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A new fourth-order integrable nonlinear equation: breather, rogue waves, other lump interaction phenomena, and conservation laws

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Abstract
In this study, we investigate a new fourth-order integrable nonlinear equation. Firstly, by means of the efficient Hirota bilinear approach, we establish novel types of solutions which include breather, rogue, and three-wave solutions. Secondly, with the aid of Lie symmetry method, we report the invariance properties of the studied equation such as the group of transformations, commutator and adjoint representation tables. A differential substitution is found by nonlinear self-adjointness (NSA) and thereafter the associated conservation laws are established. We show some dynamical characteristics of the obtained solutions through via the 3-dimensional and contour graphs.

Keywords: Fourth-order integrable nonlinear equation; Lump solutions; Interaction solutions; Invariant analysis; Conservation laws

1 Introduction

In differential equation (DE) concepts, Cauchy problem (CP) is considered as one of the most fundamental problems to analyze a solution of a DE which satisfies initial data. Classical methods, like the Laplace and Fourier transformation methods, have been introduce to solve CPs for linear partial and ordinary DEs. The isomonodromic and inverse scattering approaches were created to handle CPs for nonlinear partial and ordinary DEs, respectively [1–3]. A captivating and excellent field of study is the analysis of the exact solutions (ESs) and the problems of constructing solutions for an expansive range of nonlinear equations.

The ESs for partial DEs describe important physical and mathematical aspects. A soliton solution is an ES that is investigated by exponentially located functions that move in all directions in both time and space. Also, a lump solution can be regarded as an exact solution of a partial DE, obtained by taking long wave limits from soliton theory [3]. Nonetheless, only in space, a lump solution can be localized in all directions. Additionally, it is well known that more nonlinear phenomena can be described by interaction solutions between lump and soliton solutions. Nevertheless, the properties of the interaction are seldom debated, as the mathematical calculation involved is much more complex.

Over the last two decades, several researchers have studied solitary solutions, lump solutions, and other types of integrable equation solutions. This includes the Ishimori-I equation [4], the Davey–Stewarton equation II [3], the BKP equation [5, 6], three-dimensional three-wave resonant interaction [7], and self-consistent KP equation [8]. Many nonintegrable equations do have lump solutions, such as the generalized KP and Sawada–Kotera equations [9–12]. Through important properties of lump solutions, it can be understood that amplitudes, shapes, speeds of solitons will be preserved after collision with another soliton, and this is the elastic property of a collision. Moreover, interactions between rouge, breather, three-wave, and kink solitary wave solutions have been established in [13–15]. In addition, various researches demonstrate the existence of interaction solutions between lumps and other types of specific solutions to a nonlinear integrable equation [16–19]. Furthermore, in order to justify the existence and uniqueness, some important internal properties, as well as the integrability of a DE, computing conservation laws and symmetries are some of the best aspects many scientists employ to do the job [20–26]. Therefore, establishing lumps solutions, their interactions, as well as the conservation laws for various types of DE, are of humongous importance.

This study is aimed at using the Hirota bilinear approach [14] to construct some novel breather, rogue, and three-wave solutions to a new integrable fourth-order nonlinear equation. On the other hand, the Lie symmetry analysis [27] is going to be used to generate the conservation laws for this nonlinear equation.

The new integrable fourth-order nonlinear equation is given by [28]

$$\Xi_{tt} + \Xi_{txxx} + \alpha(\Xi_x \Xi_t)_x = 0, \tag{1}$$

where α is the coefficient of the nonlinear term $(\Xi_x \Xi_t)_x$. Nonlinearity arises when the change of the output is not proportional to the change of the input [29].

2 Lump interaction phenomena

In this section, we construct some novel breather, rogue, and three-wave solutions to Eq. (1).

Applying the Cole–Hopf transformation [28, 30]

$$\Xi(x,t) = \frac{6}{\alpha}\big(\ln f(x,t)\big)_x \tag{2}$$

to Eq. (1) yields the following bilinear form:

$$3f_{xt}f_{xx} - 3f_x f_{xxxt} - f_t(f_t + f_{xxx}) + f(f_{tt} + f_{xxxt}) = 0. \tag{3}$$

2.1 Breather waves

In this subsection, we construct the breather wave solutions to Eq. (1).

Consider the following test function [31–33] as a solution to the bilinear equation (3):

$$f(x,t) = e^{\xi_1} + m_1 \cos(\xi_2) + m_2 e^{\xi_3}, \tag{4}$$

where $\xi_1 = -p_1(a_0 t + x)$, $\xi_2 = p_0(b_0 t + x)$, and $\xi_3 = p_1(a_0 t + x)$.

Substituting Eq. (4) into Eq. (3) gives a polynomial in the powers of trigonometric and exponential functions. Collecting the coefficients of the same power and equating each sum to zero yields an algebraic system of equations. We solve this system of equations to obtain the values of the parameters involved. Substituting the values of the parameters into Eq. (2) gives the following breather wave solutions to Eq. (1):

Case-1: When

$$p_1 = \frac{\sqrt{-8a_0b_0+3a_0^2-3b_0^2}}{\sqrt{24a_0+8b_0}}, \qquad p_0 = -\frac{\sqrt{3a_0-b_0}}{2\sqrt{2}}, \qquad m_2 = -\frac{b_0m_1^2(3a_0-b_0)}{4a_0(a_0-3b_0)},$$

we get

$$\begin{aligned} f_1(x,t) = {} & -\frac{b_0m_1^2(3a_0-b_0)e^{(\frac{\sqrt{-8a_0b_0+3a_0^2-3b_0^2}(a_0t+x)}{\sqrt{24a_0+8b_0}})}}{4a_0(a_0-3b_0)} \\ & + e^{(-\frac{\sqrt{-8a_0b_0+3a_0^2-3b_0^2}(a_0t+x)}{\sqrt{24a_0+8b_0}})} + m_1\cos\left(\frac{\sqrt{3a_0-b_0}(b_0t+x)}{2\sqrt{2}}\right). \end{aligned} \tag{5}$$

Thus,

$$\begin{aligned} & \Xi_1(x,t) \\ & \quad = \frac{6(-\frac{e^{-\Theta_1}\sqrt{-8a_0b_0+3a_0^2-3b_0^2}}{\sqrt{24a_0+8b_0}} - \frac{b_0e^{\Theta_1}m_1^2(3a_0-b_0)\sqrt{-8a_0b_0+3a_0^2-3b_0^2}}{4a_0(a_0-3b_0)\sqrt{24a_0+8b_0}} - \frac{m_1\sqrt{3a_0-b_0}\sin(\Theta_2)}{2\sqrt{2}})}{\alpha(-\frac{b_0e^{\Theta_1}m_1^2(3a_0-b_0)}{4a_0(a_0-3b_0)} + e^{-\Theta_1} + m_1\cos(\Theta_2))}, \end{aligned} \tag{6}$$

where

$$\Theta_1 = \frac{\sqrt{-8a_0b_0+3a_0^2-3b_0^2}(a_0t+x)}{\sqrt{24a_0+8b_0}}$$

and

$$\Theta_2 = \frac{\sqrt{3a_0-b_0}(b_0t+x)}{2\sqrt{2}}.$$

Case-2: When

$$\begin{aligned} a_0 &= \frac{p_1^6-6p_0^2p_1^4+9p_0^4p_1^2-3\sqrt{-p_0^2p_1^2(3p_0^2-p_1^2)^2}p_1^2+\sqrt{-p_0^2p_1^2(3p_0^2-p_1^2)^2}p_0^2}{2(3p_0^2p_1^2-p_1^4)}, \\ b_0 &= \frac{p_0^4-3p_1^2p_0^2-\sqrt{-9p_1^2p_0^6+6p_1^4p_0^4-p_1^6p_0^2}}{2p_0^2}, \\ m_2 &= \bigl(-7m_1^2p_1^8+55m_1^2p_0^2p_1^6-109m_1^2p_0^4p_1^4+24m_1^2\sqrt{-p_0^2p_1^2\bigl(3p_0^2-p_1^2\bigr)^2}p_1^4 \\ &\quad + 21m_1^2p_0^6p_1^2+24m_1^2\sqrt{-p_0^2p_1^2\bigl(3p_0^2-p_1^2\bigr)^2}p_0^2p_1^2\bigr) \\ &\quad /\bigl(4\bigl(-49p_1^8+145p_0^2p_1^6+5p_0^4p_1^4+3p_0^6p_1^2\bigr)\bigr), \end{aligned}$$

we get

$$f_2(x,t) = \mathrm{e}^{-\Theta_3} + m_1\cos(\Theta_4) + \frac{\mathrm{e}^{\Theta_3}(-\Theta_5 + \Theta_6 + 21m_1^2p_1^2p_0^6 - 109m_1^2p_1^4p_0^4 + 55m_1^2p_1^6p_0^2)}{4(-49p_1^8 + 145p_0^2p_1^6 + 5p_0^4p_1^4 + 3p_0^6p_1^2)}. \tag{7}$$

Thus,

$$\Xi_2(x,t) = \frac{6(m_1p_0(-\sin(\Theta_4)) - \mathrm{e}^{-\Theta_3}p_1 + \frac{\mathrm{e}^{\Theta_3}(-\Theta_5+\Theta_6+\Theta_7)p_1}{\Theta_8})}{\alpha(\mathrm{e}^{-\Theta_3} + \frac{\mathrm{e}^{\Theta_3}(-\Theta_5+\Theta_6+\Theta_7)}{\Theta_8} + m_1\cos(\Theta_4))}, \tag{8}$$

where

$$\Theta_3 = p_1\left(\frac{(p_1^6 - 6p_0^2p_1^4 + 9p_0^4p_1^2 - 3\sqrt{-p_0^2p_1^2(3p_0^2-p_1^2)^2}p_1^2 + \sqrt{-p_0^2p_1^2(3p_0^2-p_1^2)^2}p_0^2)t}{2(3p_0^2p_1^2 - p_1^4)} + x\right),$$

$$\Theta_4 = p_0\left(\frac{(p_0^4 - 3p_1^2p_0^2 - \sqrt{-9p_1^2p_0^6 + 6p_1^4p_0^4 - p_1^6p_0^2})t}{2p_0^2} + x\right),$$

$$\Theta_5 = 7m_1^2p_1^8 - 24m_1^2p_0^2p_1^2\sqrt{-p_0^2p_1^2\left(3p_0^2-p_1^2\right)^2},$$

$$\Theta_6 = 24m_1^2p_1^4\sqrt{-p_0^2p_1^2\left(3p_0^2-p_1^2\right)^2},$$

$$\Theta_7 = 21m_1^2p_1^2p_0^6 - 109m_1^2p_1^4p_0^4 + 55m_1^2p_1^6p_0^2,$$

$$\Theta_8 = 4\left(-49p_1^8 + 145p_0^2p_1^6 + 5p_0^4p_1^4 + 3p_0^6p_1^2\right).$$

2.2 Rogue waves

In this subsection, we construct the rogue wave solutions to Eq. (1). Consider the following test function [34] as a solution to the bilinear equation (3):

$$f(x,t) = \xi_1^2 + \xi_2^2 + b_7 + T\cosh(\xi_3), \tag{9}$$

where $\xi_1 = b_1t + b_2x + b_3$, $\xi_2 = b_4t + b_5x + b_6$, and $\xi_3 = T_1t + T_2x$.

Substituting Eq. (9) into Eq. (3) gives a polynomial in the powers of x, t, and hyperbolic functions. Collecting the coefficients of the same power and equating each sum to zero yields an algebraic system of equations. We solve this system of equations to obtain the values of the parameters involved. Putting the values of the parameters into Eq. (2) gives the following rogue waves solution to Eq. (1):

When

$$b_1 = -\frac{b_2}{3T_1^2}, \qquad b_4 = -\frac{\mathrm{i}b_2}{3T_1^2}, \qquad b_5 = \mathrm{i}b_2, \qquad b_6 = \frac{\sqrt{3}TT_1^3 + 2\mathrm{i}b_2b_3}{2b_2}, \qquad T_2 = -T_1^3,$$

we have

$$f_1(x,t) = \left(b_2t - \frac{b_2x}{3T_1^2} + b_3\right)^2 + b_7 + \Theta_{10}^2 + T\cosh\left(T_1x - tT_1^3\right). \tag{10}$$

Thus,

$$\Xi_1(x,t) = \frac{6(-\frac{2b_2(b_2t-\frac{b_2x}{3T_1^2}+b_3)}{3T_1^2} - \Theta_9 + TT_1\sinh(T_1x - tT_1^3))}{\alpha((b_2t - \frac{b_2x}{3T_1^2} + b_3)^2 + b_7 + \Theta_{10}^2 + T\cosh(T_1x - tT_1^3))}, \tag{11}$$

where

$$\Theta_9 = \frac{2ib_2(ib_2t + \frac{\sqrt{3}TT_1^3+2ib_2b_3}{2b_2} - \frac{ib_2x}{3T_1^2})}{3T_1^2},$$
$$\Theta_{10} = ib_2t + \frac{\sqrt{3}TT_1^3 + 2ib_2b_3}{2b_2} - \frac{ib_2x}{3T_1^2}.$$

2.3 Three-wave solutions

In this subsection, the three-wave solutions to Eq. (1) are revealed.

Consider the following test function [35] as a solution to the bilinear equation (3):

$$f(\xi,z,t) = c_1\mathrm{e}^{\xi_1} + c_2\mathrm{e}^{-\xi_1} + c_3\sin(\xi_2) + c_4\sinh(\xi_3), \tag{12}$$

where $\xi_1 = b_1x + b_2t$, $\xi_2 = b_3x + b_4t$, and $\xi_3 = b_5x + b_6t$.

Substituting Eq. (12) into Eq. (3) gives a polynomial in the powers of trigonometric, hyperbolic, and exponential functions. Collecting the coefficients of the same power and equating each sum to zero provides an algebraic system of equations. We solve this system of equations to obtain the values of the parameters involved. Putting the values of the parameters into Eq. (2) produces the following wave solutions to Eq. (1):

Case-1: When

$$b_1 = \sqrt{4\sqrt{3}b_3^2 - 7b_3^2}, \qquad b_4 = 3\sqrt{3}b_3^3 - 5b_3^3, \qquad c_3 = 2\sqrt{7c_1c_2 - 4\sqrt{3}c_1c_2}, \qquad c_4 = 0,$$
$$b_2 = \frac{1}{4}\Big(-\big((4\sqrt{3}-7)b_3^2\big)^{3/2} - 3\sqrt{(4\sqrt{3}-7)b_3^2}b_3^2\Big),$$

we have

$$f_1(x,t) = c_1e^{\Theta_{11}} + c_2e^{-\Theta_{11}} + 2\sqrt{7c_1c_2 - 4\sqrt{3}c_1c_2}\sin(\Theta_{12}). \tag{13}$$

Thus,

$$\begin{aligned}\Xi_1(x,t) = {} & \Big(6\Big(\sqrt{4\sqrt{3}b_3^2 - 7b_3^2}c_1\mathrm{e}^{\Theta_{11}} - \sqrt{4\sqrt{3}b_3^2 - 7b_3^2}c_2\mathrm{e}^{-\Theta_{11}} \\ & + 2b_3\sqrt{7c_1c_2 - 4\sqrt{3}c_1c_2}\cos(\Theta_{12})\Big)\Big) \\ & /\Big(\alpha\big(c_1\mathrm{e}^{\Theta_{11}} + c_2\mathrm{e}^{-\Theta_{11}} + 2\sqrt{7c_1c_2 - 4\sqrt{3}c_1c_2}\sin(\Theta_{12})\big)\Big),\end{aligned} \tag{14}$$

where

$$\Theta_{11} = \frac{1}{4}\Big(-\big((4\sqrt{3}-7)b_3^2\big)^{3/2} - 3\sqrt{(4\sqrt{3}-7)b_3^2}b_3^2\Big)t + \sqrt{4\sqrt{3}b_3^2 - 7b_3^2}x$$

and

$$\Theta_{12} = \left(3\sqrt{3}b_3^3 - 5b_3^3\right)t + b_3 x.$$

Case-2: When

$$b_1 = -\sqrt[3]{2}\sqrt[3]{b_6} - b_5, \qquad b_2 = b_6, \qquad c_3 = 0, \qquad c_4 = \frac{2\sqrt{\Theta_{13}}}{\sqrt{16b_5^6 b_6 - b_6^3}},$$

we have

$$f_2(x,t) = \frac{2\sqrt{\Theta_{13}}\sinh(b_6 t + b_5 x)}{\sqrt{16b_5^6 b_6 - b_6^3}} + c_1 e^{\Theta_{14}} + c_2 e^{-\Theta_{14}}. \tag{15}$$

Thus,

$$\Xi_2(x,t) = \frac{6((-\sqrt[3]{2}\sqrt[3]{b_6} - b_5)c_1 e^{\Theta_{14}} + (\sqrt[3]{2}\sqrt[3]{b_6} + b_5)c_2 e^{-\Theta_{14}} + \frac{2b_5\sqrt{\Theta_{13}}\cosh(b_6 t + b_5 x)}{\sqrt{16b_5^6 b_6 - b_6^3}})}{\alpha(\frac{2\sqrt{\Theta_{13}}\sinh(b_6 t + b_5 x)}{\sqrt{16b_5^6 b_6 - b_6^3}} + c_1 e^{\Theta_{14}} + c_2 e^{-\Theta_{14}})}, \tag{16}$$

where

$$\begin{aligned}\Theta_{13} &= -48\sqrt[3]{2}b_5^5 b_6^{4/3} c_1 c_2 - 48\, 2^{2/3} b_5^4 b_6^{5/3} c_1 c_2 + 12\sqrt[3]{2} b_5^2 b_6^{7/3} c_1 c_2 \\ &\quad + 12\, 2^{2/3} b_5 b_6^{8/3} c_1 c_2 + 7b_6^3 c_1 c_2 - 24 b_5^3 b_6^2 c_1 c_2 - 16 b_5^6 b_6 c_1 c_2, \\ \Theta_{14} &= b_6 t + \left(-\sqrt[3]{2}\sqrt[3]{b_6} - b_5\right)x.\end{aligned}$$

3 Numerical simulations

In this section, using suitable values of parameters and different values of the coefficients of the nonlinear term in the studied equation, we present the dynamics of lump solution with the periodic and singular periodic wave solutions.

Figure 1 presents the interaction between lump, kink, and singular periodic (breather) wave solutions. Figure 1 (a,d) and (b,e) display lump-kink shape when $\alpha < 0$ and Fig. 1 (c,f) displays lump-kink shape with some singularity when $\alpha > 0$. Figure 2 presents the interaction between lump and kink (breather) solutions. Figure 2 (a,d), (b,e), and (c,f) display lump-kink solutions throughout the range of values $-10 < \alpha < 10$. Figure 3 presents the interaction between lump and periodic (rogue) wave solutions. Throughout the range of values $-10 < \alpha < 10$, Figs. 3 (a,d), (b,e), and (c,f) display the singular bell-type shape. Figure 4 presents the interaction between the lump and periodic (rogue) wave solutions. Throughout the range of values $-10 < \alpha < 10$, Figs. 4 (a,d), (b,e), and (c,f) display the lump-kink shape. Figure 5 presents the interaction between the lump and singular periodic (multiwave) wave solutions. Throughout the range of values $-10 < \alpha < 10$, Figs. 5 (a,d), (b,e), and (c,f) display the lump-period shape. Figure 6 presents the interaction between the lump, kink and periodic (multiwave) wave solutions. Throughout the range of values $-10 < \alpha < 10$, Figs. 6 (a,d), (b,e), and (c,f) display the lump-kink shape.

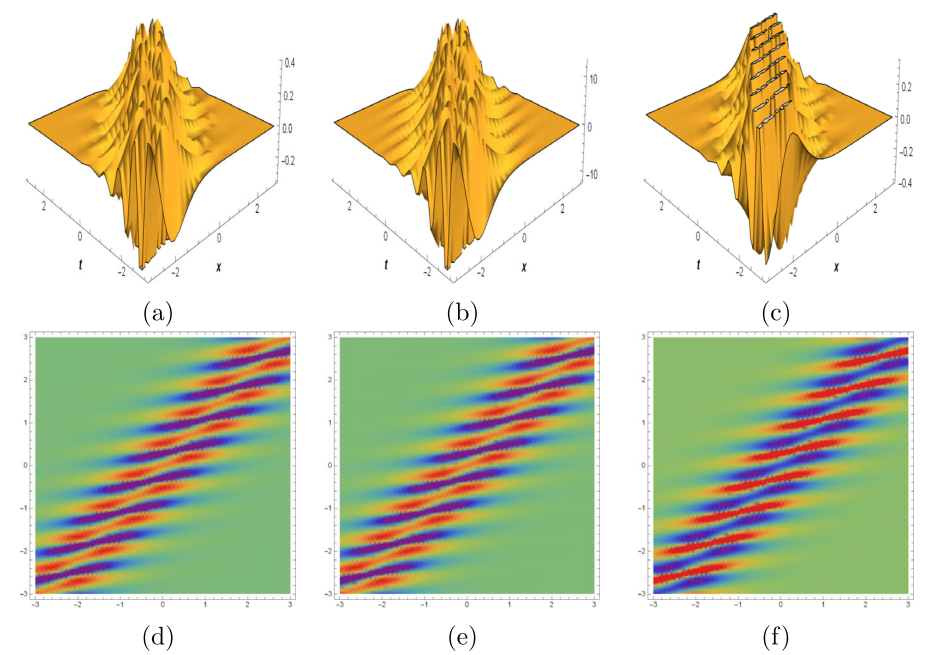

Figure 1 The 3D and density profiles of Eq. (6) for $a_0 = -10$, $b_0 = -0.92$, $m_1 = 6.1$

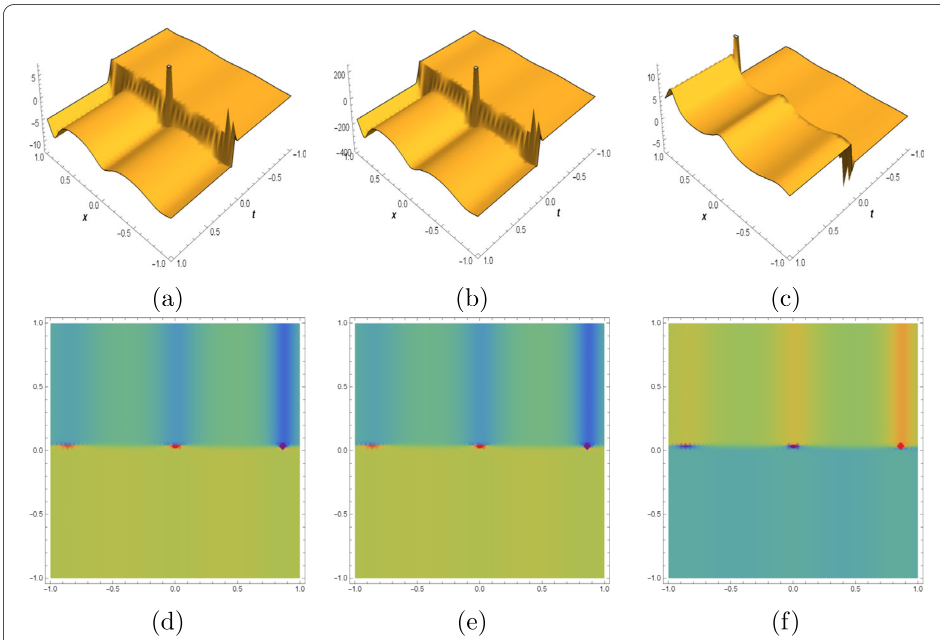

Figure 2 The 3D and density profiles of Eq. (8) for $p_0 = -7.25$, $p_1 = -0.26$, $m_1 = -7.1$

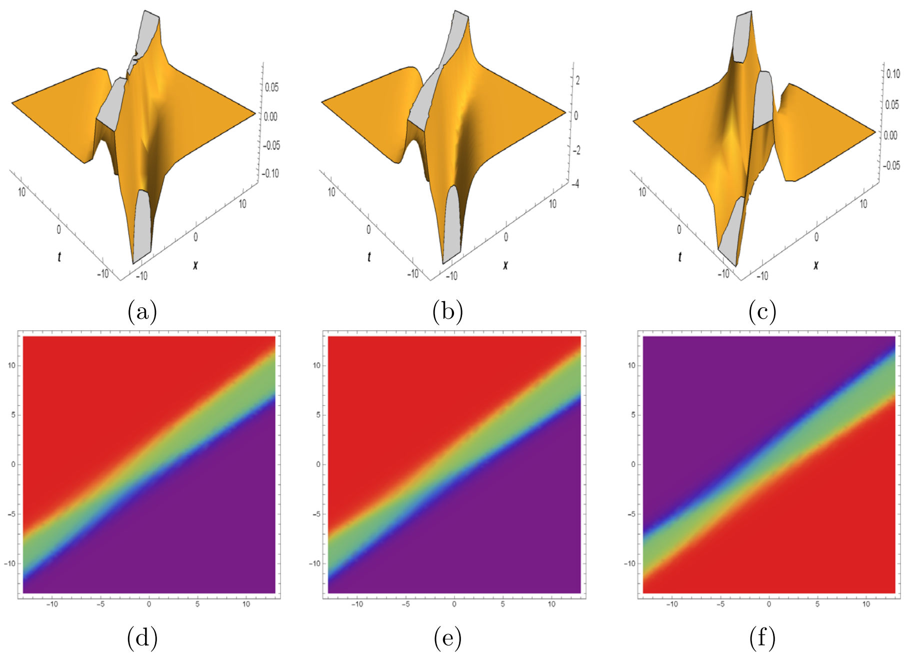

Figure 3 The 3D and density profiles of imaginary part of Eq. (11) for $b_2 = 1.23$, $b_3 = 2.04$, $b_7 = 2.19$, $T_1 = 1$, $T = 1$

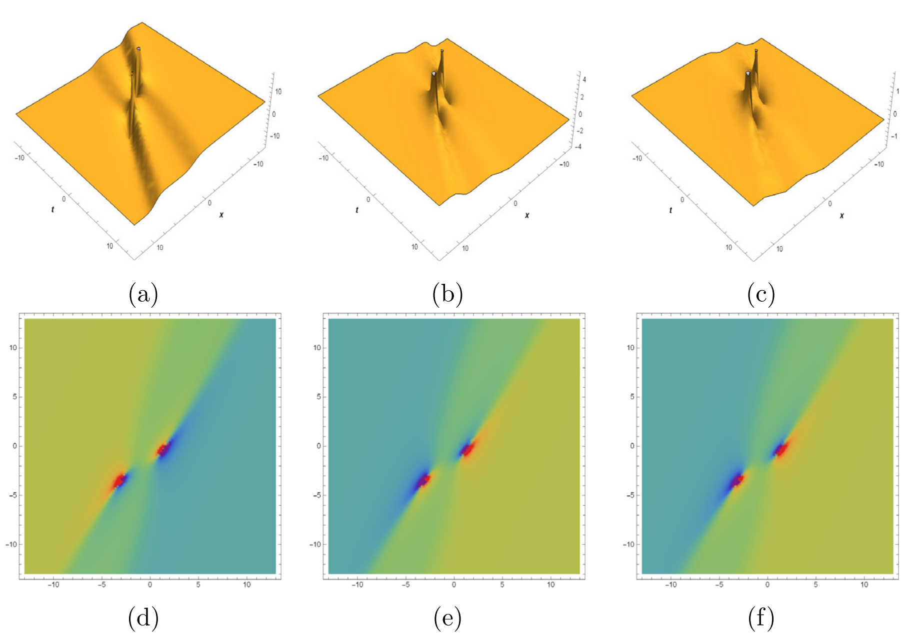

Figure 4 The 3D and density profiles of imaginary part of Eq. (11) for $b_2 = 1.23$, $b_3 = 2.04$, $b_7 = 2.19$, $T_1 = 1$, $T = 1$

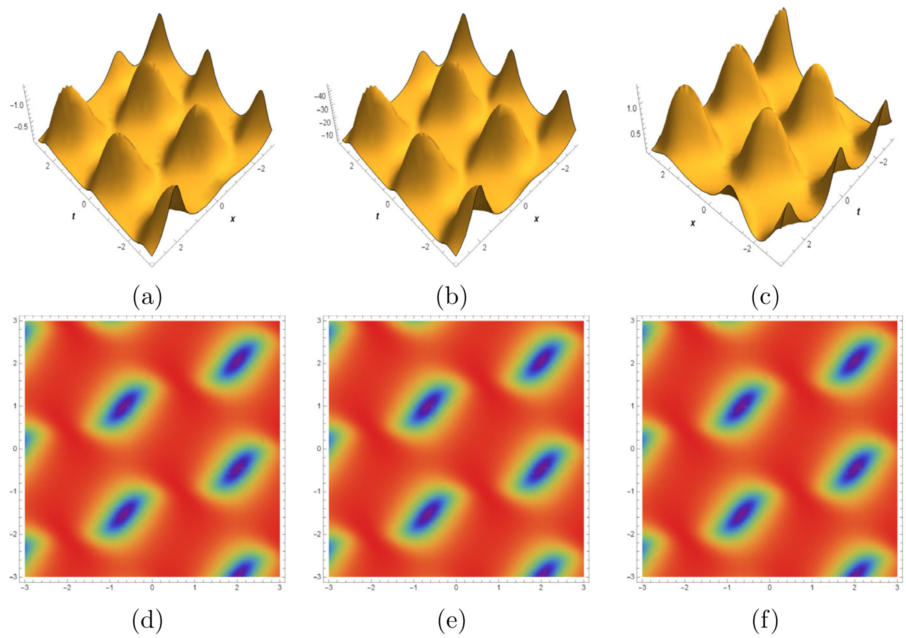

Figure 5 The 3D and density profiles of imaginary part of Eq. (14) for $c_1 = 1.04$, $b_3 = 2.62$, $c_2 = 2.38$

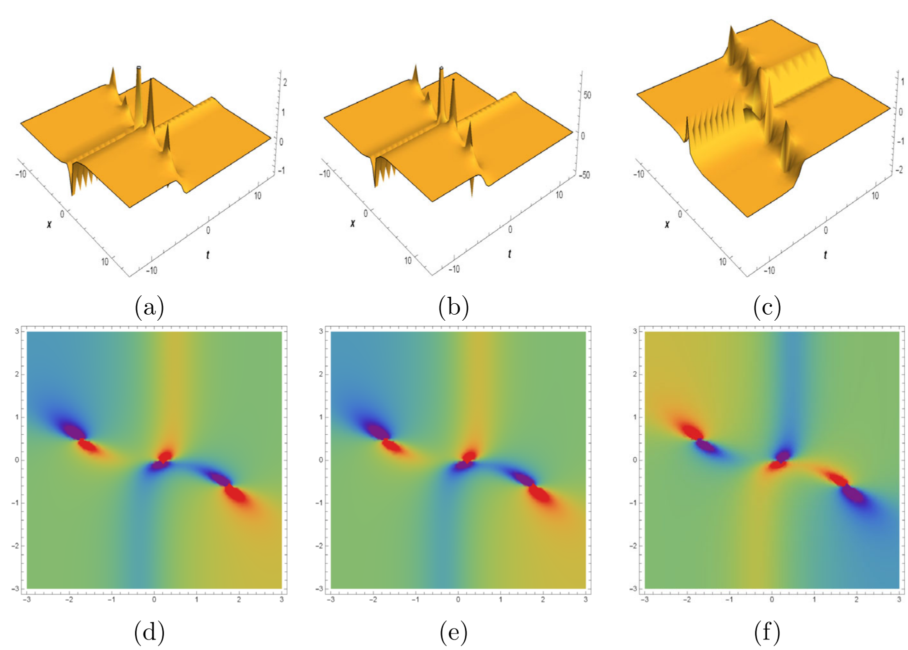

Figure 6 The 3D and density profiles of imaginary part of Eq. (16) for $c_1 = 0.74$, $b_6 = -1$, $b_5 = -1$, $c_2 = -1$

4 Invariant analysis

The symmetries of (1) are expressed in the form of a vector field as

$$\mathcal{X} = \xi_1(x,t,\Xi)\frac{\partial}{\partial x} + \xi_2(x,t,\Xi)\frac{\partial}{\partial t} + \xi_3(x,t,\Xi)\frac{\partial}{\partial \Xi}. \tag{17}$$

And the associated infinitesimals are:

$$\begin{aligned} \xi_1 &= c_1 - xc_3, \\ \xi_2 &= c_2 - 3tc_3, \\ \xi_3 &= c_4 + \Xi_3, \end{aligned} \tag{18}$$

where c_i $(i = 1,2,3,4)$ denote arbitrary constants. Consequently, (1) admits the following fields:

$$\begin{aligned} \mathcal{X}_1 &= \partial_x, \\ \mathcal{X}_2 &= \partial_\Xi, \\ \mathcal{X}_3 &= \partial_t, \\ \mathcal{X}_4 &= -3t\partial_t + \Xi\partial_\Xi - x\partial_x. \end{aligned} \tag{19}$$

4.1 Group of transformations

The governing equation can be considered as a submanifold in the jet space $J^3(\mathcal{R}^2, \mathcal{R}^2)$. Therefore, to get the group transformations that the infinitesimal generators produce, $\eta_1\partial_x + \eta_2\partial_t + \eta_3\partial_\Xi$, the following systems of differential equations must be solved:

$$\begin{aligned} \frac{d\bar{x}(\epsilon)}{d\epsilon} &= \xi_1\bigl(\bar{x}(\epsilon), \bar{t}(\epsilon), \bar{\Xi}(\epsilon)\bigr), \quad \bar{x}(0) = x, \\ \frac{d\bar{t}(\epsilon)}{d\epsilon} &= \xi_2\bigl(\bar{x}(\epsilon), \bar{t}(\epsilon), \bar{\Xi}(\epsilon)\bigr), \quad \bar{t}(0) = t, \\ \frac{d\bar{\Xi}(\epsilon)}{d\epsilon} &= \xi_3\bigl(\bar{x}(\epsilon), \bar{t}(\epsilon), \bar{\Xi}(\epsilon)\bigr), \quad \bar{\Xi}(0) = u. \end{aligned} \tag{20}$$

Taking the exponential of the obtained infinitesimal symmetries, the one-parameter groups $G_k(\epsilon)$ generated by $\mathcal{X}_k$ for $k = 1, \dots, 3$, are given by:

$$\begin{aligned} \mathcal{G}_1 &: (x,t,\Xi) \to (x+\epsilon, t, \Xi), \\ \mathcal{G}_2 &: (x,t,\Xi) \to (x, t+\epsilon, \Xi), \\ \mathcal{G}_3 &: (x,t,\Xi) \to (x, t, \Xi+\epsilon), \\ \mathcal{G}_4 &: (x,t,\Xi) \to \bigl(e^{-\epsilon}x, e^{-3\epsilon}t, e^{\epsilon}\Xi\bigr), \end{aligned} \tag{21}$$

where entries give the transform point $e^{\epsilon\mathcal{X}_i}(x,t,\Xi) = (\bar{x}, \bar{t}, \bar{\Xi})$.

Note that in general a family of solutions, called invariant solutions, must refer to each parameter subgroup of the complete symmetry group of a system. We may therefore state the following: If $\Xi = f(x,t)$ is a solution for the governing equation, then such are the

Table 1 The commutation relations of infinitesimal generators $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{X}_3$

$[\mathcal{X}_i, \mathcal{X}_j]$	$\mathcal{X}_1$	$\mathcal{X}_2$	$\mathcal{X}_3$	$\mathcal{X}_4$
$\mathcal{X}_1$	0	0	0	$-\mathcal{X}_1$
$\mathcal{X}_2$	0	0	0	$\mathcal{X}_2$
$\mathcal{X}_3$	0	0	0	$-3\mathcal{X}_3$
$\mathcal{X}_4$	$\mathcal{X}_1$	$-\mathcal{X}_2$	$3\mathcal{X}_3$	0

following functions:

$$\begin{aligned} \Xi_1 &= \mathcal{G}_1(\epsilon).f(x,t) = f(x+\epsilon, t), \\ \Xi_2 &= \mathcal{G}_2(\epsilon).f(x,t) = f(x, t+\epsilon), \\ \Xi_3 &= \mathcal{G}_3(\epsilon).f(x,t) = f\big(e^{-\epsilon}x, e^{-3\epsilon}t\big), \end{aligned} \tag{22}$$

Now, one can get the general category of symmetries by considering a general linear combination $c_1\mathcal{X}_1 + c_2\mathcal{X}_2 + c_3\mathcal{X}_3 + c_4\mathcal{X}_4$ of the given vector fields. In particular, if $\mathcal{G}$ is the symmetry group action near identity, it can be expressed in the form $\mathcal{G} = e^{\epsilon_3\mathcal{X}_3} \circ \cdots \circ e^{\epsilon_1\mathcal{X}_1}$.

4.2 Commutator table

Writing it in tabular form is the most convenient way of showing the structure of a given Lie algebra. Suppose that an r-dimensional Lie algebra is g and $\mathcal{X}_1, \dots, \mathcal{X}_r$ form a basis for g, then the commutator table for g will be the $r \times r$ table whose (i,j)th entry depicts the Lie bracket $[\mathcal{X}_i, \mathcal{X}_j]$. It should be noted that the table will remain skew-symmetric all the time since $[\mathcal{X}_i, \mathcal{X}_j] = -[\mathcal{X}_j, \mathcal{X}_i]$. Also, the structure constants can easily read off from the commutator table; namely, C_{ij}^k is the coefficient of $\mathcal{X}_k$ in the (i,j)th entry of Table 1.

4.3 Adjoint representation tables

Subsequently, the adjoint representation table is used to generate the adjoint transformations and to show the conjugacy map structure of the given Lie algebra. In the tabular form, it is useful to demonstrate conjugation relationships of each subalgebra with each other subalgebra. Define the adjoint operator as

$$\mathrm{Ad}\big(e^{\epsilon\mathcal{X}}\big)\mathcal{Y} \equiv e^{-\epsilon\mathcal{X}}\mathcal{Y}e^{\epsilon\mathcal{X}}. \tag{23}$$

According to Campbell–Hausdorff [28], we have

$$\mathrm{Ad}\big(e^{\epsilon\mathcal{X}}\big)\mathcal{Y} = \mathcal{Y} - \epsilon[\mathcal{X},\mathcal{Y}] + \frac{\epsilon^2}{2}\big[\mathcal{Y},[\mathcal{X},\mathcal{Y}]\big] - \cdots. \tag{24}$$

For an n-dimensional Lie algebra $\mathcal{L}^n$, the adjoint representation table is an $n \times n$ matrix, whose (i,j)th entry presents the adjoint action of $\mathcal{X}_i$ on $\mathcal{X}_j$ as $\mathrm{Ad}(e^{\epsilon\mathcal{X}_i})\mathcal{X}_j$. The adjoint representation table is given in Table 2.

5 Adjoint system and conditions for nonlinear self-adjointness

Take into account the following:

Theorem 5.1 *Lie–Bäcklund, nonlocal, and Lie point symmetries, given by*

$$\mathcal{X} = \xi_i\frac{\partial}{\partial\bar{x}^i} + \eta_{\bar{\alpha}}\frac{\partial}{\partial\,\bar{\Xi}^{\bar{\alpha}}}, \tag{25}$$

Table 2 The commutation relations of infinitesimal generators $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{X}_3$

$[\mathcal{X}_i, \mathcal{X}_j]$	$\mathcal{X}_1$	$\mathcal{X}_2$	$\mathcal{X}_3$	$\mathcal{X}_4$
$\mathcal{X}_1$	$\mathcal{X}_1$	$\mathcal{X}_2$	$\mathcal{X}_3$	$\epsilon\mathcal{X}_1 + \mathcal{X}_4$
$\mathcal{X}_2$	$\mathcal{X}_1$	$\mathcal{X}_2$	$\mathcal{X}_3$	$-\epsilon\mathcal{X}_2 + \mathcal{X}_4$
$\mathcal{X}_3$	$\mathcal{X}_1$	$\mathcal{X}_2$	$\mathcal{X}_3$	$3\epsilon\mathcal{X}_3 + \mathcal{X}_4$
$\mathcal{X}_4$	$e^{-\epsilon}\mathcal{X}_1$	$e^{\epsilon}\mathcal{X}_2$	$e^{-3\epsilon}\mathcal{X}_3$	$\mathcal{X}_4$

of a nonlinear partial differential equation

$$F_{\bar{\alpha}}(\bar{x}, \Xi, \ldots, \Xi_s) = 0, \quad \bar{\alpha} = 1, 2, \ldots, \bar{m}, \tag{26}$$

with m dependent variables will have an adjoint equation

$$\mathcal{F}^*_{\bar{\alpha}}(\bar{x}, \Xi, \ldots, \Xi_s) = \frac{\delta(\nu^{\bar{\beta}}\mathcal{F}_{\bar{\beta}})}{\delta\,\Xi^{\bar{\alpha}}}, \quad \bar{\alpha} = 1, 2, \ldots, \bar{m}, \tag{27}$$

the Lagrangian is thus

$$\mathcal{L} = Z^{\bar{\beta}}\mathcal{F}_{\bar{\beta}}(\bar{x}, \Xi, \Xi_{(1)}, \ldots, \Xi_{(s)}), \tag{28}$$

with Z denotes a new dependent variable.

On account of (1), we have

$$\mathcal{L} = \upsilon\big(\alpha(\Xi_x\Xi_{xt} + \Xi_t\Xi_{xx}) + \Xi_{tt} + \Xi_{xxxt}\big). \tag{29}$$

And the adjoint equation is then obtained as

$$\mathcal{F}^* = \frac{\delta\mathcal{L}}{\delta\,\Xi} = 0, \tag{30}$$

where

$$\frac{\delta\mathcal{L}}{\delta\,\Xi} = \frac{\partial\mathcal{L}}{\partial\,\Xi} - D_t\frac{\partial\mathcal{L}}{\partial\,\Xi_t} - D_x\frac{\partial\mathcal{L}}{\partial\,\Xi_x} + (D_x)^2\frac{\partial\mathcal{L}}{\partial\,\Xi_{xx}} - (D_{xxx})^3\frac{\partial\mathcal{L}}{\partial\,\Xi_{xxx}} + (D_x)^4\frac{\partial\mathcal{L}}{\partial\,\Xi_{xxxx}}. \tag{31}$$

On the basis of (29), we obtain

$$\mathcal{F}^* = 2\alpha\,\Xi_{xt}\upsilon_x + \alpha\,\Xi_x\upsilon_{xt} + \alpha\,\Xi_t\upsilon_{xx} + \upsilon_{tt} + \upsilon_{xxxt} = 0. \tag{32}$$

Definition 5.2 Equation (1) is an NSA only if

$$\mathcal{F}^*|_{\upsilon=Z(x,t,\Xi)} = \Lambda\mathcal{F} = 0, \tag{33}$$

such that not all $\upsilon = Z(x, t, \Xi)$ are zero and Λ_i $(i = 1, 2, 3)$ are undetermined coefficients.

Therefore, from the coefficients of Ξ_t, Ξ_x, Ξ_{xt}, Ξ_{xx}, Ξ_{xxt}, Ξ_{xxx}, we obtain

$$\Lambda = -Z_\Xi. \tag{34}$$

Consequently, we reach the differential substitution as

$$H = c_1 + tc_2. \tag{35}$$

Hence, (1) is an NSA.

5.1 Conservation laws

Herein, we establish the conservation laws of (1). We recall the following theorem:

Theorem 5.3 *Equation* (1) *with obtained symmetries satisfies the conservation equation*

$$D_i\big(C^i\big)|_{(1)=0} = 0, \tag{36}$$

where

$$\begin{aligned} C^i &= \xi_i \mathcal{L} + W^{\bar{\alpha}}\left[\frac{\partial \mathcal{L}}{\partial \Xi_i^{\bar{\alpha}}} - D_j\left(\frac{\partial \mathcal{L}}{\partial \Xi_{ij}^{\bar{\alpha}}}\right) + D_j D_k\left(\frac{\partial \mathcal{L}}{\partial \Xi_{ijk}^{\bar{\alpha}}}\right) - \cdots\right] \\ &\quad + D_j\big(W^{\bar{\alpha}}\big)\left[\frac{\partial \mathcal{L}}{\partial \Xi_{ij}^{\bar{\alpha}}} - D_k\left(\frac{\partial \mathcal{L}}{\partial \Xi_{ijk}^{\bar{\alpha}}}\right) + \cdots\right] + D_j D_k\big(W^{\bar{\alpha}}\big)\left[\frac{\partial \mathcal{L}}{\partial \Xi_{ijk}^{\bar{\alpha}}} + \cdots\right], \end{aligned} \tag{37}$$

and $W^{\bar{\alpha}} = \eta_{\bar{\alpha}} - \xi_j \Xi_j^{\bar{\alpha}}$. *The expression* C^i *represents the conserved vectors.*

Now, we compute the conservation laws for (1) using the obtained symmetries.

- The symmetry $\mathcal{X}_1 = \partial_x$ admits the conserved vectors:

$$\begin{aligned} C_1^x &= \frac{1}{4}\big((c_1 + c_2 t)(4\alpha \Xi_x \Xi_{xt} + 4\Xi_{tt} + \Xi_{xxxt}) + c_2\big(2\alpha \Xi_x^2 + \Xi_{xxx}\big)\big), \\ C_1^t &= u_x\big(c_2 - \alpha(c_1 + c_2 t)\Xi_{xx}\big) - \frac{1}{4}(c_1 + c_2 t)(4\Xi_{xt} + \Xi_{xxxx}). \end{aligned}$$

- The symmetry $\mathcal{X}_2 = \partial_\Xi$ admits the conserved vectors

$$\begin{aligned} C_2^x &= -\frac{1}{2}\alpha\big((c_1 + c_2 t)\Xi_{xt} + c_2 \Xi_x\big), \\ C_2^t &= \frac{1}{2}\alpha(c_1 + c_2 t)\Xi_{xx} - c_2. \end{aligned} \tag{38}$$

- The symmetry $\mathcal{X}_3 = \partial_t$ admits the conserved vectors

$$\begin{aligned} C_3^x &= \frac{1}{4}\big(\Xi_t\big(2\alpha c_2 \Xi_x - 2\alpha(c_1 + c_2 t)\Xi_{xt}\big) \\ &\quad - (c_1 + c_2 t)(2\alpha \Xi_{tt}\Xi_x + 3\Xi_{xxtt}) + c_2 \Xi_{xxt}\big), \\ C_3^t &= \frac{1}{4}\big(2\Xi_t\big(\alpha(c_1 + c_2 t)\Xi_{xx} + 2c_2\big) + (c_1 + c_2 t)(2\alpha \Xi_x \Xi_{xt} + 3\Xi_{xxxt})\big). \end{aligned} \tag{39}$$

- The symmetry $\mathcal{X}_4 = -3t\partial_t + \Xi\partial_\Xi - x\partial_x$ admits the conserved vectors

$$\begin{aligned}
C_4^x &= \frac{1}{4}\big(-2\alpha(3t\Xi_t + x\Xi_x + u)\big((c_1 + c_2t)\Xi_{xt} + c_2\Xi_x\big)\\
&\quad + 2\alpha(c_1 + c_2t)\Xi_1(4\Xi_t + 3t\Xi_{tt} + x\Xi_{xt})\\
&\quad + 4\alpha(c_1 + c_2t)\Xi_t(2\Xi_x + 3t\Xi_{xt} + x\Xi_{xx})\\
&\quad - 4x(c_1 + c_2t)(\alpha\Xi_x\Xi_{xt} + \alpha\Xi_t\Xi_{xx} + \Xi_{tt} + \Xi_{xxxt})\\
&\quad + 3(c_1 + c_2t)(6\Xi_{xxt} + 3t\Xi_{xxtt} + x\Xi_{xxxt}) - c_2(3\Xi_{xx} + 3t\Xi_{xxt} + x\Xi_{xxx})\big),\\
C_4^t &= \frac{1}{4}\big(2\alpha(c_1 + c_2t)\Xi_x(2\Xi_x + 3t\Xi_{xt} + x\Xi_{xx})\\
&\quad - (3t\Xi_t + x\Xi_x + \Xi)\big(4c_2 - 2\alpha(c_1 + c_2t)\Xi_{xx}\big)\\
&\quad - 12t(c_1 + c_2t)(\alpha\Xi_x\Xi_{xt} + \alpha\Xi_t\Xi_{xx} + \Xi_{tt} + \Xi_{xxxt})\\
&\quad + 4(c_1 + c_2t)(4\Xi_t + 3t\Xi_{tt} + x\Xi_{xt})\\
&\quad + (c_1 + c_2t)(4\Xi_{xxx} + 3t\Xi_{xxxt} + x\Xi_{xxxx})\big).
\end{aligned} \tag{40}$$

6 Concluding remarks

In this research, we investigated a new fourth-order integrable nonlinear equation by means of the efficient Hirota bilinear and Lie symmetry approaches. Consequently, we established novel types of solutions, such as breather, rogue and three-wave solutions. Wazwaz [28] investigated this new fourth-order integrable nonlinear equation. Multiple soliton solutions were reported using direct substitution. By using the tanh–coth method, topological and singular soliton solutions were constructed. On the other hand, using the tanh–coth method, singular periodic solutions were successfully reported. Comparing our solutions with the results presented in [28], one can observe that our results are new. To the best of our knowledge, the results reported in this paper have not been published, yet. Moreover, the reported solutions in this study have some physical meanings, for instance, the hyperbolic sine arises in the gravitational potential of a cylinder and the calculation of the Roche limit. The hyperbolic cosine function is the shape of a hanging cable (the so-called catenary) [36]. On the other hand, invariance properties, such as the group of transformations, as well as commutator and adjoint representation tables, have been reported. A differential substitution has been found via nonlinear self-adjointness and the associated conservation laws have been established. Using suitable values of parameters, the dynamical characteristics of the obtained solutions have been depicted via the 3-dimensional and contour graphs. To the best of our knowledge, the results and analysis presented in this study have not appeared in the literature before.

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