Yazgan and Tunç *Advances in Difference Equations* (2021) 2021:249
https://doi.org/10.1186/s13662-021-03403-y

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

The analysis of some special results of a Lasota–Wazewska model with mixed variable delays

Ramazan Yazgan[1*] and Osman Tunç[2]

*Correspondence: ryazgan503@gmail.com
[1]Department of Mathematics, Faculty of Sciences, Van Yuzuncu Yil University, 65080, Campus, Van, Turkey
Full list of author information is available at the end of the article

Abstract

This study is about getting some conditions that guarantee the existence and uniqueness of the weighted pseudo almost periodic (WPAP) solutions of a Lasota–Wazewska model with time-varying delays. Some adequate conditions have been obtained for the existence and uniqueness of the WPAP solutions of the Lasota–Wazewska model, which we dealt with using some differential inequalities, the WPAP theory, and the Banach fixed point theorem. Besides, an application is given to demonstrate the accuracy of the conditions of our main results.

MSC: 34K20; 34K14; 34C27; 92D25

Keywords: Lasota–Wazewska system; Banach fixed point theorem; Variable time delay; Weighted pseudo almost periodic

1 Introduction

In 1976, Wazewska and Lasota [1] presented the delayed logistic differential model

$$z'(t) = -\varrho(t)z(t) + \sum_{k=1}^{p} \kappa_k(t)e^{-\eta_k(t)z(t-\rho_k(t))} \tag{1.1}$$

to define the survival of red cells in an animal [2]. In (1.1) p is a positive integer, $z(t)$ stands for the number of red blood cells at time t, $\varrho(t)$ stands for the death rate of the red blood cell, $\kappa_k(t)$ and $\eta_k(t)$ are related to the production of red blood cells per unit time, and $\rho_k(t)$ represents the time to produce a red blood cell. For details, see [1, 3], also [4, 5] for logistic-type models from biological models as (1.1), but involving also diffusion and drift contributions.

Zhou [6] considered the following model:

$$z'(t) = -\delta(t)z(t) + \sum_{j=1}^{m} c_j(t)e^{-\omega_j(t)\int_{-\infty}^{0} C_j(s)z(t+s)\,ds}. \tag{1.2}$$

The author obtained some conditions on the almost periodic solution of this model using the fixed point theorem in cones. In [7], the researchers established some qualitative behaviors of PAP solutions of the following equation with constant delays:

$$z'(t) = -\alpha(t)z(t) + \sum_{j=1}^{m} A_j(t)e^{-\omega_j(t)\int_{-\infty}^{t} C_j(t-s)z(s)\,ds} + \sum_{i=1}^{n} B_i(t)e^{-z(t-\tau_i)\beta_i(t)}, \quad t \in R. \tag{1.3}$$

The study of almost periodic (AP) and pseudo almost periodic (PAP) differential equations is one of the most interesting issues for the study of almost periodic of many mathematicians: indeed, they are of great importance even in probability for investigating stochastic processes in stability problems tied to oscillatory phenomena [1, 3, 6, 8–24], and [25]. In [26], Diagana familiarized the concept of (WPAP) functions, which is a natural generalization of the concept of (PAP) functions. Since then, some interesting and remarkable results concerning composition theorem, translation invariance, and the ergodicity of (WPAP) have been obtained [26–29]. It is clear that under some limitations of weight function, many of the properties of almost periodic (AP) and pseudo almost periodic (PAP) are valid in this type of class. Thanks to the invariant property under translation, it is quite simple to investigate such solutions in delayed differential equations. For some works on the pseudo almost periodic solutions, oscillation of solutions, and so fourth of various differential equations, see [4, 5, 24, 25, 30–34].

Our main purpose is to obtain some sufficient conditions for the existence, uniqueness, and global exponential stability of (WPAP) solutions of the following Lasota–Wazewska model with mixed variable delays:

$$z'(t) = -\delta(t)z(t) + \sum_{j=1}^{m} A_j(t)e^{-\omega_j(t)\int_{-\infty}^{t} C_j(t-s)z(s)\,ds} + \sum_{i=1}^{n} B_i(t)e^{-\beta_i(t)z(t-\tau_i(t))}, \tag{1.4}$$

where $t \in \mathbb{R}$.

As far as we know, there are no studies related to the (WPAP) solutions of (1.4) with variable delays. Therefore, the results attained here are new and complementary to previous studies.

Throughout this paper, $\delta(t) \in \mathrm{AP}(\mathbb{R}, \mathbb{R}^+)$, $\tau_i(t), p_i(t) \in \mathrm{PAP}(\mathbb{R}, \mathbb{R}^+, \upsilon)$, $\tau = \max_{1\leq i\leq K}\{\sup_{t\in R} \tau_i(t)\}$, $(i = 1, 2, \dots K)$ and given $F \in BC(\mathbb{R}, \mathbb{R}^+)$, F^+ and F^- are defined as $F^+ = \sup_{t\in R} F(t)$ and $F^- = \inf_{t\in R} F(t)$. If $z(t)$ is defined on $[-\tau + t_0, \varsigma)$ with $t_0, \varsigma \in R$, then we define $z_t(\phi) \in D$, where $z_t(\phi) = z(t + \phi)$ for all $\phi \in [-\tau, 0]$ and $D = D([-\tau, 0], \mathbb{R})$ is the continuous function space supremum norm $\|\cdot\|$. For all $j = 1, 2, \dots m, C_j \in C(\mathbb{R}^+, \mathbb{R}^+)$ are integrable, $\int_0^\infty C_j(x)\,dx = 1$ and $\int_0^\infty C_j(x)e^{\xi x}\,dx < \infty$.

Let us consider the following initial condition:

$$z(s) = \varphi(s), \quad \varphi \in BC\big([-\tau, 0], \mathbb{R}^+\big) \text{ and } \varphi(0) > 0. \tag{1.5}$$

2 Preliminary results

Definition 2.1 ([8]) A function $f \in C(\mathbb{R}, \mathbb{R})$ is called almost periodic if for any $\varepsilon > 0$ there exists a trigonometric polynomial T_ε such that

$$\big|f(x) - T_\varepsilon(x)\big| < \varepsilon, \quad x \in R.$$

Definition 2.2 ([27]) A function $\eta \in C(\mathbb{R},\mathbb{R})$ is called (PAP) if it can be written as

$$\eta = \eta_1 + \eta_2,$$

with $\eta_1 \in \mathrm{AP}(\mathbb{R},\mathbb{R})$ and $\eta_2 \in \mathrm{PAP}_0(\mathbb{R},\mathbb{R})$, where space PAP_0 is defined by

$$\mathrm{PAP}_0(\mathbb{R}) := \left\{ \eta_2 \in BC(\mathbb{R},\mathbb{R}) \mid \lim_{r\to\infty} \frac{1}{2q}\int_{-q}^{q} \left\| \eta_2(t) \right\| dt = 0 \right\}.$$

Let Λ be the set of functions (weight) $\upsilon : \mathbb{R} \to (0,\infty)$ which are integrable on $(-\infty,\infty)$. If $\upsilon \in \Lambda$ and $Q := [-q,q]$ for $q > 0$, we then set

$$\upsilon(Q_q) := \int_{Q_q} \upsilon(x)\,dx.$$

The space of weights Λ_∞ is defined by

$$\Lambda_\infty := \left\{ \upsilon \in \Lambda : \inf_{x\in R} \upsilon(x) = \upsilon_0 > 0 \text{ and } \lim_{r\to\infty} \upsilon(Q_r) = \infty \right\}$$

and

$$\Lambda_\infty^+ := \left\{ \upsilon \in \Lambda_\infty : \lim_{|x|\to\infty} \sup \frac{\upsilon(\alpha x)}{\upsilon(x)} < +\infty, \lim_{|x|\to\infty} \sup \frac{\upsilon([-\alpha q,\alpha q])}{\upsilon([-q,q])} < +\infty \right\}.$$

Fix $\upsilon \in \Lambda_\infty^+$, $(\mathrm{PAP}(\mathbb{R},\upsilon), \|\cdot\|_\infty)$ is a Banach space.

Definition 2.3 ([27]) Fix $\upsilon \in \Lambda_\infty$. A continuous function is called WPAP if it can be written as

$$\eta = \eta_1 + \eta_2,$$

with $\eta_1 \in \mathrm{AP}(\mathbb{R},\mathbb{R})$ and $\eta_2 \in \mathrm{PAP}_0(\mathbb{R},\mathbb{R})$, where space PAP_0 is defined by

$$\mathrm{PAP}_0(\mathbb{R},\mathbb{R},\upsilon) = \left\{ \eta_2 \in BC(\mathbb{R},\mathbb{R}) : \lim_{r\to\infty} \frac{1}{\upsilon([-q,q])} \int_{-q}^{q} \left| \eta_2(t) \right| \upsilon(t)\,dt = 0 \right\}.$$

Lemma 2.1 ([27]) *Fix $\upsilon \in \Lambda_\infty^+$. For any $s \in (-\infty,\infty)$, assume that*

$$\lim_{t\to\infty} \sup_{t\in R} \upsilon(t+s)/\upsilon(t) < \infty,$$

the space $\mathrm{PAP}(\mathbb{R},\mathbb{R},\upsilon)$ *is translation invariant.*

Lemma 2.2 ([28]) *Let $\upsilon \in \Lambda_\infty^+$. If $\eta(t) \in \mathrm{PAP}(R,R,\upsilon)$, $\varpi(t) \in C^1(R,R)$ and $\varpi(t) \geq 0$, $\varpi'(t) \leq 1$, then $f(t-\varpi(t)) \in \mathrm{PAP}(X,\upsilon)$.*

3 Main results

Lemma 3.1 *Suppose that*

$$\sup_{T>0} \left\{ \int_{-T}^{T} e^{-\delta^-(T+t)} \upsilon(t)\,dt \right\} < \infty. \tag{3.1}$$

Define a nonlinear operator G for each $z \in \mathrm{PAP}(\mathbb{R}, \mathbb{R}, \upsilon)$

$$\begin{aligned}(Gz)(t) &= z_1(t) + z_2(t)\\ &= \int_{-\infty}^{t} e^{-\int_u^s \delta(\varsigma)\,d\varsigma} \left[\sum_{j=1}^{m} A_j(u) e^{-\omega_j(u)\int_{-\infty}^{u} C_j(u-s)z(s)\,ds} + \sum_{i=1}^{n} B_i(u) e^{-z(u-\tau_i(u))\beta_i(u)}\right] dt,\end{aligned}$$

where

$$\begin{aligned}z_1(t) &= \int_{-\infty}^{t} \sum_{j=1}^{m} A_j(u) e^{-\omega_j(u)\int_{-\infty}^{u} C_j(u-s)z(s)\,ds} e^{-\int_u^s \delta(\varsigma)\,d\varsigma}\,du,\\ z_2(t) &= \int_{-\infty}^{t} \left[\sum_{i=1}^{n} B_i(u) e^{-z(u-\tau_i(u))\beta_i(u)}\right] e^{-\int_u^s \delta(\varsigma)\,d\varsigma}\,du.\end{aligned}$$

Then $Gz \in \mathrm{PAP}(\mathbb{R}, \mathbb{R}, \upsilon)$.

Proof Because of $M[a] > 0$ in [8] and Lemma 3.1 in [7], we have that

$$z_2(t) \in \mathrm{PAP}(\mathbb{R}, \mathbb{R}, \upsilon). \tag{3.2}$$

Now we show that $z_1(t) \in \mathrm{PAP}(\mathbb{R}, \mathbb{R}, \upsilon)$.

According to Lemma 2.1, Lemma 2.2, we obtain that there are $z_{11}(t) \in \mathrm{AP}(\mathbb{R}, \mathbb{R})$ and $z_{12}(t) \in \mathrm{PAP}_0(\mathbb{R}, \mathbb{R}, \upsilon)$ such that

$$z_{11}(t) + z_{12}(t) = \sum_{j=1}^{m} A_j(u) e^{-\omega_j(u)\int_{-\infty}^{u} C_j(u-s)z(s)\,ds} \in \mathrm{PAP}(R, R, \upsilon).$$

Also noting that $M[a] > 0$, we have that

$$\int_{-\infty}^{t} e^{-\int_t^s \delta(\varsigma)\,d\varsigma} z_{11}(t)\,dt \in \mathrm{AP}(\mathbb{R}, \mathbb{R}) \tag{3.3}$$

is a solution of the following almost periodic differential equation:

$$w(t) = -\delta(t)w(t) + z_{11}(t).$$

Now, let us show that $\int_{-\infty}^{t} e^{-\int_t^s \delta(\varsigma)\,d\varsigma} z_{12}(t)\,dt$ belongs to $\mathrm{PAP}(\mathbb{R}, \mathbb{R}, \upsilon)$. By using a similar manner in the proof of Theorem 3.5 in [7], it can be displayed that $z_{12}(t) \in BC(\mathbb{R}, \mathbb{R})$. Also,

$$\begin{aligned}0 &\le \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \left(\left|\int_{-\infty}^{t} e^{-\int_s^t \delta(u)\,du} z_{12}(s)\,ds\right|\right) \upsilon(t)\,dt\\ &\le \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \left(\int_{-\infty}^{t} e^{-\int_s^t \delta(u)\,du} \left|z_{12}(s)\right|\,ds\right) \upsilon(t)\,dt\\ &\le \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \left(\int_{-\infty}^{t} e^{-\delta(t-s)} \left|z_{12}(s)\right|\,ds\right) \upsilon(t)\,dt\\ &\le L_1 + L_2,\end{aligned}$$

where

$$L_1 = \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \left(\int_{-r}^{t} e^{-\delta^-(t-s)} \left| z_{12}(s) \right| ds \right) \upsilon(t)\, dt,$$

$$L_2 = \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \left(\int_{-\infty}^{-r} e^{-\delta^-(t-s)} \left| z_{12}(s) \right| ds \right) \upsilon(t)\, dt.$$

Now, we shall prove that $L_1 = L_2 = 0$,

$$\begin{aligned} L_1 &= \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \left(\int_{-r}^{t} e^{-\delta^-(t-s)} \left| z_{12}(s) \right| ds \right) \upsilon(t)\, dt \\ &\leq \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \left(\int_{0}^{\infty} e^{-\delta^-\xi} \left| z_{12}(t-\xi) \right| d\xi \right) \upsilon(t)\, dt \\ &= \lim_{r\to\infty} \int_{0}^{+\infty} e^{-\delta^-\xi} \left(\frac{1}{2r} \int_{-r}^{r} \left| z_{12}(t-\xi) \right| \upsilon(t)\, dt \right) d\xi \\ &= \int_{0}^{+\infty} e^{-\delta^-\xi} \left(\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \left| z_{12}(t-\xi) \right| \upsilon(t)\, dt \right) d\xi. \end{aligned}$$

From Lemma 2.1 the function $z_{12}(t-\xi) \in \mathrm{PAP}_0(\mathbb{R},\mathbb{R},\upsilon)$, we obtain that

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \left| z_{12}(t-\xi) \right| \upsilon(t)\, dt.$$

Therefore,

$$L_1 = \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \left(\int_{-r}^{t} e^{-\delta^-(t-s)} \left| z_{12}(s) \right| ds \right) \upsilon(t)\, dt = 0. \tag{3.4}$$

Notice that $|z_{12}|_\infty = \sup_{t\in R} |z_{12}(t)| = M$ and by (3.1), then

$$\begin{aligned} L_2 &= \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \left(\int_{-\infty}^{-r} e^{-\delta^-(t-s)} \left| z_{12}(s) \right| ds \right) \upsilon(t)\, dt \\ &\leq \lim_{r\to\infty} \frac{1}{2r} \int_{-\infty}^{-r} e^{s\delta^-} \left| z_{12}(s) \right| ds \int_{-r}^{r} e^{-t\delta^-} \upsilon(t)\, dt \\ &= \frac{M}{\delta^-} \lim_{r\to\infty} \frac{1}{2r} \left[e^{s\delta^-} \right]_{-\infty}^{-r} \int_{-r}^{r} e^{-t\delta^-} \upsilon(t)\, dt \\ &= \frac{M}{\delta^-} \lim_{r\to\infty} \frac{1}{2r} \left[e^{-r\delta^-} - e^{-\infty\delta^-} \right] \int_{-r}^{r} e^{-t\delta^-} \upsilon(t)\, dt \\ &= \frac{M}{\delta^-} \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} e^{-\delta^-(t+r)} \upsilon(t)\, dt = 0, \end{aligned} \tag{3.5}$$

combining with (3.2), (3.3), (3.4), and (3.5), leads to $Gz \in \mathrm{PAP}(\mathbb{R},\mathbb{R},\upsilon)$. □

Theorem 3.1 *Let* $\max_{1\leq i\leq K}\{\inf_{t\in R} 1 - \tau_i'(t)\} > 0$,

$$\left(\delta^-\right)^{-1} \left(\sum_{j=1}^{m} (A_j \omega_j)^+ + \sum_{i=1}^{n} (\beta_i B_i)^+ \right) < 1, \tag{3.6}$$

and by Lemma 2.2, then (1.4) has a unique WPAP solution in the region

$$C^{*}=\{\varphi|\varphi\in\mathrm{PAP}(\mathbb{R},\mathbb{R}^{+},\upsilon),K_1\leq|\varphi(t)|\leq K_2\},$$

where $K_2=(\delta^-)^{-1}(\sum_{i=1}^{m}A_i^{+}+\sum_{i=1}^{n}B_i^{+})$ *and* $K_1=(\delta^-)^{-1}(\sum_{j=1}^{m}A_j^{-}e^{-\omega_j^{+}K_2}+\sum_{i=1}^{n}B_i^{-}e^{-\beta_j^{+}K_2})$.

Proof First, let us prove that $G\in\mathrm{PAP}(\mathbb{R},\mathbb{R}^{+},\upsilon)$ into itself. It is clear that

$$\begin{aligned}\left|(Gz)(t)\right|&=\left|\int_{-\infty}^{t}e^{-\int_t^s\delta(\varsigma)\,d\varsigma}\left[\sum_{j=1}^{m}A_j(s)e^{-\omega_j(s)\int_{-\infty}^{s}C_j(s-u)z(u)\,du}+\sum_{i=1}^{n}B_i(s)e^{-\beta_i(s)z(s-\tau_i(s))}\right]ds\right|\\&\leq\int_{-\infty}^{t}e^{-\int_t^s\delta(\varsigma)\,d\varsigma}\left(\sum_{j=1}^{m}A_j(s)+\sum_{i=1}^{n}B_i\right)ds\leq\left(\delta^-\right)^{-1}\left(\sum_{i=1}^{m}A_i^{+}+\sum_{i=1}^{n}B_i^{+}\right)=K_2\end{aligned}$$

and

$$\begin{aligned}\left|(Gz)(t)\right|&=\int_{-\infty}^{t}e^{-\int_t^s\delta(\varsigma)\,d\varsigma}\left[\sum_{j=1}^{m}A_j(t)e^{-\omega_j(t)\int_{-\infty}^{s}C_j(s-u)z(u)\,du}+\sum_{i=1}^{n}B_i(t)e^{-\beta_i(t)z(t-\tau_i(t))}\right]dt\\&\geq\int_{-\infty}^{t}e^{-\int_t^s\delta(\varsigma)\,d\varsigma}\left(\sum_{j=1}^{m}A_j^{-}e^{-\omega_j^{+}K_2}+\sum_{i=1}^{n}B_i^{-}e^{-\beta_j^{+}K_2}\right)ds\\&\geq\left(\delta^-\right)^{-1}\left(\sum_{j=1}^{m}A_j^{-}e^{-\omega_j^{+}K_2}+\sum_{i=1}^{n}B_i^{-}e^{-\beta_j^{+}K_2}\right)=K_1,\end{aligned}$$

which implies that $G\in C^{*}$.

Let $f_1,f_2\in C^{*}$. Then

$$\begin{aligned}&\left|(Gf_1)(t)-(Gf_2)(t)\right|\\&=\left|\int_{-\infty}^{t}e^{-\int_t^s\delta(\varsigma)\,d\varsigma}\left\{\sum_{j=1}^{m}A_j(s)\left[e^{-\omega_j(s)\int_{-\infty}^{s}C_j(t-s)f_1(s)\,ds}-e^{-\omega_j(s)\int_{-\infty}^{0}C_j(t-s)f_2(s)\,ds}\right]\right.\right.\\&\quad\left.\left.+\sum_{i=1}^{n}B_i(s)\left[e^{-\beta_i(s)f_1(s-\tau_i(s))}-e^{-\beta_i(s)f_2(s-\tau_i(s))}\right]\right\}ds\right|\\&\leq\sup_{t\in R}\int_{-\infty}^{t}e^{-\int_t^s\delta(\varsigma)\,d\varsigma}\left\{\sum_{j=1}^{m}\left|A_j(s)\right|\left|e^{-\omega_j(s)\int_{-\infty}^{s}C_j(s-u)f_1(u)\,ds}-e^{-\omega_j(t)\int_{-\infty}^{s}C_j(s-u)f_2(s)\,ds}\right|\right.\\&\quad\left.+\sum_{i=1}^{n}\left|B_i(s)\right|\left|e^{-\beta_i(s)f_1(s-\tau_i(s))}-e^{-\beta_i(s)f_2(s-\tau_i(s))}\right|\right\}ds.\end{aligned}$$

Obviously, for $x_1,x_1\in[0,+\infty]$,

$$\left|e^{-x_1}-e^{-x_2}\right|<|x_1-x_2|.$$

Therefore,

$$\left|(Gf_1)(t)-(Gf_2)(t)\right|$$

$$
\begin{aligned}
&\leq \sup_{t\in R}\int_{-\infty}^{t} e^{-\delta^{-}(t-s)}\{\sum_{j=1}^{m}\left|A_j(t)\right|\left|f_1(s)-f_2(s)\right|\left|\omega_i(t)\right|\int_{-\infty}^{t} C_j(t-s)\,ds \\
&= \sup_{t\in R}\int_{-\infty}^{t} e^{-\delta^{-}(t-s)}\left\{\sum_{j=1}^{m}\left|A_j(t)\right|\left|f_1(s)-f_2(s)\right|\left|\omega_i(t)\right| + \sum_{i=1}^{n}\left|B_i(t)\right||f_1-f_2|_\infty\left|\beta_i(t)\right|\right\} dt \\
&\leq \left(\delta^{-}\right)^{-1}\left(\sum_{j=1}^{m}(A_j\omega_j)^{+} + \sum_{i=1}^{n}(\beta_i B_i)^{+}\right)|f_1-f_2|_\infty.
\end{aligned}
$$

By AA we can see that $(1-(\delta^{-})^{-1}(\sum_{j=1}^{m}(A_j\omega_j)^{+}+\sum_{i=1}^{n}(\beta_i B_i)^{+}))\in(0,1)$, and hence G is a contraction mapping of C^*. Subsequently, G has a unique fixed point $z^*\in C^*$ that is $G(z^*)=z^*$. Thus, z^* is the unique WPAP solution of (1.4) in C^*. □

Theorem 3.2 *Let Theorem* 3.1 *hold, the WPAP solution of nonlinear* (1.4) *is globally exponentially stable.*

Proof Let

$$
\Pi(\varpi)=\sup_{t\in R}\left\{-\left[\delta(t)-\varpi\right]+e^{\varpi\tau}\left(\sum_{j=1}^{m}(A_j\omega_j)^{+}+\sum_{i=1}^{n}(\beta_i B_i)^{+}\right)\right\},\quad \theta\in[0,1].
$$

Then

$$
\Pi(0)=\sup_{t\in R}\left\{-\delta(t)+\left(\sum_{j=1}^{m}(A_j\omega_j)^{+}+e^{\lambda\tau}\sum_{i=1}^{n}(\beta_i B_i)^{+}\right)\right\}<0. \tag{3.7}
$$

Since $\Pi(\theta)$ is continuous, a constant $\lambda\in(0,\delta^{-}]$ can be picked out as

$$
\Pi(\lambda)=\sup_{t\in R}\left\{-\left[\delta(t)-\lambda\right]+\left(\sum_{j=1}^{m}(A_j\omega_j)^{+}+e^{\lambda\tau}\sum_{i=1}^{n}(\beta_i B_i)^{+}\right)\right\}<0. \tag{3.8}
$$

Assume $z(t)$ as an arbitrary solution of (1.4) with (1.5) and $z^*(t)$ as a WPAP solution of Theorem 3.1. Let us accept $\rho(t)=z(t)-z^*(t)$, so we obtain

$$
\begin{aligned}
\rho'(t) &= -\delta(t)\rho(t)+\sum_{j=1}^{m}A_j(t)\left[e^{-\omega_j(t)\int_{-\infty}^{t}C_j(t-s)z(s)\,ds}-e^{-\omega_j(t)\int_{-\infty}^{t}C_j(t-s)z^*(s)\,ds}\right] \\
&\quad+\sum_{i=1}^{n}B_i(t)\left[e^{-\beta_i(t)z(t-\tau_i(t))}-e^{-\beta_i(t)z^*(t-\tau_i(t))}\right].
\end{aligned} \tag{3.9}
$$

Let

$$
\left\|\rho(t)\right\|=\sup_{t\in R}\left|\varphi(t)-z^*(t)\right|.
$$

For any $\varepsilon>0$, it is trivial to show that

$$
\left\|\rho(t)\right\|<M\left(\left\|\varphi-z^*\right\|+\varepsilon\right)e^{-\lambda t}\quad \text{for all } t\in(-\infty,0],
$$

where $M > 1$ is a constant number. We show that

$$\|\rho(t)\| < M(\|\varphi - z^*\| + \varepsilon)e^{-\lambda t} \quad \text{for all } t > 0. \tag{3.10}$$

Contrarily, there must exist $\theta > 0$

$$\begin{cases} \|\rho(\theta)\| = M(\|\varphi - z^*\| + \varepsilon)e^{-\lambda\theta}, \\ \|\rho(t)\| < M(\|\varphi - z^*\| + \varepsilon)e^{-\lambda t} \quad \forall t \in (-\infty, \theta). \end{cases} \tag{3.11}$$

Given (3.9) and integrating it on $[0, \theta]$, we have

$$\begin{aligned} \rho(\theta) = {} & \rho(0)e^{-\int_0^t \delta(\varsigma)\,d\varsigma} \\ & + \int_0^t e^{-\int_s^t \delta(\varsigma)\,d\varsigma} \Biggl\{ \sum_{j=1}^m A_j(s)\bigl[e^{-\omega_j(t)\int_{-\infty}^s C_j(s-u)z(u)\,du} - e^{-\omega_j(t)\int_{-\infty}^s C_j(s-u)z^*(u)\,du}\bigr] \\ & + \sum_{i=1}^n B_i(t)\bigl[e^{-\beta_i(s)z(s-\tau_i(s))} - e^{-\beta_i(s)z^*(s-\tau_i(s))}\bigr] \Biggr\}\,ds. \end{aligned}$$

Hence

$$\begin{aligned} & \bigl|\rho(\theta)\bigr| \\ & = \Biggl| \rho(0)e^{-\int_0^\theta \delta(\varsigma)\,d\varsigma} \\ & \quad + \int_0^\theta e^{-\int_s^\theta \delta(\varsigma)\,d\varsigma} \Biggl\{ \sum_{j=1}^m A_j(s)\bigl[e^{-\omega_j(s)\int_{-\infty}^s C_j(s-u)z(u)\,du} - e^{-\omega_j(s)\int_{-\infty}^s C_j(s-u)z^*(u)\,du}\bigr] \\ & \quad + \sum_{i=1}^n B_i(s)\bigl[e^{-\beta_i(s)z(s-\tau_i(s))} - e^{-\beta_i(s)z^*(s-\tau_i(s))}\bigr] \Biggr\}\,ds \Biggr| \\ & \le M(\|\varphi - z^*\| + \varepsilon)e^{-\int_0^\theta \delta(\varsigma)\,d\varsigma} \\ & \quad + \int_0^\theta e^{-\int_s^\theta \delta(\varsigma)\,d\varsigma} \Biggl\{ \sum_{j=1}^m \bigl|A_j(s)\bigr|\bigl|\omega_j(t)\bigr| \int_{-\infty}^s C_j(s-u)\bigl|z(u) - z^*(u)\bigr|\,du \\ & \quad + \sum_{i=1}^n \bigl|B_i(s)\bigr|\bigl|\beta_i(s)\bigr|\bigl|z(s-\tau_i(s)) - z^*(s-\tau_i(s))\bigr| \Biggr\}\,ds \\ & \le M(\|\varphi - z^*\| + \varepsilon)e^{-\int_0^\theta \delta(\varsigma)\,d\varsigma} \\ & \quad + \int_0^\theta e^{-\int_s^\theta \delta(\varsigma)\,d\varsigma} \sum_{j=1}^m (A_j\omega_j)^+ \biggl| \int_{-\infty}^s C_j(s-u)e^{-\lambda u}\,du \biggr| M(\|\varphi - z^*\| + \varepsilon) \\ & \quad + \sum_{i=1}^n (\beta_i B_i)^+ M(\|\varphi - z^*\| + \varepsilon)e^{-\lambda(s-\tau_i(s))}\}\,ds \\ & = M(\|\varphi - z^*\| + \varepsilon)e^{-\int_0^\theta \delta(\varsigma)\,d\varsigma} \end{aligned}$$

$$
\begin{aligned}
&+\int_0^\theta e^{-\int_s^\theta \delta(\varsigma)\,d\varsigma}\sum_{j=1}^m (A_j\omega_j)^+\left|\int_0^\infty C_j(v)e^{\lambda v}\,dv\right|e^{-\lambda s}M\big(\|\varphi-z^*\|+\varepsilon\big)\\
&+\sum_{i=1}^n(\beta_iB_i)^+M\big(\|\varphi-z^*\|+\varepsilon\big)e^{-\lambda(s-\tau_i(s))}\}\,ds\\
&=M\big(\|\varphi-z^*\|+\varepsilon\big)e^{-\lambda\theta}e^{-\int_0^\theta(\delta(\varsigma)-\lambda)\,d\varsigma}+\int_0^\theta e^{-\int_s^\theta(\delta(\varsigma)-\lambda)\,d\varsigma}\Big(\sum_{j=1}^m(A_j\omega_j)^+\\
&+\sum_{i=1}^n(\beta_iB_i)^+e^{\tau\lambda}M\big(\|\varphi-z^*\|+\varepsilon\big)e^{-\lambda\theta}\}\,ds\\
&=(1+\int_0^\theta e^{-\int_s^\theta(\delta(\varsigma)-\lambda)\,d\varsigma}\left(\sum_{j=1}^m(A_j\omega_j)^++\sum_{i=1}^n(\beta_iB_i)^+e^{\tau\lambda}\right)dsM\big(\|\varphi-z^*\|+\varepsilon\big)e^{-\lambda\theta}\\
&\le\left(1+\int_0^\theta e^{-\int_s^\theta(\delta(\varsigma)-\lambda)\,d\varsigma}\big(\delta(s)-\lambda\big)\,ds\right)M\big(\|\varphi-z^*\|+\varepsilon\big)e^{-\lambda\theta}\\
&\le M\big(\|\varphi-z^*\|+\varepsilon\big)e^{-\lambda\theta},
\end{aligned}
$$

which contradicts (3.11). Hence, (3.10) holds. Letting $\varepsilon\to 0$, we have that

$$\|\rho(t)\|<M\|\varphi-z^*\|e^{-\lambda t}\quad \forall t>0,$$

which proves Theorem 3.1. □

Remark 3.1 Lately, Rihami [4] got some conditions for the PAP solutions of (1.3) with constant delays. WPAP functions are a generalization of the concept of PAP functions; therefore, it is noticeable that results in [4] are special cases of our results.

Example 3.1 Consider the system

$$
\begin{aligned}
z'(t)&=-\big(8+\sin^2\sqrt{2}t+\sin^2 t\big)z(t)+\big(1+0.25\sin^2\sqrt{2}t+0.25\sin^2\pi t+e^{-t}\big)\\
&\quad\times e^{-(0.25\cos^2\sqrt{2}t+0.25\cos^2\pi t+e^{-t})\int_{-\infty}^t e^{s-t}z(s)\,ds}\\
&\quad+\big(1+0.25\sin^2\sqrt{2}t+0.25\sin^2\pi t+0.5e^{-5t}\big)\\
&\quad\times e^{-(0.25\cos^2\sqrt{2}t+0.25\cos^2\pi t+e^{-t})z(t-\sin^2 t)},
\end{aligned}
\tag{3.12}
$$

where

$$
\begin{aligned}
&\delta(t)=8+\sin^2\sqrt{2}t+\sin^2 t,\qquad A_1(t)=1+0.25\cos^2\sqrt{2}t+0.25\cos^2\pi t+e^{-t},\\
&\omega_1(t)=0.25\cos^2\sqrt{2}t+0.25\cos^2\pi t+e^{-t},\\
&B_1(t)=1+0.25\cos^2\sqrt{2}t+0.25\cos^2\pi t+0.5e^{-5t},\\
&\beta_1(t)=0.25\cos^2\sqrt{2}t+0.25\cos^2\pi t+e^{-t},\qquad C_j(t)=e^{-t},\qquad \tau_1(t)=\sin^2 t.
\end{aligned}
$$

Figure 1 shows weighted pseudo almost periodic solutions of eq. (3.12).

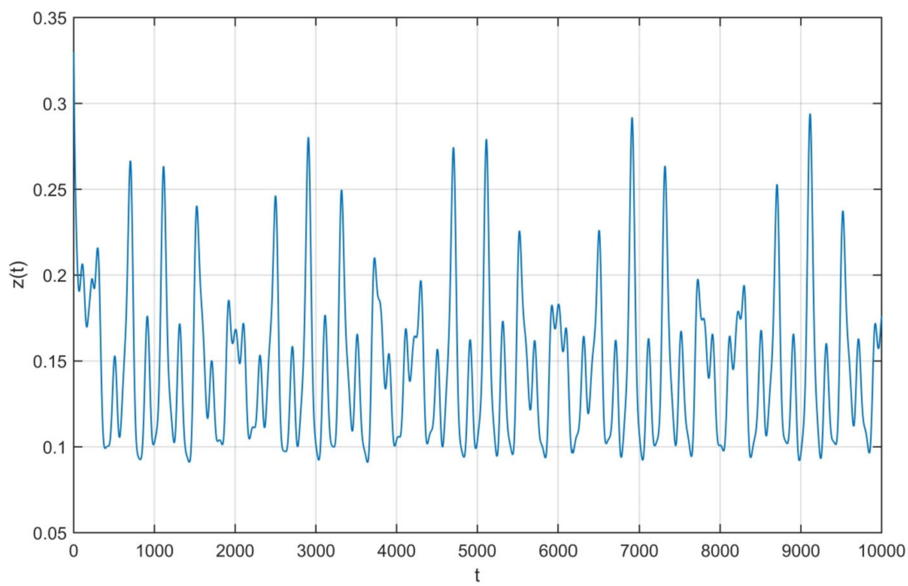

Figure 1 The trajectory $z(t)$ of (3.12) for $\varphi(s) = 0.33, s \in [-1, 0]$

And for $\upsilon(t) = e^t$ and $\delta^- = 8, \delta^+ = 10, A_1^+ = 2.5, A_1^- = 1, B_1^+ = 2.5, B_1^- = 1, \omega_1^+ = 1.5, \beta_1^+ = 1.25$, $\tau = 1$, we have

$$\left(\delta^-\right)^{-1}\left((A_1\omega_1)^+ + (\beta_1 B_1)^+\right) = \frac{6.875}{8} = 0.859375 < 1.$$

$$K_2 = \frac{2.5 + 2.5}{9} = \frac{5}{9} \approx 05555, \qquad K_1 = \frac{e^{-1.5\times 0.859375}}{10} = 0.027553,$$

$$\sup_{T>0}\left\{\int_{-T}^{T} e^{-\delta^-(T+t)}\upsilon(t)\,dt\right\} = \sup_{T>0}\left\{\int_{-T}^{T} e^{-8(T+t)}e^t\,dt\right\} = \sup_{T>0}\left\{\int_{-T}^{T} e^{-8T-7t}\,dt\right\}$$
$$= -\frac{1}{7}\sup_{T>0}\left\{\left[e^{-15T} - e^{-T}\right]\right\} < \infty.$$

All conditions of Theorems 3.1 and 3.2 are satisfied, then (3.12) has a unique WPAP solution. Therefore, this solution is globally exponentially stable with a convergence rate $\lambda = 0.02$ in the region

$$C^* = \left\{\varphi | \varphi BC(\mathbb{R}, \mathbb{R}), 0.027553 \le \left|\varphi(t)\right| \le 0.5, \text{ for all } t \in R\right\}.$$

Remark 3.2 According to the results of [4], the globally exponentially stable positive WPAP solution of (3.12) is invalid because

$$A_1(t) = 1 + 0.25\sin^2\sqrt{2}t + 0.25\sin^2\pi t + e^{-t},$$
$$B_1(t) = 1 + 0.25\sin^2\sqrt{2}t + 0.25\sin^2\pi t + 0.5e^{-5t}$$

are WPAP functions, not almost, and pseudo almost periodic. Consequently, this article is more comprehensive compared to previous studies.

Acknowledgements
Not applicable.

Funding
Not applicable.

Availability of data and materials
Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Faculty of Sciences, Van Yuzuncu Yil University, 65080, Campus, Van, Turkey. [2]Department of Computer Programing, Baskale Vocational School, Van Yuzuncu Yil University, 65080, Campus, Van, Turkey.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 5 March 2021 Accepted: 29 April 2021 Published online: 11 May 2021

References
1. Ważewska-Czyżewska, M., Lasota, A.: Mathematical problems of the dynamics of a system of red blood cells. Mat. Stosow. **3**, 23–40 (1976)
2. Hale, J.K.: Theory of Functions Differential Equations. Springer, New York (1977)
3. Kulenovic, M.R.S., Ladas, G., Sficas, Y.G.: Global attractivity in population dynamics. Comput. Math. Appl. **18**, 925–928 (1989)
4. Li, T., Viglialoro, G.: Boundedness for a nonlocal reaction chemotaxis model even in the attraction dominated regime. Differ. Integral Equ. **34**, 315–336 (2021)
5. Viglialoro, G.: Very weak global solutions to a parabolic-parabolic chemotaxis-system with logistic source. J. Math. Anal. Appl. **1**, 197–212 (2016)
6. Hui, Z., Zongfu, Z., Qi, W.: Positive almost periodic solution for a class of Lasota–Wazewska model with infinite delays. Appl. Math. Comput. **8**, 4501–4506 (2011)
7. Rihani, S., Kessab, A., Chérif, F.: Pseudo almost periodic solutions for a Lasota–Wazewska model. Electron. J. Differ. Equ. **62**, 17 (2016)
8. Amerio, L., Prouse, G.: Almost-Periodic Functions and Functional Equations. von Nostrand Reinhold Co., New York (1971)
9. Chérif, F.: A various types of almost periodic functions on Banach spaces: part I. Int. Math. Forum **6**, 921–952 (2011)
10. Chérif, F.: A various types of almost periodic functions on Banach spaces: part II. Int. Math. Forum **6**, 953–985 (2011)
11. Yazgan, R., Tunç, C.: On the almost periodic solutions of fuzzy cellular neural networks of high order with multiple time lags. Int. J. Math. Comput. Sci. **1**, 183–198 (2020)
12. Liu, G., Zhao, A., Yan, J.: Existence and global attractivity of unique positive periodic solution for a Lasota–Wazewska model. Nonlinear Anal. **64**, 1737–1746 (2006)
13. Saker, S.H.: Qualitative analysis of discrete nonlinear delay survival red blood cells model. Nonlinear Anal., Real World Appl. **9**, 471–489 (2008)
14. Huang, Z., Gong, S., Wang, L.: Positive almost periodic solution for a class of Lasota–Wazewska model with multiple time-varying delays. Comput. Math. Appl. **61**, 755–760 (2011)
15. N'Guerekata, G.M.: Almost Automorphic and Almost Periodic Functions in Abstract Spaces. Kluwer Academic, New York (2001)
16. Wang, L., Yu, M., Niu, P.: Periodic solution and almost periodic solution of impulsive Lasota–Wazewska model with multiple time-varying delays. Comput. Math. Appl. **8**, 2383–2394 (2012)
17. Yan, J.: Existence and global attractivity of positive periodic solution for an impulsive Lasota–Wazewska model. J. Math. Anal. Appl. **279**, 111–120 (2003)
18. Zhang, C.: Pseudo almost periodic solutions of some differential equations I. J. Math. Anal. Appl. **181**, 62–76 (1994)
19. Ait Dads, E., Ezzinbi, K.: Existence of positive pseudo-almost-periodic solution for some nonlinear infinite delay integral equations arising in epidemic problems. Nonlinear Anal. **41**, 1–13 (2000)
20. Amir, B., Maniar, L.: Composition of pseudo almost periodic functions and Cauchy problems with operators of non dense domain. Ann. Math. Blaise Pascal **6**, 1–11 (1999)
21. Tunç, C., Liu, B.: Global stability of pseudo almost periodic solutions for a Nicholson's blowflies model with a harvesting term. Vietnam J. Math. **44**(3), 485–494 (2016)
22. Cherif, F.: Existence and global exponential stability of pseudo almost periodic solution for SICNNs with mixed delays. J. Appl. Math. Comput. **39**, 235–251 (2012)
23. Cieutat, P., Fatajou, S., N'Guerekata, G.M.: Composition of pseudo almost periodic and pseudo almost automorphic functions and applications to evolution equations. Appl. Anal. **89**, 11–27 (2010)
24. Infusino, M., Kuna, T.: The full moment problem on subsets of probabilities and point configurations. J. Math. Anal. Appl. **1**, 123551 (2020)
25. Chiu, K.S., Li, T.: Oscillatory and periodic solutions of differential equations with piecewise constant generalized mixed arguments. Math. Nachr. **10**, 2153–2164 (2019)

26. Diagana, T.: Weighted pseudo almost periodic functions and applications. C. R. Acad. Sci. Paris, Ser. I **343**, 643–646 (2006)
27. Yazgan, R., Tunç, C.: On the weighted pseudo almost periodic solutions of Nicholson's blowflies equation. Appl. Appl. Math. **14**(2), 875–889 (2019)
28. Ding, H., N'Guerekata, H.M., Nieto, J.J.: Weighted pseudo almost periodic solutions for a class of discrete hematopoiesis model. Rev. Math. Comput. **26**, 427–443 (2016)
29. Ding, H.S., Liang, J., Hu, X.Y.: Weighted pseudo almost periodic functions and applications to evolution equations with delay. Appl. Math. Comput. **219**, 8949–8958 (2013)
30. Graef, J.R., Grace, S.R., Tunç, E.: Oscillation of even-order nonlinear differential equations with sublinear and superlinear neutral terms. Publ. Math. (Debr.) **96**, 195–206 (2020)
31. Graef, J.R., Özdemir, O., Kaymaz, A., Tunç, E.: Oscillation of damped second-order linear mixed neutral differential equations. Monatshefte Math. **194**, 85–104 (2021)
32. Liu, B., Tunç, C.: Pseudo almost periodic solutions for CNNs with leakage delays and complex deviating arguments. Neural Comput. Appl. **26**(2), 429–435 (2015)
33. Liu, B., Tunç, C.: Pseudo almost periodic solutions for a class of nonlinear Duffing system with a deviating argument. J. Appl. Math. Comput. **49**, 233–242 (2015)
34. Yazgan, R.: On the weighted pseudo almost periodic solutions for Liénard-type systems with variable delays. Mugla J. Sci. Technol. **6**, 89–93 (2020)