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An iterative scheme for split equality equilibrium problems and split equality hierarchical fixed point problem

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Abstract

This paper deals with a split equality equilibrium problem for pseudomonotone bifunctions and a split equality hierarchical fixed point problem for nonexpansive and quasinonexpansive mappings. We suggest and analyze an iterative scheme where the stepsizes do not depend on the operator norms, the so-called simultaneous projected subgradient-proximal iterative scheme for approximating a common solution of the split equality equilibrium problem and the split equality hierarchical fixed point problem. Further, we prove a weak convergence theorem for the sequences generated by this scheme. Furthermore, we discuss some consequences of the weak convergence theorem. We present a numerical example to justify the main result.

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Keywords: Split equality equilibrium problem; Split equality hierarchical fixed point problem; Simultaneous hybrid projected subgradient-proximal iterative scheme; Quasinonexpansive mapping; Pseudomonotone bifunction

1 Introduction

Let H_1, H_2, and H_3 be real Hilbert spaces with their inner products and induced norms $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$. Let C_1 and C_2 be nonempty closed convex subsets of H_1 and H_2, respectively. Recall that a mapping $U_1 : H_1 \to H_1$ is nonexpansive if $\|U_1x_1 - U_1y_1\| \leq \|x_1 - y_1\|$ for all $x_1, y_1 \in H_1$. Note that if $\mathrm{Fix}(U_1) := \{x_1 \in H_1 : U_1x_1 = x_1\} \neq \emptyset$, then $\mathrm{Fix}(U_1)$ is closed and convex.

We consider the following split equality equilibrium problem (SEEP): Find $x_1 \in C_1$ and $x_2 \in C_2$ such that

$$g_1(x_1,y_1) \geq 0, \quad y_1 \in C_1, \tag{1.1}$$

$$g_2(x_2,y_2) \geq 0, \quad y_2 \in C_2, \tag{1.2}$$

and

$$A_1x_1 = A_2x_2,$$

where $g_1 : C_1 \times C_1 \to \mathbb{R}$ and $g_2 : C_2 \times C_2 \to \mathbb{R}$ are monotone bifunctions, and $A_1 : H_1 \to H_3$ and $A_2 : H_2 \to H_3$ are bounded linear operators. When looked separately, (1.1) is called the equilibrium problem (EP). EP (1.1) was introduced and studied by Blum and Otteli [3]. We denote the solution set of EP (1.1) by Sol(EP(1.1)). The solution set of SEEP (1.1)–(1.2) is denoted by $\Omega = \{(x_1, x_2) \in C_1 \times C_1 : x_1 \in \mathrm{Sol}(\mathrm{EP}(1.1)), x_2 \in \mathrm{Sol}(\mathrm{EP}(1.2)),\ \text{and}\ A_1x_1 = A_2x_2\}$. If $H_3 = H_2$ and $A_2 = I$ (the identity operator), then SEEP (1.1)–(1.2) is reduced to the split equilibrium problem (SEP), which was initially introduced by Moudafi [26] and studied by Kazmi and Rizvi [19] for monotone bifunctions. Recently, Hieu [14] studied the strong convergence of some projected subgradient-proximal iterative schemes for solving SEP for a pseudomonotone bifunction. For further related work, see [12, 15]. As particular cases, SEP includes the split variational inequalities [7] and split feasibility problem [6], which have a wide range of applications; see [4, 5, 7, 10, 11, 21, 31, 32].

SEEP (1.1)–(1.2) has been studied by many authors; see, for instance, Ma et al. [23, 24] and Ali et al. [2] for monotone bifunctions g_1, g_2. It is interesting to study SEEP (1.1)–(1.2) when both bifunctions g_1, g_2 are pseudomonotone.

Further, we consider the split equality hierarchical fixed point problem (SEHFPP) [8]: Find $x_1 \in \mathrm{Fix}(V_1)$ and $x_2 \in \mathrm{Fix}(V_2)$ such that

$$\langle x_1 - U_1x_1, x_1 - y_1\rangle \le 0, \quad y_1 \in \mathrm{Fix}(V_1), \tag{1.3}$$

$$\langle x_2 - U_2x_2, x_2 - y_2\rangle \le 0, \quad y_2 \in \mathrm{Fix}(V_2), \tag{1.4}$$

and

$$A_1x_1 = A_2x_2,$$

where $U_1, V_1 : C_1 \to C_1$ and $U_2, V_2 : C_2 \to C_2$ are nonexpansive mappings. When we look separately, (1.3) is called a hierarchical fixed point problem (HFPP), introduced and studied by Moudafi and Mainge [29]. Since then, HFPP has been studied by many authors; see, for example, [9, 16–18, 20, 25, 29, 30, 33, 35]. The solution set of HFPP (1.3) is denoted by Sol(HFPP(1.3)). The solution set of SEHFPP (1.3)–(1.4) is denoted by $\Gamma := \{(x_1, x_2) \in \mathrm{Fix}(V_1) \times \mathrm{Fix}(V_2) : x_1 \in \mathrm{Sol}(\mathrm{HFPP}(1.3)), x_2 \in \mathrm{Sol}(\mathrm{HFPP}(1.4)),\ \text{and}\ A_1x_1 = A_2x_2\}$. If $H_3 = H_2$ and $A_2 = I$, then SEHFPP (1.3)–(1.4) reduces to a new class of problems called the split hierarchical fixed point problem. In particular, if we set $U_1 = I_1$ and $U_2 = I_2$ (the identity mappings), then SEHFPP (1.3)–(1.4) reduces to the split equality fixed point problem (SEFPP) [27]: Find $x_1 \in C_1$ and $x_2 \in C_2$ such that

$$x_1 \in \mathrm{Fix}(V_1), \qquad x_2 \in \mathrm{Fix}(V_2), \quad \text{and} \quad A_1x_1 = A_2x_2. \tag{1.5}$$

The solution set of SEFPP (1.5) is denoted by Γ_1.

SEHFPP (1.3)–(1.4) was introduced and studied by Behzad et al. [8] for nonexpansive mappings U_1, U_2, V_1, V_2. SEHFPP (1.3)–(1.4) covers the split equality variational inequality problem over the fixed point sets, and so on; see [8]. Very recently, Alansari et al. [1] suggested an iterative scheme for solving a split equilibrium problem for a monotone bifunction, a pseudomonotone bifunction, and a hierarchical fixed point problem for nonexpansive and quasinonexpansive mappings.

In 2013, Moudafi and Al-Shemas [28] proved a weak convergence theorem for a simultaneous iterative algorithm to solve SEFPP (1.5). However, to employ this algorithm, we need to know a priori the norms (or at least estimates of the norms) of the bounded linear operators A_1 and A_2, which is in general not an easy work in practice. To overcome this difficulty, López et al. [22] presented a helpful iterative method for estimating the stepsizes, which do not need a priori knowledge of the operator norms for solving the split feasibility problems. In 2015, Zhao [36] extended the iterative method [22] for SEFPP (1.5). Very recently, Behzad et al. [8] have extended the iterative method [36] for SEHFPP (1.3)–(1.4).

Inspired by the works mentioned, in this paper, we consider SEEP (1.1)-(1.2) where the both bifunctions g_1 and g_2 are pseudomonotone, and SEHFPP (1.3)–(1.4) where the U_1, U_2 are quasinonexpansive mappings and V_1, V_2 are nonexpansive mappings in real Hilbert spaces. We propose an iterative scheme where the stepsizes do not depend on the operator norms for approximating a common solution of these problems. We further prove a weak convergence theorem for the proposed iterative scheme. We present a numerical example to justify the main result.

2 Preliminaries

Let the symbols $\rightarrow$ and $\rightharpoonup$ denote strong and weak convergence, respectively.

Definition 2.1 A mapping $U_1 : C_1 \rightarrow C_1$ is said to be:

(i) *quasinonexpansive* if, for any $p_1 \in \mathrm{Fix}(U_1)$,

$$\|U_1 x_1 - p_1\| \leq \|x_1 - p_1\|, \quad x_1 \in C_1;$$

(ii) *monotone* if

$$\langle U_1 x_1 - U_1 y_1, x_1 - y_1 \rangle \geq 0, \quad x_1, y_1 \in C_1;$$

Lemma 2.1 ([13]) *Let $V_1 : C_1 \rightarrow C_1$ be a nonexpansive mapping on C_1. Then V_1 is demiclosed on C_1 in the sense that if $\{x_1^k\}$ converges weakly to $x_1 \in C_1$ and $\{x_1^k - V_1 x_1^k\}$ converges strongly to* 0, *then $x_1 \in \mathrm{Fix}(V_1)$.*

Definition 2.2 A bifunction $g_1 : C_1 \times C_1 \rightarrow \mathbb{R}$ is said to be:

(i) *strongly monotone* on C_1 if there exists a constant $\gamma_1 > 0$ such that $g_1(x_1, y_1) + g_1(y_1, x_1) \leq -\gamma \|x_1 - y_1\|^2$, $x_1, y_1 \in C_1$;

(ii) *monotone* on C_1 if $g_1(x_1, y_1) + g_1(y_1, x_1) \leq 0$, $x_1, y_1 \in C_1$;

(iii) *pseudomonotone* on C_1 if $g_1(x_1, y_1) \geq 0 \Rightarrow g_1(y_1, x_1) \leq 0$, $x_1, y_1 \in C_1$.

Note that it is evident from the definition that a strongly monotone bifunction is monotone and a monotone bifunction is pseudomonotone.

Definition 2.3 ([12]) Let $g_1 : C_1 \times C_1 \rightarrow \mathbb{R}$ be a bifunction, where $g_1(x_1, \cdot)$ is a convex function for each $x_1 \in C_1$. Then, for $\epsilon \geq 0$, the ϵ-subdifferential (ϵ-diagonal subdifferential) of g_1 at x_1, denoted by $\partial_\epsilon g_1(x_1, \cdot)(x_1)$ or $\partial_\epsilon g_1(x_1, x_1)$, is given by

$$\partial_\epsilon g_1(x_1, \cdot)(x_1) = \left\{ w_1 \in H_1 : g_1(x_1, y_1) - g_1(x_1, x_1) + \epsilon \geq \langle w_1, y_1 - x_1 \rangle, y_1 \in C_1 \right\}.$$

Assumption 2.1 For each $i = 1,2$, the bifunction $g_i : C_i \times C_i \longrightarrow \mathbb{R}$ satisfies the following assumptions:

(i) $g_i(x_i, x_i) = 0$, $x_i \in C_i$;

(ii) g_1 and g_2 are pseudomonotone, respectively, on C_1 with respect to $x_1 \in \mathrm{Sol}(\mathrm{EP}(1.1))$ and on C_2 with respect to $x_2 \in \mathrm{Sol}(\mathrm{EP}(1.2))$;

(iii) g_i satisfies the following condition, called the strict paramonotonicity property:

$$x_1 \in \mathrm{Sol}(\mathrm{EP}(1.1)), y_1 \in C_1, g_1(y_1, x_1) = 0 \quad \Rightarrow \quad y_1 \in \mathrm{Sol}(\mathrm{EP}(1.1));$$

$$x_2 \in \mathrm{Sol}(\mathrm{EP}(1.2)), y_2 \in C_1, g_2(y_2, x_2) = 0 \quad \Rightarrow \quad y_2 \in \mathrm{Sol}(\mathrm{EP}(1.2));$$

(iv) g_i is jointly weakly upper semicontinuous on $C_i \times C_i$ in the sense that if $x_i, y_i \in C_i$ and $\{x_i^k\}$, $\{y_i^k\} \subseteq C_i$ converge weakly to x_i and y_i, respectively, then $g_i(x_i^k, y_i^k) \to g_i(x_i, y_i)$ as $k \to \infty$;

(v) $g_i(x_i, \cdot)$ is convex, lower semicontinuous, and subdifferentiable on C_i for all $x_i \in C_i$;

(vi) If $\{x_i^k\}$ is bounded sequence in C_i and $\epsilon_k \to 0$, then the sequence $\{w_i^k\}$ with $w_i^k \in \partial_{\epsilon_k} g_i(x_i^k, \cdot)(x_i^k)$ is bounded.

Lemma 2.2 ([34]) *Let* $\{\delta_k\}$ *and* $\{\gamma_k\}$ *be nonnegative sequences satisfying*

$$\sum_{k=0}^{\infty} \delta_k < +\infty \quad \text{and} \quad \gamma_{k+1} \le \gamma_k + \delta_k, \quad k = 0, 1, 2, \ldots.$$

Then $\{\gamma_k\}$ *is a convergent sequence.*

3 Simultaneous projected subgradient-proximal iterative scheme

We suggest the following simultaneous projected subgradient-proximal iterative scheme for solving SEEP (1.1)–(1.2) and SEHFPP (1.3)–(1.4).

Scheme 3.1 (Initialization) For each $i = 1,2$, choose $x_i^0 \in C_i$. Take the sequences of real numbers $\{\rho_k\}$, $\{\beta_k\}$, $\{\epsilon_k\}$, $\{r_k\}$, $\{\mu_k\}$, $\{\delta_k\}$, and $\{\sigma_k\}$ such that

(i) $\rho_k \ge \rho > 0$, $\beta_k \ge 0$, $\epsilon_k > 0$, $\epsilon_k \to 0$ as $k \to \infty$, $r_k > r > 0$, $0 < a < \delta_k < b < 1$, and $0 < a' < \sigma_k < b' < 1$.

(ii) $\sum_{k=0}^{\infty} \frac{\beta_k}{\rho_k} = +\infty$, $\sum_{k=0}^{\infty} \frac{\beta_k \epsilon_k}{\rho_k} < +\infty$, and $\sum_{k=0}^{\infty} \beta_k^2 < +\infty$.

Step I. Choose $w_i^k \in H_i$ such that $w_i^k \in \partial_{\epsilon_k} g_i(x_i^k, \cdot)(x_i^k)$ and compute $\alpha_k = \frac{\beta_k}{\eta_k}$ and $\eta_k = \max\{\rho_k, \|w_i^k\|\}$.

Step II. Compute $y_i^k = P_{C_i}(x_i^k - \alpha_k w_i^k)$.

Step III. Compute $t_i^k = (1 - \delta_k)x_i^k + \delta_k V_i((1 - \sigma_k)U_i y_i^k + \sigma_k y_i^k)$.

Step IV. $x_i^{k+1} = P_{C_i}(t_i^k + \mu_k A_i^*(A_1 t_1^k - A_2 t_2^k))$ for all $k \ge 0$, where the step size μ_k is chosen in such a way that for some $\epsilon > 0$,

$$\mu_k \in (\epsilon, \gamma_k - \epsilon), \quad k \in \Lambda; \tag{3.1}$$

otherwise, $\mu_k = \mu$ ($\mu \ge 0$), where $\gamma_k := \frac{2\|A_1 t_1^k - A_2 t_2^k\|^2}{\|A_1^*(A_1 t_1^k - A_2 t_2^k)\|^2 + \|A_2^*(A_1 t_1^k - A_2 t_2^k)\|^2}$, and the index set $\Lambda := \{k : A_1 t_1^k - A_2 t_2^k \ne 0\}$.

Remark 3.1 ([36]) Condition (3.1) implies that $\inf_{k\in\Lambda}\{\gamma_k - \mu_k\} > 0$. Since $\|A_1^*(A_1t_1^k - A_2t_2^k)\| \le \|A_1^*\|\|A_1t_1^k - A_2t_2^k\|$ and $\|A_2^*(A_1t_1^k - A_2t_2^k)\| \le \|A_2^*\|\|A_1t_1^k - A_2t_2^k\|$, we observe that $\{\gamma_k\}$ is bounded below by $\frac{2}{\|A_1\|^2+\|A_2\|^2}$, and so $\inf_{k\in\Lambda}\gamma_k > 0$. Consequently, with an appropriate choice of $\epsilon > 0$ and $\gamma_n \in (\epsilon, \inf_{n\in\Lambda}\mu_n - \epsilon)$ for $k \in \Lambda$, we have $\sup_{k\in\Lambda}\mu_k < +\infty$, and hence $\{\mu_k\}$ is bounded.

Remark 3.2 ([12]) For each $i = 1, 2$, since $g_i(x_i, \cdot)$ is a lower semicontinuous convex function and $C_i \subset \operatorname{dom} g_i(x_i, \cdot)$ for every $x_i \in C_i$, the ϵ_k-diagonal subdifferential $\partial_{\epsilon_k} g_i(x_i^k, \cdot)(x_i^k) \neq \emptyset$ for every $\epsilon_k > 0$. Moreover, $\rho_k \ge \rho > 0$. Therefore each step of the scheme is well defined, implying that Scheme 3.1 is well defined.

Remark 3.3 ([12]) For each $i = 1, 2$, if g_i satisfies Assumption 2.1 ((i), (ii) and (iv)) then Sol(EP(1.1)), Sol(EP(1.2)) are closed and convex. For each $i = 1, 2$, since A_i is a linear operator, the solution set Ω of SEEP (1.1)–(1.2) is closed and convex.

4 Weak convergence theorem

We now prove the following weak convergent theorem, which shows that the sequence $\{(x_1^k, x_2^k)\}$ generated by Scheme 3.1 converges weakly to $(q_1, q_2) \in \Phi = \Omega \cap \Gamma$, a common solution of SEEP (1.1)–(1.2) and SEHFPP (1.3)–(1.4).

Assume that $\Phi \neq \emptyset$.

Theorem 4.1 *Let H_1, H_2, and H_3 be real Hilbert spaces. For each $i = 1, 2$, let $C_i \subseteq H_i$ be a nonempty closed convex set; let $A_i : H_i \to H_3$ be a bounded linear operator with its adjoint operator A_i^*; let $V_i : C_i \to C_i$ be a nonexpansive mapping, let $U_i : C_i \to C_i$ be a continuous quasinonexpansive mapping such that $I_i - U_i$ (I_i is the identity mapping on C_i) is monotone, and let $g_i : C_i \times C_i \to \mathbb{R}$ be bifunctions satisfying Assumption 2.1. Assume that* $\mathrm{Fix}(U_1) \cap \mathrm{Fix}(V_1) \neq \emptyset$, $\mathrm{Fix}(U_2) \cap \mathrm{Fix}(V_2)) \neq \emptyset$, *and* $\Theta = \Omega \cap (\mathrm{Fix}(U_1) \cap \mathrm{Fix}(V_1), \mathrm{Fix}(U_2) \cap \mathrm{Fix}(V_2) \neq \emptyset$. *Then the iterative sequence $\{(x_1^k, x_2^k)\}$ generated by Scheme 3.1 converges weakly to $(q_1, q_2) \in \Phi$.*

Proof Let $(p_1, p_2) \in \Theta$. Then $(p_1, p_2) \in \Omega$, $p_1 \in \mathrm{Fix}(U_1) \cap \mathrm{Fix}(V_1)$, and $p_2 \in \mathrm{Fix}(U_2) \cap \mathrm{Fix}(V_2)$. For each $i = 1, 2$, setting

$$z_i^k = (1 - \sigma_k)Sy_i^k + \sigma_k y_i^k \tag{4.1}$$

and using the arguments used in the proof of [1, Theorem 3.1], we obtain that

$$\begin{aligned} \|z_i^k - p_i\|^2 &\le \|y_i^k - p_i\|^2 - \sigma_k(1 - \sigma_k)\|U_i y_i^k - y_i^k\|^2 && (4.2)\\ &\le \|y_i^k - p_i\|^2, && (4.3)\\ \|t_i^k - p_i\|^2 &\le (1 - \delta_k)\|x_i^k - p_i\|^2 + \delta_k\|z_i^k - p_i\|^2 - \delta_k(1 - \delta_k)\|V_i z_i^k - x_i^k\|^2, && (4.4)\\ \lim_{k\to\infty}\|x_i^k - y_i^k\| &= 0, && (4.5) \end{aligned}$$

and

$$\|t_i^k - p_i\|^2 \le \|x_i^k - p_i\|^2 + 2\delta_k\alpha_k\langle w_i^k, p_i - x_i^k\rangle + 2\delta_k\beta_k^2 - \delta_k(1 - \delta_k)\|V_i z_i^k - x_i^k\|^2. \tag{4.6}$$

Since $x_i^k \in C_i$ and $w_i^k \in \partial_{\epsilon_k} g_i(x_i^k, \cdot)(x_i^k)$, we have

$$g_i(x_i^k, p_i) + \epsilon_k = g_i(x_i^k, p_i) - g_i(x_i^k, x_i^k) + \epsilon_k \geq \langle w_i^k, p_i - x_i^k \rangle, \tag{4.7}$$

and hence from (4.6) and (4.7) we have

$$\begin{aligned} \|t_i^k - p_i\|^2 \leq{}& \|x_i^k - p_i\|^2 + 2\delta_k\alpha_k\big(g_i(x_i^k, p_i) + \epsilon_k\big) + 2\delta_k\beta_k^2 \\ & - \delta_k(1-\delta_k)\|V_i z_i^k - x_i^k\|^2. \end{aligned} \tag{4.8}$$

Now from the definitions of α_k and η_k we obtain $\alpha_k = \frac{\beta_k}{\eta_k} \leq \frac{\beta_k}{\rho_k}$. Hence from (4.8) we have

$$\begin{aligned} \|t_i^k - p_i\|^2 \leq{}& \|x_i^k - p_i\|^2 + 2\delta_k\alpha_k g_i(x_i^k, p_i) + \frac{2\delta_k\beta_k\epsilon_k}{\rho_k} + 2\delta_k\beta_k^2 \\ & - \delta_k(1-\delta_k)\|V_i z_i^k - x_i^k\|^2. \end{aligned} \tag{4.9}$$

Again, since $p_i \in C_i$, we have

$$\begin{aligned} &\|x_1^{k+1} - p_1\|^2 \\ &\quad = \|P_{C_1}\big(t_1^k + \mu_k A_1^*(A_1 t_1^k - A_2 t_2^k)\big) - (p_1)\|^2 \\ &\quad \leq \|t_1^k - p_1\|^2 - 2\mu_k\langle A_1 t_1^k - A_1 p_1, A_1 t_1^k - A_2 t_2^k\rangle + \mu_k^2\|A_1^*(A_1 t_1^k - A_2 t_2^k)\|^2 \\ &\quad = \|t_1^k - p_1\|^2 - \mu_k\big[\|A_1 t_1^k - A_1 p_1\|^2 + \|A_1 t_1^k - A_2 t_2^k\|^2 - \|A_2 t_2^k - A_1 p_1\|^2\big] \\ &\qquad + \mu_k^2\|A_1^*(A_1 t_1^k - A_2 t_2^k)\|^2. \end{aligned} \tag{4.10}$$

Similarly, we have

$$\begin{aligned} &\|x_2^{k+1} - p_2\|^2 \\ &\quad \leq \|t_2^k - p_2\|^2 - \mu_k\big[\|A_2 t_2^k - A_2 p_2\|^2 + \|A_1 t_1^k - A_2 t_2^k\|^2 - \|A_1 t_1^k - A_2 p_2\|^2\big] \\ &\qquad + \mu_k^2\|A_2^*(A_1 t_1^k - A_2 t_2^k)\|^2. \end{aligned} \tag{4.11}$$

From (4.10), (4.11), and the fact that $A_1 p_1 = A_2 p_2$ we have

$$\begin{aligned} &\|x_1^{k+1} - p_1\|^2 + \|x_2^{k+1} - p_2\|^2 \\ &\quad \leq \|t_1^k - p_1\|^2 + \|t_2^k - p_2\|^2 - \mu_k\big[2\|A_2 t_2^k - A_2 p_2\|^2 \\ &\qquad - \mu_k\big(\|A_1^*(A_1 t_1^k - A_2 t_2^k)\|^2 + \|A_2^*(A_1 t_1^k - A_2 t_2^k)\|^2\big)\big]. \end{aligned} \tag{4.12}$$

From (4.9) and (4.12) we have

$$\begin{aligned} &\|x_1^{k+1} - p_1\|^2 + \|x_2^{k+1} - p_2\|^2 \\ &\quad \leq \|x_1^k - p_1\|^2 + \|x_2^k - p_2\|^2 + 2\delta_k\alpha_k\big(g_1(x_1^k, p_1) + g_2(x_2^k, p_2)\big) \\ &\qquad - \mu_k\big[2\|A_2 t_2^k - A_2 p_2\|^2 - \mu_k\big(\|A_1^*(A_1 t_1^k - A_2 t_2^k)\|^2 + \|A_2^*(A_1 t_1^k - A_2 t_2^k)\|^2\big)\big] \\ &\qquad - \delta_k(1-\delta_k)\big(\|V_1 z_1^k - x_1^k\|^2 + \|V_2 z_2^k - x_2^k\|^2\big) + \zeta_k, \end{aligned} \tag{4.13}$$

where $\zeta_k = 2\delta_k(\frac{\beta_k\epsilon_k}{\rho_k} + \beta_k^2)$.

Since $(p_1, p_2) \in \Omega$ and $x_i^k \in C_i$ for $i = 1,2$, $p_i \in C_i$, and hence $g_i(p_i, x_i^k) \geq 0$. By the pseudomonotonicity of g_i we have

$$g_i(x_i^k, p_i) \leq 0. \tag{4.14}$$

Hence, using condition (3.1) and $\delta_k \in (0,1)$ in (4.13), we have

$$\|x_1^{k+1} - p_1\|^2 + \|x_2^{k+1} - p_2\|^2 \leq \|x_1^k - p_1\|^2 + \|x_2^k - p_2\|^2 + \zeta_k. \tag{4.15}$$

It follows from the conditions on β_k, ϵ_k, and ρ_k that $\sum_{k=0}^{\infty} \zeta_k < +\infty$. Hence it follows from Lemma 2.2 and (4.15) that the sequence $\{\|x_1^k - p_1\|^2 + \|x_2^k - p_2\|^2\}$ is convergent, that is,

$$\lim_{k\to\infty} \left(\|x_1^k - p_1\|^2 + \|x_2^k - p_2\|^2\right) \quad \text{exists}, \tag{4.16}$$

which implies that the sequences $\{x_1^k\}$ and $\{x_2^k\}$ are bounded. Therefore it follows from (4.5) and (4.3) that, for each $i = 1,2$, the sequences $\{y_i^k\}$, $\{z_i^k\}$ are bounded.

Since $\delta_k \in (0,1)$, $\sum_{k=0}^{\infty} \zeta_k < +\infty$, and $\{\mu_k\}$ is bounded, from (4.13), (4.14), and (4.16) it follows that

$$\lim_{k\to\infty} \|A_1^*(A_1 t_1^k - A_2 t_2^k)\| = \lim_{k\to\infty} \|A_2^*(A_1 t_1^k - A_2 t_2^k)\| = 0. \tag{4.17}$$

Similarly, from (4.13) we obtain that

$$\lim_{k\to\infty} \|V_1 z_1^k - x_1^k\| = \lim_{k\to\infty} \|V_2 z_2^k - x_2^k\| = 0. \tag{4.18}$$

Now from $\sum_{k=0}^{\infty} \zeta_k < +\infty$, (4.13), (4.14), and (4.16)–(4.18) it follows that

$$\lim_{k\to\infty} \|A_1 t_1^k - A_2 t_2^k\| = 0. \tag{4.19}$$

Again, since $\delta_k \in (0,1)$, from conditions (3.1), (4.13), and (4.17)–(4.19) it follows that

$$\begin{aligned} &2\delta_k \alpha_k \left(g_1(x_1^k, p_1) + g_2(x_2^k, p_2)\right) \\ &\quad \leq \|x_1^k - p_1\|^2 - \|x_1^{k+1} - p_1\|^2 + \|x_2^k - p_2\|^2 - \|x_2^{k+1} - p_2\|^2 + \zeta_k. \end{aligned} \tag{4.20}$$

Hence, for every m, from (4.14) and (4.20) it follows that

$$\begin{aligned} 0 &\leq \sum_{k=0}^{m} 2\delta_k \alpha_k \left(g_1(x_1^k, p_1) + g_2(x_2^k, p_2)\right) \\ &\leq \|x_1^0 - p_1\|^2 - \|x_1^{m+1} - p_1\|^2 + \|x_2^0 - p_2\|^2 - \|x_2^{m+1} - p_2\|^2 + 4\sum_{k=0}^{m} \frac{\beta_k \epsilon_k}{\rho_k} + 4\sum_{k=0}^{m} \beta_k^2. \end{aligned}$$

By taking the limit as $m \to \infty$ we have

$$0 \leq 2\sum_{k=0}^{\infty} \delta_k \alpha_k \left(g_1(x_1^k, p_1) + g_2(x_2^k, p_2)\right) < +\infty,$$

which implies

$$\sum_{k=0}^{\infty} \delta_k \alpha_k g_i(x_i^k, p_i) < +\infty \tag{4.21}$$

for $i = 1, 2$. For $i = 1, 2$, the boundedness of the sequence $\{x_i^k\}$ and Assumption 2.1(vi) imply that the sequence $\{w_i^k\}$ is bounded. Further, using the conditions on the parameters, we have $\alpha_k = \frac{\beta_k}{\rho_k \max\{1, \frac{\|w^k\|}{\rho_k}\|\}} \geq \frac{\beta_k \rho}{\rho_k w}$. Since $\delta_k \in (a, b) \subset (0, 1)$, from (4.21) it follows that

$$0 \leq \frac{2\rho a}{w} \sum_{k=0}^{\infty} \frac{\beta_k}{\rho_k}\left(-g_i(x_i^k, p_i)\right) \leq 2a \sum_{k=0}^{\infty} \alpha_k\left(-g_i(x_i^k, p_i)\right) < +\infty. \tag{4.22}$$

Since $\sum_{k=0}^{\infty} \frac{\beta_k}{\rho_k} = +\infty$, from (4.14) and (4.22) it follows that

$$\limsup_{k\to\infty} g_1(x_1^k, p_1) = \limsup_{k\to\infty} g_2(x_2^k, p_2) = 0. \tag{4.23}$$

Further, from the equation in Step III of Scheme 3.1 and (4.18) it follows that

$$\lim_{k\to\infty} \|t_1^k - x_1^k\| = \lim_{k\to\infty} \|t_2^k - x_2^k\| = 0. \tag{4.24}$$

Since

$$\|y_i^k - p_i\|^2 \leq \|x_i^k - p_i\|^2 + 2\langle y_i^k - x_i^k, y_i^k - p_i\rangle \quad (i = 1, 2) \tag{4.25}$$

and $\{y_1^k\}$, $\{y_1^k\}$ are bounded, from (4.2), (4.4), and (4.12) it follows that

$$\begin{aligned}
&\delta_k \sigma_k (1 - \sigma_k)\left(\|U_1 y_1^k - y_1^k\|^2 + \|U_2 y_2^k - y_2^k\|^2\right) \\
&\quad \leq \|x_1^k - p_1\|^2 - \|x_1^{k+1} - p_1\|^2 + \|x_2^k - p_2\|^2 - \|x_2^{k+1} - p_2\|^2 \\
&\qquad + 2\delta_k\left[\|y_1^k - x_1^k\|\|y_1^k - p_1\| + \|y_2^k - x_2^k\|\|y_2^k - p_2\|\right] \\
&\qquad - \delta_k(1 - \delta_k)\left[\|V_1 z_1^k - x_1^k\|^2 + \|V_1 z_2^k - x_2^k\|^2\right] \\
&\qquad - \mu_k\left[2\|A_2 t_2^k - A_2 p_2\|^2 - \mu_k\left(\|A_1^*(A_1 t_1^k - A_2 t_2^k)\|^2\right.\right. \\
&\qquad \left.\left. + \|A_2^*(A_1 t_1^k - A_2 t_2^k)\|^2\right)\right].
\end{aligned} \tag{4.26}$$

Again, since $\delta_k \in (a, b) \subset (0, 1)$ and $\sigma_k \in (a', b') \subset (0, 1)$, from (4.5) and (4.16)–(4.19) it follows that

$$\lim_{k\to\infty} \|U_1 y_1^k - y_1^k\|^2 = \lim_{k\to\infty} \|U_2 y_2^k - y_2^k\|^2 = 0. \tag{4.27}$$

For each $i = 1, 2$, from the inequality

$$\begin{aligned}
\|V_i z_i^k - y_i^k\|^2 &\leq \|V_i z_i^k - x_i^k\|^2 + 2\langle x_i^k - y_i^k, V_i z_i^k - y_i^k\rangle \\
&\leq \|V_i z_i^k - x_i^k\|^2 + 2\|x_i^k - y_i^k\|\|V_i z_i^k - y_i^k\|,
\end{aligned} \tag{4.28}$$

the boundedness of the sequences $\{y_i^k\}$ and $\{z_i^k\}$, (4.5), and(4.18) it follows that

$$\lim_{k\to\infty} \| V_i z_i^k - y_i^k \|^2 = 0. \tag{4.29}$$

Since

$$\| U_i y_i^k - V_i z_i^k \| \leq \| U_i y_i^k - y_i^k \| + \| y_i^k - V_i z_i^k \|, \tag{4.30}$$

from (4.27), (4.29), and (4.30) it follows that

$$\lim_{k\to\infty} \| U_i y_i^k - V_i z_i^k \| = 0. \tag{4.31}$$

The equality

$$\| z_i^k - y_i^k \| = (1-\sigma_k) \| U_i y_i^k - y_i^k \|$$

implies that

$$\lim_{k\to\infty} \| z_i^k - y_i^k \| = 0. \tag{4.32}$$

The inequality

$$\| V_i z_i^k - z_i^k \| \leq \| V_i z_i^k - y_i^k \| + \| y_i^k - z_i^k \| \tag{4.33}$$

implies that

$$\lim_{k\to\infty} \| V_i z_i^k - z_i^k \| = 0. \tag{4.34}$$

Now, since the sequence $\{x_i^k\}$ is bounded in C_i for $i = 1, 2,,$ without the loss of generality, we can assume that there exists a subsequence $\{x_i^{k_l}\}$ of $\{x_i^k\}$ such that $x_i^{k_l} \rightharpoonup q_i \in C_i$ as $l \to \infty$ and $\limsup_{k\to\infty} g_i(x_i^k, p_i) = \lim_{l\to\infty} g_i(x_i^{k_l}, p_i)$. From (4.5), (4.24), and (4.32) it follows that the sequences $\{x_i^k\}$, $\{y_i^k\}$, $\{t_i^k\}$, and $\{z_i^k\}$ have the same asymptotic behavior, and hence there are subsequences $\{y_i^{k_l}\}$ of $\{y_i^k\}$, $\{t_i^{k_l}\}$ of $\{t_i^k\}$, and $\{z_i^{k_l}\}$ of $\{z_i^k\}$ such that $y_i^{k_l} \rightharpoonup q_i$, $t_i^{k_l} \rightharpoonup q_i$, and $z_i^{k_l} \rightharpoonup q_i$ as $l \to \infty$. Since A_i is continuous for $i = 1, 2$, $A_i t_i^{k_l} \rightharpoonup A_i q_i$. Further, for $i = 1, 2$, it follows from the demiclosedness of $I_i - V_i$ on C_i and (4.34) that $q_i \in \mathrm{Fix}(V_i)$. We now show that $(q_1, q_2) \in \Gamma$. From (4.1) it follows that

$$\frac{z_i^k - V_i z_i^k}{\sigma_k} = (I_i - U_i) y_i^k + \frac{1}{\sigma_k} \big(U_i y_i^k - V_i z_i^k \big). \tag{4.35}$$

Therefore, for all $z_i \in \mathrm{Fix}(V_i)$, using (4.1) and the monotonicity of $(I_i - U_i)$, we estimate

$$\begin{aligned}
&\left\langle \frac{z_i^k - V_i z_i^k}{\sigma_k}, y_i^k - z_i \right\rangle \\
&\quad = \big\langle (I_i - U_i) y_i^k - (I_i - U_i) z_i, y_i^k - z_i \big\rangle + \big\langle z_i - U_i z_i, y_i^k - z_i \big\rangle
\end{aligned}$$

$$+\frac{1}{\sigma_k}\langle U_i y_i^k - V_i z_i^k, y_i^k - z_i\rangle$$
$$\geq \langle z_i - U_i z_i, y_i^k - z_i\rangle + \frac{1}{\sigma_k}\langle U_i y_i^k - V_i z_i^k, y_i^k - z_i\rangle. \tag{4.36}$$

Since $\{y_i^k\}$ is bounded and $\sigma_k \in (a', b') \subset (0,1)$, from (4.31), (4.34), and (4.36) it follows that

$$\limsup_{k\to\infty}\langle z_i - U_i z_i, y_i^k - z_i\rangle \leq 0, \quad z_i \in \operatorname{Fix}(V_i). \tag{4.37}$$

Replacing k with k_l in (4.37) and then taking the limit as $l \to \infty$, we have

$$\langle (I_i - U_i)z_i, q_i - z_i\rangle \leq 0, \quad z_i \in \operatorname{Fix}(V_i). \tag{4.38}$$

Since $\operatorname{Fix}(V_i)$ is convex, $\lambda z_i + (1-\lambda)q_i \in \operatorname{Fix}(V_i)$ for $\lambda \in (0,1)$, and hence

$$\langle (I_i - U_i)\big(\lambda z_i + (1-\lambda)q_i\big), q_i - z_i\rangle \leq 0, \quad z_i \in \operatorname{Fix}(V_i). \tag{4.39}$$

Since $(I_i - U_i)$ is continuous, by taking the limit as $\lambda \to 0_+$, we have

$$\langle (I_i - U_i)q_i, q_i - z_i\rangle \leq 0, \quad z_i \in \operatorname{Fix}(V_i), \tag{4.40}$$

that is, $q_1 \in \operatorname{Sol}(\mathrm{HFPP}(1.3))$ and $q_1 \in \operatorname{Sol}(\mathrm{HFPP}(1.3))$. Further, since $\|\cdot\|^2$ is weakly lower semicontinuous, from (4.19) it follows that

$$\|A_1 q_1 - A_2 q_2\|^2 \leq \liminf_{k\to\infty}\|A_1 t_1^{k_l} - A_2 t_2^{k_l}\|^2 = 0, \tag{4.41}$$

that is, $A_1 q_1 = A_2 q_2$. Hence $(q_1, q_2) \in \Gamma$. Next, we show that $(q_1, q_2) \in \Omega$. Since $x_i^{k_l} \rightharpoonup q_i$ and $\limsup_{k\to\infty} g_i(x_i^k, p_i) = \lim_{l\to\infty} g_i(x_i^{k_l}, p_i)$, by the weak upper semicontinuity of $g_i(\cdot, p_i)$ and (4.23) we have

$$g_i(q_i, p_i) \geq \limsup_{l\to\infty} g_i\big(x_i^{k_l}, p_i\big) = \lim_{l\to\infty} g_i\big(x_i^{k_l}, p_i\big) = \limsup_{k\to\infty} g_i\big(x_i^k, p_i\big) = 0. \tag{4.42}$$

Since $(p_1, p_2) \in \Omega$ and $q_i \in C_i$, we have $g_i(p_i, q_i) \geq 0$, and hence from Assumption 2.1(ii) it follows that $g_i(q_i, p_i) \leq 0$. Consequently, $g_i(q_i, p_i) = 0$, and therefore by Assumption 2.1(iv) we have $q_1 \in \operatorname{Sol}(\mathrm{EP}(1.1))$ and $q_2 \in \operatorname{Sol}(\mathrm{EP}(1.2))$. Hence $(q_1, q_2) \in \Omega$, and thus $(q_1, q_2) \in \Phi$.

From (4.16) it follows that $\lim_{k\to\infty}\|x_i^k - p_i\|$ exists for $i = 1, 2$. Therefore since the Hilbert space H_i satisfies the Opial condition, it follows that the sequence $\{x_i^k\}$ has only one weak cluster point, and hence $\{(x_1^k, x_2^k)\}$ converges weakly to $(q_1, q_2) \in \Phi$. □

5 Consequences

Now, we give some consequences of Theorem 4.1.

(I). The following theorem shows that the sequence $\{(x_1^k, x_2^k)\}$ generated by Scheme 3.1 with $U_i = I_i$ $(i = 1, 2)$ converges weakly to $(q_1, q_2) \in \Phi_1 = \Omega \cap \Gamma_1$, a common solution of SEEP (1.1)–(1.2) and SEFPP (1.5).

Assume that $\Phi_1 \neq \emptyset$.

Theorem 5.1 *Let H_1, H_2, and H_3 be real Hilbert spaces. For $i = 1,2$, let $C_i \subseteq H_i$ be a nonempty closed convex set, let $A_i : H_i \to H_3$ be a bounded linear operator with its adjoint operator A_i^*, let $V_i : C_i \to C_i$ be a nonexpansive mapping, and let $g_i : C_i \times C_i \to \mathbb{R}$ be a bifunction satisfying Assumption 2.1. Assume that $\mathrm{Fix}(V_1) \neq \emptyset$, $\mathrm{Fix}(V_2)) \neq \emptyset$, and $\Theta_1 = \Omega \cap (\mathrm{Fix}(V_1), \mathrm{Fix}(V_2) \neq \emptyset$. Then the iterative sequence $\{(x_1^k, x_2^k)\}$ generated by Scheme 3.1 with $U_i = I_i$ ($i = 1,2$) converges weakly to $(q_1, q_2) \in \Phi_1$.*

(II). The following theorem shows that the sequence $\{x_1^k\}$ generated by Scheme 3.1 with $H_1 = H_2$, $U_1 = U_2$, $V_1 = V_2$, $C_1 = C_2 = Q_2 = Q_1$, and $A_i = B_i = I_i$ ($i = 1,2$) converges weakly to $q_1 \in \Phi_2 = \mathrm{Sol}(\mathrm{EP}(1.1)) \cap \mathrm{Sol}(\mathrm{HFPP}(1.3))$, a common solution of EP (1.1) and HFPP (1.3).

Assume that $\Phi_2 \neq \emptyset$.

Theorem 5.2 *Let H_1 and H_3 be real Hilbert spaces. Let $C_1 \subseteq H_1$ be a nonempty closed convex set, let $V_1 : C_1 \to C_1$ be a nonexpansive mapping, let $U_1 : C_1 \to C_1$ be a continuous quasi-onexpansive mapping such that $I_1 - U_1$ (I_1 is the identity mapping on C_1) is monotone, and let $g_1 : C_1 \times C_1 \to \mathbb{R}$ be a bifunction satisfying Assumption 2.1. Assume that $\mathrm{Fix}(U_1) \cap \mathrm{Fix}(V_1) \neq \emptyset$ and $\Theta_2 = \mathrm{Sol}(\mathrm{EP}(1.1)) \cap \mathrm{Fix}(U_1) \cap \mathrm{Fix}(V_1) \neq \emptyset$. Then the iterative sequence $\{x_1^k\}$ generated by Scheme 3.1 with $H_1 = H_2$, $U_1 = U_2$, $V_1 = V_2$, $C_1 = C_2 = Q_2 = Q_1$, and $A_i = B_i = I_i$ ($i = 1,2$) converges weakly to $q_1 \in \Phi_2$.*

6 Numerical example

Finally, we give a numerical example for Scheme 3.1.

Example 6.1 Let $H_1 = H_2 = H_3 = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x, y\rangle = xy$, $x, y \in \mathbb{R}$, and induced usual norm $|\cdot|$. Let $C_1 = [-\pi, 0]$ and $C_2 = [0, \pi]$, let $g_1 : C_1 \times C_1 \to \mathbb{R}$ and $g_2 : C_2 \times C_2 \to \mathbb{R}$ be defined by $g_1(x_1, y_1) = 2x_1y_1(y_1 - x_1) + x_1y_1|y_1 - x_1|$, $x_1, y_1 \in C_1$, and $g_2(x_2, y_2) = x_2^2(y_2 - x_2)$, $x_2, y_2 \in C_2$. Let the mappings $A_1 : \mathbb{R} \to \mathbb{R}$ and $A_2 : \mathbb{R} \to \mathbb{R}$ be defined by $A_1(x_1) = 2x_1$, $x_1 \in \mathbb{R}$, and $A_2(x_2) = -2x_2$, $x_2 \in \mathbb{R}$. Let the mappings $V_1 : C_1 \to C_1$ and $U_1 : C_1 \to C_1$ be defined by $V_1x_1 = \frac{x_1}{2}$, $U_1x_1 = x_1 \cos x_1$, $x_1 \in C_1$, and $V_2 : C_2 \to C_2$ and $U_2 : C_2 \to C_2$ be defined by $V_2x_2 = \frac{x_2}{3}$, $U_2x_2 = -x_2 \cos x_2$, $x_2 \in C_2$. Setting $\delta_k = \frac{1}{2k}$, $\sigma_k = \frac{1}{2k}$, $\rho_k = 1$, $\epsilon_k = 0$, $\alpha_k = \frac{1}{2}$, $\beta_k = \frac{1}{k}$, $k \geq 1$. Then the sequences $\{x_1^k\}$ and $\{x_2^k\}$ generated by Scheme 3.1 converge to $q_1 = 0$ and $q_2 = 0$, respectively, so that $(q_1, q_2) = (0, 0) \in \Phi$.

Proof It is easy to prove that the bifunctions g_1 and g_2 are pseudomonotone on C_1 and C_2, respectively. Note that $g_1(x_1, \cdot)$ and $g_1(x_2, \cdot)$ are convex for $x_1 \in C_1$ and $x_2 \in C_2$ and $\partial g_1(x, \cdot)x_1 = [x_1^2, 3x_1^2]$ and $\partial g_2(x_2, \cdot)x_2 = [x_2^2]$ by taking $\epsilon_k = 0$ for all $k \in \mathbb{N}$. A_1 and A_2 are bounded linear operators on $\mathbb{R}$ with adjoint operators A_1^* and A_2^*, $\|A_1\| = \|A_1^*\| = 2$, $\|A_2\| = \|A_2^*\| = 2$, and hence $\mu_k \in (\epsilon, \frac{1}{9} - \epsilon)$. Therefore, for $\epsilon = \frac{1}{100}$, we choose $\mu_k = 0.02 + \frac{0.02}{k}$ for all k. The mappings V_1 and V_2 are nonexpansive with $\mathrm{Fix}(V_1) = \{0\}$ and $\mathrm{Fix}(V_2) = \{0\}$. Further, U_1 and U_2 are continuous with $\mathrm{Fix}(U_1) = \{0\}$ and $\mathrm{Fix}(U_2) = \{0\}$, and $(I - U_1)$ and $(I - U_2)$ are monotone. The mappings U_1 and U_2 are quasinonexpansive but not nonexpansive. After computation, we obtain $\Gamma = \mathrm{Sol}(\mathrm{SEHFPP}(1.3)\text{–}(1.4)) = \{0\}$ and $\Omega = \{0\}$. Therefore

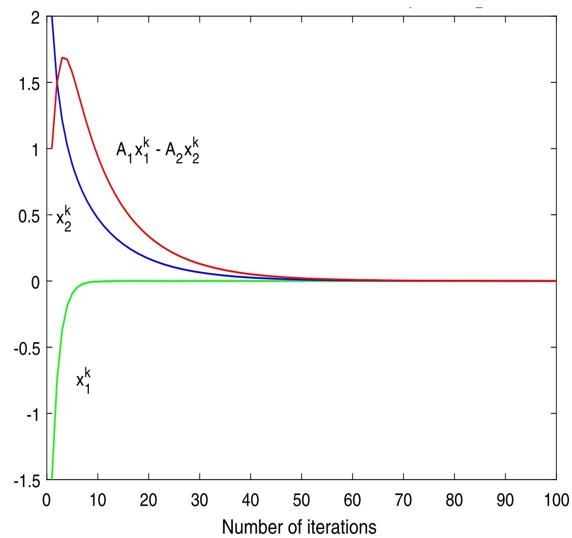

Figure 1 Convergence for initial values $x_1^0 = -3, x_2^0 = 3$

$\Phi = \Omega \cap \Gamma = \{0\} \neq \emptyset$. After simplification, Scheme 3.1 is reduced to the following:

$$\begin{cases} w_1^k \in H_1, w_2^k \in H_2 \quad \text{such that } w_1^k \in \partial_{\epsilon_k} g_1(x_1^k, \cdot)(x_1^k) = [(x_1^k)^2, 3(x_1^k)^2] \\ \text{and} \quad w_2^k \in \partial_{\epsilon_k} g_2(x_2^k, \cdot)(x_2^k) = [(x_2^k)^2]; \\ y_1^k = \begin{cases} 0 & \text{if } x_1 < 0, \\ 1 & \text{if } x_1 > 1, \\ x_1^k - \alpha_k w_1^k & \text{otherwise}; \end{cases} \\ y_2^k = \begin{cases} 0 & \text{if } x_2 < 0, \\ 1 & \text{if } x_2 > 1, \\ x_2^k - \alpha_k w_2^k & \text{otherwise}; \end{cases} \\ t_1^k = (1-\delta_k)x_1^k + \delta_k V_1\big((1-\sigma_k)y_1^k \cos y_1^k + \sigma_k y_1^k\big); \\ t_2^k = (1-\delta_k)x_2^k + \delta_k V_2\big(-(1-\sigma_k)y_2^k \cos y_2^k + \sigma_k y_2^k\big); \\ x_1^{k+1} = P_{C_1}\big(t_1^k + \mu_k A_1^*\big(A_1 t_1^k - A_2 t_2^k\big)\big); \\ x_2^{k+1} = P_{C_2}\big(t_2^k + \mu_k A_2^*\big(A_2 t_1^k - A_2 t_2^k\big)\big). \end{cases} \tag{6.1}$$

Finally, using the software Matlab 7.8.0, we have Fig. 1, which shows that $\{x_1^k\}$ and $\{x_2^k\}$ converge to $q_1 = 0$ and $q_2 = 0$, respectively, so that $(q_1, q_2) = (0, 0) \in \Phi$. □

7 Conclusion

We have proved a weak convergence theorem for an iterative scheme called the simultaneous projected subgradient-proximal iterative scheme, where the stepsizes do not depend on the operator norms, for solving the split equality equilibrium problem SEEP (1.1)–(1.2) for pseudomonotone bifunctions and the split equality hierarchical fixed point problem SEHFPP (1.3)–(1.4) for nonexpansive and quasinonexpansive mappings. Further, we have discussed some consequences of Theorem 4.1. Finally, we presented a numerical example to justify Theorem 4.1. Further research is needed to extend the presented work to the setting of Banach spaces.

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