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Two problems of binomial sums involving harmonic numbers

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Abstract

Two open problems recently proposed by Xi and Luo (Adv. Differ. Equ. 2021:38, 2021) are resolved by evaluating explicitly three binomial sums involving harmonic numbers, that are realized mainly by utilizing the generating function method and symmetric functions.

MSC: Primary 05A10; secondary 05A19; 11B65

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1 Introduction and outline

Denote by $\mathbb{N}$ the set of natural numbers with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For an indeterminate x, define the rising and falling factorials by $(x)_0 = \langle x\rangle_0 \equiv 1$ and

$$(x)_n = x(x+1)\cdots(x+n-1) \quad \text{for } n \in \mathbb{N},$$
$$\langle x\rangle_n = x(x-1)\cdots(x-n+1) \quad \text{for } n \in \mathbb{N}.$$

The harmonic numbers of higher order are given by

$$H_0^{(\lambda)} = 1 \quad \text{and} \quad H_n^{(\lambda)} = \sum_{k=1}^{n} \frac{1}{k^\lambda} \quad \text{for } n, \lambda \in \mathbb{N}.$$

In order to reduce lengthy expressions, we shall employ the notations of elementary and complete symmetric functions. For a finite set S of real numbers, we define these functions by $\Phi_0(x|S) = \Psi_0(x|S) \equiv 1$ and

$$\Phi_n(x|S) = \sum_{\substack{\sum_{\alpha\in S} k_\alpha = n \\ 0\le k_\alpha \le 1}} \prod_{\alpha\in S} \frac{1}{(x+\alpha)^{k_\alpha}} \quad \text{for } n \in \mathbb{N}, \tag{1}$$

$$\Psi_n(x|S) = \sum_{\substack{\sum_{\alpha\in S} k_\alpha = n \\ 0\le k_\alpha \le n}} \prod_{\alpha\in S} \frac{1}{(x+\alpha)^{k_\alpha}} \quad \text{for } n \in \mathbb{N}. \tag{2}$$

We shall also need the signless Stirling numbers of the first kind (see [6]) which are determined by the connection coefficient of expanding the shifted factorials into monomials

$$(y)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} y^k. \tag{3}$$

There exist numerous summation formulae involving harmonic numbers (cf. [1–3, 7, 8]). In a recent paper [9], Xi and Luo proposed the following two open problems.

Problem I Let x be an indeterminate. For $m, n \in \mathbb{N}_0$ with $m > n$, how to calculate the combinatorial sums

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{m+k}{k} \quad \text{and} \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{m+k}{k} \frac{x}{x+k}?$$

Problem II Let x be an indeterminate. For $m, n, \lambda, \rho \in \mathbb{N}_0$, what are the combinatorial sums

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{m+k}{k} \left(\frac{x}{x+k}\right)^{\lambda} \quad \text{and} \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{m+k}{k} \{H_{\rho+k}^{(\lambda)} - H_k^{(\lambda)}\}?$$

The first binomial sum in Problem I can easily be evaluated by the Chu–Vandermonde convolution formula as follows:

$$\begin{aligned} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{m+k}{k} &= \sum_{k=0}^{n} \binom{n}{n-k} \binom{-m-1}{k} \\ &= \binom{n-m-1}{n} = (-1)^n \binom{m}{n}. \end{aligned}$$

As the primary motivation, the aim of the present paper is to resolve these problems and evaluate the remaining three sums explicitly in the following theorems.

Theorem 1 *Let x be an indeterminate. Then for $m, n \in \mathbb{N}_0$, the following algebraic identity holds*:

$$\begin{aligned} \frac{n!}{(x)_{n+1}} \binom{m-x}{m} = {} & \sum_{k=0}^{n} \frac{(-1)^k}{x+k} \binom{n}{k} \binom{m+k}{k} \\ & + \sum_{k=1}^{m-n} (-1)^{n+k} \binom{m}{n+k} \frac{(x+n+1)_{k-1}}{(n+1)_k}. \end{aligned}$$

We remark that when $m > n$, this theorem evaluates the second sum in Problem I by determining the polynomial part of the rational function explicitly as in the last line, which vanishes for $m \le n$, instead.

Theorem 2 *Let x be an indeterminate. Then for $m,n,\lambda\in\mathbb{N}_0$, the following algebraic identity holds*:

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}\binom{m+k}{k}\frac{(-1)^k}{(x+k)^{\lambda}}\\&\quad=\frac{n!}{(x)_{n+1}}\binom{m-x}{m}\sum_{k=1}^{\lambda}\frac{\Phi_{k-1}(-x|[1,m])}{(k-1)!}\Psi_{\lambda-k}\big(x|[0,n]\big)\\&\qquad+\sum_{k=1}^{m-n}\frac{(-1)^{n+k+\lambda}}{(\lambda-1)!}\binom{m}{n+k}\frac{(x+n+1)_{k-1}}{(n+1)_k}\Phi_{\lambda-1}\big(x+n|[1,k-1]\big).\end{aligned}$$

Theorem 3 *Let x be an indeterminate. Then for $m,n,\lambda,\rho\in\mathbb{N}_0$, the following algebraic identity holds*:

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{m+k}{k}\big\{H_{\rho+k}^{(\lambda)}-H_k^{(\lambda)}\big\}\\&\quad=\frac{n!}{m!}\sum_{k=1}^{\lambda}\sum_{j=1}^{m}\sum_{i=1}^{k}\frac{(-1)^{i+j}}{(j)_{n+1}}\begin{bmatrix}j\\i\end{bmatrix}\begin{bmatrix}m-j+1\\k-i\end{bmatrix}\frac{\Psi_{\lambda-k}(j|[0,n])}{(k-1)!}\\&\qquad+\sum_{k=1}^{\lambda}\sum_{j=m+1}^{\rho}\frac{n!}{(j)_{n+1}}\frac{\Psi_{\lambda-k}(j|[0,n])}{(k-1)!}\binom{m-j}{m}\Phi_{k-1}\big(-j|[1,m]\big)\\&\qquad+\sum_{k=1}^{m-n}\sum_{j=1}^{\rho}\frac{(-1)^{n+k+\lambda}}{(\lambda-1)!}\binom{m}{n+k}\frac{(j+n+1)_{k-1}}{(n+1)_k}\Phi_{\lambda-1}\big(j+n|[1,k-1]\big).\end{aligned}$$

The rest paper will be organized as follows. In the next section, we shall prove Theorem 1 by determining explicitly the polynomial part of a rational function when its numerator degree is greater than that of the denominator. Then Theorems 2 and 3 will be shown in Sect. 3 by establishing two analytical formulae of the derivatives of higher order for a polynomial function of the rising factorial and its reciprocal. The informed reader will notice that by employing symmetric functions Φ and Ψ, several involved expressions become simpler than those appearing in [9], where the Bell polynomials were employed.

2 Proof of Theorem 1

Observe that the rational function below can be decomposed into partial fractions

$$\frac{n!}{(x)_{n+1}}\binom{m-x}{m}=P_n^m(x)+\sum_{k=0}^{n}\frac{A_k}{x+k},$$

where $P_n^m(x)$ is a polynomial of degree $m-n-1$ in x which reduces to zero when $m\le n$, and the coefficients A_k are determined by the limits

$$A_k=\lim_{x\to-k}(x+k)\left\{\frac{n!}{(x)_{n+1}}\binom{m-x}{m}\right\}=(-1)^k\binom{n}{k}\binom{m+k}{m}.$$

Therefore, we have found the equality

$$\frac{n!}{(x)_{n+1}}\binom{m-x}{m} = P_n^m(x) + \sum_{k=0}^{n} \frac{(-1)^k}{x+k}\binom{n}{k}\binom{m+k}{k}. \tag{4}$$

By scaling down m and then making use of

$$\frac{m-x}{m} = \frac{m+k}{m} - \frac{k+x}{m},$$

we can rewrite the last equality as

$$\begin{aligned}
&\frac{n!}{(x)_{n+1}}\binom{m-x}{m} \\
&\quad = \frac{m-x}{m} \times \frac{n!}{(x)_{n+1}}\binom{m-1-x}{m-1} \\
&\quad = \frac{m-x}{m}\left\{P_n^{m-1}(x) + \sum_{k=0}^{n} \frac{(-1)^k}{x+k}\binom{n}{k}\binom{m-1+k}{k}\right\} \\
&\quad = \frac{m-x}{m} P_n^{m-1}(x) + \sum_{k=0}^{n} \frac{(-1)^k}{x+k}\binom{n}{k}\binom{m+k}{k} - \sum_{k=0}^{n} \frac{(-1)^k}{m}\binom{n}{k}\binom{m-1+k}{k}.
\end{aligned}$$

Evaluating the last sum by means of the Chu–Vandemonde formula and then comparing the resultant expression with (4), we get the following recurrence relation:

$$P_n^m(x) = \frac{m-x}{m} P_n^{m-1}(x) - \frac{(-1)^n}{m}\binom{m-1}{n}. \tag{5}$$

In order to find an explicit expression for $P_n^m(x)$, let $Q_m := P_n^{m+n}(x)$. Then the equality corresponding to (5) becomes

$$Q_m = \frac{m+n-x}{m+n} Q_{m-1} - \frac{(-1)^n}{m+n}\binom{m+n-1}{n}. \tag{6}$$

It is routine to figure out the initial values $Q_0 = 0$ and $Q_1 = \frac{(-1)^{n+1}}{n+1}$. Then we can manipulate the generating function

$$\begin{aligned}
Q(y) &:= \sum_{m=1}^{\infty} Q_m y^{m+n} \\
&= \sum_{m=1}^{\infty} \left(1 - \frac{x}{m+n}\right) Q_{m-1} y^{m+n} \\
&\quad - \sum_{m=1}^{\infty} (-1)^n \binom{m+n-1}{n} \frac{y^{m+n}}{m+n}.
\end{aligned}$$

By differentiating the last equation with respect to y,

$$Q'(y) = \frac{d}{dy}\big\{yQ(y)\big\} - xQ(y) - \sum_{m=1}^{\infty} (-1)^n \binom{m+n-1}{n} y^{m+n-1},$$

and then evaluating the binomial series on the right, we find, after some simplification, that $Q(y)$ satisfies the following differential equation:

$$(1-y)Q'(y)-(1-x)Q(y)=\frac{y^n}{(y-1)^{n+1}}. \tag{7}$$

It is trivial to check that the corresponding homogeneous equation

$$\frac{Q'(y)}{Q(y)}=\frac{1-x}{1-y}$$

has the binomial solution

$$Q(y)=\Omega(1-y)^{x-1} \tag{8}$$

where Ω is an arbitrary constant. When $\Omega:=\Omega(y)$ is considered as a function of y, substituting the above solution into (7) gives rise to

$$\Omega'(y)=(-1)^{n+1}y^n(1-y)^{-x-n-1}.$$

Therefore, we have the integral representation

$$\Omega(y)=(-1)^{n+1}\int_0^y T^n(1-T)^{-x-n-1}\,dT.$$

Define for simplicity

$$J_n:=\int_0^y T^n(1-T)^{-x-n-1}\,dT \quad \text{with } J_0=\frac{(1-y)^{-x}-1}{x}.$$

According to integration by parts, we can calculate J_n as follows:

$$\begin{aligned}J_n&=\int_0^y T^n(1-T)^{-x-n-1}\,dT=\frac{y^n}{x+n}(1-y)^{-x-n}-\frac{n}{x+n}J_{n-1}\\&=\frac{y^n}{x+n}(1-y)^{-x-n}-\frac{ny^{n-1}}{\langle x+n\rangle_2}(1-y)^{1-x-n}+\frac{\langle n\rangle_2}{\langle x+n\rangle_2}J_{n-2}.\end{aligned}$$

By means of the induction principle, we can show that

$$\begin{aligned}J_n&=\sum_{k=0}^{n-1}\frac{(-1)^k\langle n\rangle_k}{\langle x+n\rangle_{k+1}}y^{n-k}(1-y)^{k-x-n}+\frac{(-1)^n\langle n\rangle_n}{\langle x+n\rangle_n}J_0\\&=\sum_{k=0}^{n}\frac{(-1)^k\langle n\rangle_k}{\langle x+n\rangle_{k+1}}y^{n-k}(1-y)^{k-x-n}+\frac{(-1)^{n+1}n!}{(x)_{n+1}},\end{aligned}$$

which is equivalent to the expression

$$\Omega(y)=\frac{n!}{(x)_{n+1}}-\sum_{k=0}^{n}\frac{(-1)^{n+k}\langle n\rangle_k}{\langle x+n\rangle_{k+1}}y^{n-k}(1-y)^{k-x-n}.$$

Substituting this into (8), we obtain the explicit generating function

$$Q(y)=\frac{n!}{(x)_{n+1}}(1-y)^{x-1}-\sum_{k=0}^{n}\frac{(-1)^{n+k}\langle n\rangle_k}{\langle x+n\rangle_{k+1}}y^{n-k}(1-y)^{k-1-n}.$$

Extracting the coefficient of y^{m+n} across the last equation yields

$$Q_m=\left[y^{m+n}\right]Q(y)=\binom{m+n-x}{m+n}\frac{n!}{(x)_{n+1}}-\sum_{k=0}^{n}\frac{(-1)^{n+k}\langle n\rangle_k}{\langle x+n\rangle_{k+1}}\binom{m+n}{n-k}.$$

By reformulating the last sum with respect to k as

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{m+n}{n-k}\frac{(-1)^{n-k}\langle n\rangle_k}{\langle x+n\rangle_{k+1}}\\
&\quad=\frac{n!}{(x)_{n+1}}\sum_{k=0}^{n}\binom{m+n}{m+k}\binom{-x}{n-k}\\
&\quad=\frac{n!}{(x)_{n+1}}\left\{\sum_{k=-m}^{n}\binom{m+n}{m+k}\binom{-x}{n-k}-\sum_{k=-m}^{-1}\binom{m+n}{m+k}\binom{-x}{n-k}\right\}\\
&\quad=\frac{n!}{(x)_{n+1}}\binom{m+n-x}{m+n}-\frac{n!}{(x)_{n+1}}\sum_{k=1}^{m}\binom{m+n}{m-k}\binom{-x}{n+k},
\end{aligned}$$

we find finally the binomial expression

$$Q_m=\frac{n!}{(x)_{n+1}}\sum_{k=1}^{m}\binom{m+n}{m-k}\binom{-x}{n+k}=\sum_{k=1}^{m}(-1)^{n+k}\binom{m+n}{n+k}\frac{(x+n+1)_{k-1}}{(n+1)_k}.$$

This gives consequently the desired formula stated in Theorem 1:

$$P_n^m(x)=Q_{m-n}(x)=\sum_{k=1}^{m-n}(-1)^{n+k}\binom{m}{n+k}\frac{(x+n+1)_{k-1}}{(n+1)_k}.$$

3 Proofs of Theorems 2 and 3

For the derivative operator $\mathcal{D}$ with respect to x, we have the following analytical formulae of higher order derivatives:

$$\mathcal{D}^n\prod_{\alpha\in S}(x+\alpha)=\Phi_n(x|S)\prod_{\alpha\in S}(x+\alpha), \tag{9}$$

$$\mathcal{D}^n\prod_{\alpha\in S}\frac{1}{x+\alpha}=\Psi_n(x|S)\frac{n!(-1)^n}{\prod_{\alpha\in S}(x+\alpha)}. \tag{10}$$

The first one in (9) can be evaluated easily by induction on n. In order to prove the second one in (10), define

$$R(x):=\prod_{\alpha\in S}\frac{1}{x+\alpha}\quad\text{and}\quad\mathcal{D}^nR(x)=R(x)G_n, \tag{11}$$

where the function G_n remains to be determined with the initial values

$$G_0 = 1 \quad \text{and} \quad G_1 = -\Psi_1(x|S).$$

Then by making use of the Leibniz rule, we have

$$\begin{aligned}
\mathcal{D}^{\lambda+1}R(x) &= -\mathcal{D}^{\lambda}\bigl\{R(x)\Psi_1(x|S)\bigr\} \\
&= -\sum_{k=0}^{\lambda}\binom{\lambda}{k}\mathcal{D}^{\lambda-k}R(x)\mathcal{D}^{k}\Psi_1(x|S) \\
&= -R(x)\sum_{k=0}^{\lambda}\binom{\lambda}{k}G_{\lambda-k}\mathcal{D}^{k}\Psi_1(x|S),
\end{aligned}$$

which leads us to the binomial recursion

$$G_{\lambda+1} = -\sum_{k=0}^{\lambda}\binom{\lambda}{k}G_{\lambda-k}\mathcal{D}^{k}\Psi_1(x|S). \tag{12}$$

In order to find an explicit expression for G_λ, we examine the exponential generating function defined by

$$G(y) := \sum_{\lambda=0}^{\infty}\frac{y^\lambda}{\lambda!}G_\lambda.$$

According to (12), its derivative with respect to y can be expressed as

$$\begin{aligned}
G'(y) &= \sum_{\lambda=0}^{\infty}\frac{y^\lambda}{\lambda!}G_{\lambda+1} = -\sum_{\lambda=0}^{\infty}\frac{y^\lambda}{\lambda!}\sum_{k=0}^{\lambda}\binom{\lambda}{k}G_{\lambda-k}\mathcal{D}^{k}\Psi_1(x|S) \\
&= -\sum_{k=0}^{\infty}\frac{y^k}{k!}\mathcal{D}^{k}\Psi_1(x|S)\sum_{\lambda=k}^{\infty}\frac{y^{\lambda-k}}{(\lambda-k)!}G_{\lambda-k}.
\end{aligned}$$

We therefore get the differential equation

$$G'(y) = -G(y)\sum_{k=0}^{\infty}\frac{y^k}{k!}\mathcal{D}^{k}\Psi_1(x|S)$$

whose solution is given by the exponential function

$$G(y) = \exp\left\{-\int_0^y\sum_{k=0}^{\infty}\frac{y^k}{k!}\mathcal{D}^{k}\Psi_1(x|S)\,dy\right\} = \exp\left\{-\sum_{k=0}^{\infty}\frac{y^{k+1}}{(k+1)!}\mathcal{D}^{k}\Psi_1(x|S)\right\}.$$

Evaluating the last sum with respect to k explicitly as

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{y^{k+1}}{(k+1)!}\mathcal{D}^k\Psi_1(x|S)&=\sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}\sum_{\alpha\in S}\frac{y^{k+1}}{(x+\alpha)^{k+1}}\\&=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\sum_{\alpha\in S}\frac{y^k}{(x+\alpha)^k}\\&=\sum_{\alpha\in S}\ln\left(1+\frac{y}{x+\alpha}\right),\end{aligned}$$

we find the simplified generating function

$$G(y)=\prod_{\alpha\in S}\left(1+\frac{y}{x+\alpha}\right)^{-1}. \tag{13}$$

By extracting the coefficient of y^n, we confirm the formula (10) as follows:

$$G_n=n!\left[y^n\right]G(y)=n!(-1)^n\Psi_n(x|S).$$

3.1 Proof of Theorem 2

This can be done by differentiating $\lambda-1$ times the equality displayed in Theorem 1. Firstly, it is trivial to have

$$\mathcal{D}^{\lambda-1}\frac{1}{x+k}=(-1)^{\lambda-1}\frac{(\lambda-1)!}{(x+k)^{\lambda}}.$$

Then by making use of the Leibniz rule, we can compute

$$\begin{aligned}\mathcal{D}^{\lambda-1}\frac{(1-x)_m}{(x)_{n+1}}&=(-1)^m\mathcal{D}^{\lambda-1}\frac{\langle x-1\rangle_m}{(x)_{n+1}}\\&=(-1)^m\sum_{k=1}^{\lambda}\binom{\lambda-1}{k-1}\mathcal{D}^{k-1}\langle x-1\rangle_m\mathcal{D}^{\lambda-k}\frac{1}{(x)_{n+1}}\\&=\frac{(1-x)_m}{(x)_{n+1}}\sum_{k=1}^{\lambda}(-1)^{\lambda-k}\frac{(\lambda-1)!}{(k-1)!}\Phi_{k-1}\big(x|[-m,-1]\big)\Psi_{\lambda-k}\big(x|[0,n]\big)\\&=(-1)^{\lambda-1}\frac{(1-x)_m}{(x)_{n+1}}\sum_{k=1}^{\lambda}\frac{(\lambda-1)!}{(k-1)!}\Phi_{k-1}\big(-x|[1,m]\big)\Psi_{\lambda-k}\big(x|[0,n]\big),\end{aligned}$$

where we have invoked two derivative formulae (9) and (10). Finally,

$$\begin{aligned}\mathcal{D}^{\lambda-1}(x+n+1)_{k-1}&=(x+n+1)_{k-1}\Phi_{\lambda-1}\big(x|[n+1,n+k-1]\big)\\&=(x+n+1)_{k-1}\Phi_{\lambda-1}\big(x+n|[1,k-1]\big).\end{aligned}$$

Substituting the above three expressions into the equality of Theorem 1 and then making some simplifications, we find the algebraic identity in Theorem 2.

3.2 Proof of Theorem 3

Recalling (3), we can deduce, for the signless Stirling numbers, the symmetric function expression (see [4, Chap. V] and [5, §6.1])

$$\begin{aligned}\begin{bmatrix} n+1 \\ k+1 \end{bmatrix} = \left[y^k\right](1+y)_n &= \sum_{1\le i_1<i_2<\cdots<i_{n-k}\le n} i_1 i_2\cdots i_{n-k} \\ &= n! \sum_{1\le j_1<j_2<\cdots<j_k\le n} \frac{1}{j_1 j_2\cdots j_k}.\end{aligned}$$

This gives rise to the following identity:

$$\Phi_k\big(0|[1,n]\big) = \frac{1}{n!}\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}. \tag{14}$$

Let ρ be a natural number. When $x\to j$ with $1\le j\le\rho$, the limiting case of the equation displayed in Theorem 2 reads as

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}\binom{m+k}{k}\frac{(-1)^k}{(j+k)^\lambda} \\ &\quad= \frac{n!}{(j)_{n+1}}\sum_{k=1}^{\lambda}\frac{\Psi_{\lambda-k}(j|[0,n])}{(k-1)!}\lim_{x\to j}\binom{m-x}{m}\Phi_{k-1}\big(-x|[1,m]\big) \\ &\qquad+\sum_{k=1}^{m-n}\frac{(-1)^{n+k+\lambda}}{(\lambda-1)!}\binom{m}{n+k}\frac{(j+n+1)_{k-1}}{(n+1)_k}\Phi_{\lambda-1}\big(j+n|[1,k-1]\big).\end{aligned} \tag{15}$$

When $j>m$, the limit in the middle line is given directly by letting $x=j$

$$\lim_{x\to j}\binom{m-x}{m}\Phi_{k-1}\big(-x|[1,m]\big) = \binom{m-j}{m}\Phi_{k-1}\big(-j|[1,m]\big)$$

since the two factors on the right are well defined. Instead, for $1\le j\le m$, that limit can be determined as

$$\begin{aligned}&\lim_{x\to j}\binom{m-x}{m}\Phi_{k-1}\big(-x|[1,m]\big) \\ &\quad= \lim_{x\to j}\frac{(1-x)_m}{m!}\Phi_{k-1}\big(-x|[1,m]\big) \\ &\quad= \frac{(-1)^{j-1}}{j\binom{m}{j}}\sum_{i=1}^{k-1}\Phi_{i-1}\big(-j|[1,j-1]\big)\Phi_{k-i-1}\big(-j|[j+1,m]\big) \\ &\quad= \frac{(-1)^{j}}{j\binom{m}{j}}\sum_{i=1}^{k-1}(-1)^i\Phi_{i-1}\big(0|[1,j-1]\big)\Phi_{k-i-1}\big(0|[1,m-j]\big) \\ &\quad= \frac{(-1)^{j}}{m!}\sum_{i=1}^{k-1}(-1)^i\begin{bmatrix} j \\ i \end{bmatrix}\begin{bmatrix} m-j+1 \\ k-i \end{bmatrix},\end{aligned}$$

where the last line is justified by (14). Finally summing equation (15) over j from 1 to ρ, we obtain the following equality involving harmonic numbers:

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{m+k}{k}\big\{H_{\rho+k}^{(\lambda)}-H_k^{(\lambda)}\big\}\\
&\quad=\frac{n!}{m!}\sum_{j=1}^{m}\frac{(-1)^j}{(j)_{n+1}}\sum_{k=1}^{\lambda}\frac{\Psi_{\lambda-k}(j|[0,n])}{(k-1)!}\sum_{i=1}^{k-1}(-1)^i\begin{bmatrix}j\\ i\end{bmatrix}\begin{bmatrix}m-j+1\\ k-i\end{bmatrix}\\
&\qquad+\sum_{j=m+1}^{\rho}\frac{n!}{(j)_{n+1}}\sum_{k=1}^{\lambda}\frac{\Psi_{\lambda-k}(j|[0,n])}{(k-1)!}\binom{m-j}{m}\Phi_{k-1}\big(-j|[1,m]\big)\\
&\qquad+\sum_{k=1}^{m-n}\sum_{j=1}^{\rho}\frac{(-1)^{n+k+\lambda}}{(\lambda-1)!}\binom{m}{n+k}\frac{(j+n+1)_{k-1}}{(n+1)_k}\Phi_{\lambda-1}\big(j+n|[1,k-1]\big),
\end{aligned}$$

which is equivalent to the formula stated in Theorem 3.

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Authors' contributions
Both authors contributed equally to the writing of this paper, who read and approved the final manuscript.

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