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A note on two-term exponential sum and the reciprocal of the quartic Gauss sums

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Abstract

The main purpose of this article is by using the properties of the fourth character modulo a prime p and the analytic methods to study the calculating problem of a certain hybrid power mean involving the two-term exponential sums and the reciprocal of quartic Gauss sums, and to give some interesting calculating formulae of them.

MSC: 11L05

Keywords: Reciprocal of the quartic Gauss sums; Two-term exponential sums; Hybrid power means; Analytic methods

1 Introduction

Let $q \geq 3$ be a fixed integer. For any integers k and m with $k \geq 2$ and $(m,q) = 1$, the kth Gauss sums $G(m,k;q)$ and the two-term exponential sums $H(m,k;q)$ in [1] are defined as

$$G(m,k;q) = \sum_{a=0}^{q-1} e\left(\frac{ma^k}{q}\right) \quad \text{and} \quad H(m,k;q) = \sum_{a=0}^{q-1} e\left(\frac{ma^k + a}{q}\right),$$

where $e(y) = e^{2\pi iy}$ and $i^2 = -1$.

We all know that these sums occupy a very important position in the study of analytic number theory, and many number theory problems are related to them. Therefore, many scholars have studied their various properties and obtained a series of meaningful results. We will not repeat it here. Interested readers can refer to references [1–14].

Recently, Zhang Wenpeng and Chen Zhuoyu [1] studied the hybrid power mean involving $H(m,3;p)$ and the reciprocal of the quartic Gauss sums $G(m,4;p)$, and they obtained two interesting results as follows.

If p is a prime with $p \equiv 5 (\bmod 8)$, then one has the identity

$$\sum_{m=1}^{p-1} \left| \frac{H(m,3;p)}{G(m,4;p)} \right|^2 = \begin{cases} \frac{3p(p-2)-2\sqrt{p}\alpha}{9p-4\alpha^2} & \text{if } 3 \mid (p-1), \\ \frac{3p^2+2\sqrt{p}\alpha}{9p-4\alpha^2} & \text{if } 3 \nmid (p-1); \end{cases} \tag{1}$$

If p is a prime with $p \equiv 5(\bmod 8)$ for any real number $k \geq 0$, then one has

$$\sum_{m=1}^{p-1} \frac{1}{|G(m,4;p)|^{2k}} = \frac{p-1}{2} \cdot \frac{(3\sqrt{p}+2\alpha)^k + (3\sqrt{p}-2\alpha)^k}{p^{\frac{k}{2}} \cdot (9p-4\alpha^2)^k}, \tag{2}$$

where $\alpha = \alpha(p) = \sum_{a=1}^{\frac{p-1}{2}} (\frac{a+\overline{a}}{p})$ is an integer, $(\frac{*}{p})$ denotes the Legendre symbol mod$p$, and $\overline{a}$ denotes the solution of the equation $ax \equiv 1(\bmod p)$.

These results are significant, because dealing with the reciprocal of the trigonometric sums is not common to us. But the methods in their article cannot handle the case of $p \equiv 1(\bmod 8)$, thus leaving it as an open problem.

Of course, the integer $\alpha = \alpha(p)$ in (1) and (2) is closely related to p. In fact, if $p \equiv 1(\bmod 4)$, then we have (see Theorems 4–11 in [15])

$$p = \alpha^2 + \beta^2,$$

where $\beta = \sum_{b=1}^{\frac{p-1}{2}} (\frac{b+r\cdot\overline{b}}{p})$, and r is any quadratic non-residue modulo p.

In this paper, we consider a generalized problem: For any prime p with $p \equiv 1(\bmod 8)$ and number-theoretic function $F(m)$, whether there is an exact calculating formula for the hybrid power mean

$$V_k(p) = \sum_{m=1}^{p-1} \frac{F(m)}{G^k(m,4;p)}, \tag{3}$$

where $k \geq 0$ is an integer.

We use the analytic methods and the properties of the fourth character modulo p to give an interesting fourth-order linear recursive formula for $V_k(p)$.

Theorem 1 *Let p be a prime with $p \equiv 1(\bmod 8)$. Then, for any number-theoretic function $F(m)$, we have the fourth-order linear recursive formula*

$$V_k(p) = \frac{8\alpha}{p-4\alpha^2} \cdot V_{k-1}(p) + \frac{6}{p-4\alpha^2} \cdot V_{k-2}(p) - \frac{1}{p(p-4\alpha^2)} \cdot V_{k-4}(p).$$

For all integers $k \geq 4$ with the initial values

$$V_j(p) = \sum_{m=1}^{p-1} \frac{F(m)}{G^j(m,4;p)}, \quad j = 0,1,2,3.$$

Obviously, in order to obtain all values of $V_j(p)$ for any integer $k \geq 0$, we need to compute $V_0(p)$, $V_1(p)$, $V_2(p)$, and $V_3(p)$, then we can compute all the values of $V_k(p)$ using this fourth-order linear recursion formula. In general, the first four terms of $V_j(p)$ do not always get the exact value, but for some special function $F(m)$, we can compute the exact value of $V_j(p)$ with $j = 0,1,2,3$, and we can get all the terms of the recursive sequence $V_j(p)$.

Especially for $F(m) = 1$ and $W_k(p) = \sum_{m=1}^{p-1} \frac{1}{G^k(m,4;p)}$ in Theorem 1, we have the following result.

Theorem 2 *If p is a prime with $p \equiv 1(\bmod 8)$, then we have the fourth-order linear recursive formula*

$$W_k(p) = \frac{8\alpha}{p-4\alpha^2} \cdot W_{k-1}(p) + \frac{6}{p-4\alpha^2} \cdot W_{k-2}(p) - \frac{1}{p(p-4\alpha^2)} \cdot W_{k-4}(p)$$

for all integers $k \geq 4$ with the initial values $W_0(p) = p-1$; $W_1(p) = \frac{2(p-1)\alpha}{p-4\alpha^2}$;

$$W_2(p) = \frac{(p-1)(3p+4\alpha^2)}{(p-4\alpha^2)^2} \quad \text{and} \quad W_3(p) = \frac{4(p-1)\alpha(9p-4\alpha^2)}{(p-4\alpha^2)^3}.$$

If we take $F(m) = H^2(m,3;p)$ and $\overline{W}_k(p) = \sum_{m=1}^{p-1} \frac{H^2(m,3;p)}{G^k(m,4;p)}$, we prove the following result.

Theorem 3 *If p is a prime with $p \equiv 17(\bmod 24)$, then we have*

$$\overline{W}_k(p) = \frac{8\alpha}{p-4\alpha^2} \cdot \overline{W}_{k-1}(p) + \frac{6}{p-4\alpha^2} \cdot \overline{W}_{k-2}(p) - \frac{1}{p(p-4\alpha^2)} \cdot \overline{W}_{k-4}(p)$$

for all integer $k \geq 4$ with $\overline{W}_0(p) = p^2$; $\overline{W}_1(p) = \frac{2p^2\alpha+2p\alpha\beta-p^{\frac{3}{2}}\beta-p^{\frac{3}{2}}}{p-4\alpha^2}$;

$$\overline{W}_2(p) = \frac{3p^3+4p^2\alpha^2+8p\alpha^2\beta-10p^{\frac{3}{2}}\alpha+2p^2\beta-8p^{\frac{3}{2}}\alpha\beta+8p^{\frac{1}{2}}\alpha^3}{(p-4\alpha^2)^2}$$

and

$$\overline{W}_3(p) = \frac{36p^3\alpha-16p^2\alpha^3+16p\alpha^3\beta-48p^{\frac{3}{2}}\alpha^2+28p^2\alpha\beta}{(p-4\alpha^2)^3} + \frac{48p^{\frac{1}{2}}\alpha^4-48p^{\frac{3}{2}}\alpha^2\beta-5p^{\frac{5}{2}}\beta-7p^{\frac{5}{2}}+16p^{\frac{1}{2}}\alpha^4\beta}{(p-4\alpha^2)^3},$$

where $\beta = \tau(\psi) + \tau(\overline{\psi})$, it satisfies the identity $\beta^2 = 2\sqrt{p}\alpha + 2p$. And ψ denotes any fourth-order character modulo p.

Theorem 4 *If p is a prime with $p \equiv 1(\bmod 24)$, then we have*

$$\overline{W}_k(p) = \frac{8\alpha}{p-4\alpha^2} \cdot \overline{W}_{k-1}(p) + \frac{6}{p-4\alpha^2} \cdot \overline{W}_{k-2}(p) - \frac{1}{p(p-4\alpha^2)} \cdot \overline{W}_{k-4}(p)$$

for all $k \geq 4$ with $\overline{W}_0(p) = p(p-2)$; $\overline{W}_1(p) = \frac{2p(p-2)\alpha-2p\alpha\beta+p^{\frac{3}{2}}\beta+p^{\frac{3}{2}}}{p-4\alpha^2}$;

$$\overline{W}_2(p) = \frac{p(p-2)(3p+4\alpha^2)-8p\alpha^2\beta+10p^{\frac{3}{2}}\alpha-2p^2\beta+8p^{\frac{3}{2}}\alpha\beta-8p^{\frac{1}{2}}\alpha^3}{(p-4\alpha^2)^2}$$

and

$$\overline{W}_3(p) = \frac{4p(p-2)(9p-4\alpha^2)\alpha-16p\alpha^3\beta+48p^{\frac{3}{2}}\alpha^2-28p^2\alpha\beta}{(p-4\alpha^2)^3} + \frac{48p^{\frac{3}{2}}\alpha^2\beta-48p^{\frac{1}{2}}\alpha^4+5p^{\frac{5}{2}}\beta+7p^{\frac{5}{2}}-16p^{\frac{1}{2}}\alpha^4\beta}{(p-4\alpha^2)^3}.$$

Taking $k = 2$ or 4, from these theorems we have the following corollaries.

Corollary 1 *If p is a prime with $p \equiv 1(\bmod 8)$, then we have the identity*

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right|^{-4} = \frac{p-1}{p(p-4\alpha^2)^4}\cdot\left(17p^3+252p^2\alpha^2-272p\alpha^4+64\alpha^6\right).$$

Corollary 2 *If p is a prime with $p \equiv 17(\bmod 24)$, then we have the identity*

$$\begin{aligned}\sum_{m=1}^{p-1}\frac{H^2(m,3;p)}{G^2(m,4;p)} &= \sum_{m=1}^{p-1}\left|\frac{\sum_{a=0}^{p-1} e(\frac{ma^3+a}{p})}{\sum_{a=0}^{p-1} e(\frac{ma^4}{p})}\right|^2 \\ &= \frac{3p^3+4p^2\alpha^2+8p\alpha^2\beta-10p^{\frac{3}{2}}\alpha+2p^2\beta-8p^{\frac{3}{2}}\alpha\beta+8p^{\frac{1}{2}}\alpha^3}{(p-4\alpha^2)^2}.\end{aligned}$$

Corollary 3 *If p is a prime with $p \equiv 1(\bmod 24)$, then we have the identity*

$$\begin{aligned}\sum_{m=1}^{p-1}\frac{H^2(m,3;p)}{G^2(m,4;p)} &= \sum_{m=1}^{p-1}\left|\frac{\sum_{a=0}^{p-1} e(\frac{ma^3+a}{p})}{\sum_{a=0}^{p-1} e(\frac{ma^4}{p})}\right|^2 \\ &= \frac{p(p-2)(3p+4\alpha^2)-24p\alpha^2\beta+6p^{\frac{3}{2}}\alpha+2p^2\beta+8p^{\frac{3}{2}}\alpha\beta+8p^{\frac{1}{2}}\alpha^3}{(p-4\alpha^2)^2}.\end{aligned}$$

2 Several lemmas

To complete the proofs of our theorems, we need to give some basic lemmas. Of course, the proofs of these lemmas need some knowledge of elementary and analytic number theory. They can be found in many number theory books, such as [15–18]. First we have the following.

Lemma 1 *Let p be an odd prime with $p \equiv 1(\bmod 4)$, ψ be any fourth-order character* mod *p. Then we have the identity*

$$\tau^2(\psi)+\tau^2(\overline{\psi}) = \sqrt{p}\cdot\sum_{a=1}^{p-1}\left(\frac{a+\overline{a}}{p}\right) = 2\sqrt{p}\cdot\alpha.$$

Proof This is Lemma 2.2 in [2]. □

Lemma 2 *If p is a prime with $p \equiv 1(\bmod 8)$, then we have the identities*

$$\sum_{m=1}^{p-1}\left(\frac{m}{p}\right)\left(\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right)^2 = -\left(\frac{3}{p}\right)\cdot p$$

and

$$\sum_{m=1}^{p-1}\psi(m)\left(\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right)^2 = \left(\frac{3}{p}\right)\cdot p^{\frac{3}{2}},$$

where $(\frac{*}{p})$ denotes the Legendre symbol, and ψ is any fourth-order character mod*p.*

Proof We only prove the second formula in Lemma 2. Similarly, we can deduce the first one. Let $(\frac{*}{p}) = \chi_2$, note the identities $\psi^2 = \chi_2$, $\chi_2(-1) = \chi_2(2) = 1$, $\psi^3 = \overline{\psi}$, $\tau(\chi_2) = \sqrt{p}$, and $\psi(-1) = 1$. From the definition and properties of the classical Gauss sums $\bmod p$, we have

$$\begin{aligned}
&\sum_{m=1}^{p-1} \psi(m) \left(\sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p} \right) \right)^2 \\
&\quad = \sum_{m=1}^{p-1} \psi(m) \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p} \right) + \sum_{m=1}^{p-1} \psi(m) \sum_{a=0}^{p-1} \sum_{b=1}^{p-1} e\left(\frac{ma^3 + mb^3 + a + b}{p} \right) \\
&\quad = \tau(\psi) \sum_{a=1}^{p-1} \overline{\psi}^3(a) e\left(\frac{a}{p} \right) + \tau(\psi) \sum_{a=0}^{p-1} \overline{\psi}\left(a^3 + 1\right) \sum_{b=1}^{p-1} \psi(b) e\left(\frac{b(a+1)}{p} \right) \\
&\quad = \tau^2(\psi) + \tau^2(\psi) \sum_{a=0}^{p-1} \overline{\psi}\left(a^3 + 1\right) \overline{\psi}(a+1) \\
&\quad = \tau^2(\psi) + \tau^2(\psi) \sum_{a=1}^{p-1} \overline{\psi}\left(a^3 - 3a^2 + 3a\right) \overline{\psi}(a) \\
&\quad = \tau^2(\psi) + \tau^2(\psi) \sum_{a=1}^{p-1} \overline{\psi}\left(1 - 3\overline{a} + 3\overline{a}^2\right) \\
&\quad = \tau^2(\psi) + \chi_2(2) \tau^2(\psi) \sum_{a=0}^{p-1} \overline{\psi}\left(12a^2 - 12a + 4\right) - \chi_2(2) \tau^2(\psi) \\
&\quad = \tau^2(\psi) \sum_{a=0}^{p-1} \overline{\psi}\left(3(2a-1)^2 + 1\right) = \tau^2(\psi) \sum_{a=0}^{p-1} \overline{\psi}\left(3a^2 + 1\right).
\end{aligned} \tag{4}$$

Note that $\tau(\psi)\tau(\overline{\psi}) = p$, and for any integer k with $(k,p) = 1$, we have the identity

$$\sum_{a=0}^{p-1} e\left(\frac{ka^2}{p} \right) = \left(\frac{k}{p} \right) \tau(\chi_2). \tag{5}$$

From (5) and the properties of the Gauss sums, we have

$$\begin{aligned}
\sum_{a=0}^{p-1} \overline{\psi}\left(3a^2 + 1\right) &= \frac{1}{\tau(\psi)} \sum_{a=0}^{p-1} \sum_{b=1}^{p-1} \psi(b) e\left(\frac{b(3a^2 + 1)}{p} \right) \\
&= \frac{1}{\tau(\psi)} \sum_{b=1}^{p-1} \psi(b) e\left(\frac{b}{p} \right) \sum_{a=0}^{p-1} e\left(\frac{3ba^2}{p} \right) = \left(\frac{3}{p} \right) \frac{\sqrt{p}}{\tau(\psi)} \sum_{b=1}^{p-1} \psi(b) \chi_2(b) e\left(\frac{b}{p} \right) \\
&= \left(\frac{3}{p} \right) \frac{\sqrt{p}}{\tau(\psi)} \cdot \tau(\overline{\psi}) = \left(\frac{3}{p} \right) \cdot \frac{\tau^2(\overline{\psi})}{\sqrt{p}}.
\end{aligned} \tag{6}$$

Combining (4) and (6), we have the identity

$$\sum_{m=1}^{p-1} \psi(m) \cdot \left(\sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p} \right) \right)^2 = \left(\frac{3}{p} \right) \cdot p^{\frac{3}{2}}.$$

This proves Lemma 2. □

Lemma 3 *If p is a prime with $p \equiv 1(\bmod 8)$, then we have the identity*

$$G^4(m,4;p) = 4p\alpha^2 - p^2 + 8p\alpha G(m,4;p) + 6pG^2(m,4;p).$$

Proof From Lemma 1 and the properties of the Gauss sums, we have

$$\begin{aligned} G(m,4;p) &= \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) = 1 + \sum_{a=1}^{p-1}\left(1+\psi(a)+\psi\left(a^2\right)+\psi\left(a^3\right)\right)e\left(\frac{ma}{p}\right) \\ &= \chi_2(m)\sqrt{p} + \overline{\psi}(m)\tau(\psi) + \psi(m)\tau(\overline{\psi}). \end{aligned} \tag{7}$$

Note that $\psi^2(m) = \chi_2(m)$ and $\tau(\psi)\tau(\overline{\psi}) = p$. From (7) we have

$$\begin{aligned} G^2(m,4;p) &= 3p + 2\chi_2(m)\sqrt{p}\left(\overline{\psi}(m)\tau(\psi) + \psi(m)\tau(\overline{\psi})\right) \\ &\quad + \chi_2(m)\left(\tau^2(\psi) + \tau^2(\overline{\psi})\right) \\ &= p + 2\chi_2(m)\sqrt{p}\alpha + 2\chi_2(m)\sqrt{p}\cdot G(m,4;p) \end{aligned}$$

and

$$\left(G^2(m,4;p) - p\right)^2 = 4p\left(\alpha + G(m,4;p)\right)^2,$$

which implies that

$$G^4(m,4;p) = 4p\alpha^2 - p^2 + 8p\alpha G(m,4;p) + 6pG^2(m,4;p).$$

This proves Lemma 3. □

Lemma 4 *If p is a prime with $p \equiv 1(\bmod 8)$, then we have the identities*

$$\sum_{m=1}^{p-1} \frac{1}{G(m,4;p)} = \frac{2(p-1)\alpha}{p-4\alpha^2};$$

$$\sum_{m=1}^{p-1} \frac{1}{G^2(m,4;p)} = \frac{(p-1)(3p+4\alpha^2)}{(p-4\alpha^2)^2}$$

and

$$\sum_{m=1}^{p-1} \frac{1}{G^3(m,4;p)} = \frac{4(p-1)\alpha(9p-4\alpha^2)}{(p-4\alpha^2)^3}.$$

Proof If $p = 8k+1$, then from (7) and Lemma 1 we have

$$G^2(m,4;p) = 3p + 2\sqrt{p}\left(\psi(m)\tau(\psi) + \overline{\psi}(m)\tau(\overline{\psi})\right) + \chi_2(m)2\sqrt{p}\alpha; \tag{8}$$

$$\begin{aligned} G^3(m,4;p) &= 6p\alpha + 7\chi_2(m)p^{\frac{3}{2}} + 6p\left(\overline{\psi}(m)\tau(\psi) + \psi(m)\tau(\overline{\psi})\right) \\ &\quad + \psi(m)\tau^3(\psi) + \overline{\psi}(m)\tau^3(\overline{\psi}). \end{aligned} \tag{9}$$

Therefore, from (7)–(9) and the orthogonality of the characters $\bmod p$, we have

$$\sum_{m=1}^{p-1} G(m,4;p) = 0; \tag{10}$$

$$\sum_{m=1}^{p-1} G^2(m,4;p) = 3p(p-1) \tag{11}$$

and

$$\sum_{m=1}^{p-1} G^3(m,4;p) = 6p(p-1)\alpha. \tag{12}$$

From (10), (12), and Lemma 3 we have the identity

$$\begin{aligned}\sum_{m=1}^{p-1} \frac{1}{G(m,4;p)} &= \frac{1}{4p\alpha^2-p^2}\sum_{m=1}^{p-1}\bigl(G^3(m,4;p)-8p\alpha-6pG(m,4;p)\bigr)\\ &= \frac{1}{4p\alpha^2-p^2}\cdot\bigl(6p(p-1)\alpha-8p\alpha(p-1)\bigr) = \frac{2(p-1)\alpha}{p-4\alpha^2}.\end{aligned} \tag{13}$$

From (11), (13), and Lemma 3, we have

$$\begin{aligned}\sum_{m=1}^{p-1} \frac{1}{G^2(m,4;p)} &= \frac{1}{4p\alpha^2-p^2}\sum_{m=1}^{p-1}\left(G^2(m,4;p)-\frac{8p\alpha}{G(m,4;p)}-6p\right)\\ &= \frac{1}{4p\alpha^2-p^2}\cdot\left(3p(p-1)-\frac{16p(p-1)\alpha^2}{p-4\alpha^2}-6p(p-1)\right)\\ &= \frac{(p-1)(3p+4\alpha^2)}{(p-4\alpha^2)^2}.\end{aligned} \tag{14}$$

From (10), (13), (14), and Lemma 3, we have

$$\begin{aligned}\sum_{m=1}^{p-1} \frac{1}{G^3(m,4;p)} &= \frac{1}{4p\alpha^2-p^2}\sum_{m=1}^{p-1}\left(G(m,4;p)-\frac{8p\alpha}{G^2(m,4;p)}-\frac{6p}{G(m,4;p)}\right)\\ &= \frac{1}{p-4\alpha^2}\cdot\left(\frac{8\alpha(p-1)(3p+4\alpha^2)}{(p-4\alpha^2)^2}+\frac{12(p-1)\alpha}{p-4\alpha^2}\right)\\ &= \frac{4(p-1)\alpha(9p-4\alpha^2)}{(p-4\alpha^2)^3}.\end{aligned} \tag{15}$$

Now Lemma 4 follows from (13), (14), and (15). □

Lemma 5 *If p is a prime with $p\equiv 1 \pmod 8$, then we have the identity*

$$\sum_{m=1}^{p-1}\frac{H^2(m,3;p)}{G(m,4;p)} = \begin{cases}\frac{2p(p-2)\alpha-2p\alpha\beta+p^{\frac{3}{2}}\beta+p^{\frac{3}{2}}}{p-4\alpha^2}, & \text{if } p\equiv 1 \pmod{24};\\ \frac{2p^2\alpha+2p\alpha\beta-p^{\frac{3}{2}}\beta-p^{\frac{3}{2}}}{p-4\alpha^2}, & \text{if } p\equiv 17 \pmod{24},\end{cases}$$

where $\beta=\tau(\psi)+\tau(\overline{\psi})$ and $\beta^2=2\sqrt{p}\alpha+2p$, ψ is any fourth-order character $\bmod p$.

Proof From (7) and Lemma 2, we have the identity

$$\begin{aligned}
&\sum_{m=1}^{p-1} G(m,4;p)\cdot H^2(m,3;p)\\
&\quad=\sqrt{p}\sum_{m=1}^{p-1}\chi_2(m)\cdot H^2(m,3;p)\\
&\qquad+\sum_{m=1}^{p-1}\bigl(\overline{\psi}(m)\tau(\psi)+\psi(m)\tau(\overline{\psi})\bigr)\cdot H^2(m,3;p)\\
&\quad=-\left(\frac{3}{p}\right)\cdot p^{\frac{3}{2}}+\left(\frac{3}{p}\right)\cdot p^{\frac{3}{2}}\bigl(\tau(\psi)+\tau(\overline{\psi})\bigr), \qquad (16)\\
&\sum_{m=1}^{p-1} G^2(m,4;p)\cdot H^2(m,3;p)\\
&\quad=2\sqrt{p}\alpha\sum_{m=1}^{p-1}\chi_2(m)\cdot H^2(m,3;p)\\
&\qquad+3p\sum_{m=1}^{p-1}H^2(m,3;p)+2\sqrt{p}\sum_{m=1}^{p-1}\bigl(\psi(m)\tau(\psi)+\overline{\psi}(m)\tau(\overline{\psi})\bigr)\cdot H^2(m,3;p)\\
&\quad=2\left(\frac{3}{p}\right)p^2\bigl(\tau(\psi)+\tau(\overline{\psi})\bigr)-2\left(\frac{3}{p}\right)p^{\frac{3}{2}}\alpha+3p\sum_{m=1}^{p-1}H^2(m,3;p). \qquad (17)
\end{aligned}$$

From Lemma 1 we have the identity

$$\begin{aligned}
2\sqrt{p}\alpha\bigl(\tau(\psi)+\tau(\overline{\psi})\bigr)&=\bigl(\tau^2(\psi)+\tau^2(\overline{\psi})\bigr)\bigl(\tau(\psi)+\tau(\overline{\psi})\bigr)\\
&=\tau^3(\psi)+\tau^3(\overline{\psi})+p\bigl(\tau(\psi)+\tau(\overline{\psi})\bigr).
\end{aligned}$$

Applying (9), Lemma 1, and Lemma 2, we have

$$\begin{aligned}
&\sum_{m=1}^{p-1} G^3(m,4;p)\cdot H^2(m,3;p)\\
&\quad=6p\alpha\sum_{m=1}^{p-1}H^2(m,3;p)\\
&\qquad+7p^{\frac{3}{2}}\sum_{m=1}^{p-1}\chi_2(m)H^2(m,3;p)+6p\sum_{m=1}^{p-1}\bigl(\overline{\psi}(m)\tau(\psi)+\psi(m)\tau(\overline{\psi})\bigr)H^2(m,3;p)\\
&\qquad+\sum_{m=1}^{p-1}\bigl(\psi(m)\tau^3(\psi)+\overline{\psi}(m)\tau^3(\overline{\psi})\bigr)\cdot H^2(m,3;p)\\
&\quad=-7\left(\frac{3}{p}\right)p^{\frac{5}{2}}+6\left(\frac{3}{p}\right)p^{\frac{5}{2}}\bigl(\tau(\psi)+\tau(\overline{\psi})\bigr)+\left(\frac{3}{p}\right)p^{\frac{3}{2}}\bigl(\tau^3(\psi)+\tau^3(\overline{\psi})\bigr)\\
&\qquad+6p\alpha\sum_{m=1}^{p-1}H^2(m,3;p)
\end{aligned}$$

$$
\begin{aligned}
&= -7\left(\frac{3}{p}\right)p^{\frac{5}{2}} + \left(\frac{3}{p}\right)p^2(2\alpha + 5\sqrt{p})\big(\tau(\psi) + \tau(\overline{\psi})\big) \\
&\quad + 6p\alpha \sum_{m=1}^{p-1} H^2(m,3;p).
\end{aligned} \tag{18}
$$

From the trigonometrical identity

$$
\sum_{a=0}^{p-1} e\left(\frac{na}{p}\right) = \begin{cases} 0 & \text{if } p \nmid n, \\ p & \text{if } p \mid n, \end{cases} \tag{19}
$$

we can deduce that

$$
\sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right)\right)^2 = \begin{cases} p^2 & \text{if } 3 \nmid (p-1), \\ p^2 - 2p & \text{if } 3 \mid (p-1). \end{cases} \tag{20}
$$

If $p = 24k + 1$, then $(\frac{3}{p}) = 1$. From (16), (17), (18), (20), and Lemma 3, we have

$$
\begin{aligned}
&\sum_{m=1}^{p-1} \frac{H^2(m,3;p)}{G(m,4;p)} \\
&= \frac{1}{4p\alpha^2 - p^2} \sum_{m=1}^{p-1}\big(G^3(m,4;p) - 8p\alpha - 6pG(m,4;p)\big) \cdot H^2(m,3;p) \\
&= \frac{1}{4p\alpha^2 - p^2}\big(-7p^{\frac{5}{2}} + p^2(2\alpha + 5\sqrt{p})\beta + 6p\big(p^2 - 2p\big)\alpha\big) \\
&\quad - \frac{1}{4p\alpha^2 - p^2}\big(8p\alpha\big(p^2 - 2p\big) - 6p^{\frac{5}{2}} + 6p^{\frac{5}{2}}\beta\big) \\
&= \frac{2p(p-2)\alpha - 2p\alpha\beta + p^{\frac{3}{2}}\beta + p^{\frac{3}{2}}}{p - 4\alpha^2}.
\end{aligned} \tag{21}
$$

If $p = 24k + 17$, then $(\frac{3}{p}) = -1$. From (16), (17), (18), (20), and Lemma 3, we have

$$
\begin{aligned}
&\sum_{m=1}^{p-1} \frac{H^2(m,3;p)}{G(m,4;p)} \\
&= \frac{1}{4p\alpha^2 - p^2} \sum_{m=1}^{p-1}\big(G^3(m,4;p) - 8p\alpha - 6pG(m,4;p)\big) \cdot H^2(m,3;p) \\
&= \frac{1}{4p\alpha^2 - p^2}\big(7p^{\frac{5}{2}} - p^2(2\alpha + 5\sqrt{p})\beta + 6p^3\alpha - 8p^3\alpha - 6p^{\frac{5}{2}} + 6p^{\frac{5}{2}}\beta\big) \\
&= \frac{2p^2\alpha + 2p\alpha\beta - p^{\frac{3}{2}}\beta - p^{\frac{3}{2}}}{p - 4\alpha^2}.
\end{aligned} \tag{22}
$$

Now Lemma 5 follows from (21) and (22). □

Lemma 6 *If p is a prime with $p \equiv 17 (\bmod 24)$, then we have*

$$\begin{aligned}\overline{W}_2(p) &= \sum_{m=1}^{p-1} \frac{H^2(m,3;p)}{G^2(m,4;p)} \\ &= \frac{3p^3 + 4p^2\alpha^2 + 8p\alpha^2\beta - 10p^{\frac{3}{2}}\alpha + 2p^2\beta - 8p^{\frac{3}{2}}\alpha\beta + 8p^{\frac{1}{2}}\alpha^3}{(p-4\alpha^2)^2};\end{aligned}$$

If p is a prime with $p \equiv 1 (\bmod 24)$, then we have

$$\begin{aligned}\overline{W}_2(p) &= \sum_{m=1}^{p-1} \frac{H^2(m,3;p)}{G^2(m,4;p)} \\ &= \frac{p(p-2)(3p+4\alpha^2) - 8p\alpha^2\beta + 10p^{\frac{3}{2}}\alpha - 2p^2\beta + 8p^{\frac{3}{2}}\alpha\beta - 8p^{\frac{1}{2}}\alpha^3}{(p-4\alpha^2)^2}.\end{aligned}$$

Proof If p is a prime with $p \equiv 17(\bmod 24)$, then note that $(\frac{3}{p}) = -1$, from (17), (22), and Lemma 3, we have the identity

$$\begin{aligned}\overline{W}_2(p) &= \sum_{m=1}^{p-1} \frac{H^2(m,3;p)}{G^2(m,4;p)} \\ &= \frac{1}{4p\alpha^2 - p^2} \sum_{m=1}^{p-1} \left(G^2(m,4;p) - \frac{8p\alpha}{G(m,4;p)} - 6p \right) \cdot H^2(m,3;p) \\ &= \frac{1}{4p\alpha^2 - p^2} \left(3p^3 - 2p^2\beta + 2p^{\frac{3}{2}}\alpha - 6p^3\right) - \frac{8p\alpha}{4p\alpha^2 - p^2} \sum_{m=1}^{p-1} \frac{H^2(m,3;p)}{G(m,4;p)} \\ &= \frac{3p^2 + 2p\beta - 2\sqrt{p}\alpha}{p-4\alpha^2} + \frac{8\alpha}{p-4\alpha^2} \cdot \frac{2p^2\alpha + 2p\alpha\beta - p^{\frac{3}{2}}\beta - p^{\frac{3}{2}}}{p-4\alpha^2} \\ &= \frac{3p^3 + 4p^2\alpha^2 + 8p\alpha^2\beta - 10p^{\frac{3}{2}}\alpha + 2p^2\beta - 8p^{\frac{3}{2}}\alpha\beta + 8p^{\frac{1}{2}}\alpha^3}{(p-4\alpha^2)^2}.\end{aligned}$$

This proves the first formula in Lemma 6.

Similarly, if $p = 24k + 1$, then $(\frac{3}{p}) = 1$. From (17), (21), and Lemma 3, we can also deduce the second formula. □

Lemma 7 *If p is a prime with $p \equiv 17 (\bmod 24)$, then we have the identity*

$$\begin{aligned}\overline{W}_3(p) = \sum_{m=1}^{p-1} \frac{H^2(m,3;p)}{G^3(m,4;p)} &= \frac{36p^3\alpha - 16p^2\alpha^3 + 16p\alpha^3\beta - 48p^{\frac{3}{2}}\alpha^2}{(p-4\alpha^2)^3} \\ &+ \frac{28p^2\alpha\beta - 48p^{\frac{3}{2}}\alpha^2\beta + 48p^{\frac{1}{2}}\alpha^4 - 5p^{\frac{5}{2}}\beta - 7p^{\frac{5}{2}} + 16p^{\frac{1}{2}}\alpha^4\beta}{(p-4\alpha^2)^3}.\end{aligned}$$

Proof Since $p \equiv 17(\bmod 24)$, so we have $(\frac{3}{p}) = -1$. From (16), Lemma 3, Lemma 5, Lemma 6, and the complex calculations, we can get identity

$$\begin{aligned}
\overline{W}_3(p) &= \sum_{m=1}^{p-1} \frac{H^2(m,3;p)}{G^3(m,4;p)} \\
&= \frac{1}{4p\alpha^2 - p^2} \sum_{m=1}^{p-1} \left(G(m,4;p) - \frac{8p\alpha}{G^2(m,4;p)} - \frac{6p}{G(m,4;p)} \right) \cdot H^2(m,3;p) \\
&= \frac{1}{4p\alpha^2 - p^2} \sum_{m=1}^{p-1} G(m,4;p) \cdot H^2(m,3;p) - \frac{8p\alpha}{4p\alpha^2 - p^2} \sum_{m=1}^{p-1} \frac{H^2(m,3;p)}{G^2(m,4;p)} \\
&\quad - \frac{6p}{4p\alpha^2 - p^2} \cdot \sum_{m=1}^{p-1} \frac{H^2(m,3;p)}{G^(m,4;p)} \\
&= \frac{36p^3\alpha - 16p^2\alpha^3 + 16p\alpha^3\beta - 48p^{\frac{3}{2}}\alpha^2 + +28p^2\alpha\beta - 48p^{\frac{3}{2}}\alpha^2\beta}{(p-4\alpha^2)^3} \\
&\quad + \frac{48p^{\frac{1}{2}}\alpha^4 - 5p^{\frac{5}{2}}\beta - 7p^{\frac{5}{2}} + 16p^{\frac{1}{2}}\alpha^4\beta}{(p-4\alpha^2)^3}.
\end{aligned}$$

This proves Lemma 7. □

Lemma 8 *If p is a prime with* $p \equiv 1(\bmod 24)$, *then we have the identity*

$$\begin{aligned}
\overline{W}_3(p) &= \sum_{m=1}^{p-1} \frac{H^2(m,3;p)}{G^3(m,4;p)} \\
&= \frac{4p(p-2)(9p-4\alpha^2)\alpha - 16p\alpha^3\beta + 48p^{\frac{3}{2}}\alpha^2 - 28p^2\alpha\beta}{(p-4\alpha^2)^3} \\
&\quad + \frac{48p^{\frac{3}{2}}\alpha^2\beta - 48p^{\frac{1}{2}}\alpha^4 + 5p^{\frac{5}{2}}\beta + 7p^{\frac{5}{2}} - 16p^{\frac{1}{2}}\alpha^4\beta}{(p-4\alpha^2)^3}.
\end{aligned}$$

Proof Note that $(\frac{3}{p}) = 1$. From (16), Lemma 3, Lemma 5, Lemma 6, and the complex calculations, we can get identity

$$\begin{aligned}
\overline{W}_3(p) &= \sum_{m=1}^{p-1} \frac{H^2(m,3;p)}{G^3(m,4;p)} \\
&= \frac{1}{4p\alpha^2 - p^2} \sum_{m=1}^{p-1} \left(G(m,4;p) - \frac{8p\alpha}{G^2(m,4;p)} - \frac{6p}{G(m,4;p)} \right) \cdot H^2(m,3;p) \\
&= \frac{1}{4p\alpha^2 - p^2} \sum_{m=1}^{p-1} G(m,4;p) \cdot H^2(m,3;p) - \frac{8p\alpha}{4p\alpha^2 - p^2} \sum_{m=1}^{p-1} \frac{H^2(m,3;p)}{G^2(m,4;p)} \\
&\quad - \frac{6p}{4p\alpha^2 - p^2} \cdot \sum_{m=1}^{p-1} \frac{H^2(m,3;p)}{G^(m,4;p)}
\end{aligned}$$

$$= \frac{4p(p-2)(9p-4\alpha^2)\alpha - 16p\alpha^3\beta + 48p^{\frac{3}{2}}\alpha^2 - 28p^2\alpha\beta}{(p-4\alpha^2)^3}$$
$$+ \frac{48p^{\frac{3}{2}}\alpha^2\beta - 48p^{\frac{1}{2}}\alpha^4 + 5p^{\frac{5}{2}}\beta + 7p^{\frac{5}{2}} - 16p^{\frac{1}{2}}\alpha^4\beta}{(p-4\alpha^2)^3}.$$

This proves Lemma 8. □

3 Proofs of the theorems

Now we prove our theorems. In fact, if $p \equiv 1(\bmod 8)$, then for any number-theoretic function $F(m)$ and integer $k \geq 4$, from Lemma 3 we have

$$\frac{1}{G^4(m,4;p)} = \frac{1}{4p\alpha^2 - p^2}\left(1 - \frac{8p\alpha}{G^3(m,4;p)} - \frac{6p}{G^2(m,4;p)}\right). \tag{23}$$

For any integer $k \geq 4$, from formula (23) we have

$$\begin{aligned} V_k(p) &= \sum_{m=1}^{p-1} \frac{F(m)}{G^k(m,4;p)} = \sum_{m=1}^{p-1} \frac{F(m)}{G^{k-4}(m,4;p)} \cdot \frac{1}{G^4(m,4;p)} \\ &= \frac{1}{4p\alpha^2 - p^2} \sum_{m=1}^{p-1} \frac{F(m)}{G^{k-4}(m,4;p)} \cdot \left(1 - \frac{8p\alpha}{G^3(m,4;p)} - \frac{6p}{G^2(m,4;p)}\right) \\ &= \frac{8\alpha}{p-4\alpha^2} \cdot V_{k-1}(p) + \frac{6}{p-4\alpha^2} \cdot V_{k-2}(p) - \frac{1}{p(p-4\alpha^2)} \cdot V_{k-4}(p). \end{aligned}$$

This proves Theorem 1.

Note that $W_0(p) = p - 1$, so Theorem 2 follows from Theorem 1 and Lemma 4.

Theorem 3 follows from Lemma 5, Lemma 6, and Lemma 7.

Theorem 4 follows from Lemma 5, Lemma 6, and Lemma 8.

This completes the proofs of our all results.

4 Conclusion

The main purpose of this article is by using the properties of the fourth character modulo a prime p and the analytic methods to study the calculating problems of a certain hybrid power mean involving the two-term exponential sums and the reciprocal of quartic Gauss sums, and to give a series of fourth-order linear recursive formulae. These results not only give the exact values of some special Gauss sums, but they are also some new contribution to the research in related fields.

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References

1. Zhang, W.P., Chen, Z.Y.: A hybrid power mean of the two-term exponential sums and the reciprocal of the quartic Gauss sums. Bull. Math. Soc. Sci. Math. Roum. **110**, 191–198 (2019)
2. Chen, Z.Y., Zhang, W.P.: On the fourth-order linear recurrence formula related to classical Gauss sums. Open Math. **15**, 1251–1255 (2017)
3. Zhang, W.P., Han, D.: On the sixth power mean of the two-term exponential sums. J. Number Theory **136**, 403–413 (2014)
4. Zhang, H., Zhang, W.P.: The fourth power mean of two-term exponential sums and its application. Mathem. Rep. **19**, 75–81 (2017)
5. Shen, S.M., Zhang, W.P.: On the quartic Gauss sums and their recurrence property. Adv. Differ. Equ. **2017**, 43 (2017)
6. Han, D.: A hybrid mean value involving two-term exponential sums and polynomial character sums. Czechoslov. Math. J. **64**, 53–62 (2014)
7. Liu, X.Y., Zhang, W.P.: On the high-power mean of the generalized Gauss sums and Kloosterman sums. Mathematics **7**, 907 (2019)
8. Li, X.X., Hu, J.Y.: The hybrid power mean quartic Gauss sums and Kloosterman sums. Open Math. **15**, 151–156 (2017)
9. Chen, L., Chen, Z.Y.: Some new hybrid power mean formulae of trigonometric sums. Adv. Differ. Equ. **2020**, 220 (2020)
10. Chen, L., Hu, J.Y.: A linear recurrence formula involving cubic Gauss sums and Kloosterman sums. Acta Math. Sinica (Chin. Ser.) **61**, 67–72 (2018)
11. Chowla, S., Cowles, J., Cowles, M.: On the number of zeros of diagonal cubic forms. J. Number Theory **9**, 502–506 (1977)
12. Zhang, W.P., Hu, J.Y.: The number of solutions of the diagonal cubic congruence equation $\bmod p$. Mathem. Rep. **20**, 70–76 (2018)
13. Kim, T., Kim, D.S.: Degenerate polyexponential functions and degenerate Bell polynomials. J. Math. Anal. Appl. **487**(2), 124017 (2020)
14. Kim, D.S., Kim, T.: A note on a new type of degenerate Bernoulli numbers. Russ. J. Math. Phys. **27**, 227–235 (2020)
15. Zhang, W.P., Li, H.L.: Elementary Number Theory. Shaanxi Normal University Press, Xi'an (2013)
16. Apostol, T.M.: Introduction to Analytic Number Theory. Springer, New York (1976)
17. Pan, C.D., Pan, C.B.: Goldbach Conjecture. Science Press, Beijing (1992)
18. Ireland, K., Rosen, M.: A Classical Introduction to Modern Number Theory. Springer, New York (1982)