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# The Parseval identity for $q$ -Sturm–Liouville problems with transmission conditions

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## Abstract

In the present study, we investigate the existence of spectral functions and obtain the Parseval identity and expansion formula in eigenfunctions for the singular  $q$ -Sturm–Liouville problem with transmission conditions.

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**Keywords:**  $q$ -Sturm–Liouville operator; Parseval identity; Spectral function

## 1 Introduction and background

The development of applications of the  $q$ -calculus has been of great interest recently. It has an especially important role in the different areas of mathematics, such as orthogonal polynomials, mathematical physics, the calculus of variations, statistical mechanics, theory of various operators of convergence, quantum theory. More detailed concepts of the  $q$ -calculus can be obtained from books [1, 2]. For a general introduction to the  $q$ -calculus, very useful studies have been done in [3–9].

It is well known that eigenfunction expansion problems are important for solving various problems, and there are lots of techniques for obtained (see [3, 10, 11]). On the other hand, many researchers have focused on certain generalizations of Sturm–Liouville problems. In particular,  $q$ -Sturm–Liouville eigenvalue problems were studied and a self-adjoint  $q$ -difference operator was formulated in a Hilbert space in [7, 12].

In [13] authors also studied the eigenfunction expansion for a certain  $q$ -Sturm–Liouville problem by using Titchmarsh’s technique and defined some concepts for deriving eigenfunction expansion problems.

Sturm–Liouville problems with transmission conditions have been investigated by many authors [5, 14, 15] who gave asymptotic formulas for eigenvalues and the corresponding eigenfunction for these problems.

The existence of a spectral function for singular  $q$ -Sturm–Liouville operators on semi-unbounded interval was proved in [4], and also they obtained the Parseval identity and on the expansion formula.

In this study, we consider a  $q$ -Sturm–Liouville expression as follows:

$$\ell(y) := -\frac{1}{q}D_{q^{-1}}D_q y(\zeta) + u(\zeta)y(\zeta), \quad \zeta \in J := [-1, 0) \cup (0, 1].$$

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Here, we denote  $J_1 := [-1, 0)$ ,  $J_2 := (0, 1]$ , and so  $J := J_1 \cup J_2$ . Suppose that the points  $-1$ ,  $1$  and  $0$  are  $q$ -regular for the differential expression  $\ell$ .  $u$  is a real, Lebesgue measurable function in  $J$  and  $u \in L^1_q(J_k)$ ,  $k = 1, 2$ . Recall that  $0$  is a  $q$ -regular point of the function  $u$  which belongs to  $L^1_q[0 - \varepsilon, 0 + \varepsilon]$  for some  $\varepsilon > 0$ .

The rest of the study is arranged as follows. In Sect. 2, we give some preliminaries for  $q$ -calculus. In Sect. 3, we investigate the existence of a spectral function. The Parseval identity and an expansion formula with eigenfunctions for a singular  $q$ -Sturm–Liouville problem with transmission conditions are obtained.

### 2 Preliminaries

We begin with some preliminary facts and notations for quantum calculus (see [2, 6]). Our main tools are  $q$ -derivative and  $q$ -integral. Let  $q$  be any fixed constant with  $q \in (0, 1)$ . We say that a set  $A \subset \mathbb{R}$  is  $q$ -geometric if, for every  $a \in A$ ,  $qa \in A$ . Also a  $q$ -difference equation is an equation which includes  $q$ -derivatives of a function defined on a  $q$ -geometric set  $A$ . The  $q$ -difference operator is denoted by  $D_q$ , the Jackson  $q$ -derivative of a function  $\varphi : A \rightarrow \mathbb{C}$  is defined by

$$D_q\varphi(a) = \frac{\varphi(qa) - \varphi(a)}{qa - a}, \quad \forall a \in A \setminus \{0\}.$$

We say that the point  $0$  in  $A$  is the Jackson  $q$ -derivative at zero if the limit

$$D_q\varphi(0) = \lim_{n \rightarrow \infty} \frac{\varphi(q^n a) - \varphi(0)}{q^n a} \quad (a \in A)$$

exists and belongs to  $\mathbb{C}$ . Here, note that the value of the limit is independent of  $a$  (see [8]). The Jackson  $q$ -integral is given by

$$\int_0^a \varphi(\zeta) d_q \zeta = a(1 - q) \sum_{k=0}^{\infty} q^k \varphi(q^k a) \quad (a \in A),$$

where the series is convergent in [9]. Additionally, the following result is satisfied:

$$\int_a^b \varphi(\zeta) d_q \zeta = \int_0^b \varphi(\zeta) d_q \zeta - \int_0^a \varphi(\zeta) d_q \zeta, \quad \forall a, b \in A.$$

The Jackson  $q$ -integration of  $\varphi$  on  $[0, \infty)$  is defined in [16] by the formula

$$\int_0^{\infty} \varphi(\zeta) d_q \zeta = \sum_{n=-\infty}^{\infty} q^n \varphi(q^n a), \quad (a \in A)$$

provided that some converges absolutely. A function  $\varphi$  is  $q$ -regular at the point zero if the limit

$$\lim_{n \rightarrow \infty} \varphi(aq^n) = \varphi(0), \quad (a \in A \cup \{0\})$$

exists. Throughout the study, the functions will be assumed to be  $q$ -regular at zero. If  $\varphi$  and  $\psi$  are  $q$ -regular at zero, then the following equality holds:

$$\int_0^a \psi(\zeta) D_q \varphi(\zeta) d_q \zeta + \int_0^a \varphi(q\zeta) D_q \psi(\zeta) d_q \zeta = \varphi(a)\psi(a) - \varphi(0)\psi(0).$$

The separable Hilbert space is  $L_q^2(0, \infty) := \{\varphi \mid \int_0^\infty |\varphi(\zeta)|^2 d_q\zeta < \infty, \varphi : (0, \infty) \rightarrow \mathbb{C}\}$  with the norm

$$\|\varphi\| := \left( \int_0^\infty |\varphi(\zeta)|^2 d_q\zeta \right)^{\frac{1}{2}} < \infty$$

and given with the inner product

$$\langle \varphi, \psi \rangle := \int_0^\infty \varphi(\zeta) \overline{\psi(\zeta)} d_q\zeta, \quad \varphi, \psi \in L_q^2(0, \infty)$$

(see [13]).

We call the  $q$ -Wronskian of  $\varphi, \psi$  on  $A$  if

$$W_q[\varphi, \psi](a) := \varphi(a)D_q\psi(a) - \psi(a)D_q\varphi(a) \tag{2.1}$$

exists.

### 3 Main results

In this section, we begin with the  $q$ -Sturm–Liouville equations as follows:

$$\ell(y) := -\frac{1}{q}D_{q^{-1}}D_q y(\zeta) + u(\zeta)y(\zeta) = \lambda y, \quad \zeta \in J \tag{3.1}$$

with the boundary conditions

$$y(-1) \cos \alpha + D_{q^{-1}}y(-1) \sin \alpha = 0, \tag{3.2}$$

$$y(1) \cos \beta + D_{q^{-1}}y(1) \sin \beta = 0 \tag{3.3}$$

and the transmission conditions

$$y(0+) - \gamma_1 y(0-) - \gamma_2 D_q y(0-) = 0, \tag{3.4}$$

$$D_q y(0+) - \gamma_3 y(0-) - \gamma_4 D_q y(0-) = 0, \tag{3.5}$$

where  $\lambda$  is a complex eigenparameter, the potential function  $u \in L_q^1(J)$ , and notice that it guarantees that  $y(0\mp)$  and  $D_q y(0\mp)$  in (3.4)–(3.5) make sense; here we assume that

$$\gamma = \begin{vmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{vmatrix} > 0. \tag{3.6}$$

Furthermore, the class  $H_q = L_q^2(J_1) \oplus L_q^2(J_2)$  is introduced as a Hilbert space with the inner product

$$\langle \varphi, \psi \rangle_{H_q} := \int_{-1}^0 \varphi_1 \overline{\psi_1} d_q\zeta + \frac{1}{\gamma} \int_0^1 \varphi_2 \overline{\psi_2} d_q\zeta,$$

where

$$\varphi(\zeta) = \begin{cases} \varphi_1(\zeta), & \zeta \in J_1, \\ \varphi_2(\zeta), & \zeta \in J_2, \end{cases} \quad \psi(\zeta) = \begin{cases} \psi_1(\zeta), & \zeta \in J_1, \\ \psi_2(\zeta), & \zeta \in J_2. \end{cases}$$

It can be easily obtained by direct manipulation from [17], (p.217). A compact resolvent of the regular self-adjoint boundary value problem (3.1)–(3.3), (3.4)–(3.5) with transmission was proved by authors in [18, 19], and they also showed that it has a completely discrete spectrum.

Let  $\lambda_{m;1}$ , ( $m \in \mathbb{N}$ ) denote the eigenvalues of this problem and by

$$\phi_{m;1}(\zeta) = \begin{cases} \phi_{m;1}^{(1)}(\zeta), & \zeta \in J_1, \\ \phi_{m;1}^{(2)}(\zeta), & \zeta \in J_2, \end{cases}$$

$$\phi_{m;1}(\zeta) := \phi(\zeta, \lambda_{m;1})$$

the corresponding real-valued eigenfunctions which satisfy conditions (3.2)–(3.5).

If  $\varphi \in H_q$  is a real-valued function with

$$\varphi(\zeta) = \begin{cases} \varphi_1(\zeta), & \zeta \in J_1, \\ \varphi_2(\zeta), & \zeta \in J_2, \end{cases}$$

then

$$\begin{aligned} \|\varphi\|_{H_q}^2 &= \int_{-1}^0 (\varphi_1(\zeta))^2 d_q \zeta + \frac{1}{\gamma} \int_0^1 (\varphi_2(\zeta))^2 d_q \zeta \\ &= \sum_{m=1}^{\infty} \frac{1}{\alpha_{m;1}^2} \left[ \int_{-1}^0 \varphi_1(\zeta) \phi_{m;1}^{(1)}(\zeta) d_q \zeta + \frac{1}{\gamma} \int_0^1 \varphi_2(\zeta) \phi_{m;1}^{(2)}(\zeta) d_q \zeta \right]^2, \end{aligned} \tag{3.7}$$

where

$$\alpha_{m;1}^2 = \int_{-1}^0 (\phi_{m;1}^{(1)})^2 d_q \zeta + \frac{1}{\gamma} \int_0^1 (\phi_{m;1}^{(2)})^2 d_q \zeta$$

is obtained. Here equality (3.7) is called the Parseval identity (see [5]).

Now we give a monotone increasing step function on  $\mathbb{R}$ :

$$\sigma_1(\lambda) = \begin{cases} -\sum_{\lambda < \lambda_{m;1} < 0} \frac{1}{\alpha_{m;1}^2} & \text{for } \lambda < 0, \\ \sum_{0 \leq \lambda_{m;1} < \lambda} \frac{1}{\alpha_{m;1}^2} & \text{for } \lambda \geq 0. \end{cases} \tag{3.8}$$

Thus, we can write (3.7) as

$$\int_{-1}^0 (\varphi_1(\zeta))^2 d_q \zeta + \frac{1}{\gamma} \int_0^1 (\varphi_2(\zeta))^2 d_q \zeta = \int_{-\infty}^{+\infty} \Phi^2(\lambda) d\sigma_1(\lambda), \tag{3.9}$$

where

$$\Phi(\lambda) = \int_{-1}^0 \varphi_1(\zeta) \phi_{m;1}^{(1)}(\zeta) d_q \zeta + \frac{1}{\gamma} \int_0^1 \varphi_2(\zeta) \phi_{m;1}^{(2)}(\zeta) d_q \zeta.$$

We will obtain the Parseval identity for (3.1)–(3.5) from (3.9) by letting  $1 \rightarrow \infty$ .

The function  $\varphi$  is said to be of bounded variation on the interval  $[a, b]$  if and only if there exists a positive constant  $M$  such that

$$\sum_{k=1}^n |\varphi(\zeta_k) - \varphi(\zeta_{k-1})| \leq M$$

for all finite partitions  $\mathbb{P} = \{\zeta_0, \zeta_1, \dots, \zeta_n\}$  of  $[a, b]$ .

If  $\varphi : [a, b] \rightarrow \mathbb{R}$  is of bounded variation on  $[a, b]$ , then the total variation of  $\varphi$  on  $[a, b]$  is defined to be

$$V_a^b(\varphi) := \sup \sum_{k=1}^n |\varphi(\zeta_k) - \varphi(\zeta_{k-1})|,$$

where the supremum is taken over all partitions of  $[a, b]$  (see [5]).

**Lemma 3.1** *For arbitrary  $M$ , the formula*

$$V_{-M}^M(\sigma_1(\lambda)) = \sum_{-M < \lambda_{m;1} < M} \frac{1}{\alpha_{m;1}^2} = \sigma_1(M) - \sigma_1(-M) < \Upsilon \tag{3.10}$$

*holds for a positive constant  $\Upsilon = \Upsilon(M)$ .*

*Proof* Assume first that  $\sin \alpha \neq 0$ . Since  $\phi(\zeta, \lambda)$  is continuous on the domain  $[-1, 0] \times [-M, M]$  with the condition  $\phi(-1, \lambda) = \sin \alpha$ , there exists a positive number  $k$  such that

$$\left( \frac{1}{k} \int_{-1}^k \phi_{m;1}^{(1)}(\zeta, \lambda) d_q \zeta \right)^2 > \frac{\sin^2 \alpha}{2}. \tag{3.11}$$

Let us define

$$\varphi_k(\zeta) := \begin{cases} \frac{1}{k}, & -1 \leq \zeta < k, \\ 0, & \zeta \geq k. \end{cases}$$

From (3.9) and (3.11), we get

$$\begin{aligned} \int_{-1}^k \varphi_k^2(\zeta) d_q \zeta &= \frac{k+1}{k^2} \\ &= \int_{-\infty}^{\infty} \left( \frac{1}{k} \int_{-1}^k \phi_{m;1}^{(1)}(\zeta, \lambda) d_q \zeta \right)^2 d\sigma_1(\lambda) \\ &\geq \int_{-M}^M \left( \frac{1}{k} \int_{-1}^k \phi_{m;1}^{(1)}(\zeta, \lambda) d_q \zeta \right)^2 d\sigma_1(\lambda) \\ &> \frac{1}{2} \sin^2 \alpha \int_{-M}^M d\sigma_1(\lambda) \\ &= \frac{1}{2} \sin^2 \alpha [\sigma_1(M) - \sigma_1(-M)], \end{aligned}$$

so it satisfies inequality (3.10).

If  $\sin \alpha = 0$ , we give a formula for the function  $\varphi_k(\zeta)$  by

$$\varphi_k(\zeta) := \begin{cases} \frac{1}{k^2}, & -1 \leq \zeta < k, \\ 0, & \zeta \geq k. \end{cases}$$

Thus, we obtain (3.10) by applying the Parseval identity. □

Let us now mention the following well-known Helly’s first and second theorems in [20].

Firstly, recall Helly’s first theorem that given a uniformly bounded sequence  $\{\psi_n\}$  of monotone increasing real functions on  $[a, b]$ , there exists a subsequence  $\{\psi_{n_k}\}$  of  $\{\psi_n\}$  converging to a monotone increasing real function  $\psi$  on  $[a, b]$ .

Secondly, given a sequence  $\{\psi_n\}$  of monotone increasing real functions on  $[a, b]$ , converging to a monotone increasing real function  $\psi$ , for every continuous function  $\varphi$  on  $[a, b]$ , we have

$$\lim_{n \rightarrow \infty} \int_a^b \varphi(\lambda) d\psi_n(\lambda) = \int_a^b \varphi(\lambda) d\psi(\lambda).$$

We introduce the Hilbert space  $H := L^2_q(J_1) \oplus L^2_q(J_3)$ , ( $J_1 := [-1, 0), J_3 := (0, \infty)$ ) with the inner product

$$\langle \varphi, \psi \rangle_H := \int_{-1}^0 \varphi_1 \overline{\psi_1} d_q \zeta + \frac{1}{\gamma} \int_0^\infty \varphi_2 \overline{\psi_2} d_q \zeta,$$

where

$$\varphi(\zeta) = \begin{cases} \varphi_1(\zeta), & \zeta \in J_1, \\ \varphi_2(\zeta), & \zeta \in J_3, \end{cases} \quad \psi(t) = \begin{cases} \psi_1(\zeta), & \zeta \in J_1, \\ \psi_2(\zeta), & \zeta \in J_3. \end{cases}$$

We assume that  $\sigma$  is any nondecreasing function for  $-\infty < \lambda < \infty$ . Let us define all measurable real functions of Hilbert space by  $L^2_\sigma(\mathbb{R})$  which satisfies

$$\int_{-\infty}^\infty \varphi^2(\lambda) d\sigma(\lambda) < \infty,$$

with the inner product

$$\langle \varphi, \psi \rangle_\sigma := \int_{-\infty}^\infty \varphi(\lambda) \psi(\lambda) d\sigma(\lambda).$$

The fundamental result of the study is given as follows.

**Theorem 3.2** *The nondecreasing function  $\sigma(\lambda)$  on  $-\infty < \lambda < \infty$  for  $q$ -Sturm–Liouville problem (3.1)–(3.5) satisfies the following properties:*

(i) *If*

$$\varphi(\zeta) = \begin{cases} \varphi_1(\zeta), & \zeta \in J_1, \\ \varphi_2(\zeta), & \zeta \in J_3 \end{cases}$$

is a real-valued function and  $\varphi$  belongs to  $H$ , then there is a function  $\Phi \in L^2_\sigma(\mathbb{R})$  such that

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \left[ \Phi(\lambda) - \int_{-1}^0 \varphi_1(\zeta) \phi_{m;1}^{(1)}(\zeta, \lambda) d_q \zeta - \frac{1}{\gamma} \int_0^n \varphi_2(\zeta) \phi_{m;1}^{(2)}(\zeta, \lambda) d_q \zeta \right]^2 d\sigma(\lambda) = 0 \tag{3.12}$$

and the Parseval identity

$$\|\varphi\|_H^2 = \int_{-1}^0 (\varphi_1(\zeta))^2 d_q \zeta + \frac{1}{\gamma} \int_0^\infty (\varphi_2(\zeta))^2 d_q \zeta = \int_{-\infty}^\infty \Phi^2(\lambda) d\sigma(\lambda). \tag{3.13}$$

(ii) The integral  $\int_{-\infty}^\infty \Phi(\lambda) \phi(\zeta, \lambda) d\sigma(\lambda)$  converges to  $\varphi$  in  $H$ ; that is,

$$\lim_{n \rightarrow \infty} \left[ \int_{-1}^0 \left( \varphi_1(\zeta) - \int_{-n}^n \Phi(\lambda) \phi_{m;1}^{(1)}(\zeta, \lambda) d\sigma(\lambda) \right)^2 d_q \zeta + \frac{1}{\gamma} \int_0^\infty \left( \varphi_2(\zeta) - \int_{-n}^n \Phi(\lambda) \phi_{m;1}^{(2)}(\zeta, \lambda) d\sigma(\lambda) \right)^2 d_q \zeta \right] = 0.$$

It should be known that the function  $\sigma$  is said to be a spectral function for boundary value problem (3.1)–(3.5).

*Proof* We may assume that

$$\varphi_\xi(\zeta) = \begin{cases} \varphi_{1;\xi}(\zeta), & \zeta \in [-1, 0), \\ \varphi_{2;\xi}(\zeta), & \zeta \in (0, \xi] \end{cases}$$

satisfies three conditions as follows:

- (a) Let  $\varphi_\xi(\zeta)$  be identically zero outside the set  $[-1, 0) \cup (0, \xi]$  with  $\xi < 1$ .
- (b) Let  $\varphi_\xi(\zeta)$  and  $D_q \varphi_\xi(\zeta)$  be  $q$ -regular functions at zero.
- (c) Let  $\varphi_\xi(\zeta)$  satisfy boundary conditions (3.1)–(3.5).

Applying Parseval identity (3.9) to the function  $\varphi_\xi(\zeta)$ , we obtain

$$\int_{-1}^0 (\varphi_{1;\xi}(\zeta))^2 d_q \zeta + \frac{1}{\gamma} \int_0^\xi (\varphi_{2;\xi}(\zeta))^2 d_q \zeta = \int_{-\infty}^\infty \Phi_\xi^2(\lambda) d\sigma(\lambda), \tag{3.14}$$

where

$$\Phi_\xi(\lambda) = \int_{-1}^0 \varphi_{1;\xi}(\zeta) \phi_{m;1}^{(1)}(\zeta, \lambda) d_q \zeta + \frac{1}{\gamma} \int_0^\xi \varphi_{2;\xi}(\zeta) \phi_{m;1}^{(2)}(\zeta, \lambda) d_q \zeta. \tag{3.15}$$

Because  $\phi(\zeta, \lambda)$  satisfies (3.1), it is clear that

$$\phi(\zeta, \lambda) = \frac{1}{\lambda} \left[ -\frac{1}{q} D_{q^{-1}} D_q \phi(\zeta, \lambda) + u(\zeta) \phi(\zeta, \lambda) \right].$$

From (3.14) we get

$$\begin{aligned} \Phi_\xi(\lambda) &= \frac{1}{\lambda} \int_{-1}^0 \varphi_{1;\xi}(\zeta) \left[ -\frac{1}{q} D_{q^{-1}} D_q \phi_{m;1}^{(1)}(\zeta, \lambda) + u(\zeta) \phi_{m;1}^{(1)}(\zeta, \lambda) \right] d_q \zeta \\ &\quad + \frac{1}{\gamma} \frac{1}{\lambda} \int_0^\xi \varphi_{2;\xi}(\zeta) \left[ -\frac{1}{q} D_{q^{-1}} D_q \phi_{m;1}^{(2)}(\zeta, \lambda) + u(\zeta) \phi_{m;1}^{(2)}(\zeta, \lambda) \right] d_q \zeta. \end{aligned}$$

Since  $\varphi_\xi(\zeta)$  is identically zero in a neighborhood of the point 1 and both  $\varphi_\xi(\zeta)$  and  $\phi(\zeta, \lambda)$  satisfy boundary conditions (3.1)–(3.3), taking by  $q$ -integration by parts, we obtain

$$\begin{aligned} \Phi_\xi(\lambda) &= \frac{1}{\lambda} \int_{-1}^0 \phi_{m;1}^{(1)}(\zeta, \lambda) \left[ -\frac{1}{q} D_{q^{-1}} D_q \varphi_{1;\xi}(\zeta) + u(\zeta) \varphi_{1;\xi}(\zeta) \right] d_q \zeta \\ &\quad + \frac{1}{\gamma} \frac{1}{\lambda} \int_0^1 \phi_{m;1}^{(2)}(\zeta, \lambda) \left[ -\frac{1}{q} D_{q^{-1}} D_q \varphi_{2;\xi}(\zeta) + u(\zeta) \varphi_{2;\xi}(\zeta) \right] d_q \zeta. \end{aligned}$$

For any finite  $M > 0$ , from (3.9) we have

$$\begin{aligned} &\int_{|\lambda|>M} \Phi_\xi^2(\lambda) d\sigma_1(\lambda) \\ &\leq \frac{1}{M^2} \int_{|\lambda|>M} \left\{ \int_{-1}^0 \phi_{m;1}^{(1)}(\zeta, \lambda) \left[ -\frac{1}{q} D_{q^{-1}} D_q \varphi_{1;\xi}(\zeta) + u(\zeta) \varphi_{1;\xi}(\zeta) \right] d_q \zeta \right. \\ &\quad \left. + \frac{1}{\gamma} \int_0^1 \phi_{m;1}^{(2)}(\zeta, \lambda) \left[ -\frac{1}{q} D_{q^{-1}} D_q \varphi_{2;\xi}(\zeta) + u(\zeta) \varphi_{2;\xi}(\zeta) \right] d_q \zeta \right\}^2 d\sigma_1(\lambda) \\ &\leq \frac{1}{M^2} \int_{-\infty}^{+\infty} \left\{ \int_{-1}^0 \phi_{m;1}^{(1)}(\zeta, \lambda) \left[ -\frac{1}{q} D_{q^{-1}} D_q \varphi_{1;\xi}(\zeta) + u(\zeta) \varphi_{1;\xi}(\zeta) \right] d_q \zeta \right. \\ &\quad \left. + \frac{1}{\gamma} \int_0^1 \phi_{m;1}^{(2)}(\zeta, \lambda) \left[ -\frac{1}{q} D_{q^{-1}} D_q \varphi_{2;\xi}(\zeta) + u(\zeta) \varphi_{2;\xi}(\zeta) \right] d_q \zeta \right\}^2 d\sigma_1(\lambda) \\ &= \frac{1}{M^2} \int_{-1}^0 \left[ -\frac{1}{q} D_{q^{-1}} D_q \varphi_{1;\xi}(\zeta) + u(\zeta) \varphi_{1;\xi}(\zeta) \right]^2 d_q \zeta \\ &\quad + \frac{1}{M^2} \frac{1}{\gamma} \int_0^\xi \left[ -\frac{1}{q} D_{q^{-1}} D_q \varphi_{2;\xi}(\zeta) + u(\zeta) \varphi_{2;\xi}(\zeta) \right]^2 d_q \zeta. \end{aligned}$$

From (3.14), we obtain that

$$\begin{aligned} &\left| \int_{-1}^0 (\varphi_{1;\xi}(\zeta))^2 d_q \zeta + \frac{1}{\gamma} \int_0^\xi (\varphi_{2;\xi}(\zeta))^2 d_q \zeta - \int_{-M}^M \Phi_\xi^2(\lambda) d\sigma_1(\lambda) \right| \\ &< \frac{1}{M^2} \int_{-1}^0 \left[ -\frac{1}{q} D_{q^{-1}} D_q \varphi_{1;\xi}(\zeta) + u(\zeta) \varphi_{1;\xi}(\zeta) \right]^2 d_q \zeta \\ &\quad + \frac{1}{M^2} \frac{1}{\gamma} \int_0^\xi \left[ -\frac{1}{q} D_{q^{-1}} D_q \varphi_{2;\xi}(\zeta) + u(\zeta) \varphi_{2;\xi}(\zeta) \right]^2 d_q \zeta. \end{aligned} \tag{3.16}$$

We know that the set  $\{\sigma_1(\lambda)\}$  is bounded from Lemma 3.1. A sequence  $\{\psi_k\}$  ( $\psi_k \rightarrow \infty$ ) such that the function  $\sigma_{1;\psi_k}(\lambda)$  converges to a monotone function  $\sigma(\lambda)$  can be found from

Helly’s first and second theorems. By taking limit with respect to  $\{\psi_k\}$  in (3.16), we get

$$\begin{aligned} & \left| \int_{-1}^0 (\varphi_{1;\xi}(\zeta))^2 d_q \zeta + \frac{1}{\gamma} \int_0^\xi (\varphi_{2;\xi}(\zeta))^2 d_q \zeta - \int_{-M}^M \Phi_\xi^2(\lambda) d\sigma(\lambda) \right| \\ & < \frac{1}{M^2} \int_{-1}^0 \left[ -\frac{1}{q} D_{q^{-1}} \varphi_{1;\xi}(\zeta) + u(\zeta) \varphi_{1;\xi}(\zeta) \right]^2 d_q \zeta \\ & \quad + \frac{1}{M^2} \frac{1}{\gamma} \int_0^\xi \left[ -\frac{1}{q} D_{q^{-1}} \varphi_{2;\xi}(\zeta) + u(\zeta) \varphi_{2;\xi}(\zeta) \right]^2 d_q \zeta. \end{aligned}$$

Therefore, letting  $M \rightarrow \infty$ , we get

$$\int_{-1}^0 (\varphi_{1;\xi}(\zeta))^2 d_q \zeta + \frac{1}{\gamma} \int_0^\xi (\varphi_{2;\xi}(\zeta))^2 d_q \zeta = \int_{-\infty}^{+\infty} \Phi_\xi^2(\lambda) d\sigma(\lambda).$$

Let  $\varphi$  be any real function on  $H$ . It is known that there is a sequence of functions  $\{\varphi_\xi(\zeta)\}$  satisfying conditions (3.1)–(3.5) and such that

$$\lim_{\xi \rightarrow \infty} \left[ \int_{-1}^0 (\varphi_1(\zeta) - \varphi_{1;\xi}(\zeta))^2 d_q \zeta + \frac{1}{\gamma} \int_0^\xi (\varphi_2(\zeta) - \varphi_{2;\xi}(\zeta))^2 d_q \zeta \right] = 0.$$

Let

$$\Phi_\xi(\lambda) = \int_{-1}^\infty \varphi_{1;\xi}(\zeta) \phi_{m;1}^{(1)}(\zeta, \lambda) d_q \zeta + \frac{1}{\gamma} \int_{-1}^\infty \varphi_{2;\xi}(\zeta) \phi_{m;1}^{(2)}(\zeta, \lambda) d_q \zeta.$$

Then from this we can get

$$\int_{-1}^0 (\varphi_{1;\xi}(\zeta))^2 d_q \zeta + \frac{1}{\gamma} \int_0^\infty (\varphi_{2;\xi}(\zeta))^2 d_q \zeta = \int_{-\infty}^{+\infty} \Phi_\xi^2(\lambda) d\sigma(\lambda).$$

Since

$$\int_{-1}^0 (\varphi_{1;\xi_1}(\zeta) - \varphi_{1;\xi_2}(\zeta))^2 d_q \zeta + \frac{1}{\gamma} \int_0^\infty (\varphi_{2;\xi_1}(\zeta) - \varphi_{2;\xi_2}(\zeta))^2 d_q \zeta \rightarrow 0$$

as  $\xi_1, \xi_2 \rightarrow \infty$ , we have

$$\int_{-\infty}^{+\infty} (\Phi_{\xi_1}(\lambda) - \Phi_{\xi_2}(\lambda))^2 d\sigma(\lambda) \rightarrow 0$$

as  $\xi_1, \xi_2 \rightarrow \infty$ . Accordingly there is a limit function  $\Phi$  such that

$$\int_{-1}^0 (\varphi_1(\zeta))^2 d_q \zeta + \frac{1}{\gamma} \int_0^\infty (\varphi_2(\zeta))^2 d_q \zeta = \int_{-\infty}^{+\infty} \Phi^2(\lambda) d\sigma(\lambda)$$

holds by the completeness of the space  $L^2_\sigma(\mathbb{R})$ .

Our next aim is to see that

$$K_\xi(\lambda) = \int_{-1}^0 \varphi_1(\zeta) \phi_{m;1}^{(1)}(\zeta, \lambda) d_q \zeta + \frac{1}{\gamma} \int_0^\xi \varphi_2(\zeta) \phi_{m;1}^{(2)}(\zeta, \lambda) d_q \zeta$$

converges to  $\Phi$  as  $\xi \rightarrow \infty$  in the space  $L^2_\sigma(\mathbb{R})$ . Let  $\psi$  be another real-valued function in  $H$ . By the same way, let  $\Psi(\lambda)$  be defined by  $\psi$ . It is clear that for

$$\psi(\zeta) := \begin{cases} \psi_1(\zeta), & \zeta \in [-1, 0) \cup [0, \xi], \\ \psi_2(\zeta), & \zeta \in (\xi, \infty), \end{cases}$$

$$\int_{-1}^0 (\varphi_1(\zeta) - \psi_1(\zeta))^2 d_q \zeta + \frac{1}{\gamma} \int_0^\infty (\varphi_2(\zeta) - \psi_2(\zeta))^2 d_q \zeta = \int_{-\infty}^{+\infty} [\Phi(\lambda) - \Psi(\lambda)]^2 d\sigma(\lambda).$$

Let us define

$$\psi(\zeta) = \begin{cases} \varphi(\zeta), & \zeta \in [-1, 0) \cup [0, \xi], \\ 0, & \zeta \in (\xi, \infty) \end{cases}$$

then

$$\int_{-\infty}^{+\infty} [\Phi(\lambda) - K_\xi(\lambda)]^2 d\sigma(\lambda) = \frac{1}{\gamma} \int_\xi^\infty (\varphi_2(\zeta))^2 d_q \zeta \rightarrow 0,$$

as  $\xi \rightarrow \infty$ , which proves that  $K_\xi$  converges to  $\Phi$  in  $L^2_\sigma(\mathbb{R})$  as  $\xi \rightarrow \infty$ . This gives us (i).

Now let us give the proof of (ii). Assume that the real functions  $\varphi, \psi \in H$  and  $\Phi(\lambda), \Psi(\lambda)$  are their Fourier transforms respectively (see [10]). Then  $\Phi \mp \Psi$  are transforms of  $\varphi \mp \psi$ . Consequently, by (3.13) we have

$$\int_{-1}^0 [\varphi_1(\zeta) + \psi_1(\zeta)]^2 d_q \zeta + \frac{1}{\gamma} \int_0^\infty [\varphi_2(\zeta) + \psi_2(\zeta)]^2 d_q \zeta = \int_{-\infty}^{+\infty} [\Phi(\lambda) + \Psi(\lambda)]^2 d\sigma(\lambda),$$

$$\int_{-1}^0 [\varphi_1(\zeta) - \psi_1(\zeta)]^2 d_q \zeta + \frac{1}{\gamma} \int_0^\infty [\varphi_2(\zeta) - \psi_2(\zeta)]^2 d_q \zeta = \int_{-\infty}^{+\infty} [\Phi(\lambda) - \Psi(\lambda)]^2 d\sigma(\lambda).$$

Let us subtract the resulting two equations side by side, we have

$$\int_{-1}^0 \varphi_1(\zeta)\psi_1(\zeta) d_q \zeta + \frac{1}{\gamma} \int_0^\infty \varphi_2(\zeta)\psi_2(\zeta) d_q \zeta = \int_{-\infty}^{+\infty} \Phi(\lambda)\Psi(\lambda) d\sigma(\lambda), \tag{3.17}$$

which may be called the generalized Parseval identity. Set

$$\varphi_\tau^{(j)}(\zeta) = \int_{-\tau}^{+\tau} \Phi(\lambda)\phi_{m;1}^{(j)}(\zeta, \lambda) d\sigma(\lambda), \quad j = 1, 2,$$

where  $\Phi$  is the function defined in (3.12). Let  $\psi \in H$  be a real function which equals zero outside the set  $[-1, 0) \cup (0, \mu]$ . Thus we obtain

$$\int_{-1}^0 \varphi_\tau^{(1)}(\zeta)\psi_1(\zeta) d_q \zeta + \frac{1}{\gamma} \int_0^\mu \varphi_\tau^{(2)}(\zeta)\psi_2(\zeta) d_q \zeta$$

$$= \int_{-1}^0 \left[ \int_{-\tau}^{\tau} \Phi(\lambda)\phi_{m;1}^{(1)}(\zeta, \lambda) d\sigma(\lambda) \right] \psi_1(\zeta) d_q \zeta$$

$$+ \frac{1}{\gamma} \int_0^\mu \left[ \int_{-\tau}^{\tau} \Phi(\lambda)\phi_{m;1}^{(2)}(\zeta, \lambda) d\sigma(\lambda) \right] \psi_2(\zeta) d_q \zeta \tag{3.18}$$

$$\begin{aligned}
 &= \int_{-\tau}^{\tau} \Phi(\lambda) \left[ \int_{-1}^0 \phi_{m;1}^{(1)}(\zeta, \lambda) \psi_1(\zeta) d_q \zeta + \frac{1}{\gamma} \int_0^{\mu} \phi_{m;1}^{(2)}(\zeta, \lambda) \psi_2(\zeta) d_q \zeta \right] d\sigma(\lambda) \\
 &= \int_{-\tau}^{\tau} \Phi(\lambda) \Psi(\lambda) d\sigma(\lambda).
 \end{aligned}$$

Subtracting (3.17) and (3.18), we have

$$\begin{aligned}
 &\int_{-1}^0 (\varphi_1(\zeta) - \varphi_{\tau}^{(1)}(\zeta)) \psi_1(\zeta) d_q \zeta + \frac{1}{\gamma} \int_0^{\infty} (\varphi_2(\zeta) - \varphi_{\tau}^{(2)}(\zeta)) \psi_2(\zeta) d_q \zeta \\
 &= \int_{|\lambda|>\tau} \Phi(\lambda) \Psi(\lambda) d\sigma(\lambda).
 \end{aligned}$$

Using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
 &\left( \int_{-1}^0 (\varphi_1(\zeta) - \varphi_{\tau}^{(1)}(\zeta)) \psi_1(\zeta) d_q \zeta + \frac{1}{\gamma} \int_0^{\infty} (\varphi_2(\zeta) - \varphi_{\tau}^{(2)}(\zeta)) \psi_2(\zeta) d_q \zeta \right)^2 \\
 &\leq \int_{|\lambda|>\tau} \Phi^2(\lambda) d\sigma(\lambda) \int_{|\lambda|>\tau} \Psi^2(\lambda) d\sigma(\lambda).
 \end{aligned}$$

If we carry out this inequality to the function

$$\psi(\zeta) = \begin{cases} \varphi_{\tau}(\zeta) - \varphi(\zeta), & \zeta \in [-1, 0) \cup (0, \mu], \\ 0, & \zeta \in (\mu, \infty), \end{cases}$$

we get

$$\begin{aligned}
 &\int_{-1}^0 (\varphi_1(\zeta) - \varphi_{\tau}^{(1)}(\zeta))^2 d_q \zeta + \frac{1}{\gamma} \int_0^{\infty} (\varphi_2(\zeta) - \varphi_{\tau}^{(2)}(\zeta))^2 d_q \zeta \\
 &\leq \int_{|\lambda|>\tau} \Phi^2(\lambda) d\sigma(\lambda).
 \end{aligned}$$

Since the above inequality is not dependent on  $\mu$ , the result is achieved by letting  $\tau \rightarrow \infty$ . □

#### 4 Conclusion

In this study, we investigate the existence of a spectral function for the singular  $q$ -Sturm–Liouville problem with transmission conditions on a closed interval. We prove the Parseval identity with the help of the inner product in class  $H_q = L_q^2(J_1) \oplus L_q^2(J_2)$  as Hilbert space. We also give the expansion formula in the eigenfunctions.

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**Authors' contributions**

The author carried out the research of this paper, and she read and approved the final version of the manuscript.

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