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Global bifurcation and constant sign solutions of discrete boundary value problem involving p -Laplacian

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Abstract

We study the unilateral global bifurcation result for the one-dimensional discrete p -Laplacian problem

$$\begin{cases} -\Delta[\varphi_p(\Delta u(t-1))] = \lambda a(t)\varphi_p(u(t)) + g(t, u(t), \lambda), & t \in [1, T+1]_Z, \\ \Delta u(0) = u(T+2) = 0, \end{cases}$$

where $\Delta u(t) = u(t+1) - u(t)$ is a forward difference operator, $\varphi_p(s) = |s|^{p-2}s$ ($1 < p < +\infty$) is a one-dimensional p -Laplacian operator. λ is a positive real parameter, $a: [1, T+1]_Z \rightarrow [0, +\infty)$ and $a(t_0) > 0$ for some $t_0 \in [1, T+1]_Z$, $g: [1, T+1]_Z \times \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfies the Carathéodory condition in the first two variables. We show that $(\lambda_1, 0)$ is a bifurcation point of the above problem, and there are two distinct unbounded continua \mathcal{C}^+ and \mathcal{C}^- , consisting of the bifurcation branch \mathcal{C} from $(\lambda_1, 0)$, where λ_1 is the principal eigenvalue of the eigenvalue problem corresponding to the above problem. Let $T > 1$ be an integer, Z denote the integer set for $m, n \in Z$ with $m < n$, $[m, n]_Z := \{m, m+1, \dots, n\}$.

As the applications of the above result, we prove more details about the existence of constant sign solutions for the following problem:

$$\begin{cases} -\Delta[\varphi_p(\Delta u(t-1))] = \lambda a(t)f(u(t)), & t \in [1, T+1]_Z, \\ \Delta u(0) = u(T+2) = 0, \end{cases}$$

where $f \in C(\mathbb{R}, \mathbb{R})$ with $sf(s) > 0$ for $s \neq 0$.

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1 Introduction

In this paper, we establish a Dancer-type unilateral global bifurcation result for one-dimensional discrete p -Laplacian problem

$$\begin{cases} -\Delta[\varphi_p(\Delta u(t-1))] = \lambda a(t)\varphi_p(u(t)) + g(t, u(t), \lambda), & t \in [1, T+1]_Z, \\ \Delta u(0) = u(T+2) = 0, \end{cases} \quad (1.1)$$

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where $a : [1, T + 1]_Z \rightarrow [0, +\infty)$ and $a(t_0) > 0$ for some $t_0 \in [1, T + 1]_Z$, $g : [1, T + 1]_Z \times \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfies the Carathéodory condition in the first two variables and

$$\lim_{|s| \rightarrow 0} \frac{g(t, s, \lambda)}{|s|^{p-1}} = 0$$

uniformly for a.e. $t \in [1, T + 1]_Z$ and λ on bounded sets. Under these assumptions, we shall show that $(\lambda_1, 0)$ is a bifurcation point of (1.1) and there are two distinct unbounded continua \mathcal{C}^+ and \mathcal{C}^- , consisting of the bifurcation branch \mathcal{C} from $(\lambda_1, 0)$, where λ_1 is the principal eigenvalue of the eigenvalue problem corresponding to (1.1).

When $p = 2$, Ma and Ma [17] considered the optimal intervals of r , for which they gave a complete description of the global behavior of solutions set for the problem

$$\begin{cases} -\Delta^2 u(t-1) = rg(t)f(u(t)), & t \in [1, T]_Z, \\ u(0) = u(T), \quad \Delta u(0) = \Delta u(T) \end{cases} \quad (1.2)$$

under some suitable assumptions on f and g . Using the bifurcation theory of Rabinowitz [5, 18], they proved that if $\frac{\lambda_1}{f_\infty} < r < \frac{\lambda_1}{f_0}$ or $\frac{\lambda_1}{f_0} < r < \frac{\lambda_1}{f_\infty}$, (1.2) has two solutions u^+ and u^- such that u^+ is positive in $[0, T]_Z$ and u^- is negative in $[0, T]_Z$, where $f_0 = \lim_{s \rightarrow 0^+} \frac{f(s)}{s}$, $f_\infty = \lim_{s \rightarrow +\infty} \frac{f(s)}{s}$, λ_1 is the first eigenvalue of the following linear eigenvalue problem:

$$\begin{cases} -\Delta^2 u(t-1) = rg(t)u(t), & t \in [1, T]_Z, \\ u(0) = u(T), \quad \Delta u(0) = \Delta u(T). \end{cases}$$

The idea of using bifurcation methods to study the solvability of nonlinear boundary value problems has been applied to study some Dirichlet, Sturm–Liouville, and periodic boundary value problems, for instance, [10, 17, 21, 24].

When $p \neq 2$, many authors discussed the existence and multiplicity of solutions for the one-dimensional discrete p -Laplacian problem by exploiting various methods, including the method of upper and lower solutions, Leray–Schauder degree, fixed point theory, critical theory, and variational methods, which can be seen in D’Aguì [6], Bereanu [3], Wang et al. [23], and [8, 11–13, 22] and the references therein. In particular, He [11] established the existence of one or two positive solutions for the equation

$$\begin{cases} \Delta[\varphi_p(\Delta u(t-1))] + a(t)f(u(t)) = 0, & t \in [1, T+1]_Z, \\ \Delta u(0) = u(T+2) = 0. \end{cases}$$

In the paper, he assumed that the nonlinear term f is positive and f is superlinear at 0, sublinear at infinity. In 2015, Bai and Chen [2] discussed the discrete p -Laplacian boundary value problem

$$\begin{cases} -\Delta[\varphi_p(\Delta u(t-1))] = \lambda p(t)g(u(t)), & t \in [1, T]_Z, \\ u(0) = u(T+1) = 0, \end{cases}$$

where $\lim_{s \rightarrow \infty} \frac{g(s)}{\varphi_p(s)} = 0$ and $\frac{g(s)}{\varphi_p(s)}$ is strictly decreasing on $(0, \infty)$. They obtained an unbounded continuum \mathcal{C} of positive solutions emanating from $(\lambda, u) = (0, 0)$; while the existence of global continuum of solutions comes from [19].

There is no report on the global structure of solution sets by the bifurcation theory for discrete p -Laplacian problem (1.1). Although there are a great amount of papers researching the bifurcation phenomenon of p -Laplacian problem, but those results are not unilateral, and their conclusions are all obtained in the differential case. As we know, the proofs are based on the local properties of solutions of (1.1) bifurcating from $(\lambda_1, 0)$ (see Lemma 3.4). Although the proof of the above result follows the same steps as for the semilinear case from [7], his methods cannot be applied directly to the quasilinear discrete problem. In addition, the main reason is that the spectrum of the discrete p -Laplacian eigenvalue problem is unknown.

In 2002, Anane [1] discussed the spectra of the following differential p -Laplacian problem:

$$\begin{cases} -(\varphi_p(u'))' = \lambda m(t)\varphi_p(u), & t \in (a, b), \\ u(a) = u(b) = 0, \end{cases} \quad (1.3)$$

where $m \in M(I) := \{m \in L^\infty \mid \text{meas}\{t \in I, m(t) > 0\} \neq 0, I = [a, b]\}$, λ is the spectral parameter. They obtained the following result.

Theorem A *Assume that $m \in M(I)$ such that $m \not\equiv 1$, $p \neq 2$, we have:*

- (i) *Every eigenfunction corresponding to the k th eigenvalues $\lambda_k(m, I)$ has exactly $k - 1$ zeros.*
- (ii) *For every k , $\lambda_k(m, I)$ is simple and verifies the strict monotonicity property with respect to the weight m and the domain I .*
- (iii) *(1.3) has infinite real eigenvalues, the eigenvalues are ordered as*

$$0 < \lambda_1(m, I) < \lambda_2(m, I) < \cdots < \lambda_k(m, I) < \cdots \rightarrow +\infty \quad \text{as } k \rightarrow +\infty.$$

The first eigenvalue λ_1 is of special importance. However, the method in [1] has no effect on the spectral study of the discrete p -Laplacian problem, whether the first eigenvalue of the discrete p -Laplacian problem is simple or not, and the properties of the corresponding eigenfunction are both unknown.

Of course, the next natural question is: does the unilateral bifurcation version exist for quasilinear difference problem (1.1)? Furthermore, what is the existence of a positive solution or a negative one for a nonlinear p -Laplacian difference problem (1.1)? In this paper, we give a positive answer for those questions.

This paper is organized as follows. In Sect. 2, we show the existence of the principal eigenvalue and the sign of the corresponding eigenfunctions for the one-dimensional discrete p -Laplacian eigenvalue problem, which will be of interest for us. In Sect. 3, we establish the unilateral global bifurcation theory for (1.1). In Sect. 4, as an application, we prove that there exist constant sign solutions for problem (4.1) (see Sect. 4) according to the different behavior of nonlinear term f at 0 and ∞ .

2 The existence of the principal eigenvalue

In this section, we consider the existence of the principal eigenvalue and the sign of the corresponding eigenfunction for the discrete p -Laplacian eigenvalue problem

$$\begin{cases} -\Delta[\varphi_p(\Delta u(t-1))] = \lambda a(t)\varphi_p(u(t)), & t \in [1, T+1]_Z, \\ \Delta u(0) = u(T+2) = 0. \end{cases} \tag{2.1}$$

Let E be defined by

$$E = \{u \mid u : [1, T+1]_Z \rightarrow \mathbb{R} \text{ and } \Delta u(0) = u(T+2) = 0\}$$

equipped with the norm $\|u\| = \max_{t \in [0, T+2]_Z} |u(t)|$, then $(E, \|\cdot\|)$ is a Banach space.

Lemma 2.1 *Let $u \in E$. Then*

$$-\sum_{t=1}^{T+1} \Delta[\varphi_p(\Delta u(t-1))] \cdot u(t) = \sum_{t=1}^{T+1} |\Delta u(t)|^p.$$

Proof We have

$$\begin{aligned} & \sum_{t=1}^{T+1} \Delta[\varphi_p(\Delta u(t-1))] \cdot u(t) \\ &= \sum_{t=1}^{T+1} \Delta(|\Delta u(t-1)|^{p-2} \Delta u(t-1)) \cdot u(t) \\ &= \sum_{j=0}^T u(j+1) \Delta(|\Delta u(j)|^{p-2} \Delta u(j)) \quad (j = t-1) \\ &= \sum_{j=0}^T u(j+1) (|\Delta u(j+1)|^{p-2} \Delta u(j+1) - |\Delta u(j)|^{p-2} \Delta u(j)) \\ &= \sum_{j=0}^T u(j+1) |\Delta u(j+1)|^{p-2} \Delta u(j+1) - \sum_{j=0}^T u(j+1) |\Delta u(j)|^{p-2} \Delta u(j) \\ &= \sum_{k=1}^{T+1} u(k) |\Delta u(k)|^{p-2} \Delta u(k) - \sum_{j=0}^T u(j+1) |\Delta u(j)|^{p-2} \Delta u(j) \quad (k = j+1) \\ &= \sum_{k=1}^{T+1} u(k) |\Delta u(k)|^{p-2} \Delta u(k) - \sum_{j=1}^T u(j+1) |\Delta u(j)|^{p-2} \Delta u(j) - u(1) |\Delta u(0)|^{p-2} \Delta u(0) \\ &= \sum_{k=1}^T u(k) |\Delta u(k)|^{p-2} \Delta u(k) - \sum_{k=1}^T u(k+1) |\Delta u(k)|^{p-2} \Delta u(k) \\ &\quad + u(T+1) |\Delta u(T+1)|^{p-2} \Delta u(T+1) - u(1) |\Delta u(0)|^{p-2} \Delta u(0) \\ &= - \sum_{k=1}^T |\Delta u(k)|^p + u(T+1) |\Delta u(T+1)|^{p-2} \Delta u(T+1) - u(1) |\Delta u(0)|^{p-2} \Delta u(0) \end{aligned}$$

$$\begin{aligned}
 &= - \sum_{k=1}^T |\Delta u(k)|^p - |\Delta u(T+1)|^p - |\Delta u(0)|^p \quad (\Delta u(0) = u(T+2) = 0) \\
 &= - \sum_{k=1}^{T+1} |\Delta u(k)|^p.
 \end{aligned}$$

□

Lemma 2.2 $\lambda_1(a)$ is the first eigenvalue of (2.1), then the first eigenvalue $\lambda_1(a)$ is the minimum of the Rayleigh quotient, that is,

$$\lambda_1(a) = \inf \left\{ \frac{\sum_{t=1}^{T+1} |\Delta u(t)|^p}{\sum_{t=1}^{T+1} a(t)|u(t)|^p} : u \in E \right\}.$$

Furthermore, $\lambda_1(a) < \lambda(a)$, where $\lambda(a)$ is some other eigenvalue of (2.1).

Proof Combining the equation of (2.1) with Lemma 2.1, the conclusion is clearly established. □

Applying a similar method to prove [4, Proposition 1.10] with obvious changes, we can obtain the following theorem.

Lemma 2.3 The first eigenvalue $\lambda_1(a)$ is simple. Let ϕ_1 be the eigenfunction corresponding to $\lambda_1(a)$, then ϕ_1 does not change sign in $[0, T + 2]_Z$. Moreover, ϕ_1 does not vanish in $[0, T + 2]_Z$.

3 Unilateral global bifurcation results for (1.1)

In this section, we establish the unilateral global bifurcation theory for (1.1).

We consider the following auxiliary problem:

$$\begin{cases} \Delta[\varphi_p(\Delta u(t-1))] = h(t), & t \in [1, T+1]_Z, \\ \Delta u(0) = u(T+2) = 0, \end{cases} \tag{3.1}$$

where $h : [1, T + 1]_Z \rightarrow \mathbb{R}$. It can be easily seen that problem (3.1) is equivalently written as

$$u(t) = G_p(h)(t) := u(1) + \sum_{s=1}^{t-1} \varphi_p^{-1} \left[\sum_{\tau=1}^s h(\tau) \right], \quad t \in [1, T + 2]_Z,$$

where $G_p : \mathbb{R} \rightarrow E$ maps bounded sets of \mathbb{R} into relative compacts of E .

We define the operator $T_\lambda^p : E \rightarrow E$ by

$$\begin{aligned}
 T_\lambda^p(u)(t) &= u(1) + \sum_{s=1}^{t-1} \varphi_p^{-1} \left[- \sum_{\tau=1}^s \lambda a \varphi_p(u)(\tau) \right] \\
 &= G_p(-\lambda a \varphi_p(u))(t),
 \end{aligned}$$

then $T_\lambda^p : E \rightarrow E$ is completely continuous and (2.1) is equivalent to

$$u = T_\lambda^p(u).$$

Next, we use Brouwer degree theory to calculate its topological degree. Let $\text{deg}(I - T_\lambda^p, B_r, 0)$ be the Leray–Schauder degree for $I - T_\lambda^p$ on B_r , where $B_r = \{u \in E : \|u\| < r\}$.

Lemma 3.1 *Let λ be a constant, then for arbitrary $r > 0$,*

$$\text{deg}(I - T_\lambda^p, B_r, 0) = \begin{cases} 1, & 0 < \lambda < \lambda_1(a), \\ -1, & \lambda \in (\lambda_1(a), \lambda_2(a)), \end{cases}$$

where $\lambda_2(a)$ is the second eigenvalue of problem (2.1).

Proof Using the similar method of [15, Lemma 2.8], we can get the conclusion of this theorem. □

Define Nemytskii operators $H : \mathbb{R} \times E \rightarrow \mathbb{R}$ by

$$H(\lambda, u)(t) = -\lambda a(t)\varphi_p(u(t)) - g(t, u(t), \lambda).$$

Then it is clear that H is a continuous operator which maps bounded sets of $\mathbb{R} \times E$ into the bounded sets of \mathbb{R} . Obviously, (1.1) can be equivalently written as

$$u = G_p \circ H(\lambda, u) = F(\lambda, u).$$

It is easy to see that $F : \mathbb{R} \times E \rightarrow E$ is completely continuous and $F(\lambda, 0) = 0, \forall \lambda \in \mathbb{R}$.

For convenience, we abbreviate $\lambda_1(a)$ as λ_1 . Our first main result for (1.1) is the following theorem.

Theorem 3.1 *For $p > 1$, λ_1 is a bifurcation point of (1.1) and the associated bifurcation branch \mathcal{C} in $\mathbb{R} \times E$ whose closure contains $(\lambda_1, 0)$, then either*

- (i) \mathcal{C} is unbounded in $\mathbb{R} \times E$, or
- (ii) \mathcal{C} contains a pair $(\bar{\lambda}, 0)$, where $\bar{\lambda}$ is an eigenvalue of (2.1) and $\bar{\lambda} \neq \lambda_1$.

Proof Suppose on the contrary that $(\lambda_1, 0)$ is not a bifurcation point of (1.1). Then there exist $\varepsilon > 0, \rho_0 > 0$ such that, for $|\lambda - \lambda_1| \leq \varepsilon$ and $0 < \rho < \rho_0$, there is no nontrivial solution of the equation

$$u - F(\lambda, u) = 0$$

with $\|u\| = \rho$. By the invariance of the degree under a compact homotopy, we obtain

$$\text{deg}(I - F(\lambda, u), B_\rho, 0) \equiv \text{constant} \tag{3.2}$$

for $\lambda \in [\lambda_1 - \varepsilon, \lambda_1 + \varepsilon]$.

Take ε small enough such that there is no eigenvalue of (2.1) in $(\lambda_1, \lambda_1 + \varepsilon]$. Fix $\lambda \in (\lambda_1, \lambda_1 + \varepsilon]$, we claim that the equation

$$u - G_p(-\lambda a(t)\varphi_p(u(t)) - \alpha g(t, u(t), \lambda)) = 0 \tag{3.3}$$

has no solution u such that $\|u\| = \rho$ for every $\alpha \in [0, 1]$, where ρ is sufficiently small. Suppose that $\{u_n\}$ is the solution of (3.3) with $\|u_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Let $v_n = \frac{u_n}{\|u_n\|}$, then v_n satisfies

$$\begin{aligned} v_n &= G_p \left(-\lambda a(t) \varphi_p(v_n(t)) - \alpha \frac{g(t, u_n(t), \lambda)}{\|u_n\|^{p-1}} \right) \\ &= u(1) + \sum_{s=1}^{t-1} \varphi_p^{-1} \left[\sum_{\tau=1}^s h_n(\tau) \right], \end{aligned}$$

where $h_n(t) = -\lambda a(t) \varphi_p(v_n(t)) - \alpha \frac{g(t, u_n(t), \lambda)}{\|u_n\|^{p-1}}$. Let

$$\tilde{g}(t, u, \lambda) = \max_{0 \leq |s| \leq u} |g(t, s, \lambda)|,$$

then \tilde{g} is nondecreasing with respect to u and

$$\lim_{|u| \rightarrow 0} \frac{\tilde{g}(t, u, \lambda)}{|u|^{p-1}} = 0. \tag{3.4}$$

Furthermore, (3.4) implies

$$\frac{g(t, u, \lambda)}{\|u\|^{p-1}} \leq \frac{\tilde{g}(t, |u|, \lambda)}{\|u\|^{p-1}} \leq \frac{\tilde{g}(t, \|u\|, \lambda)}{\|u\|^{p-1}} \rightarrow 0, \quad \|u\| \rightarrow 0, \tag{3.5}$$

uniformly for a.e. $t \in [1, T + 1]_{\mathbb{Z}}$ and λ on bounded sets.

It is easy to see that $\{h_n\}$ is a bounded sequence in \mathbb{R} , thus we can assume that

$$v_n \rightarrow v_0 \quad \text{and} \quad \|v_0\| = 1 \quad \text{as } n \rightarrow \infty,$$

v_0 satisfies

$$-\Delta[\varphi_p(\Delta v_0(t - 1))] = \lambda a(t) \varphi_p(v_0(t)).$$

This implies that λ is an eigenvalue of (2.1), this is a contradiction. From the homotopic invariance of the degree (refer to the proof method of Theorem 2.10 in [15]) and Lemma 3.1, we conclude that

$$\deg(I - F(\lambda, \cdot), B_r, 0) = \deg(I - T_\lambda^p, B_r, 0) = -1. \tag{3.6}$$

Similarly, for $\lambda \in [\lambda_1 - \varepsilon, \lambda_1)$, it follows that

$$\deg(I - F(\lambda, \cdot), B_r, 0) = 1. \tag{3.7}$$

It is easy to see that (3.6) and (3.7) contradict (3.2). Thus $(\lambda_1, 0)$ is a bifurcation point of (1.1). By the global bifurcation theory [18], we can get the existence of a global branch of solutions of (1.1) emanating from $(\lambda_1, 0)$. \square

Let $S^+ = \{u \in E \mid u(t) > 0, t \in [1, T + 1]_{\mathbb{Z}}\}$, and set $S^- = -S^+$, $S = S^+ \cup S^-$. It is clear that S^+ and S^- are disjoint and open in E . Let $\Phi^\pm = \mathbb{R} \times S^\pm$ and $\Phi = \mathbb{R} \times S$ under the product topology.

Lemma 3.2 *If $\mathcal{C} \subset (\Phi \cup \{(\lambda_1, 0)\})$, the second alternative of Theorem 3.1 is impossible.*

Proof Suppose on the contrary that there exists $\{(\lambda_m, u_m)\} \subset \mathcal{C}$ such that $u_m \not\equiv 0$, $(\lambda_m, u_m) \rightarrow (\lambda_k, 0)$, $k \neq 1$. With the help of the definition of λ_1 , we see that $\lambda_k > \lambda_1$. According to the Sturm comparison theorem, if $\lambda > \lambda_1$, then the eigenfunction u corresponding to λ must change sign in $[0, T + 2]_Z$. Let $v_m = \frac{u_m}{\|u_m\|}$, then v_m satisfies

$$v_m = G_p \left(-\lambda_m a(t) \varphi_p(v_m(t)) - \frac{g(t, u_m(t), \lambda_m)}{\|u_m\|^{p-1}} \right). \tag{3.8}$$

From (3.5), (3.8), and the compactness of G_p , we can get

$$v_m \rightarrow v_0 \quad \text{and} \quad \|v_0\| = 1.$$

In addition, v_0 satisfies

$$-\Delta[\varphi_p(\Delta v_0(t - 1))] = \lambda_k a(t) \varphi_p(v_0(t)).$$

Hence, $v_0 \in S_k$, where S_k denotes the set of functions in E which must change sign in $[0, T + 2]_Z$. Therefore, when m is sufficiently large, $v_m \in S_k$, combining the definition of S_k with Lemma 2.3, it can be seen that this is a contradiction. \square

Lemma 3.3 *Let (λ, u) be a solution of (1.1). If there exists $t_0 \in [0, T + 1]_Z$ such that one of the following cases holds:*

- (i) $u(t_0) = 0, \Delta u(t_0) = 0$;
- (ii) $u(t_0) = 0, u(t_0 - 1)u(t_0 + 1) \geq 0$.

Then $u \equiv 0$ in $[0, T + 2]_Z$.

Proof (i) By virtue of the equation of (1.1), we have

$$\varphi_p(\Delta u(t_0 - 1)) - \varphi_p(\Delta u(t_0)) = \lambda a(t_0) \varphi_p(u(t_0)) + g(t, u(t_0), \lambda).$$

Combining $u(t_0) = 0$ with the assumption of g , we obtain

$$\begin{aligned} \varphi_p(\Delta u(t_0 - 1)) - \varphi_p(\Delta u(t_0)) &= 0, \\ \varphi_p(u(t_0) - u(t_0 - 1)) - \varphi_p(u(t_0 + 1) - u(t_0)) &= 0. \end{aligned}$$

Thus, $u(t_0 - 1) = 0$. Similarly, in view of

$$\varphi_p(\Delta u(t_0)) - \varphi_p(\Delta u(t_0 + 1)) = 0,$$

we can get $u(t_0 + 2) = 0$. Further,

$$\Delta u(t_0 - 1) = \Delta u(t_0 + 1) = 0,$$

step by step, it follows that $u \equiv 0$.

(ii) Using the same method, we conclude that

$$|u(t_0 - 1)|^{p-2} u(t_0 - 1) = |u(t_0 + 1)|^{p-2} u(t_0 + 1),$$

since $u(t_0 - 1)u(t_0 + 1) \geq 0$, we can only get

$$u(t_0 - 1) = u(t_0 + 1) = 0.$$

Hence, $u \equiv 0$. □

Theorem 3.2 $(\lambda_1, 0)$ bifurcates an unbounded continuum \mathcal{C} of solutions to problem (1.1), with the solutions on \mathcal{C} not changing sign.

Proof In view of the conclusion of Theorem 3.1 and Lemma 3.2, we only need to prove that $\mathcal{C} \subset (\Phi \cup \{(\lambda_1, 0)\})$.

Suppose on the contrary that $\mathcal{C} \not\subset (\Phi \cup \{(\lambda_1, 0)\})$. Then there exists $(\lambda, u) \in \mathcal{C} \cap (\mathbb{R} \times \partial S)$ such that

$$(\lambda, u) \neq (\lambda_1, 0), \quad u \notin S,$$

and $(\lambda_n, u_n) \in \mathcal{C} \cap (\mathbb{R} \times \partial S)$ with

$$(\lambda_n, u_n) \rightarrow (\lambda, u).$$

Since $u \in \partial S$, by Lemma 3.3, we obtain $u \equiv 0$. Let $\omega_n = \frac{u_n}{\|u_n\|}$, then ω_n satisfies

$$\omega_n = G_p \left(-\lambda_n a(t) \varphi_p(\omega_n(t)) - \frac{g(t, u_n(t), \lambda_n)}{\|u_n\|^{p-1}} \right). \tag{3.9}$$

Combining (3.5), (3.9) with the compactness of G_p , we obtain $\omega_n \rightarrow \omega_0$ and $\|\omega_0\| = 1$. Obviously, there is

$$-\Delta[\varphi_p(\Delta\omega_0(t-1))] = \lambda a(t) \varphi_p(\omega_0(t)).$$

Hence there exists $\lambda = \lambda_k$ ($k \neq 1$). Furthermore,

$$(\lambda_n, u_n) \rightarrow (\lambda_k, 0).$$

This contradicts Lemma 3.2. □

Now, we will show more details about the bifurcation from Theorem 3.2. Let $\mathbb{E} = \mathbb{R} \times E$, $\Phi(\lambda, u) = u - F(\lambda, u)$ and $S := \overline{\{(\lambda, u) \in \mathbb{E} : \Phi(\lambda, u) = 0, u \neq 0\}}^{\mathbb{E}}$. For convenience, let us introduce a few notations. Given any $\lambda \in \mathbb{R}$ and $0 < s < +\infty$, we consider an open neighborhood of $(\lambda_1, 0)$ in \mathbb{E} defined by

$$\mathbb{E}_s(\lambda_1, 0) := \{(\lambda, u) \in \mathbb{E} \mid \|u\| + |\lambda - \lambda_1| < s\}.$$

And $B_s(0)$ denotes $\{u \in E \mid \|u\| < s\}$. Let E_0 be a closed subspace of E such that

$$E = \text{span}\{\phi_1\} \oplus E_0.$$

By the Hahn–Banach theorem [26], there exists a linear functional $l \in E^*$, where E^* denotes the dual space of E such that

$$l(\phi_1) = 1 \quad \text{and} \quad E_0 = \{u \in E \mid l(u) = 0\}.$$

Finally, for any $0 < \eta < 1$, we define

$$K_\eta = \{(\lambda, u) \in \mathbb{E} \mid |l(u)| > \eta \|u\|\}.$$

Since $u \mapsto |l(u)| - \|u\|$ is continuous, K_η is an open subset of \mathbb{E} consisting of two disjoint components K_η^+ and K_η^- , where

$$K_\eta^+ = \{(\lambda, u) \in \mathbb{E} \mid l(u) > \eta \|u\|\}, \quad K_\eta^- = \{(\lambda, u) \in \mathbb{E} \mid l(u) < -\eta \|u\|\}.$$

In particular, both K_η^+ and K_η^- are convex cones, $K_\eta^+ = -K_\eta^-$.

Before proving our main result, we need the following lemma, which localizes the possible solutions of (1.1) bifurcating from $(\lambda_1, 0)$.

Lemma 3.4 *For every $\eta \in (0, 1)$, there is $\delta_0 > 0$ such that, for each $0 < \delta < \delta_0$,*

$$((S \setminus \{(\lambda_1, 0)\}) \cap \overline{\mathbb{B}_\delta(\lambda_1, 0)}) \subset K_\eta.$$

Moreover, for each

$$(\lambda, u) \in (S \setminus \{(\lambda_1, 0)\}) \cap (\overline{\mathbb{B}_\delta(\lambda_1, 0)}),$$

there are $s \in \mathbb{R}$ and $y \in E_0$ (unique) such that

$$u = s\phi_1 + y \quad \text{and} \quad |s| > \eta \|u\|.$$

Furthermore, for these solutions (λ, u) ,

$$\lambda = \lambda_1 + o(1) \quad \text{and} \quad y = o(s)$$

as $s \rightarrow 0$.

Proof This conclusion can be obtained by using the similar method in López-Gómez [16, Lemma 6.4.1]. □

Remark 3.1 From the proof of Lemma 6.4.1 of [14], we can see that if $g(t, u, \lambda)$ is replaced by $g_n(t, u, \lambda)$, which satisfies

$$\lim_{\|u\| \rightarrow 0} \frac{g_n(t, u, \lambda)}{\|u\|^{p-1}} = 0$$

uniformly for all $n \in \mathbb{N}$, then δ_0 can be chosen uniformly with respect to n .

Let $\delta > 0$ be the constant from Lemma 3.4. For $0 < \varepsilon \leq \delta$, we define $\mathcal{D}_{\lambda_1, \varepsilon}^v$ to be the component of $\{(\lambda_1, 0)\} \cup (S \cap \overline{B}_\varepsilon \cap K_\eta^v)$ containing $(\lambda_1, 0)$, $\mathcal{C}_{\lambda_1, \varepsilon}^v$ to be the component of $\overline{\mathcal{C}} \setminus \overline{\mathcal{D}_{\lambda_1, \varepsilon}^v}$ containing $(\lambda_1, 0)$, and \mathcal{C}^v to be the closure of $\bigcup_{0 < \varepsilon \leq \delta} \mathcal{C}_{\lambda_1, \varepsilon}^v$. Obviously, \mathcal{C}^v is connected. By Lemma 3.4, the definition of \mathcal{C}^v is independent of the choice of η and $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-$.

Similar to Dancer’s result [9, Theorem 2], we can obtain the following unilateral global bifurcation result.

Theorem 3.3 *Either \mathcal{C}^+ and \mathcal{C}^- are both unbounded, or else $\mathcal{C}^+ \cap \mathcal{C}^- \neq \{(\lambda_1, 0)\}$.*

Remark 3.2 Although the proof of the above result is similar to that of the semilinear case in [9], the method cannot be used directly in this paper. In fact, the proof of Lemma 3 in [9] strictly depends on the linear characteristics of L . Therefore, we use Lemma 3.4 and functional analysis to prove the above result.

To show Theorem 3.3, we need the following three lemmas.

Lemma 3.5 *Suppose $\delta_1, \delta_2 > 0$ such that $0 < \delta_1 + \delta_2 < \delta$ and $\Phi(\lambda, u) \neq 0$ if $\|u\| = \delta_1$ and $|\lambda - \lambda_1| \leq \delta_2$. If $0 < \sigma < \delta_2$ and $\beta(\sigma)$ is sufficiently small and positive, then*

$$\deg(\Phi(\lambda_1 + \sigma, \cdot), W^v, 0) - \deg(\Phi(\lambda_1 - \sigma, \cdot), W^v, 0) = 1,$$

where $W^v = \{u \in E \mid (\lambda, u) \in K_\eta^v, \beta(\sigma) < \|u\| < \delta_1\}$.

Proof Recall that $u = l(u)\varphi_1 + y$. We define

$$\widehat{g}(t, u, \lambda) = \begin{cases} g(t, u, \lambda), & l(u) \leq -\eta\|u\|, \\ \frac{-l(u)}{\eta\|u\|}g(t, -\eta\|u\|\varphi_1 + y, \lambda), & -\eta\|u\| < l(u) \leq 0, \\ -g(t, -u, \lambda), & l(u) > 0. \end{cases}$$

It is easy to verify that \widehat{g} is odd with respect to u . Let

$$\widehat{\Phi}(\lambda, u) = u - G_p(-\lambda a(t)\varphi_p(u(t)) - \widehat{g}(t, u(t), \lambda)),$$

then the mapping $\widehat{\Phi}(\lambda, u)$ is odd with respect to u .

By Lemma 3.4 and our assumptions, the equation $\Phi(\lambda_1 + \sigma, u) = 0$ has no solution in $B_{\delta_1} \setminus (W^+ \cup W^- \cup B_\beta)$. By Lemma 3.4, $\widehat{\Phi}(\lambda_1 + \sigma, u) = \Phi(\lambda_1 + \sigma, u)$ on $\partial B_{\delta_1} \cup \partial B_\beta$. It follows that

$$\begin{aligned} &\deg(\widehat{\Phi}(\lambda_1 + \sigma, \cdot), B_{\delta_1}, 0) - \deg(\widehat{\Phi}(\lambda_1 + \sigma, \cdot), B_\beta, 0) \\ &= \deg(\widehat{\Phi}(\lambda_1 + \sigma, \cdot), W^+, 0) + \deg(\widehat{\Phi}(\lambda_1 + \sigma, \cdot), W^-, 0). \end{aligned}$$

The oddness of $\widehat{\Phi}(\lambda_1 + \sigma, \cdot)$ and the definition of the degree in Schwartz [20] ensure that

$$\deg(\widehat{\Phi}(\lambda_1 + \sigma, \cdot), W^+, 0) = \deg(\widehat{\Phi}(\lambda_1 + \sigma, \cdot), W^-, 0).$$

Then the definition of $\widehat{\Phi}$ implies

$$\deg(\Phi(\lambda_1 + \sigma, \cdot), W^\nu, 0) = \deg(\widehat{\Phi}(\lambda_1 + \sigma, \cdot), W^\nu, 0).$$

Thus

$$2 \deg(\Phi(\lambda_1 + \sigma, \cdot), W^\nu, 0) = \deg(\widehat{\Phi}(\lambda_1 + \sigma, \cdot), B_{\delta_1}, 0) - \deg(\widehat{\Phi}(\lambda_1 + \sigma, \cdot), B_\beta, 0). \tag{3.10}$$

A similar result holds with $\lambda_1 + \sigma$ replaced by $\lambda_1 - \sigma$,

$$2 \deg(\Phi(\lambda_1 - \sigma, \cdot), W^\nu, 0) = \deg(\widehat{\Phi}(\lambda_1 - \sigma, \cdot), B_{\delta_1}, 0) - \deg(\widehat{\Phi}(\lambda_1 - \sigma, \cdot), B_\beta, 0). \tag{3.11}$$

Connecting Lemma 3.4 with the definition of $\widehat{\Phi}$, we obtain

$$\deg(\widehat{\Phi}(\lambda_1 + \sigma, \cdot), B_\beta, 0) = \deg(\Phi(\lambda_1 + \sigma, \cdot), B_\beta, 0)$$

and

$$\deg(\widehat{\Phi}(\lambda_1 - \sigma, \cdot), B_\beta, 0) = \deg(\Phi(\lambda_1 - \sigma, \cdot), B_\beta, 0).$$

From the proof of Theorem 3.1, it is obvious that

$$\deg(\Phi(\lambda_1 - \sigma, \cdot), B_\beta, 0) = 1 \quad \text{and} \quad \deg(\Phi(\lambda_1 + \sigma, \cdot), B_\beta, 0) = -1. \tag{3.12}$$

By Lemma 3.4 and the definition of $\widehat{\Phi}$, it follows that

$$\deg(\widehat{\Phi}(\lambda_1 + \sigma, \cdot), B_{\delta_1}, 0) = \deg(\Phi(\lambda_1 + \sigma, \cdot), B_{\delta_1}, 0)$$

and

$$\deg(\widehat{\Phi}(\lambda_1 - \sigma, \cdot), B_{\delta_1}, 0) = \deg(\Phi(\lambda_1 - \sigma, \cdot), B_{\delta_1}, 0).$$

By our hypothesis, for $\lambda \in [\lambda_1 - \sigma, \lambda_1 + \sigma]$, the homotopy $\Phi(\lambda, \cdot)$ is admissible on B_{δ_1} . The homotopy invariance of the degree ensures that

$$\deg(\Phi(\lambda_1 + \sigma, \cdot), B_{\delta_1}, 0) = \deg(\Phi(\lambda_1 - \sigma, \cdot), B_{\delta_1}, 0).$$

Subtracting (3.10) from (3.11) and using (3.12), we have

$$\deg(\Phi(\lambda_1 + \sigma, \cdot), W^\nu, 0) - \deg(\Phi(\lambda_1 - \sigma, \cdot), W^\nu, 0) = 1.$$

The proof of this lemma is complete. □

Define $T_{\lambda_1, \varepsilon}^-$ to be the component of $\mathcal{C} \setminus (\mathbb{B}_\varepsilon(\lambda_1, 0) \cap K_\eta^+)$ containing $(\lambda_1, 0)$.

Lemma 3.6 *If $0 < \varepsilon < \delta$, zero is an isolated solution of $\Phi(\lambda_1, u) = 0$ and $T_{\lambda_1, \varepsilon}^-$ is bounded in \mathbb{E} , then*

$$\partial \mathbb{B}_\varepsilon(\lambda_1, 0) \cap K_\eta^+ \cap T_{\lambda_1, \varepsilon}^- \neq \emptyset.$$

Proof The proof of Lemma 2 in Dancer [9] is also valid for the quasilinear case, so we omit the proof. \square

Lemma 3.7 *Lemma 3.6 holds without the assumption that zero is an isolated solution of $\Phi(\lambda_1, u) = 0$.*

Proof Because when $\|u\| \rightarrow 0$, there is

$$\frac{g(t, u, \lambda_1)}{\|u\|^{p-1}} \rightarrow 0$$

uniformly for a.e. $t \in [1, T + 1]_{\mathbb{Z}}$. So we can choose a continuous function $\rho : [0, +\infty) \rightarrow \mathbb{R}$ such that $\rho(0) = 0$, and for any $0 < \iota < \delta$,

$$\rho(\iota) \|\phi_1\| > \sup \left\{ \frac{\|g(t, u, \lambda_1)\|}{\|u\|^{p-1}} : \|u\| = \iota \right\}.$$

For every integer n , we can choose continuous functions $f_n : [0, +\infty) \rightarrow [0, 1]$ such that

$$f_n(s) = \begin{cases} \varphi_p(s), & 0 \leq |s| \leq \frac{1}{2n}, \\ 0, & |s| \geq \frac{1}{n}. \end{cases}$$

Define

$$\Phi_n(\lambda, u) := u - G_p(-\lambda a(t)\varphi_p(u(t)) - g(t, u(t), \lambda) - f_n(\|u\|)\rho(\|u\|)\phi_1).$$

Since $\lim_{\|u\| \rightarrow 0} \frac{g(t, u, \lambda)}{\|u\|^{p-1}} = 0$, we can see that

$$\lim_{\|u\| \rightarrow 0} \frac{g(t, u, \lambda) + f_n(\|u\|)\rho(\|u\|)\phi_1}{\|u\|^{p-1}} = 0, \quad n \in \mathbb{N} \tag{3.13}$$

uniformly for a.e. $t \in [1, T + 1]_{\mathbb{Z}}$ and λ on bounded sets. Let

$$S_n := \overline{\{(\lambda, u) \in \mathbb{E} : \Phi_n(\lambda, u) = 0, u \neq 0\}}^{\mathbb{E}},$$

using Remark 3.1 and (3.13), S_n can be chosen such that

$$((S_n \setminus \{(\lambda_1, 0)\}) \cap \overline{\mathbb{B}_\delta(\lambda_1, 0)}) \subset K_\eta.$$

We claim that zero is an isolated solution of $\Phi_n(\lambda_1, u) = 0$ for each $n \in \mathbb{N}^+$.

Suppose on the contrary that u is a nontrivial solution of $\Phi_n(\lambda_1, u) = 0$ such that

$$0 < \|u\| := \iota < \delta.$$

We divide the proof into two cases.

Case 1. $u \in \text{span} \{ \phi_1 \}$.

$\Phi_n(\lambda_1, u) = 0$ implies

$$u = G_p(-\lambda_1 a(t) \varphi_p(u(t)) - g(t, u(t), \lambda_1) - f_n(\|u\|) \rho(\|u\|) \phi_1),$$

i.e.,

$$-\Delta[\varphi_p(\Delta u(t-1))] = \lambda_1 a(t) \varphi_p(u(t)) + g(t, u(t), \lambda_1) + f_n(\|u\|) \rho(\|u\|) \phi_1.$$

Obviously, there is

$$g(t, u, \lambda_1) + f_n(\|u\|) \rho(\|u\|) \phi_1 = 0. \tag{3.14}$$

However,

$$\begin{aligned} \|g(t, u, \lambda_1) + f_n(\|u\|) \rho(\|u\|) \phi_1\| &\geq \|f_n(\|u\|) \rho(\|u\|) \phi_1\| - \|g(t, u, \lambda_1)\| \\ &\geq \iota^{p-1} \rho(\iota) \|\phi_1\| - \|g(t, u, \lambda_1)\| \\ &> 0. \end{aligned}$$

This contradicts (3.14).

Case 2. $u \notin \text{span} \{ \phi_1 \}$.

By $\Phi_n(\lambda_1, u) = 0$ and Lemma 2.1, we have

$$\sum_{t=1}^{T+1} |\Delta u(t)|^p = \lambda_1 \sum_{t=1}^{T+1} a(t) |u|^p + \sum_{t=1}^{T+1} g(t, u, \lambda_1) u + f_n(\|u\|) \rho(\|u\|) \sum_{t=1}^{T+1} \phi_1 u.$$

Let

$$f(u) = \sum_{t=1}^{T+1} |\Delta u(t)|^p - \lambda_1 \sum_{t=1}^{T+1} a(t) |u|^p - \sum_{t=1}^{T+1} g(t, u, \lambda_1) u - f_n(\|u\|) \rho(\|u\|) \sum_{t=1}^{T+1} \phi_1 u,$$

then $f(u) = 0$. $u \notin \text{span} \{ \phi_1 \}$ implies that there is $\gamma > 0$ such that

$$\left| \sum_{t=1}^{T+1} |\Delta u(t)|^p - \lambda_1 \sum_{t=1}^{T+1} a(t) |u|^p \right| \geq \gamma \|u\|^{p-1}.$$

From (3.13), we know that

$$\lim_{\|u\| \rightarrow 0} \frac{g(t, u, \lambda_1) u + f_n(\|u\|) \rho(\|u\|) \phi_1 u}{\|u\|^{p-1}} = 0, \quad n \in \mathbb{N} \tag{3.15}$$

uniformly for a.e. $t \in [1, T + 1]_{\mathbb{Z}}$. By (3.15), it can be easily seen that when ι is sufficiently small,

$$\left| \sum_{t=1}^{T+1} g(t, u, \lambda_1) u + f_n(\|u\|) \rho(\|u\|) \sum_{t=1}^{T+1} \phi_1 u \right| < \gamma \|u\|^{p-1}.$$

Thus $|f(u)| > 0$. This contradicts $f(u) = 0$.

For any $0 < \varepsilon < \delta$, we assume that $T_{\lambda_1, \varepsilon}^-$ is bounded in \mathbb{E} . Define T_n to be the component of $S_n \setminus (\mathbb{B}_\varepsilon(\lambda_1, 0) \cap K_\eta^+)$ containing $(\lambda_1, 0)$. It is easy to see that the limit of T_n is $T_{\lambda_1, \varepsilon}^-$, so T_n is bounded in \mathbb{E} . Suppose that Lemma 3.7 is false, then

$$\partial \mathbb{B}_\varepsilon(\lambda_1, 0) \cap K_\eta^+ \cap T_{\lambda_1, \varepsilon}^- = \emptyset.$$

The definition of $T_{\lambda_1, \varepsilon}^-$ implies

$$\mathbb{B}_\varepsilon(\lambda_1, 0) \cap K_\eta^+ \cap T_{\lambda_1, \varepsilon}^- = \emptyset.$$

Since $T_{\lambda_1, \varepsilon}^-$ is bounded, we can find a constant $R > 0$ such that $T_{\lambda_1, \varepsilon}^- \subset \mathbb{B}_R(\lambda_1, 0)$.

By these facts and the classical topological result from Whyburn [21], we obtain that

$$K := (S \cap \overline{\mathbb{B}_R(\lambda_1, 0)}) \setminus (\mathbb{B}_\varepsilon(\lambda_1, 0) \cap K_\eta^+) = k_1 \cup k_2,$$

where k_1, k_2 are disjoint compact subsets of K and $T_{\lambda_1, \varepsilon}^- \subset k_1$,

$$(S \cap \partial \mathbb{B}_R(\lambda_1, 0)) \cup (S \cap \partial \mathbb{B}_\varepsilon(\lambda_1, 0) \cap K_\eta^+) \subset k_2.$$

Therefore, there exists a bounded open set U in \mathbb{E} such that $k_1 \subset U, k_2 \cap \overline{U} = \emptyset, (\lambda_1, 0) \in U, (\partial U \cap S) \subset (\mathbb{B}_\varepsilon(\lambda_1, 0) \cap K_\eta^+)$ and $\partial \mathbb{B}_\varepsilon(\lambda_1, 0) \cap K_\eta^+ \cap U = \emptyset$.

Applying Lemma 3.6 to Φ_n , we can see that

$$\partial \mathbb{B}_\varepsilon(\lambda_1, 0) \cap K_\eta^+ \cap T_n \neq \emptyset, \quad n \in \mathbb{N}.$$

By the connectedness of T_n , there exists $(\lambda_n, u_n) \in \partial U \cap T_n$. We assume that there are $u_n \rightarrow u^*$ in E and $\lambda_n \rightarrow \lambda^*$ in \mathbb{R} . Letting $n \rightarrow +\infty$ on the both of $\Phi_n(\lambda_n, u_n) = 0$ and using the compact and continuous properties of G_p , we can show that $\Phi_n(\lambda^*, u^*) = 0$. It is easy to see that

$$(\lambda^*, u^*) \in (S \cap \partial U) \setminus (\mathbb{B}_\varepsilon(\lambda_1, 0) \cap K_\eta^+),$$

this contradicts the definition of U . We have completed the proof. □

Proof of Theorem 3.3 Define $T_{\lambda_1}^-$ to be the closure of $\bigcup_{0 < \varepsilon \leq \delta} T_{\lambda_1, \varepsilon}^-$, then $T_{\lambda_1}^- \subseteq \mathcal{C}^-$. Suppose that \mathcal{C}^- is bounded. Then, by Lemma 3.7, for any $0 < \varepsilon \leq \delta$, we obtain

$$\partial \mathbb{B}_\varepsilon(\lambda_1, 0) \cap K_\eta^+ \cap T_{\lambda_1}^- \neq \emptyset.$$

It follows that

$$(T_{\lambda_1}^- \setminus (\mathbb{B}_\delta(\lambda_1, 0) \cap K_\eta^-)) \cap \partial \mathbb{B}_\varepsilon(\lambda_1, 0) \neq \emptyset. \tag{3.16}$$

Furthermore, for every open set U in \mathbb{E} , which satisfies $(\lambda_1, 0) \in U$ and $U \subseteq \mathbb{B}_\delta(\lambda_1, 0)$, (3.16) implies

$$(T_{\lambda_1}^- \setminus (\mathbb{B}_\delta(\lambda_1, 0) \cap K_\eta^-)) \cap \partial U \neq \emptyset. \tag{3.17}$$

Let $\mathcal{E} = T_{\lambda_1}^- \setminus (\mathbb{B}_\delta(\lambda_1, 0) \cap K_\eta^-)$, T be the component of $T_{\lambda_1}^- \setminus (\mathbb{B}_\delta(\lambda_1, 0) \cap K_\eta^-)$ containing $(\lambda_1, 0)$. It is easy to know that \mathcal{E} is a compact metric space under the induced topology of \mathbb{E} and T is a closed subset of \mathcal{E} .

We claim that $T \cap \partial\mathbb{B}_\delta(\lambda_1, 0) \neq \emptyset$.

Suppose on the contrary that $T \cap \partial\mathbb{B}_\delta(\lambda_1, 0) = \emptyset$. From [25], we know that $K = K_1 \cup K_2$, where K_1, K_2 are disjoint compact subsets of K containing T and $\partial\mathbb{B}_\delta(\lambda_1, 0) \cap K$, respectively. There exists a bounded open neighborhood \mathcal{O} in \mathbb{E} of K_1 such that $\mathcal{O} \subseteq \mathbb{B}_\delta(\lambda_1, 0)$ and $\partial\mathcal{O} \cap K_2 = \emptyset$. This contradicts (3.17).

Combining the definition of \mathcal{C}^+ with $T \cap \partial\mathbb{B}_\delta(\lambda_1, 0) \neq \emptyset$, we obtain $T \setminus \{(\lambda_1, 0)\} \neq \emptyset$ and

$$\mathcal{C}_{\lambda_1}^+ \supseteq \mathcal{C}_{\lambda_1, \delta}^+ \supseteq T.$$

Therefore, $\mathcal{C}^+ \cap \mathcal{C}^- \neq \{(\lambda_1, 0)\}$. Since a similar argument could be used for \mathcal{C}^- , this completes the proof of Theorem 3.3. □

Combining Theorem 3.2 and Theorem 3.3, we can obtain the following unilateral global bifurcation result.

Theorem 3.4 *Let $v \in \{+, -\}$, then \mathcal{C}^v is unbounded in $\mathbb{R} \times E$ and*

$$\mathcal{C}^v \subset \{(\lambda_1, 0)\} \cup (\mathbb{R} \times S^v). \tag{3.18}$$

Proof We can find a bounded neighborhood \mathbb{O} of $(\lambda_1, 0)$ such that

$$(\mathbb{O} \cap \mathcal{C}^v) \subset \{(\lambda_1, 0)\} \cup (\mathbb{R} \times S^v) \quad \text{or} \quad (\mathbb{O} \cap \mathcal{C}^v) \subset \{(\lambda_1, 0)\} \cup (\mathbb{R} \times S^{-v}).$$

Without loss of generality, we may suppose that

$$(\mathbb{O} \cap \mathcal{C}^v) \subset \{(\lambda_1, 0)\} \cup (\mathbb{R} \times S^v).$$

By Theorem 3.2, $\mathcal{C}^v \setminus \mathbb{O}$ in $\mathbb{R} \times S^v$, so (3.18) holds. Next, we only need to prove that both \mathcal{C}^+ and \mathcal{C}^- are unbounded. Suppose on the contrary that \mathcal{C}^+ is bounded, the case for \mathcal{C}^- is similar. By Theorem 3.3, we know that

$$(\mathcal{C}^+ \cap \mathcal{C}^-) \setminus \{(\lambda_1, 0)\} \neq \emptyset,$$

in view of (3.18), there exists $(\lambda_*, u_*) \in \mathcal{C}^+ \cap \mathcal{C}^-$ such that

$$(\lambda_*, u_*) \neq (\lambda_1, 0) \quad \text{and} \quad u_* \in S^+ \cap S^-.$$

This contradicts the definitions of S^+ and S^- . □

4 Constant sign solutions for nonlinear discrete p -Laplacian problem

In this section, we use Theorem 3.4 to prove the existence of constant sign solutions for the discrete p -Laplacian problem

$$\begin{cases} -\Delta[\varphi_p(\Delta u(t-1))] = \lambda a(t)f(u(t)), & t \in [1, T+1]_{\mathbb{Z}}, \\ \Delta u(0) = u(T+2) = 0, \end{cases} \tag{4.1}$$

where $a : [1, T + 1]_{\mathbb{Z}} \rightarrow [0, +\infty)$ and $a(t_0) > 0$ for some $t_0 \in [1, T + 1]_{\mathbb{Z}}$, $f \in C(\mathbb{R}, \mathbb{R})$ with $sf(s) > 0$ for $s \neq 0$. We will discuss the existence of constant sign solutions according to the different behavior of nonlinear term f at 0 and ∞ . Denote

$$f_0 = \lim_{s \rightarrow 0} \frac{f(s)}{\varphi_p(s)}, \quad f_\infty = \lim_{s \rightarrow \infty} \frac{f(s)}{\varphi_p(s)}.$$

Definition 4.1 (see [17]) Let X be a Banach space, $\{C_n \mid n = 1, 2, 3, \dots\}$ be a family of subsets of X . Then the superior D of C_n is defined by

$$D := \limsup_{n \rightarrow \infty} C_n = \{x \in X \mid \exists n_i \subset N \text{ and } x_{n_i} \in C_{n_i} \text{ such that } x_{n_i} \rightarrow x\}.$$

Definition 4.2 (see [20]) The component of M is the largest connected subset in M .

Lemma 4.1 (see [20]) Suppose that X is a compact metric space, A and B are non-intersecting closed subsets of X , and no component of X intersects both A and B . Then there exist two disjoint compact subsets X_A and X_B , such that $X = X_A \cup X_B$, $A \subset X_A$, $B \subset X_B$.

Lemma 4.2 (see [19]) Let X be a Banach space, C_n is a component of X , assume that:

- (i) There exist $z_n \in C_n$ ($n = 1, 2, \dots$) and $z^* \in X$ such that $z_n \rightarrow z^*$;
- (ii) $\lim_{n \rightarrow \infty} r_n = \infty$, where $r_n = \sup\{\|x\| : x \in C_n\}$;
- (iii) For every $R > 0$, $(\bigcup_{n=1}^\infty C_n) \cap \Omega_R$ is a relative compact set of X , where $\Omega_R = \{x \in X : \|x\| \leq R\}$.

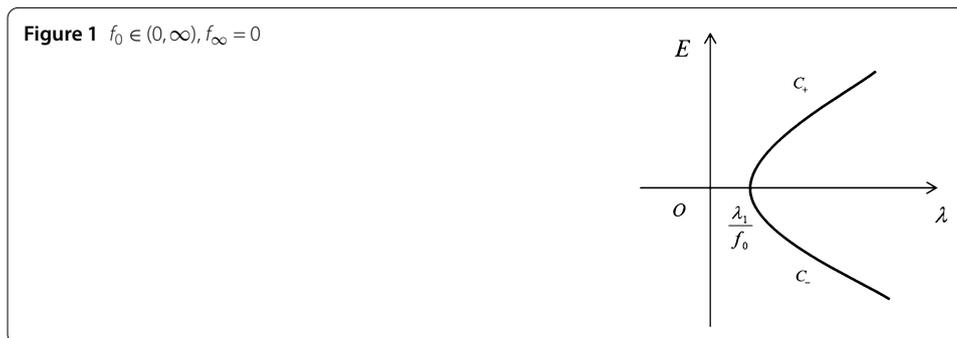
Then $D := \limsup_{n \rightarrow \infty} C_n$ contains an unbounded component C such that $z^* \in C$.

The main results of this section are the following.

Theorem 4.1 If $f_0 \in (0, \infty)$ and $f_\infty = 0$, then (4.1) has at least two solutions u^+ and u^- for $\lambda \in (\frac{\lambda_1}{f_0}, +\infty)$, where u^+ is positive in $[0, T + 1]_{\mathbb{Z}}$ and u^- is negative in $[0, T + 1]_{\mathbb{Z}}$ (see Fig. 1).

Proof Let $\zeta \in C(\mathbb{R}, \mathbb{R})$ such that $f(s) = f_0\varphi_p(s) + \zeta(s)$, where ζ satisfies

$$\lim_{s \rightarrow 0} \frac{\zeta(s)}{\varphi_p(s)} = 0.$$



Applying Theorem 3.4 to problem (4.1), it can be seen that there exist two different unbounded connected components C^+ and C^- , which bifurcate from $(\frac{\lambda_1}{f_0}, 0)$, and

$$C^\nu \subset \left(\left\{ \left(\frac{\lambda_1}{f_0}, 0 \right) \right\} \cup (\mathbb{R} \times S^\nu) \right),$$

where $\nu = +$ or $-$. Obviously, $C^\nu \cap (\{0\} \times E) = \emptyset$.

Next, we prove that C^ν is unbounded in the direction of the λ axis. Assume on the contrary that

$$\sup\{\lambda : (\lambda, y) \in C^\nu\} < \infty.$$

Then there exists a sequence $\{(\mu_k, y_k)\} \subset C^\nu$ such that

$$\lim_{k \rightarrow \infty} \|y_k\| = \infty, \quad |\mu_k| \leq C_0$$

for some positive constant C_0 independent of k . This implies that

$$\lim_{k \rightarrow \infty} y_k(t) = \infty \quad \text{uniformly on } t \in [0, T + 2]_{\mathbb{Z}}. \tag{4.2}$$

Since $\{(\mu_k, y_k)\} \subset C^\nu$, we have that

$$\begin{cases} \Delta[\varphi_p(\Delta y_k(t-1))] + \mu_k a(t) f(y_k(t)) = 0, & t \in [1, T + 1]_{\mathbb{Z}}, \\ \Delta y_k(0) = y_k(T + 2) = 0. \end{cases} \tag{4.3}$$

Set $v_k(t) = \frac{y_k(t)}{\|y_k\|}$, then

$$\|v_k\| = 1.$$

Choosing a subsequence and relabeling if necessary, it follows that there exists $(\mu_*, v_*) \in (0, C_0] \times E$ with

$$\|v_*\| = 1 \tag{4.4}$$

such that

$$\lim_{k \rightarrow \infty} (\mu_k, v_k) = (\mu_*, v_*) \quad \text{in } \mathbb{R} \times E.$$

Moreover, from (4.2), (4.3), and $f_\infty = 0$, it follows that

$$\begin{cases} \Delta[\varphi_p(\Delta v_*(t-1))] + \mu_* a(t) \cdot 0 = 0, & t \in [1, T + 1]_{\mathbb{Z}}, \\ \Delta v_*(0) = v_*(T + 2) = 0. \end{cases}$$

Hence, $v_*(t) \equiv 0$ for $t \in [0, T + 2]_{\mathbb{Z}}$. This contradicts (4.4). Therefore,

$$\sup\{\lambda : (\lambda, y) \in C^\nu\} = \infty. \tag{□}$$



Theorem 4.2 *If $f_0 \in (0, \infty)$ and $f_\infty = +\infty$, then (4.1) has at least two solutions u^+ and u^- for $\lambda \in (0, \frac{\lambda_1}{f_0})$, where u^+ is positive in $[0, T + 1]_Z$ and u^- is negative in $[0, T + 1]_Z$ (see Fig. 2).*

Proof In this case, we show that C^v joins $(\frac{\lambda_1}{f_0}, 0)$ with $(0, \infty)$.

Similar to Theorem 4.1, we know that there exist two different unbounded connected components C^+ and C^- , which bifurcate from $(\frac{\lambda_1}{f_0}, 0)$.

Let $\{(\mu_k, y_k)\} \subset C^v$ be such that

$$|\mu_k| + \|y_k\| \rightarrow \infty, \quad k \rightarrow \infty.$$

Then

$$\begin{cases} \Delta[\varphi_p(\Delta y_k(t - 1))] + \mu_k a(t) f(y_k(t)) = 0, & t \in [1, T + 1]_Z, \\ \Delta y_k(0) = y_k(T + 2) = 0. \end{cases}$$

If $\{\|y_k\|\}$ is bounded, i.e., there exists a constant M_1 depending not on k such that $\|y_k\| \leq M_1$, then we may assume that

$$\lim_{k \rightarrow \infty} \mu_k = \infty.$$

Combining this with the fact

$$\frac{f(y_k(t))}{\varphi_p(y_k(t))} \geq \inf \left\{ \frac{f(s)}{\varphi_p(s)} \mid 0 \leq s \leq M_1 \right\} > 0,$$

we obtain

$$\lim_{k \rightarrow \infty} \mu_k \frac{f(y_k(t))}{\varphi_p(y_k(t))} = \infty, \quad \forall t \in [0, T + 2]_Z,$$

using the relation

$$\Delta[\varphi_p(\Delta y_k(t - 1))] + \mu_k a(t) \frac{f(y_k(t))}{\varphi_p(y_k(t))} \varphi_p(y_k(t)) = 0, \quad t \in [1, T + 1]_Z. \tag{4.5}$$

We deduce that y_k must change its sign on $[0, T + 2]_Z$ if k is large enough, and this contradicts the fact that y_k does not change sign. Hence,

$$\|y_k\| \rightarrow \infty, \quad k \rightarrow \infty.$$

Now, taking $\{(\mu_k, y_k)\} \subset C^v$ such that

$$\|y_k\| \rightarrow \infty, \quad k \rightarrow \infty,$$

we show that $\lim_{k \rightarrow \infty} \mu_k = 0$.

Suppose on the contrary that choosing a subsequence and relabeling if necessary, $\mu_k \geq M_2$ for some constant $M_2 > 0$. By (4.2), we obtain

$$\lim_{k \rightarrow \infty} y_k(t) = \infty, \quad \text{uniformly on } t \in [0, T + 2]_Z.$$

Consequently, we have

$$\lim_{k \rightarrow \infty} \mu_k \frac{f(y_k(t))}{\varphi_p(y_k(t))} = \infty, \quad \forall t \in [0, T + 2]_Z.$$

Hence, we have from (4.5) that y_k must change its sign on $[0, T + 2]_Z$ for k large enough. However, this is impossible. Thus, $\lim_{k \rightarrow \infty} \mu_k = 0$.

So, we prove that C^v joins $(\frac{\lambda_1}{f_0}, 0)$ with $(0, \infty)$. □

Theorem 4.3 *If $f_0 = 0$ and $f_\infty \in (0, +\infty)$, then (4.1) has at least two solutions u^+ and u^- for $\lambda \in (\frac{\lambda_1}{f_\infty}, +\infty)$, where u^+ is positive in $[0, T + 1]_Z$ and u^- is negative in $[0, T + 1]_Z$ (see Fig. 3).*

Proof If (λ, u) is a solution of (4.1) and $\|u\| \neq 0$, dividing (4.1) by $\|u\|^{2(p-1)}$ and setting $\omega = \frac{u}{\|u\|^2}$, we obtain

$$\begin{cases} -\Delta[\varphi_p(\Delta\omega(t-1))] = \lambda a(t) \frac{f(u(t))}{\|u\|^{2(p-1)}}, & t \in [1, T + 1]_Z, \\ \Delta\omega(0) = \omega(T + 2) = 0. \end{cases} \tag{4.6}$$

Define

$$\tilde{f}(\omega) = \begin{cases} \|\omega\|^{2(p-1)} f(\frac{\omega}{\|\omega\|^2}), & \omega \neq 0, \\ 0, & \omega = 0. \end{cases}$$



Clearly, (4.6) is equivalent to

$$\begin{cases} -\Delta[\varphi_p(\Delta\omega(t-1))] = \lambda a(t)\tilde{f}(\omega(t)), & t \in [1, T+1]_{\mathbb{Z}}, \\ \Delta\omega(0) = \omega(T+2) = 0. \end{cases} \tag{4.7}$$

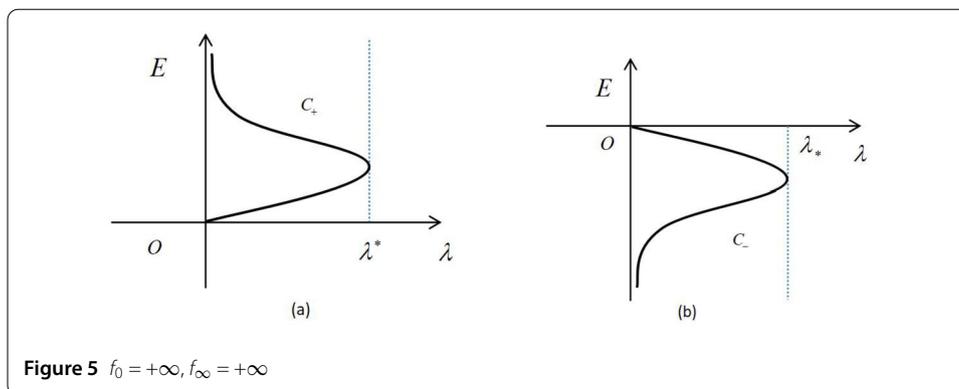
It is obvious that $(\lambda, 0)$ is always a solution of (4.7).

By simple calculation, we know $\tilde{f}_0 = f_\infty, \tilde{f}_\infty = f_0$. By applying Theorem 4.1, we can get the conclusions of this theorem under the inversion $\omega \rightarrow \frac{\omega}{\|\omega\|^2} = u$. \square

Theorem 4.4 *If $f_0 = +\infty$ and $f_\infty \in (0, +\infty)$, then (4.1) has at least two solutions u^+ and u^- for $\lambda \in (0, \frac{\lambda_1}{f_\infty})$, where u^+ is positive in $[0, T+1]_{\mathbb{Z}}$ and u^- is negative in $[0, T+1]_{\mathbb{Z}}$ (see Fig. 4).*

Proof Using the conclusion of Theorem 4.2 and the method of Theorem 4.3, we can easily prove the conclusion of this theorem. \square

Theorem 4.5 *If $f_0 = +\infty$ and $f_\infty = +\infty$, then there exists $\lambda^* > 0$ such that (4.1) has at least two solutions u_1^+ and u_2^+ for $\lambda \in (0, \lambda^*)$; in addition, u_1^+ and u_2^+ are positive in $[0, T+1]_{\mathbb{Z}}$ (see Fig. 5(a)). Similarly, there exists $\lambda_* > 0$ such that (4.1) has at least two solutions u_1^- and u_2^- for $\lambda \in (0, \lambda_*)$; in addition, u_1^- and u_2^- are negative in $[0, T+1]_{\mathbb{Z}}$ (see Fig. 5(b)).*



Proof Define

$$f_n(s) = \begin{cases} n\varphi_p(s), & s \in [-\frac{1}{n}, \frac{1}{n}], \\ (f(\frac{2}{n}) - \frac{1}{n^{p-2}})ns + \frac{2}{n^{p-2}} - f(\frac{2}{n}), & s \in (\frac{1}{n}, \frac{2}{n}), \\ -(f(-\frac{2}{n}) + \frac{1}{n^{p-2}})ns - \frac{2}{n^{p-2}} - f(-\frac{2}{n}), & s \in (-\frac{2}{n}, -\frac{1}{n}), \\ f(s), & s \in (-\infty, -\frac{2}{n}] \cup [\frac{2}{n}, +\infty). \end{cases} \tag{4.8}$$

Let us consider

$$\begin{cases} \Delta[\varphi_p(\Delta u(t-1))] + \lambda a(t)f_n(u(t)) = 0, & t \in [1, T+1]_{\mathbb{Z}}, \\ \Delta u(0) = u(T+2) = 0. \end{cases} \tag{4.9}$$

There are clearly $\lim_{n \rightarrow +\infty} f_n(s) = f(s)$, $(f_n)_0 = n$, and $(f_n)_\infty = f_\infty = +\infty$. Theorem 4.2 implies that there exists a sequence of unbounded continuum $(C^v)_n$ of solutions of problem (4.9) emanating from $(\frac{\lambda_1}{n}, 0)$ such that $(C^v)_n$ joins $(\frac{\lambda_1}{n}, 0)$ to $(0, \infty)$.

Taking $z_n = (\frac{\lambda_1}{n}, 0)$ and $z^* = (0, 0)$, we have that $z_n \rightarrow z^*$. So condition (i) in Lemma 4.2 is satisfied with $z^* = (0, 0)$.

Obviously, $r_n = \sup\{\lambda + \|u\| : (\lambda, u) \in (C^v)_n\} \rightarrow \infty$, and accordingly, (ii) in Lemma 4.2 holds. (iii) can be deduced directly from the Arzela–Ascoli theorem and the definition of f_n . Therefore, by Lemma 4.2, $\limsup_{n \rightarrow +\infty} (C^v)_n$ contains unbounded connected components C^v with

$$(0, 0) \in C^v \subset \limsup_{n \rightarrow +\infty} (C^v)_n \quad \text{and} \quad (0, \infty) \in C^v \subset \limsup_{n \rightarrow +\infty} (C^v)_n.$$

For any $\lambda_0 > 0$, it is easy to know that there is at most one n_0 such that $(\lambda_0, 0) \in (C^v)_{n_0}$. By the definition of upper bound set, we obtain $(\lambda_0, 0) \in C^v \subset \limsup_{n \rightarrow +\infty} (C^v)_n$. Naturally, $(\lambda_0, 0) \notin C^v$.

Consequently, $C^v \cap (\mathbb{R} \times \{0\}) = \{(0, 0)\}$. □

Theorem 4.6 *If $f_0 = 0$ and $f_\infty = 0$, then there exists $\lambda^r > 0$ such that (4.1) has at least two solutions u_1^+ and u_2^+ for $\lambda \in (\lambda^r, +\infty)$; in addition, u_1^+ and u_2^+ are positive in $[0, T+1]_{\mathbb{Z}}$ (see Fig. 6(a)). Similarly, there exists $\lambda_r > 0$ such that (4.1) has at least two solutions u_1^- and u_2^- for $\lambda \in (\lambda_r, +\infty)$; in addition, u_1^- and u_2^- are negative in $[0, T+1]_{\mathbb{Z}}$ (see Fig. 6(b)).*

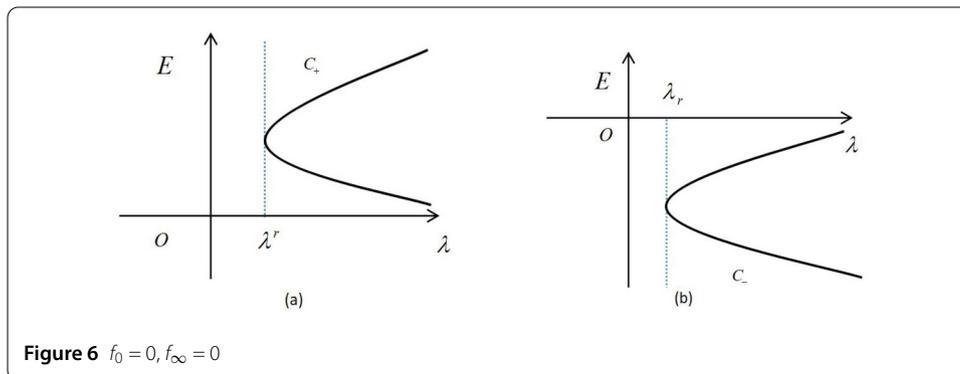


Figure 6 $f_0 = 0, f_\infty = 0$

Proof Define

$$g_n(s) = \begin{cases} \frac{1}{n}\varphi_p(s), & s \in [-\frac{1}{n}, \frac{1}{n}], \\ (f(\frac{2}{n}) - \frac{1}{n^p})ns + \frac{2}{n^p} - f(\frac{2}{n}), & s \in (\frac{1}{n}, \frac{2}{n}), \\ -(f(-\frac{2}{n}) + \frac{1}{n^p})ns - \frac{2}{n^p} - f(-\frac{2}{n}), & s \in (-\frac{2}{n}, -\frac{1}{n}), \\ f(s), & s \in (-\infty, -\frac{2}{n}] \cup [\frac{2}{n}, +\infty). \end{cases} \tag{4.10}$$

Let us consider

$$\begin{cases} \Delta[\varphi_p(\Delta u(t-1))] + \lambda a(t)g_n(u(t)) = 0, & t \in [1, T+1]_{\mathbb{Z}}, \\ \Delta u(0) = u(T+2) = 0. \end{cases}$$

Using the conclusion of Theorem 4.5 and the method of Theorem 4.3, we can easily prove the conclusion of this theorem. \square

Theorem 4.7 *If $f_0 = 0$ and $f_\infty = +\infty$, then (4.1) has at least two solutions u^+ and u^- for $\lambda \in (0, +\infty)$, where u^+ is positive in $[0, T+1]_{\mathbb{Z}}$ and u^- is negative in $[0, T+1]_{\mathbb{Z}}$ (see Fig. 7).*

Proof Define g_n as (4.10). Let us consider

$$\begin{cases} \Delta[\varphi_p(\Delta u(t-1))] + \lambda a(t)g_n(u(t)) = 0, & t \in [1, T+1]_{\mathbb{Z}}, \\ \Delta u(0) = u(T+2) = 0. \end{cases} \tag{4.11}$$

Clearly, $\lim_{n \rightarrow +\infty} g_n(s) = f(s)$, $(g_n)_0 = \frac{1}{n}$, and $(g_n)_\infty = f_\infty = +\infty$. By Theorem 4.2, there exists a sequence of unbounded continuum $(C^v)_n$ of solutions of (4.11) emanating from $(n\lambda_1, 0)$ such that $(C^v)_n$ joins $(n\lambda_1, 0)$ to $(0, \infty)$.

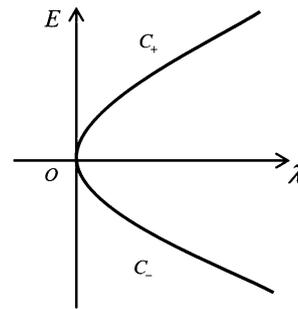
Taking $z_n = (n\lambda_1, 0)$ and $z^* = (\infty, 0)$, we have that $z_n \rightarrow z^*$. So condition (i) in Lemma 4.2 is satisfied with $z^* = (\infty, 0)$.

Obviously, $r_n = \sup\{\lambda + \|u\| : (\lambda, u) \in (C^v)_n\} \rightarrow \infty$, and accordingly, (ii) in Lemma 4.2 holds.

(iii) can be deduced directly from the Arzela–Ascoli theorem and the definition of g_n . Therefore, by Lemma 4.2, $\limsup_{n \rightarrow +\infty} (C^v)_n$ contains unbounded connected components C^v with

$$(\infty, 0) \in C^v \subset \limsup_{n \rightarrow +\infty} (C^v)_n \quad \text{and} \quad (0, \infty) \in C^v \subset \limsup_{n \rightarrow +\infty} (C^v)_n. \quad \square$$



Figure 8 $f_0 = +\infty, f_\infty = 0$ 

Theorem 4.8 *If $f_0 = +\infty$ and $f_\infty = 0$, then (2) has at least two solutions u^+ and u^- for $\lambda \in (0, +\infty)$ such that u^+ is positive in $[0, T + 1]_Z$ and u^- is negative in $[0, T + 1]_Z$ (see Fig. 8).*

Proof Define f_n as (4.8). Let us consider

$$\begin{cases} \Delta[\varphi_p(\Delta u(t-1))] + \lambda a(t)f_n(u(t)) = 0, & t \in [1, T+1]_Z, \\ \Delta u(0) = u(T+2) = 0. \end{cases}$$

Using the conclusion of Theorem 4.7 and the method of Theorem 4.3, we can easily prove the conclusion of this theorem. \square

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The author claim that the research was realized in collaboration with the same responsibility. All authors read and approved the last of the manuscript.

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