]}\,d\rho\right)^n \\
&= \left(\int_0^1 G(u,\rho)\,d\rho\right)^n \lambda 2^p\theta\big[\sigma_b\big(h(\rho),k(\rho)\big)\big] \\
&\leq \left(\sup_{u\in[0,1]}\int_0^1 G(u,\rho)\,d\rho\right)^n \lambda 2^p\theta\big[\sigma_b(h,k)\big].
\end{aligned} \tag{33}$$

Since $G(u,\rho) = \frac{u}{2} - \frac{u^2}{2}$ and $\sup_{u\in[0,1]} \int_0^1 G(u,\rho)\,d\rho \le \frac{1}{8}$ (in that case, the coefficient $p = 3 > 1$), then inequality (33) can be written

$$\sigma_b\big(fh(u),gk(u)\big) \le \frac{1}{2^{np}} \cdot \frac{\lambda}{2^{-p}}\theta\big[\sigma_b(h,k)\big] \le \frac{\lambda}{2^{(n-1)p}}\theta\big[\sigma_b(h,k)\big] = \frac{\lambda}{s^p}\theta\big[\sigma_b(h,k)\big].$$

Hence,

$$\max_{u\in[0,1]} \big(\big|fh(u)\big| + \big|gk(u)\big|\big)^n \le \frac{\lambda}{s^p}\theta\big[\sigma_b(h,k)\big],$$

and we convert the result to

$$\alpha(h,k)\sigma_b\big(fh(u),gk(u)\big) \le \lambda\theta\big[\sigma_b(h,k)\big]. \tag{34}$$

Thus, taking $\psi(x) = x$, and $\beta,\gamma,\delta \in \Gamma$ as $\beta(x) = \lambda$, $\gamma(x) = 0$, $\delta(x) = 0$, where $\lambda \in (0,1)$, from inequality (34) we deduce

$$\begin{aligned}\psi\big(\alpha(h,k)\sigma_b(fh,gk)\big) \le{}& \beta\big[\sigma_b(h,k)\big]\theta\big[\sigma_b(h,k)\big]\\ &+ \gamma\big[\sigma_b(h,k)\big]\theta\big[\sigma_b(h,fh)\big]\\ &+ \delta\big[\sigma_b(h,k)\big]\theta\big[\sigma_b(k,gk)\big].\end{aligned}$$

Therefore, Theorem 2.10 can be applied to obtain a solution of the system of boundary value problems (31). □

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Author details
[1]Department of Mathematics and Computer Sciences, Faculty of Natural Sciences, University of Gjirokastra, Gjirokastra 6001, Albania. [2]Institut Supérieur d'Informatique et des Techniques de Communication, Université de Sousse, H. Sousse, 4000, Tunisia. [3]Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, South Africa. [4]China Medical University Hospital, China Medical University, Taichung, 40402, Taiwan. [5]College of Business Administration-Finance Department, Dar Al Uloom University Saudi Arabia, Riyadh, Saudi Arabia.

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