Gao et al. *Advances in Difference Equations* (2021) 2021:171
https://doi.org/10.1186/s13662-021-03331-x

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH

Open Access

A Kirchhoff-type problem involving concave-convex nonlinearities

Yuan Gao[1], Lishan Liu[1*], Shixia Luan[1] and Yonghong Wu[2]

*Correspondence:
mathlls@163.com
[1]School of Mathematical Sciences, Qufu Normal University, Qufu, Shandong 273165, People's Republic of China
Full list of author information is available at the end of the article

Abstract

A Kirchhoff-type problem with concave-convex nonlinearities is studied. By constrained variational methods on a Nehari manifold, we prove that this problem has a sign-changing solution with least energy. Moreover, we show that the energy level of this sign-changing solution is strictly larger than the double energy level of the ground state solution.

MSC: 35J20; 35J65; 35A15; 35J60

Keywords: Concave-convex nonlinearities; Kirchhoff-type problem; Nehari manifolds; Ground state sign-changing solutions

1 Introduction

We study the following Kirchhoff-type equation with concave-convex nonlinearities:

$$\begin{cases} (a + \lambda \int_{\mathbb{R}^3} |\nabla u|^2 + \lambda b \int_{\mathbb{R}^3} u^2)(-\Delta u + bu) \\ \quad = Q(x)|u|^{p-1}u + \kappa G(x)|u|^{q-1}u, \quad x \in \mathbb{R}^3, \\ u \in H_r^1(\mathbb{R}^3), \end{cases} \tag{1.1}$$

where $a > 0$, $b > 0$, $\lambda > 0$, $\kappa < 0$, $p \in (3,5)$, $q \in (0,1)$, and $Q, G \in C(\mathbb{R}^3, \mathbb{R}^+)$ satisfying the following conditions:

(Q_1) There exists $\beta \in [0, p-2)$ such that $\limsup_{x\to+\infty} \frac{Q(x)}{|x|^\beta} < +\infty$;

(G_1) $G(x) \in L^2(\mathbb{R}^3, \mathbb{R}^+)$.

In recent years, the following elliptic problem has been investigated by many researchers [1, 3, 6, 9, 17, 20]:

$$\begin{cases} -(a + b\int_{\mathbb{R}^3} |\nabla u|^2)\Delta u = f(x,u), \quad x \in \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases} \tag{1.2}$$

where $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and $a > 0$, $b > 0$. The term $\int_{\mathbb{R}^3} |\nabla u|^2$ in (1.2) has an interesting physical application. Moreover, this problem is related to the stationary analogue of the following equation proposed by Kirchhoff [10]:

$$u_{tt} - \left(a + b \int_{\Omega} |\nabla u|^2 \right) \Delta u = f(x,u). \tag{1.3}$$

Inspired by the variational framework given by Lions [12], problem (1.3) has been investigated by many researchers, and the reader is referred to [5, 7, 11, 13, 19, 22] and the references therein for more details.

Shuai [16] studied the ground state sign-changing solution of problem (1.2) by using Brouwer degree theory, where $f(x,u)$ is replaced with $f(u)$ with the following hypotheses:

(f_1'): $f(s) = o(|s|)$ as $s \to 0$;

(f_2'): For some constant $p \in (4, 2^*)$, $\lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = 0$, where $2^* = +\infty$ for $N = 1, 2$ and $2^* = 6$ for $N = 3$;

(f_3'): $\lim_{s\to\infty} \frac{F(s)}{s^4} = +\infty$, where $F(s) = \int_0^s f(t)\,dt$;

(f_4'): $\frac{f(s)}{|s|^3}$ is an increasing function with respect to $s \in \mathbb{R} \setminus \{0\}$.

Huang and Liu [8] obtained the ground state sign-changing solutions of problem (1.4) with accurately two nodal domains

$$-\left(1 + \lambda \int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2 \right) \right) \left[\Delta u + V(x)u \right] = |u|^{p-1}u, \quad x \in \mathbb{R}^N, \tag{1.4}$$

where $p \in (3,5)$, $\lambda > 0$ and $V \in C(\mathbb{R}^N, \mathbb{R})$ is to ensure the establishment of compactness.

Deng et al. [4] showed the existence of radial sign-changing solutions u_k^b of problem (1.5)

$$\begin{cases} -(a + b \int_{\mathbb{R}^3} |\nabla u|^2) \Delta u + V(x)u = f(x,u), & x \in \mathbb{R}^3, \\ u \in H_r^1(\mathbb{R}^3), \end{cases} \tag{1.5}$$

by constrained minimization on the Nehari manifold, where k is any positive integer. Ye [21] studied the existence of least energy sign-changing solutions for problem (1.5), where $f(x,u)$ is replaced with $f(u)$.

Shao and Mao [15] got at least one sign-changing solution of problem (1.6) with concave-convex nonlinearities

$$\begin{cases} -(a + b \int_{\Omega} |\nabla u|^2) \Delta u = \mu g(x,u) + f(x,u), & \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

by using the method of invariant sets of descending flow.

Motivated by the aforementioned works, we prove the existence of sign-changing solutions with least energy for problem (1.1) with concave-convex nonlinearities and unbounded potential by constrained variational methods on a Nehari manifold.

Now we will give the main results by Theorems 1.1 and 1.2.

Theorem 1.1 *Assume that* (Q_1) *and* (G_1) *hold, then, for* $a > 0$, $b > 0$, $\lambda > 0$, *and* $\kappa < 0$, *problem* (1.1) *has one least energy sign-changing solution with accurately two nodal domains.*

Theorem 1.2 *Assume that* (Q_1) *and* (G_1) *hold, then, for* $a > 0$, $b > 0$, $\lambda > 0$, *and* $\kappa < 0$, *problem* (1.1) *has one least energy solution. Moreover* $m_\lambda > 2c_\lambda$, *where* m_λ *and* c_λ *are defined by* (2.3) *and* (2.5) *respectively.*

Remark 1.3 Comparing with Shuai [16], Huang and Liu [8], Deng et al. [4], and Ye [21], the difference is to consider Kirchhoff-type equation with concave and convex terms, where $Q(x)$ is unbounded at infinity. Moreover, since $H_r^1(\mathbb{R}^3) \hookrightarrow L^{q+1}(\mathbb{R}^3)$ is not compact for $q \in (0,1)$, this means that the appearance of concave and convex terms has greatly increased the difficulty of problem (1.1). Shao and Mao [15] got sign-changing solutions for Kirchhoff equation with concave and convex terms by using the method of invariant sets of descending flow. However, we want to obtain ground state sign-changing solutions of (1.1) by variational methods and constrained minimization on the sign-changing Nehari manifold. It should be addressed that our methods are different to those in [15].

The rest of the paper is organized as follows. In Sect. 2 we give some notations and the main lemmas related to the proof of our main results. Sections 3 and 4 give the proofs of Theorems 1.1 and 1.2, respectively.

2 Some notations and preliminary lemmas

Here are some notations to be used in this paper.

- C denotes a positive constant;
- $H^1(\mathbb{R}^3)$ denotes the usual Sobolev space with the norm $\|u\|^2 = \int_{\mathbb{R}^3}(|\nabla u|^2 + b|u|^2)$;
- $|\cdot|$ denotes the usual norm $L^{\bar{q}}(\mathbb{R}^3)$ for $\bar{q} \in [1,\infty)$;
- $H_r^1(\mathbb{R}^3) := \{u : u \in H^1(\mathbb{R}^3), u(x) = u(|x|)\}$;
- $u^+ := \max\{u,0\}$ and $u^- := \min\{u,0\}$.

Lemma 2.1 (see Berestycki and Lions [2]) *Let $N \geq 2$ and $u \in H_r^1(\mathbb{R}^N)$, Then*

$$\left|u(r)\right| \leq C_0\|u\|r^{\frac{1-N}{2}} \quad \textit{for } r \geq 1,$$

where $C_0 > 0$ is only related to N.

Remark 2.2 For any $u \in H_r^1(\mathbb{R}^3)$, by (Q_1), (G_1), and Lemma 2.1, we have

$$0 \leq \int_{\mathbb{R}^3} Q(x)|u|^{p+1} \leq C_1\|u\|^{p+1}$$

and

$$\left|\int_{\mathbb{R}^3} G(x)|u|^{q+1}\right| \leq \int_{\mathbb{R}^3}\left|G(x)\right||u|^{q+1} \leq \left|G(x)\right|_2|u|_{2(q+1)}^{q+1} \leq C_1\|u\|^{q+1}.$$

The energy functional $J_\lambda \in C^1(H_r^1(\mathbb{R}^3),\mathbb{R})$ is well defined by

$$J_\lambda(u) = \frac{1}{2}a\|u\|^2 + \frac{1}{4}\lambda\|u\|^4 - \frac{1}{p+1}\int_{\mathbb{R}^3} Q(x)|u|^{p+1} - \frac{1}{q+1}\kappa\int_{\mathbb{R}^3} G(x)|u|^{q+1}. \tag{2.1}$$

For each $u, v \in H_r^1(\mathbb{R}^3)$,

$$\left\langle J_\lambda'(u), v\right\rangle = a(u,v) + \lambda\|u\|^2(u,v) - \int_{\mathbb{R}^3} Q(x)|u|^{p-1}uv - \kappa\int_{\mathbb{R}^3} G(x)|u|^{q-1}uv. \tag{2.2}$$

In order to get a sign-changing solution $u^{\pm} \neq 0$ of (1.1), the following functionals need to be established:

$$J_\lambda(u) = J_\lambda(u^+) + J_\lambda(u^-) + \frac{\lambda}{2}\|u^+\|^2\|u^-\|^2,$$
$$\langle J'_\lambda(u), u^+\rangle = \langle J'_\lambda(u^+), u^+\rangle + \lambda\|u^-\|^2\|u^+\|^2,$$
$$\langle J'_\lambda(u), u^-\rangle = \langle J'_\lambda(u^-), u^-\rangle + \lambda\|u^+\|^2\|u^-\|^2.$$

Let us define

$$\mathcal{M}_\lambda = \{u \in H_r^1(\mathbb{R}^3) : u^{\pm} \neq 0, \langle J'_\lambda(u), u^+\rangle = \langle J'_\lambda(u), u^-\rangle = 0\}$$

and

$$m_\lambda := \inf\{J_\lambda(u) : u \in \mathcal{M}_\lambda\}. \tag{2.3}$$

In addition, we define

$$\mathcal{N}_\lambda = \{u \in H_r^1(\mathbb{R}^3) \setminus \{0\} : \langle J'_\lambda(u), u\rangle = 0\} \tag{2.4}$$

and

$$c_\lambda := \inf\{J_\lambda(u) : u \in \mathcal{N}_\lambda\}. \tag{2.5}$$

Lemma 2.3 *Assume that* (Q_1), (G_1), *and* $u_n \rightharpoonup u$ *in* $H_r^1(\mathbb{R}^3)$ *hold, then*

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} G(x)|u_n|^{q+1} = \int_{\mathbb{R}^3} G(x)|u|^{q+1}.$$

In particular,

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} G(x)|u_n^{\pm}|^{q+1} = \int_{\mathbb{R}^3} G(x)|u^{\pm}|^{q+1}.$$

Proof If $u_n \rightharpoonup u$ in $H_r^1(\mathbb{R}^3)$, then $u_n \to u$ in $L^{\bar{q}}(\mathbb{R}^3)$ for $\bar{q} \in (2,6)$. According to [18, Theorem A.4, p. 134], we can obtain that $|u_n|^{q+1} \to |u|^{q+1}$ in $L^2(\mathbb{R}^3)$. By the Hölder inequality, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} G(x)|u_n|^{q+1} - \int_{\mathbb{R}^3} G(x)|u|^{q+1}\right| \\
&\quad\le \int_{\mathbb{R}^3} |G(x)|\left||u_n|^{q+1} - |u|^{q+1}\right| \\
&\quad\le |G(x)|_2\left||u_n|^{q+1} - |u|^{q+1}\right|_2 \to 0.
\end{aligned}$$

Thus, $\lim_{n\to\infty}\int_{\mathbb{R}^3} G(x)|u_n|^{q+1} = \int_{\mathbb{R}^3} G(x)|u|^{q+1}$. Similarly, $\lim_{n\to\infty}\int_{\mathbb{R}^3} G(x)|u_n^{\pm}|^{q+1} = \int_{\mathbb{R}^3} G(x)|u^{\pm}|^{q+1}$. □

Lemma 2.4 *Under the assumptions of Theorem 1.1. If $u \in H_r^1(\mathbb{R}^3)$ with $u^{\pm} \neq 0$, there exists a unique pair $(s_u, t_u) \in (0,+\infty) \times (0,+\infty)$ such that $s_u u^+ + t_u u^- \in \mathcal{M}_\lambda$. Moreover,*

$$J_\lambda\left(s_u u^+ + t_u u^-\right) = \max_{s,t\geq 0} J_\lambda\left(su^+ + tu^-\right).$$

Proof Let $u \in H^1(\mathbb{R}^3)$ with $u^{\pm} \neq 0$. Define

$$\begin{aligned} g_1(s,t) &= as^2\left\|u^+\right\|^2 + \lambda s^4\left\|u^+\right\|^4 + \lambda s^2t^2\left\|u^+\right\|^2\left\|u^-\right\|^2 \\ &\quad - s^{p+1}\int_{\mathbb{R}_3} Q(x)\left|u^+\right|^{p+1} - \kappa s^{q+1}\int_{\mathbb{R}_3} G(x)\left|u^+\right|^{q+1}, \qquad (2.6) \\ g_2(s,t) &= at^2\left\|u^-\right\|^2 + \lambda t^4\left\|u^-\right\|^4 + \lambda s^2t^2\left\|u^-\right\|^2\left\|u^+\right\|^2 \\ &\quad - t^{p+1}\int_{\mathbb{R}_3} Q(x)\left|u^-\right|^{p+1} - \kappa t^{q+1}\int_{\mathbb{R}_3} G(x)\left|u^-\right|^{q+1}. \qquad (2.7) \end{aligned}$$

According to Remark 2.2, for $\kappa < 0$, we have $g_i(s,s) > 0$ as $s > 0$ small and $g_i(t,t) < 0$ as $t > 0$ large, where $i = 1,2$. Then there exists $0 < \mu < \nu$ such that

$$g_i(\mu,\mu) > 0, \qquad g_i(\nu,\nu) < 0. \tag{2.8}$$

By (2.6), (2.7), (2.8), we have that

$$\begin{aligned} &g_1(\mu,t) > 0, \qquad g_1(\nu,t) < 0, \quad t \in [\mu,\nu], \\ &g_2(s,\mu) > 0, \qquad g_2(s,\nu) < 0, \quad s \in [\mu,\nu]. \end{aligned}$$

From Miranda's theorem [14], there exists a pair (s_u, t_u) such that

$$g_1(s_u,t_u) = 0, \qquad g_2(s_u,t_u) = 0, \quad \mu < s_u, t_u < \nu.$$

Thus, $s_u u^+ + t_u u^- \in \mathcal{M}_\lambda$.

Secondly, we prove the uniqueness. Let both (s_1,t_1) and (s_2,t_2) satisfy $u_i = s_i u^+ + t_i u^- \in \mathcal{M}_\lambda$ $(i = 1,2)$ and $u_1 = s_1u^+ + t_1u^- = ms_2u^+ + nt_2u^- = mu_2^+ + nu_2^-$, where $m = \frac{s_1}{s_2}$, $n = \frac{t_1}{t_2}$. By (2.6) and (2.7),

$$\begin{aligned} &g_1^{u_1}(1,1) = g_1^{u_2}(m,n) = g_1^{u_2}(1,1) = 0, \qquad (2.9) \\ &g_2^{u_1}(1,1) = g_2^{u_2}(m,n) = g_2^{u_2}(1,1) = 0. \qquad (2.10) \end{aligned}$$

We only need to prove that $m = n = 1$. Now, assume that $0 < m \leq n$. By (2.9) and (2.10),

$$g_1^{u_2}(1,1) - \frac{g_1^{u_2}(m,n)}{m^4} = 0 \tag{2.11}$$

and

$$g_2^{u_2}(1,1) - \frac{g_2^{u_2}(m,n)}{n^4} = 0. \tag{2.12}$$

If $m < 1$, then

$$\left(1-\frac{1}{m^2}\right)a\|u_2^+\|^2+\left(1-\frac{n^2}{m^2}\right)\lambda\|u_2^-\|^2\|u_2^+\|^2$$
$$=\left(1-m^{p-3}\right)\int_{\mathbb{R}^3}Q(x)|u_2^+|^{p+1}+\left(1-m^{q-3}\right)\kappa\int_{\mathbb{R}^3}G(x)|u_2^+|^{q+1},$$

this is impossible for $\kappa < 0$. Then $m \geq 1$. Similarly, if $n > 1$, (2.12) is impossible. Then $n \leq 1$. Thus $m = n = 1$.

At last, let

$$\begin{aligned}H_\lambda(s,t)&=J_\lambda\left(su^++tu^-\right)\\&=\frac{a}{2}s^2\|u^+\|^2+\frac{\lambda}{4}s^4\|u^+\|^4-\frac{s^{p+1}}{p+1}\int_{\mathbb{R}^3}Q(x)|u^+|^{p+1}-\frac{s^{q+1}}{q+1}\kappa\int_{\mathbb{R}_3}G(x)|u^+|^{q+1}\\&\quad+\frac{a}{2}t^2\|u^-\|^2+\frac{\lambda}{4}t^4\|u^-\|^4-\frac{t^{p+1}}{p+1}\int_{\mathbb{R}^3}Q(x)|u^-|^{p+1}-\frac{t^{q+1}}{q+1}\kappa\int_{\mathbb{R}_3}G(x)|u^-|^{q+1}\\&\quad+\frac{\lambda}{2}s^2t^2\|u^-\|^2\|u^+\|^2.\end{aligned}$$

Then, for $\kappa < 0$, we have $H_\lambda(s,t) > 0$ as $|(s,t)| \to 0$, $H_\lambda(s,t) < 0$ as $|(s,t)| \to \infty$, and H_λ cannot achieve the maximum point on $\partial\mathbb{R}^{+2}$. Without loss of generality, we only prove that $(0,t_0)$ is not a maximum point of H_λ. For $s > 0$ small enough,

$$\begin{aligned}\frac{\partial H_\lambda}{\partial s}(s,t_0)&=as\|u^+\|^2+\lambda s^3\|u^+\|^4+\lambda st_0^2\|u^-\|^2\|u^+\|^2\\&\quad-s^p\int_{\mathbb{R}^3}Q(x)|u^+|^{p+1}-s^q\kappa\int_{\mathbb{R}_3}G(x)|u^+|^{q+1}>0,\end{aligned}$$

this implies that $H_\lambda(s,t_0)$ is an increasing function with respect to s, where $s > 0$ is small enough, then $(0,t_0)$ is not a maximum point of H_λ. Thus, there exists $(s_u,t_u) \in \mathbb{R}^{+2}$ such that

$$J_\lambda\left(s_uu^++t_uu^-\right)=\max_{s,t\geq 0}J_\lambda\left(su^++tu^-\right).\qquad\square$$

Lemma 2.5 *Under the assumptions of Theorem 1.1. If $\langle J'_\lambda(u),u^\pm\rangle \leq 0$, there exists $(s_u,t_u) \in (0,1]\times(0,1]$ such that $s_uu^++t_uu^- \in \mathcal{M}_\lambda$ for $u \in H_r^1(\mathbb{R}^3)$ with $u^\pm \neq 0$.*

Proof Let $u \in H_r^1(\mathbb{R}^3)$ with $u^\pm \neq 0$, by Lemma 2.4, there exists a pair (s_u,t_u) such that

$$\begin{aligned}&s_u^2a\|u^+\|^2+s_u^4\lambda\|u^+\|^4+s_u^2t_u^2\lambda\|u^-\|^2\|u^+\|^2\\&\quad-s_u^{p+1}\int_{\mathbb{R}^3}Q(x)|u^+|^{p+1}-s_u^{q+1}\kappa\int_{\mathbb{R}^3}G(x)|u^+|^{q+1}=0.\end{aligned}\tag{2.13}$$

Since $\langle J'_\lambda(u),u^\pm\rangle \leq 0$, we have that

$$a\|u^+\|^2+\lambda\|u^+\|^4+\lambda\|u^-\|^2\|u^+\|^2-\int_{\mathbb{R}^3}Q(x)|u^+|^{p+1}-\kappa\int_{\mathbb{R}^3}G(x)|u^+|^{q+1}\leq 0.\tag{2.14}$$

Now, assume that $0 < t_u \le s_u$. If $s_u > 1$, by (2.13) and (2.14),

$$\left(1-\frac{1}{s_u^2}\right)a\left\|u^+\right\|^2+\left(1-\frac{t_u^2}{s_u^2}\right)\lambda\left\|u^-\right\|^2\left\|u^+\right\|^2$$
$$\le\left(1-s_u^{p-3}\right)\int_{\mathbb{R}^3}Q(x)\left|u^+\right|^{p+1}+\left(1-s_u^{q-3}\right)\kappa\int_{\mathbb{R}^3}G(x)\left|u^+\right|^{q+1},$$

which is contradictory for $\kappa < 0$. Then $s_u \le 1$. From $0 < t_u \le s_u$, we obtain that $0 < t_u \le s_u \le 1$. □

Lemma 2.6 *Under the assumptions of Theorem 1.1, $m_\lambda > 0$ can be achieved.*

Proof For all $u \in \mathcal{M}_\lambda$, by the Sobolev embedding theorem, we have

$$a\|u\|^2\le a\|u\|^2+\lambda\|u\|^4=\int_{\mathbb{R}^3}Q(x)|u|^{p+1}+\kappa\int_{\mathbb{R}^3}G(x)|u|^{q+1}\le C_1\|u\|^{p+1}.$$

Then there exists $C \ge C_1$ such that $\|u\| \ge (\frac{a}{C})^{\frac{1}{p-1}} > 0$. Since

$$\begin{aligned}
J_\lambda(u)&=J_\lambda(u)-\frac{1}{4}\langle J'_\lambda(u),u\rangle\\
&=\frac{a}{2}\|u\|^2+\frac{\lambda}{4}\|u\|^4-\frac{1}{p+1}\int_{\mathbb{R}^3}Q(x)|u|^{p+1}-\frac{1}{q+1}\kappa\int_{\mathbb{R}^3}G(x)|u|^{q+1}\\
&\quad-\frac{a}{4}\|u\|^2-\frac{\lambda}{4}\|u\|^4+\frac{1}{4}\int_{\mathbb{R}^3}Q(x)|u|^{p+1}+\frac{1}{4}\kappa\int_{\mathbb{R}^3}G(x)|u|^{q+1}\\
&=\frac{a}{4}\|u\|^2+\left(\frac{1}{4}-\frac{1}{p+1}\right)\int_{\mathbb{R}^3}Q(x)|u|^{p+1}-\left(\frac{1}{q+1}-\frac{1}{4}\right)\kappa\int_{\mathbb{R}^3}G(x)|u|^{q+1}\\
&\ge\frac{a}{8}\|u\|^2 \qquad (2.15)
\end{aligned}$$

for $\kappa < 0$. Then

$$m_\lambda=\inf_{u\in\mathcal{M}_\lambda}J_\lambda(u)>0.$$

Let $\{u_n\} \subset \mathcal{M}_\lambda$ and $J_\lambda(u_n) \to m_\lambda$. By Remark 2.2, we have

$$1+m_\lambda\ge J_\lambda(u_n)-\frac{1}{p+1}\langle J'_\lambda(u_n),u_n\rangle\ge\frac{a}{8}\|u_n\|^2.$$

This shows that $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^3)$. Then there exists $u_\lambda \in H_r^1(\mathbb{R}^3)$ such that $u_n^\pm \rightharpoonup u_\lambda^\pm$ in $H_r^1(\mathbb{R}^3)$, $u_n^\pm \to u_\lambda^\pm$ in $L^q(\mathbb{R}^3)$ for $q \in (2,6)$ and $u_n^\pm(x) \to u_\lambda^\pm(x)$ a.e. on $\mathbb{R}^3$. Since $\{u_n\} \subset \mathcal{M}_\lambda$, we have

$$0<C\le a\left\|u_n^\pm\right\|^2+\lambda\left\|u_n^\pm\right\|^4+\lambda\left\|u_n^+\right\|^2\left\|u_n^-\right\|^2=\int_{\mathbb{R}^3}Q(x)\left|u_n^\pm\right|^{p+1}+\kappa\int_{\mathbb{R}^3}G(x)\left|u_n^\pm\right|^{q+1}.$$

By Fatou's lemma and Lemma 2.3,

$$a\left\|u_\lambda^\pm\right\|^2+\lambda\left\|u_\lambda^\pm\right\|^4+\lambda\left\|u_\lambda^+\right\|^2\left\|u_\lambda^-\right\|^2\le\int_{\mathbb{R}^3}Q(x)\left|u_\lambda^\pm\right|^{p+1}+\kappa\int_{\mathbb{R}^3}G(x)\left|u_\lambda^\pm\right|^{q+1},$$

this implies that

$$\langle J'_\lambda(u_\lambda), u_\lambda^{\pm}\rangle \le 0.$$

By Lemmas 2.4 and 2.5, there exists $(s_{u_\lambda}, t_{u_\lambda}) \in (0,1] \times (0,1]$ such that $\widetilde{u}_\lambda = s_{u_\lambda} u_\lambda^+ + t_{u_\lambda} u_\lambda^- \in \mathcal{M}_\lambda$. Then

$$\begin{aligned}
m_\lambda &\le J_\lambda(\widetilde{u}_\lambda) - \frac{1}{p+1}\langle J'_\lambda(\widetilde{u}_\lambda), \widetilde{u}_\lambda\rangle \\
&= \left(\frac{1}{2} - \frac{1}{p+1}\right) a\|\widetilde{u}_\lambda\|^2 + \left(\frac{1}{4} - \frac{1}{p+1}\right)\lambda\|\widetilde{u}_\lambda\|^4 - \left(\frac{1}{q+1} - \frac{1}{p+1}\right)\kappa \int_{\mathbb{R}^3} G(x)|\widetilde{u}_\lambda|^{q+1} \\
&\le \frac{p-1}{2(p+1)} a\|u_\lambda\|^2 + \frac{p-3}{4(p+1)}\lambda\|u_\lambda\|^4 - \frac{p-q}{(q+1)(p+1)}\kappa \int_{\mathbb{R}^3} G(x)|u_\lambda|^{q+1} \\
&\le \liminf_n \left\{ \frac{p-1}{2(p+1)} a\|u_n\|^2 + \frac{p-3}{4(p+1)}\lambda\|u_n\|^4 - \frac{p-q}{(q+1)(p+1)}\kappa \int_{\mathbb{R}^3} G(x)|u_n|^{q+1} \right\} \\
&= \liminf_n \left(J_\lambda(u_n) - \frac{1}{p+1}\langle J'_\lambda(u_n), u_n\rangle \right) \\
&= m_\lambda,
\end{aligned}$$

this implies that $s_{u_\lambda} = t_{u_\lambda} = 1$. Thus, $\widetilde{u}_\lambda = u_\lambda$ and $J_\lambda(u_\lambda) = m_\lambda$. □

3 Sign-changing solutions

Lemma 3.1 *Under the assumptions of Theorem 1.1. If $u_\lambda \in \mathcal{M}_\lambda$ and $J_\lambda(u_\lambda) = m_\lambda$, then $J'_\lambda(u_\lambda) = 0$.*

Proof Suppose that $J'_\lambda(u_\lambda) \ne 0$, then there are $\sigma, \delta > 0$ such that

$$\|J'_\lambda(u)\| \ge \sigma, \quad \forall \|u - u_\lambda\| \le 3\delta.$$

Let $D = (0.5, 1.5) \times (0.5, 1.5)$. By Lemma 2.4, we obtain that

$$\iota := \max_{(s,t)\in\partial D} J_\lambda\left(su_\lambda^+ + tu_\lambda^-\right) < m_\lambda. \tag{3.1}$$

For $\varepsilon := \min\{(m_\lambda - \iota)/2, \sigma\delta/8\}$ and $S := B(u_\lambda, \delta)$, Willem [18, Lemma 2.3] produce a deformation η such that

(i) $\eta(1,u) = u$ if $u \notin J_\lambda^{-1}([m_\lambda - 2\varepsilon, m_\lambda + 2\varepsilon]) \cap S_{2\delta}$;

(ii) $\eta(1, J_\lambda^{m_\lambda+\varepsilon} \cap S) \subset J_\lambda^{m_\lambda-\varepsilon}$;

(iii) $J_\lambda(\eta(1,u)) \le J_\lambda(u)$ for all $u \in H_r^1(\mathbb{R}^3)$.

At first, we show that

$$\max_{(s,t)\in\bar{D}} J_\lambda\left(\eta\left(1, su_\lambda^+ + tu_\lambda^-\right)\right) < m_\lambda.$$

For all $(s,t) \in \bar{D}$, by Lemma 2.4, we obtain $J_\lambda(su_\lambda^+ + tu_\lambda^-) \le m_\lambda < m_\lambda + \varepsilon$, that is, $su_\lambda^+ + tu_\lambda^- \in J_\lambda^{m_\lambda+\varepsilon}$. Therefore, $J_\lambda(\eta(1, su_\lambda^+ + tu_\lambda^-)) \le m_\lambda - \varepsilon$.

Next, we prove that

$$\eta\left(1, su_\lambda^+ + tu_\lambda^-\right) \cap \mathcal{M}_\lambda \ne \emptyset, \quad \forall (s,t) \in \bar{D}.$$

Define $h(s,t) = \eta(1, su_\lambda^+ + tu_\lambda^-)$ and $\psi : [0,1] \times \bar{D} \to \mathbb{R}^2$, for any $\vartheta \in [0,1]$, we have

$$\begin{aligned}\psi\big(\vartheta,(s,t)\big) = {} & \big(\big\langle J_\lambda'\big(\eta\big(\vartheta, su_\lambda^+ + tu_\lambda^-\big)\big), \big(\eta\big(\vartheta, su_\lambda^+ + tu_\lambda^-\big)\big)^+\big\rangle, \\ & \big\langle J_\lambda'\big(\eta\big(\vartheta, su_\lambda^+ + tu_\lambda^-\big)\big), \big(\eta\big(\vartheta, su_\lambda^+ + tu_\lambda^-\big)\big)^-\big\rangle\big).\end{aligned}$$

Let

$$\begin{aligned}\psi_0 &= \psi_0(1,\cdot) = \big\langle J_\lambda'\big(su_\lambda^+ + tu_\lambda^-\big)su_\lambda^+, J_\lambda'\big(su_\lambda^+ + tu_\lambda^-\big)tu_\lambda^-\big\rangle, \\ \psi_1 &= \psi_1(1,\cdot) = \big\langle J_\lambda'\big(h(s,t)\big)h^+(s,t), J_\lambda'\big(h(s,t)\big)h^-(s,t)\big\rangle.\end{aligned}$$

By a simple calculation, $\deg(\psi_0, D, 0) = 1$. According to (3.1), we obtain that $u_\lambda = h$ on ∂D and from homotopy invariance that

$$\deg(\psi_1, D, 0) = \deg(\psi_0, D, 0) = 1.$$

Then there exists a pair $(s_0, t_0) \in D$ such that $\psi_1(s_0, t_0) = 0$ and $\eta(1, s_0u_\lambda^+ + t_0u_\lambda^-) = h(s_0, t_0) \in \mathcal{M}_\lambda$, which contradicts (3.1). Therefore, u_λ is a critical point of J_λ, and so a sign-changing solution of (1.1). □

Proof of Theorem 1.1 Firstly, by the preceding lemmas, there exists $u_\lambda \in \mathcal{M}_\lambda$ such that $J_\lambda(u_\lambda) = m_\lambda$ and $J_\lambda'(u_\lambda) = 0$. Thus, problem (1.1) has one least energy sign-changing solution u_λ.

Secondly, we prove that u_λ has only two nodal domains. Assume that $u_\lambda = u_1 + u_2 + u_3$ with

$$\begin{aligned}& u_i \not\equiv 0, \qquad u_1 \geq 0, \qquad u_2 \leq 0, \\ & \operatorname{supp}(u_i) \cap \operatorname{supp}(u_j) = \emptyset, \quad i \neq j, i,j = 1,2,3.\end{aligned}$$

Setting $w = u_1 + u_2$ with $w^+ = u_1$ and $w^- = u_2$, i.e., $w^\pm \neq 0$. Since $J_\lambda'(u_\lambda) = 0$, we get

$$\begin{aligned}\big\langle J_\lambda'(w), w^+\big\rangle &= \big\langle J_\lambda'(u_1 + u_2), u_1\big\rangle \leq \big\langle J_\lambda'(u_\lambda), u_1\big\rangle = 0, \\ \big\langle J_\lambda'(w), w^-\big\rangle &= \big\langle J_\lambda'(u_1 + u_2), u_2\big\rangle \leq \big\langle J_\lambda'(u_\lambda), u_2\big\rangle = 0.\end{aligned}$$

By Lemma 2.5, there exists $(s_w, t_w) \in (0,1] \times (0,1]$ such that

$$s_ww^+ + t_ww^- = s_wu_1 + t_wu_2 \in \mathcal{M}_\lambda, \quad m_\lambda \leq J_\lambda(s_wu_1 + t_wu_2).$$

Note that $\langle J_\lambda'(u_\lambda), u_\lambda\rangle = 0$ and $\langle J_\lambda'(s_wu_1 + t_wu_2), s_wu_1 + t_wu_2\rangle = 0$, we have

$$\begin{aligned}m_\lambda &= J_\lambda(u_\lambda) - \frac{1}{p+1}\big\langle J_\lambda'(u_\lambda), u_\lambda\big\rangle \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)a\|u_\lambda\|^2 + \left(\frac{1}{4} - \frac{1}{p+1}\right)\lambda\big(\|u_\lambda\|^2\big)^2 \\ &\quad - \left(\frac{1}{q+1} - \frac{1}{p+1}\right)\kappa\int_{\mathbb{R}^3} G(x)|u_\lambda|^{q+1}\end{aligned}$$

$$
\begin{aligned}
&> \left(\frac{1}{2}-\frac{1}{p+1}\right)a\left(\|u_1\|^2+\|u_2\|^2\right)\\
&\quad+\left(\frac{1}{4}-\frac{1}{p+1}\right)\lambda\left(\|u_1\|^4+2\|u_1\|^2\|u_2\|^2+\|u_2\|^4\right)\\
&\quad-\left(\frac{1}{q+1}-\frac{1}{p+1}\right)\kappa\int_{\mathbb{R}^3}G(x)\left(|u_1|^{q+1}+|u_2|^{q+1}\right)\\
&\geq \left(\frac{1}{2}-\frac{1}{p+1}\right)a\left(\|s_wu_1\|^2+\|t_wu_2\|^2\right)\\
&\quad+\left(\frac{1}{4}-\frac{1}{p+1}\right)\lambda\left(\|s_wu_1\|^4+2\|s_wu_1\|^2\|t_wu_2\|^2+\|t_wu_2\|^4\right)\\
&\quad-\left(\frac{1}{q+1}-\frac{1}{p+1}\right)\kappa\int_{\mathbb{R}^3}G(x)\left(|s_wu_1|^{q+1}+|t_wu_2|^{q+1}\right)\\
&= \left(\frac{1}{2}-\frac{1}{p+1}\right)a\|s_wu_1+t_wu_2\|^2+\left(\frac{1}{4}-\frac{1}{p+1}\right)\lambda\left(\|s_wu_1+t_wu_2\|^2\right)^2\\
&\quad-\left(\frac{1}{q+1}-\frac{1}{p+1}\right)\kappa\int_{\mathbb{R}^3}G(x)|s_wu_1+t_wu_2|^{q+1}\\
&= J_\lambda(s_wu_1+t_wu_2)-\frac{1}{p+1}\left\langle J'_\lambda(s_wu_1+t_wu_2),s_wu_1+t_wu_2\right\rangle\\
&= J_\lambda(s_wu_1+t_wu_2)\\
&\geq m_\lambda,
\end{aligned}
$$

which is a contradiction. □

4 Ground state solutions

Lemma 4.1 (Mountain pass theorem [18]) *Let X be a Banach space, $I\in C^1(X,\mathbb{R})$, $e\in X$, and $\rho>0$ such that $\|e\|>\rho$ and*

$$\inf_{\|u\|=\rho} I(u)>I(0)\geq I(e).$$

If I satisfies the $(PS)_c$ condition with

$$
\begin{aligned}
&c:=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I\big(\gamma(t)\big),\\
&\Gamma:=\big\{\gamma\in C\big([0,1],X\big):\gamma(0)=0,\gamma(1)=e\big\},
\end{aligned}
$$

then c is a critical value of I.

Lemma 4.2 *Under the assumptions of Theorem 1.2, there exist $e\in H^1_r(\mathbb{R}^3)$ and $\rho>0$ such that $\|e\|>\rho$ and $\inf_{\|u\|=\rho}J_\lambda(u)>J_\lambda(0)>J_\lambda(e)$.*

Proof For all $u\in H^1_r(\mathbb{R}^3)$, by Remark 2.2,

$$
\begin{aligned}
J_\lambda(u)&=\frac{a}{2}\|u\|^2+\frac{\lambda}{4}\|u\|^4-\frac{1}{p+1}\int_{\mathbb{R}^3}Q(x)|u|^{p+1}-\frac{\kappa}{q+1}\int_{\mathbb{R}^3}G(x)|u|^{q+1}\\
&\geq\frac{a}{2}\|u\|^2+\frac{\lambda}{4}\|u\|^4-\frac{C_1}{p+1}\|u\|^p,
\end{aligned}
$$

then there exists $\rho > 0$ such that

$$b := \inf_{\|u\|=\rho} J_\lambda(u) > 0 = J_\lambda(0).$$

Let $t \geq 0$, we have

$$J_\lambda(tu) = \frac{t^2}{2}a\|u\|^2 + \frac{t^4}{4}\lambda\|u\|^4 - \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^3} Q(x)|u|^{p+1} - \frac{t^{q+1}}{q+1}\kappa\int_{\mathbb{R}^3} G(x)|u|^{q+1},$$

then there exists $e := tu$ such that $\|e\| > \rho$ and $J_\lambda(e) < 0$. □

Lemma 4.3 *Under the assumptions of Theorem 1.2. J_λ satisfies the $(PS)_c$ condition.*

Proof Let $\{u_n\} \subset H_r^1(\mathbb{R}^3)$ and $J_\lambda(u_n) \to c$, $J_\lambda(u_n) \to 0$ as $n \to \infty$. By (2.15) in Lemma 2.6 above, it is easy to see that $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^3)$. Going if necessary to a subsequence, $u_n \rightharpoonup u$ in $H_r^1(\mathbb{R}^3)$, $u_n \to u$ in $L^s(\mathbb{R}^3)$ for $s \in (2,6)$, and $u_n(x) \to u(x)$ a.e. on $\mathbb{R}^3$, then by (G_1) we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} G(x)|u_n|^q(u_n - u)\right| \\
&\quad\leq \int_{\mathbb{R}^3}\big|G(x)\big|\big||u_n|^q|u_n - u|\big| \\
&\quad\leq \left(\int_{\mathbb{R}^3}\big|G(x)\big|^2\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}|u_n|^{2q}|u_n - u|^2\right)^{\frac{1}{2}} \\
&\quad\leq \big|G(x)\big|_2\left(\int_{\mathbb{R}^3}|u_n|^{2q+2}\right)^{\frac{q}{2q+2}}\left(\int_{\mathbb{R}^3}|u_n - u|^{2q+2}\right)^{\frac{1}{2q+2}} \\
&\quad\leq C\big|G(x)\big|_2\|u_n\|^q|u_n - u|_{2q+2} \to 0.
\end{aligned}$$

Since

$$\begin{aligned}
&\langle J'_\lambda(u_n) - J'_\lambda(u), u_n - u\rangle \to 0, \\
&\int_{\mathbb{R}^3} Q(x)\big(|u_n|^p - |u|^p\big)(u_n - u) \to 0
\end{aligned}$$

and

$$\begin{aligned}
&\big(a + \lambda\|u_n\|^2\big)\|u_n - u\|^2 \\
&\quad= \langle J'_\lambda(u_n) - J'_\lambda(u), u_n - u\rangle + \lambda\big(\|u\|^2 - \|u_n\|^2\big)\langle u, u_n - u\rangle \\
&\qquad+ \int_{\mathbb{R}^3} Q(x)\big(|u_n|^p - |u|^p\big)(u_n - u) + \int_{\mathbb{R}^3} G(x)\big(|u_n|^p - |u|^p\big)(u_n - u).
\end{aligned}$$

Thus, $u_n \to u$ in $H_r^1(\mathbb{R}^3)$. □

Set

$$c_1 = \inf_{u \in H_r^1(\mathbb{R}^3)\setminus\{0\}} \max_{t\geq 0} J_\lambda(tu).$$

Lemma 4.4 *Under the assumptions of Theorem 1.2, we have $c = c_\lambda = c_1$.*

Proof Similar to the proof of Lemma 2.4, for all $u \in H_r^1(\mathbb{R}^3) \setminus \{0\}$, there exists unique $t_u u \in \mathcal{N}$ such that $J_\lambda(t_u u) = \max_{t\geq 0} J_\lambda(tu)$, this implies that $c_\lambda \leq c_1$.

For each $\gamma \in \Gamma$, it follows from the property of $\mathcal{N}$ that $\gamma(t)$ crosses $\mathcal{N}$ as t varying over $[0,1]$. Since $\gamma(0) = 0$, $J_\lambda(\gamma(1)) < 0$, then

$$\max_{t\in[0,1]} J_\lambda\bigl(\gamma(t)\bigr) \geq \inf_{u\in\mathcal{N}} J_\lambda(u) = c_\lambda.$$

Therefore $c \geq c_\lambda$. On the other hand, for $u \in H_r^1(\mathbb{R}^3) \setminus \{0\}$, we have that $J_\lambda(tu) < 0$ for t large enough, and then

$$\max_{t\geq 0} J_\lambda(tu) \geq \max_{t\in[0,1]} J_\lambda(tu) \geq \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\lambda\bigl(\gamma(t)\bigr) = c.$$

Therefore $c_1 \geq c$. □

Proof of Theorem 1.2 According to Lemmas 4.1, 4.2, 4.3, and 4.4, we obtain that problem (1.1) has one least energy solution.

Now we prove $m_\lambda > 2c_\lambda$. By the proof of Theorem 1.1, there exists $u_\lambda \in \mathcal{M}_\lambda$ such that $J_\lambda(u_\lambda) = m_\lambda$. By Lemmas 2.4 and 4.4, we have

$$\begin{aligned} m_\lambda &= J_\lambda(u_\lambda) \\ &\geq J_\lambda\bigl(su_\lambda^+ + tu_\lambda^-\bigr) \\ &= J_\lambda\bigl(su_\lambda^+\bigr) + J_\lambda\bigl(tu_\lambda^-\bigr) + \frac{s^2t^2}{2}\lambda\bigl\|u_\lambda^+\bigr\|^2\bigl\|u_\lambda^-\bigr\|^2 \\ &> J_\lambda\bigl(su_\lambda^+\bigr) + J_\lambda\bigl(tu_\lambda^-\bigr) \\ &\geq 2c_\lambda. \end{aligned}$$

□

Acknowledgements
The authors would like to thank the referees for their useful suggestions which have significantly improved the paper.

Funding
This work was supported financially by the National Natural Science Foundation of China (11871302).

Availability of data and materials
Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Consent for publication
Not applicable.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Author details
[1]School of Mathematical Sciences, Qufu Normal University, Qufu, Shandong 273165, People's Republic of China.
[2]Department of Mathematics and Statistics, Curtin University, Perth, WA 6845, Australia.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 4 January 2021 Accepted: 4 March 2021 Published online: 19 March 2021

References

1. Alves, C.O., Corrêa, F.J.S.A., Ma, T.F.: Positive solutions for a quasilinear elliptic equation of Kirchhoff type. Comput. Math. Appl. **49**, 85–93 (2005)
2. Berestycki, H., Lions, P.L.: Nonlinear scalar field equations. I. Existence of a ground state. Arch. Ration. Mech. Anal. **82**(4), 313–345 (1983)
3. Cheng, B.T., Wu, X.: Existence results of positive solutions of Kirchhoff problems. Nonlinear Anal. **71**, 4883–4892 (2009)
4. Deng, Y.B., Peng, S.J., Shuai, W.: Existence and asymptotic behavior of nodal solutions for the Kirchhoff-type problems in $\mathbb{R}^3$. J. Funct. Anal. **269**, 3500–3527 (2015)
5. Duan, Y.L., Zhou, Y.G.: Existence of solutions for Kirchhoff-type equations with unbounded potential. Electron. J. Differ. Equ. **2017**, 184 (2017)
6. He, H.M., Zou, W.M.: Existence and concentration behavior of positive solutions for a Kirchhoff equation in $\mathbb{R}^3$. J. Differ. Equ. **2**, 1813–1834 (2012)
7. He, Y., Li, G.B., Peng, S.J.: Concentrating bound states for Kirchhoff-type problems in $\mathbb{R}^3$ involving critical Sobolev exponents. Adv. Nonlinear Stud. **14**(2), 483–510 (2014)
8. Huang, Y.S., Liu, Z.: On a class of Kirchhoff type problems. Arch. Math. **102**, 127–139 (2014)
9. Jeanjean, L.: On the existence of bounded Palais–Smale sequences and application to a Landsman–Lazer-type problem set on $\mathbb{R}^N$. Proc. R. Soc. Edinb., Sect. A **129**(2), 787–809 (1999)
10. Kirchhoff, G.: Mechanik. Teubner, Leipzig (1883)
11. Liang, S.H., Zhang, J.H.: Existence of solutions for Kirchhoff-type problems with critical nonlinearity in $\mathbb{R}^3$. Nonlinear Anal., Real World Appl. **17**, 126–136 (2014)
12. Lions, J.L.: On some questions in boundary value problems of mathematical physics. In: Contemporary Developments in Continuum Mechanics and Partial Differential Equations, Proc. Internat. Sympos., Inst. Mat., Univ. Fed. Rio de Janeiro, 1977. North-Holland Math. Stud., vol. 30, pp. 284–346. North-Holland, Amsterdam (1978)
13. Mao, A.M., Luan, S.X.: Sign-changing solutions of a class of nonlocal quasilinear elliptic boundary value problems. J. Math. Anal. Appl. **383**(1), 239–243 (2011)
14. Miranda, C.: Un'osservazione su un teorema di Brouwer. Boll. Unione Mat. Ital. **3**(2), 5–7 (1940)
15. Shao, M.Q., Mao, A.M.: Signed and sign-changing solutions of Kirchhoff type problems. J. Fixed Point Theory Appl. **20**(1), 2 (2018)
16. Shuai, W.: Sign-changing solutions for a class of Kirchhoff-type problem in bounded domains. J. Differ. Equ. **259**(4), 1256–1274 (2015)
17. Sun, J.T., Tang, C.L.: Existence and multiplicity of solutions for Kirchhoff type equations. Nonlinear Anal. **74**, 1212–1222 (2011)
18. Willem, M.: Minimax Theorems. Birkhäuser, Boston (1996)
19. Wu, X.: Existence of nontrivial solutions and high energy solutions for Schrödinger–Kirchhoff-type equations in $\mathbb{R}^3$. Nonlinear Anal., Real World Appl. **12**, 1278–1287 (2011)
20. Xie, Q.L., Wu, X.P., Tang, C.L.: Existence and multiplicity of solutions for Kirchhoff type problems with critical exponent. Commun. Pure Appl. Anal. **12**(6), 2773–2786 (2013)
21. Ye, H.Y.: The existence of least energy nodal solutions for some class of Kirchhoff equations and Choquard equations in $\mathbb{R}^N$. J. Math. Anal. Appl. **431**(2), 935–954 (2015)
22. Zhang, Z.T., Perera, K.: Sign-changing solutions of Kirchhoff-type problems via invariant sets of descent flow. J. Math. Anal. Appl. **317**(2), 456–463 (2006)