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On oscillation of higher-order advanced trinomial differential equations

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Abstract

We study the oscillatory property of the higher-order trinomial differential equation with advanced effects

$$x^{(n)}(t)+p(t)x'(t)+q(t)x(\sigma(t))=0, \quad \sigma(t)\geq t.$$

Suppose that all solutions of the corresponding $(n-1)$th-order two-term differential equation

$$y^{(n-1)}(t)+p(t)y(t)=0$$

are non-oscillatory. In order to supplement the research in the theory of oscillation proposed by (Džurina et al. in Electron. J. Differ. Equ. 2015:70, 2015), two types of clearly confirmable criteria for oscillatory behavior of the investigated equation are obtained. Some examples are offered to describe our main results.

MSC: 34K11; 34C10

Keywords: Higher-order differential equations; Oscillation; Advanced argument

1 Introduction

This paper focuses on the oscillatory behavior of solutions to a higher-order trinomial differential equation with advanced effects

$$x^{(n)}(t)+p(t)x'(t)+q(t)x(\sigma(t))=0, \quad \sigma(t)\geq t \tag{1.1}$$

for all $t\geq t_0$. Throughout the remaining parts of this article, we need to establish some hypotheses as follows:

(H_1) $p(t)$ and $q(t)\in C([t_0,\infty))$, $p(t)$ is nonnegative, $q(t)$ is positive;

(H_2) $\sigma(t)\in C([t_0,\infty))$, $\sigma(t)\geq t$.

The oscillatory behavior of ordinary differential equations (ODEs) is one of the significant branching problems of differential equations. The oscillatory problems to the wings of the plane can be modeled by the oscillatory problems of ODEs. As a matter of fact, differential equations with deviating arguments have numerous applications in engineering and natural sciences (see [9, 12, 26] for more details). By a solution of Eq. (1.1), we mean a nontrivial real function $x(t)$, $x \in C^1([T_x,\infty),\mathbb{R})$, $T_x \geq t_0$, which satisfies (1.1) on $[T_x,\infty)$. We investigate only proper solutions $x(t)$ to Eq. (1.1) with the property $\sup\{|x(t)| : T < t < \infty\} > 0$, for any $T \geq T_x$ and existing on some half-line $[T_x,\infty)$. Tacitly, we suppose that for Eq. (1.1) there exists such a solution. As is customary, if a solution of Eq. (1.1) possesses arbitrarily many zeros on the interval $[T_x,\infty)$, we called it an oscillatory solution. Otherwise it is said to be non-oscillatory. If all solutions to a higher-order functional and differential equation, such as Eq. (1.1), are oscillatory, it is common to call it an oscillatory equation.

With the social development and the progress of all fields of modern technology and science, such as economics, aerospace and modern physics, delay differential equations (DDEs) have received more and more consideration in the past decades. It is well known that DDEs involve the dependency of the previous state, which can help us to predict the future state with efficiency and reliability. Meanwhile, many qualitative properties such as boundedness, stability or periodicity can be explained. If we incorporate the delay effect into models, it will play a significant role when representing time taken to finish some veiled procedure. On the contrary, advanced differential equations (ADEs), different from genetic systems, can also be applied in almost all real-world areas. Population dynamics in mathematical biology, mechanical control in engineering or problems in economics are instances of areas where we can discover applications of such differential equations [15].

Regrettably, many oscillation works in the field of ODEs are considered only to finite order because several systems in engineering are naturally described by ODEs with finite order [8, 18]. Thus, the main goal of researchers in the area of ODEs is to obtain some qualitative theories as regards those equations, such as existence, uniqueness, boundedness, periodicity, and stability. Meanwhile, some criteria for the asymptotic behavior and oscillation of such equations are also important for the investigation of ODEs; see, for instance, [4, 18, 27]. We suggest the reader to consult the outstanding treatises of Elias [14], Chanturia and Kiguradze [19] and Swanson [29] for scientific research of the most important efforts made in this theory. However, several researchers, such as the authors in [2–5, 17, 25], in the field of mathematics have obtained some sufficient conditions to guarantee that all solutions of the nth-order equations

$$y^{(n)}(t) + q(t)y\bigl(\sigma(t)\bigr) = 0 \tag{1.2}$$

are oscillatory. In 2015 and 2017, Baculíková, Džurina and Jadlovská [11, 13] discussed the oscillatory behaviors of solutions of the two equations

$$x^{(4)}(t) + p(t)x'(t) + q(t)x\bigl(\tau(t)\bigr) = 0 \tag{1.3}$$

and

$$x^{(4)}(t) - p(t)x'(t) + q(t)x\bigl(\tau(t)\bigr) = 0, \tag{1.4}$$

respectively. In addition, [30, 31] were concerned with the oscillatory properties of solutions to the Swift–Hohenberg differential equation (1.3). At the end of [13], they proposed an interesting problem for further investigation: how these equations can be higher-order trinomial delay equations of the form

$$x^{(n)}(t) \pm p(t)x^{(n-3)}(t) + q(t)x\bigl(\tau(t)\bigr) = 0, \quad n \geq 4, \tag{1.5}$$

and

$$x^{(n)}(t) \pm p(t)x'(t) + q(t)x\bigl(\tau(t)\bigr) = 0, \quad n \geq 3, \tag{1.6}$$

respectively. Recently, the authors in [1] have established some improved and generalized criteria for the even-order ADE

$$\bigl(r(t)\bigl(y^{(n-1)}(t)\bigr)^{\alpha}\bigr)' + q(t)y^{\alpha}\bigl(\sigma(t)\bigr) = 0. \tag{1.7}$$

The study of oscillation theorems to trinomial fourth-order equations without or with deviating effects is investigated in [7, 8]. Other authors were concerned with the higher-order cases; see, among others, [1, 5, 17, 21, 27]. In particular, some attempts were made for an analogue of (1.3), of the form

$$x^{(n+3)}(t) + p(t)x^{(n)}(t) + q(t)x\bigl(\tau(t)\bigr) = 0, \quad \tau(t) \leq t. \tag{1.8}$$

In 2014, Liang [27] has investigated oscillation and asymptotic properties of Eq. (1.8). He deduced some sufficient conditions to guarantee all solutions to tend to zero as $t \to \infty$ or to oscillate by using a Philos-type integral averaging technique and a generalized Riccati transformation. On the other hand, no references about the oscillation of differential equations as Eq. (1.1) can be found. The main reason of this phenomenon is the increasing of the difference between the order of the two highest derivatives. In the meantime, we must recognize the complicatedness of the structure of non-oscillatory solutions to Eq. (1.1). If we pay attention to the first-order derivative of a non-oscillatory solution of Eq. (1.1), we can discover that it can be either negative or positive or even it may oscillate.

In this article, we focus on the asymptotic properties of the nth-order trinomial advanced linear differential equation

$$x^{(n)}(t) + p(t)x'(t) + q(t)x\bigl(\sigma(t)\bigr) = 0.$$

By deducing some novel comparison criteria together with integral criteria for Eq. (1.1), we establish some sufficient conditions to fill the holes in the oscillation theory of ODEs. We note that Eq. (1.1) with $p(t) = 0$ is exactly Eq. (1.2), Eq. (1.1) with $n = 4$ is exactly Eq. (1.3) with advanced argument. Thus, we argue that it will be useful and interesting to consider the oscillatory behavior of Eq. (1.1) because it can extend the former investigations and will offer a profitable new breakthrough for the oscillatory behavior of ODEs widely applied in the domain of economics, technology and ecology.

This article is organized into six sections. We propose some preliminary lemmas and definitions that are used in the proof of our main theorems in Sect. 2. In Sect. 3, we establish some useful preliminary results which will be applied in our main theorems. In Sect. 4, a generalized comparison criteria for the oscillation of Eq. (1.1) is deduced. Two examples are provided to check the efficiency of our main results. Several innovative integral criteria are added in Sect. 5 to provide several verifiable and calculable oscillation criteria for the system of Eq. (1.1). Finally, we propose some conclusions in Sect. 6.

2 Preliminaries

As we have proposed before, this section will introduce some notations applied in the article, discuss some preliminaries which are essential in the proofs of our main results and state the assumptions imposed on Eq. (1.1). Then we have the following three lemmas proposed by Kiguradze and Chaturia [19].

Lemma 1 ([19], Corollary 2.8) *Suppose that either*

$$\limsup_{t\to\infty} t^n p(t) < -m_n^*$$

or

$$\liminf_{t\to\infty} t^n p(t) > m_{*n}.$$

Then the higher-order two-term differential equation $x^{(n)}(t) + p(t)x(t) = 0$ is oscillatory, where m_{*n} and m_n^* stand for, respectively, the smallest local maxima of the two n-degree polynomials

$$\mathcal{P}_{*n} = (x-n+1)(x-n+2)\cdots(x-1)x$$

and

$$\mathcal{P}_n^* = -(x-n+1)(x-n+2)\cdots(x-1)x.$$

Lemma 2 ([19], Corollary 2.8) *Assume that $t_0 \geq 0$ and*

$$m_{*n} \geq t^n p(t) \geq -m_n^*,$$

for $t \geq t_0$, then the higher-order two-term differential equation $x^{(n)}(t) + p(t)x(t) = 0$ is non-oscillatory.

Lemma 3 ([19]) *Let $x(t) \in C^n([t_0,\infty),\mathbb{R}^+)$. If $x^{(n)}(t)$ is eventually of one sign for all large t then there exist a $t_x \geq t_0$ and an integer ℓ, $0 \leq \ell \leq n$ with $n+\ell$ even for $x^{(n)}(t) \geq 0$, or $n+\ell$ odd for $x(n)(t) \leq 0$ such that $\ell > 0$ implies that $x(k)(t) > 0$, for $t \geq t_x$, $k = 0,1,\ldots,\ell-1$, and $\ell \leq n-1$ implies that $(-1)^{\ell+k}x^{(k)}(t) > 0$, for $t \geq t_x$, $k = \ell,\ell+1,\ldots,n-1$.*

Definition 1 ([19]) Suppose that all nontrivial solutions of Eq. (1.1) for n odd either are oscillatory or satisfy the condition

$$x(t)x'(t) < 0$$

for $t \geq t_0$, where t_0 is a positive number depending on the solution, and for n even are oscillatory. Then Eq. (1.1) possesses *property A*.

Definition 2 ([19], Definition 13.1) Suppose that $x(t)$, defined on $[t_1,+\infty) \subset [t_0,+\infty)$, represents a nontrivial solution of Eq. (1.1). If it satisfies, for $t \geq t_0$ and $i = 0,1,\ldots,n-1$,

$$(-1)^i x^{(i)}(t)x(t) \geq 0,$$

then we term it a *Kneser solution*.

Suppose that $\tau(t) < t$, $\lim_{t\to\infty}\tau(t) = +\infty$ and $p(t), \tau(t) \in C[\mathbb{R}^+, \mathbb{R}^+]$. With retarded argument, the remaining part of this section will investigate the oscillatory property of solutions to the following two-term differential inequalities and equation:

$$x'(t) + p(t)x\big(\tau(t)\big) \le 0, \tag{2.1}$$

$$x'(t) + p(t)x\big(\tau(t)\big) \ge 0, \tag{2.2}$$

and

$$x'(t) + p(t)x\big(\tau(t)\big) = 0. \tag{2.3}$$

It is significant to study that the oscillation of (2.1)–(2.3) holds for higher-order differential equations or not. We derive the following lemma based on Ladas [22] (see also [10, 20, 23, 28]).

Lemma 4 *Suppose that*

$$\liminf_{t\to\infty} \int_t^{\tau(t)} -p(s)\,ds > \frac{1}{e}. \tag{2.4}$$

Then
(i) *the eventually positive solutions of* (2.1) *are nonexistent*;
(ii) *the eventually negative solutions of* (2.2) *are nonexistent*;
(iii) *the non-oscillatory solutions of* (2.3) *are nonexistent.*

There are two important lemmas proposed by Fukagai and Kusano [16], we recall them.

Lemma 5 ([16]) *Suppose that* $\sigma(t) \ge t$, $\sigma(t), p(t) \in C[\mathbb{R}^+, \mathbb{R}^+]$ *and*

$$\liminf_{t\to\infty} \int_t^{\sigma(t)} p(s)\,ds > \frac{1}{e}.$$

Then $0 \le \operatorname{sgn}(x(t))x'(t) - p(t)|x(\sigma(t))|$ *is oscillatory.*

Lemma 6 ([16], Theorem 1) *Assume that* $\sigma(t) \ge t$ *and* $p(t) \le 0$ *for* $t \ge t_0$. *If*

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} p(s)\,ds > \frac{1}{e},$$

then the equation

$$x'(t) + p(t)x\big(\sigma(t)\big) = 0 \tag{2.5}$$

is oscillatory. In addition, if,

$$\int_t^{\sigma(t)} p(s)\,ds \ge \frac{1}{e},$$

for all large enough t, *then Eq.* (2.5) *has an eventually non-oscillatory solution.*

3 Preliminary results

We will introduce and deduce some preliminary results specific to higher-order differential equations in this section, which are important to the proof of our main theorems. At the beginning, we devote our study to the decomposition of the positive solutions of Eq. (1.1). Based on the theory of disconjugate operators, all positive solutions of Eq. (1.1) have been decomposed by supposing that $y^{(n-1)}(t) + p(t)y(t) = 0$ is non-oscillatory. We argue that $x(t)$ of Eq. (1.1), a non-oscillatory solution, is a Kneser solution if it can only obey the following condition: $0 > x'(t)x(t)$.

We recall a significant lemma proposed by Kiguradze and Chaturia [19] with a modification.

Theorem 1 *Assume that all solutions of*

$$y^{(n-1)}(t) + p(t)y(t) = 0 \tag{3.1}$$

are non-oscillatory. Then every positive solution $x(t)$ of Eq. (1.1) (*for n even*) *satisfies one of the following*:

$$\begin{aligned}
&x(t) \in \mathcal{N}_1 \\
&\quad\Longleftrightarrow\quad x(t) > 0, x'(t) > 0, x''(t) < 0, x'''(t) > 0, x^{(4)}(t) < 0, \ldots, x^{(n-1)}(t) > 0, x^{(n)}(t) \le 0; \\
&x(t) \in \mathcal{N}_3 \\
&\quad\Longleftrightarrow\quad x(t) > 0, x'(t) > 0, x''(t) > 0, x'''(t) > 0, x^{(4)}(t) < 0, \ldots, x^{(n-1)}(t) > 0, x^{(n)}(t) \le 0; \\
&\cdots\cdots \\
&x(t) \in \mathcal{N}_{n-1} \\
&\quad\Longleftrightarrow\quad x(t) > 0, x'(t) > 0, x''(t) > 0, x'''(t) > 0, x^{(4)}(t) < 0, \ldots, x^{(n-1)}(t) > 0, x^{(n)}(t) \le 0;
\end{aligned}$$

and every positive solution $x(t)$ of Eq. (1.1) (*for n odd*) *satisfies one of the following*:

$$\begin{aligned}
&x(t) \in \mathcal{N}_2 \\
&\quad\Longleftrightarrow\quad x(t) > 0, x'(t) > 0, x''(t) > 0, x'''(t) < 0, x^{(4)}(t) > 0, \ldots, x^{(n-1)}(t) > 0, x^{(n)}(t) \le 0; \\
&x(t) \in \mathcal{N}_4 \\
&\quad\Longleftrightarrow\quad x(t) > 0, x'(t) > 0, x''(t) > 0, x'''(t) > 0, x^{(4)}(t) > 0, \ldots, x^{(n-1)}(t) > 0, x^{(n)}(t) \le 0; \\
&\cdots\cdots \\
&x(t) \in \mathcal{N}_{n-1} \\
&\quad\Longleftrightarrow\quad x(t) > 0, x'(t) > 0, x''(t) > 0, x'''(t) > 0, x^{(4)}(t) > 0, \ldots, x^{(n-1)}(t) > 0, x^{(n)}(t) \le 0.
\end{aligned}$$

$\mathcal{N}_0$ (*Kneser solution*) *satisfies $x(t)x'(t) < 0$.*

Proof Suppose that $x(t) > 0$ is a non-oscillatory solution of Eq. (1.1). Without loss of generality, we can suppose that $y(t) > 0$ for $t \ge t_0$. Due to $p(t) \ge 0$, we have $y^{(n-1)}(t) \le 0$. It is obvious that $y(t) = x'(t)$ is a solution of

$$y^{(n-1)}(t) + p(t)y(t) + q(t)x\big(\sigma(t)\big) = 0.$$

Based on Lemma 3, the situation of $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_{n-1}$ is obvious. In what follows we only need to consider the situation of $\mathcal{N}_0$. We denote

$$F\big(x\big(\sigma(t)\big), x'(t), t\big) = -q(t)x\big(\sigma(t)\big) - p(t)x'(t).$$

For $t \geq t_1$, we set $F(t, 0, 0) = 0$. We derive the existence result to $\mathcal{N}_0$ by applying a similar discussion to that in the proof of Theorem 13.1 of [19]. Thus, it is omitted. □

Remark 1 That all solutions of $y^{(n-1)} + py = 0$ are non-oscillatory is equivalent to saying that this equation is eventually disconjugate.

We deduce the following corollary in view of the non-oscillation criterion of Eq. (3.1).

Corollary 1 *Suppose that*

$$m_{*n-1} > \liminf p(t) \cdot t^{n-1}, \tag{3.2}$$

and for Eq. (1.1) *the positive solution class $\mathcal{N}_0$ is an empty set. Then the non-oscillatory set $\mathcal{N}_{\text{even}}$ for Eq.* (1.1) (*for n even*) *has the following decomposition*:

$$\mathcal{N}_{\text{even}} = \mathcal{N}_1 \cup \mathcal{N}_3 \cup \cdots \cup \mathcal{N}_{n-1},$$

and the non-oscillatory set $\mathcal{N}_{\text{odd}}$ for Eq. (1.1) (*for n odd*) *has the following decomposition*:

$$\mathcal{N}_{\text{odd}} = \mathcal{N}_2 \cup \mathcal{N}_4 \cup \cdots \cup \mathcal{N}_{n-1}.$$

Theorem 2 *Suppose that* (3.2) *and*

$$\int_{t_1}^{\infty} \left[p(s) + \big(\sigma(s) - t_1\big)q(s) - \frac{x(t_1)}{l_0}q(s)\right](s - t_1)^{n-1}\,ds \geq 0 \tag{P_0}$$

hold. If $x(t) \in \mathcal{N}_0$ is a positive solution of (1.1), *then*

$$\int_t^{\infty} \frac{[p(s)x'(s) + q(s)x(\sigma(s))](s-t)^{n-2}}{(n-2)!}\,ds \leq -x'(t)$$

and, furthermore, $x(t)$ is decreasing.

Proof We suppose that Eq. (1.1) has a positive solution $x(t) \in \mathcal{N}_0$. We argue that (P_0) means $\lim_{t\to\infty} x'(t) = 0$. If not, we suppose on the contrary that $\lim_{t\to\infty} -x'(t) = l_0 > 0$ and $l_0 \leq -x'(t)$, and so

$$x\big(\sigma(t)\big) \leq x(t_1) - l_0\big(\sigma(t) - t_1\big).$$

Setting the estimates into Eq. (1.1), we obtain

$$x^{(n)}(t) - l_0 p(t) + q(t)\big[x(t_1) - l_0\big(\sigma(t) - t_1\big)\big] \geq 0.$$

It is easy to see that

$$\begin{aligned} x^{(n)}(t) &\geq l_0 p(t) + q(t)\big[l_0\big(\sigma(t) - t_1\big) - x(t_1)\big] \\ &\geq l_0\big[p(t) + \big(\sigma(t) - t_1\big)q(t)\big] - x(t_1)q(t), \end{aligned}$$

and then integrating n times, we obtain

$$\begin{aligned} -x(t_1) &\geq l_0 \int_{t_1}^{\infty}\int_{s_{n-1}}^{\infty}\cdots\int_{s_1}^{\infty}\left[p(s) + \big(\sigma(s) - t_1\big)q(s) - \frac{x(t_1)}{l_0}q(s)\right] ds\, ds_1 \cdots ds_{n-1} \\ &= \frac{l_0}{(n-1)!}\int_{t_1}^{\infty}(s - t_1)^{n-1}\left[p(s) + \big(\sigma(s) - t_1\big)q(s) - \frac{x(t_1)}{l_0}q(s)\right] ds. \end{aligned}$$

In view of (P_0) of Theorem 2, we obtain a contradiction. Hence, $\lim_{t\to\infty} x'(t) = 0$. Furthermore, based on $0 < x(t)$ and $0 > x'(t)$, we can effortlessly discover that $x(t)$ is decreasing. We check

$$\int_t^{\infty}\frac{[p(s)x'(s) + q(s)x(\sigma(s))](s-t)^{n-2}}{(n-2)!}\,ds \leq -x'(t),$$

letting us integrate Eq. (1.1) from t to ∞ to get

$$\int_t^{\infty}\big[x^{(n)}(s) + p(s)x'(s) + q(s)x\big(\sigma(s)\big)\big]\,ds = 0,$$

so we have

$$x^{(n-1)}(t) \geq \int_t^{\infty}\big[p(s)x'(s) + q(s)x\big(\sigma(s)\big)\big]\,ds.$$

Integrating $n-2$ times from t to ∞, one gets

$$\int_t^{\infty}\frac{[p(s)x'(s) + q(s)x(\sigma(s))](s-t)^{n-2}}{(n-2)!}\,ds \leq -x'(t).$$

Therefore, we finish the proof of this theorem. □

Theorem 3 *Suppose that* (3.2) *and*

$$\int_{t_1}^{\infty}\left[p(s)\frac{(s-t_1)^{k-1}}{(k-1)!} + q(s)\frac{(\sigma(s)-t_1)^k}{k!}\right]\frac{(s-t_1)^{n-k-1}}{(n-k-1)!}\,ds = \infty \qquad (P_k)$$

hold. If $x(t) \in \mathcal{N}_k$ *is a positive solution of* (1.1), *then*

$$\begin{aligned} &tx^{(k)}(t) \leq x^{(k-1)}(t); \\ &\frac{t^{k-1}x(\sigma(t))}{(k-1)!}\int_t^{\infty}q(s)\frac{(s-t)^{n-k-1}}{(n-k-1)!}\,ds \leq x'(t), \end{aligned} \qquad (3.3)$$

and, furthermore, $\frac{x^{(k-1)}(t)}{t}$ *is decreasing.*

Proof We suppose that Eq. (1.1) possesses a positive solution $x(t) \in \mathcal{N}_k$. We argue that (P_k) means $\lim_{t\to\infty} x^{(k)}(t) = 0$. If not, we suppose on the contrary that $\lim_{t\to\infty} x^{(k)}(t) = l > 0$ and $l \le x^{(k)}(t)$, and so $l\frac{(t-t_1)^k}{k!} \le x(t)$ and $l\frac{(t-t_1)^{k-1}}{(k-1)!} \le x'(t)$. In view of the last two estimates, we, together with Eq. (1.1), obtain

$$x^{(n)}(t) + l\frac{(t-t_1)^{k-1}}{(k-1)!}p(t) + l\frac{(\sigma(t)-t_1)^k}{k!}q(t) \le 0.$$

Integrating the last inequality $n-k$ times, one gets

$$\begin{aligned} x^{(k)}(t_1) &\ge \int_{t_1}^{\infty} \bigl(-x^{(k+1)}(s_{n-k-1})\bigr)\, ds_{n-k-1} \\ &\ge \cdots\cdots \\ &\ge \int_{t_1}^{\infty}\int_{s_{n-k-1}}^{\infty} \cdots \int_{s_2}^{\infty} x^{(n-1)}(s_1)\, ds_1\, ds_2 \cdots ds_{n-k-1} \\ &\ge \int_{t_1}^{\infty}\int_{s_{n-k-1}}^{\infty} \cdots \int_{s_2}^{\infty}\int_{s_1}^{\infty} \bigl(-x^{(n)}(s)\bigr)\, ds\, ds_1\, ds_2 \cdots ds_{n-k-1} \\ &\ge l\int_{t_1}^{\infty} \cdots \int_{s_1}^{\infty} \left[p(s)\frac{(s-t_1)^{k-1}}{(k-1)!} + q(s)\frac{(\sigma(s)-t_1)^k}{k!}\right] ds \cdots ds_{n-k-1} \\ &= l\int_{t_1}^{\infty} \left[p(s)\frac{(s-t_1)^{k-1}}{(k-1)!} + q(s)\frac{(\sigma(s)-t_1)^k}{k!}\right]\frac{(s-t_1)^{n-k-1}}{(n-k-1)!}\, ds. \end{aligned}$$

Based on the condition (P_k) of Theorem 3, we obtain a contradiction. Hence, $\lim_{t\to\infty} x^{(k)}(t) = 0$. Furthermore, in view of $0 < x^{(k)}(t)$ and $0 > x^{(k+1)}(t)$, we have $tx^{(k)}(t) - t_1x^{(k)}(t) \le \int_{t_1}^t x^{(k)}(s)\, ds$, $0 \le x^{(k-1)}(t_1) - t_1x^{(k)}(t)$ and

$$\begin{aligned} tx^{(k)}(t) &\le tx^{(k)}(t) + x^{(k-1)}(t_1) - t_1x^{(k)}(t) \\ &\le x^{(k-1)}(t_1) + \int_{t_1}^{t} x^{(k)}(s)\, ds \\ &= x^{(k-1)}(t). \end{aligned}$$

Replacing t_1 by t in $0 \le x^{(k-1)}(t_1) - t_1x^{(k)}(t)$, one gets

$$0 \le x^{(k-1)}(t) - tx^{(k)}(t) = -t^2\left(\frac{tx^{(k)}(t) - x^{(k-1)}(t)}{t^2}\right) = -t^2\left(\frac{x^{(k-1)}(t)}{t}\right)'.$$

Clearly, the function $\frac{x^{(k-1)}(t)}{t}$ is a decreasing function. We check

$$x^{(k)}(t) \ge x\bigl(\sigma(t)\bigr)\int_t^{\infty} q(u)\frac{(u-t)^{n-k-1}}{(n-k-1)!}\, du,$$

letting us integrate Eq. (1.1) from t to ∞ to get

$$\int_t^{\infty} \bigl[x^{(n)}(s) + p(s)x'(s) + q(s)x\bigl(\sigma(s)\bigr)\bigr]\, ds = 0,$$

i.e.,

$$x^{(n-1)}(t) \geq \int_t^{\infty} \bigl[p(s)x'(s) + q(s)x\bigl(\sigma(s)\bigr)\bigr]\,ds, \tag{3.4}$$

which based on the monotonicity of $x(t)$ means

$$\begin{aligned} x^{(n-1)}(t) &\geq \int_t^{\infty} \bigl[p(s)x'(s) + q(s)x\bigl(\sigma(s)\bigr)\bigr]\,ds \\ &\geq \int_t^{\infty} q(s)x\bigl(\sigma(s)\bigr)\,ds \geq x\bigl(\sigma(t)\bigr)\int_t^{\infty} q(s)\,ds. \end{aligned} \tag{3.5}$$

Integrating the last inequality with $n-k-1$ times from t to ∞, we have

$$x^{(k)}(t) \geq x\bigl(\sigma(t)\bigr)\int_t^{\infty} q(s)\frac{(s-t)^{n-k-1}}{(n-k-1)!}\,ds.$$

And according to the monotonicity of

$$x(t), x'(t), \dots, x^{(k-1)}(t),$$

we have

$$x'(t) \geq \frac{x(\sigma(t))}{(n-k-1)!}\cdot\frac{t^{k-1}}{(k-1)!}\int_t^{\infty}(s-t)^{n-k-1}q(s)\,ds.$$

Therefore, we finish the proof of Theorem 3. □

4 Comparison criteria of Eq. (1.1)

In this section, we will state our new technique and generalized comparison criteria, which can reduce the difficulty of the oscillation investigation of higher-order trinomial equation.

Primarily, we propose some sufficient conditions which can guarantee that all positive solutions classes $\mathcal{N}_k$ are empty. For simplicity of notation, we write

$$\begin{aligned} Z_0(t) &= \int_t^{\infty} \frac{q(s)(s-t)^{n-2}}{(n-2)!}\,ds, \\ Z_k(t) &= \int_t^{\infty} \frac{(P_k(s)+q(s))(s-t)^{n-2}}{(n-2)!}\,ds, \\ P_k(t) &= \frac{p(t)\cdot t^{k-1}}{(k-1)!}\int_t^{\infty}\frac{q(s)(s-t)^{n-k-1}}{(n-k-1)!}\,ds. \end{aligned}$$

Theorem 4 *Suppose that all solutions of the first-order two-term advanced differential equation*

$$x'(t) - \gamma Z_k(t)x\bigl(\sigma(t)\bigr) = 0 \tag{E_k}$$

are oscillatory for some positive constant $\gamma \in (0,1)$ and that (3.2) holds. Then the positive solution class $\mathcal{N}_k = \varnothing$ for Eq. (1.1). In addition, if

$$\frac{1}{e} \geq \int_t^{\sigma(t)} \gamma Z_k(s)\,ds$$

for all large enough t, then for Eq. (E_k) there exists an eventually non-oscillatory solution.

Proof The second part of the theorem can be omitted due to the proof proposed by Kusano and Fukagai [16]. We have the following proof to the first part of this theorem.

By the principle of the reduction to absurdity, we can suppose on the contrary that for Eq. (1.1) there exists $x(t) \in \mathcal{N}_k$, which is an eventually positive solution. Integrating Eq. (1.1) from t to ∞ leads to (3.4). In view of the estimate of $x'(t)$:

$$x'(t) \geq \frac{x(\sigma(t))}{(n-k-1)!} \cdot \frac{t^{k-1}}{(k-1)!} \int_t^\infty (s-t)^{n-k-1} q(s)\,ds,$$

we can deduce that

$$x^{(n-1)}(t) \geq \int_t^\infty \bigl(P_k(s) + q(s)\bigr) x\bigl(\sigma(s)\bigr)\,ds.$$

Integrating $n-2$ times from t to ∞, we derive

$$\begin{aligned} x'(t) &\geq \int_t^\infty \bigl(P_k(s) + q(s)\bigr) \frac{(s-t)^{n-2}}{(n-2)!} x\bigl(\sigma(s)\bigr)\,ds \\ &\geq x\bigl(\sigma(t)\bigr) \int_t^\infty \bigl(P_k(s) + q(s)\bigr) \frac{(s-t)^{n-2}}{(n-2)!}\,ds. \end{aligned}$$

In addition, we can discover that $x(t)$ satisfies the following first-order differential inequality:

$$x'(t) - Q_k(t) x\bigl(\sigma(t)\bigr) \geq 0. \tag{4.1}$$

However, we can check that Lemma 3 of [6] can guarantee that for Eq. (E_k) there exists an eventually positive solution. This contradiction ends the proof of Theorem 4. □

Theorem 5 *Suppose that all solutions of the first-order two-term advanced differential equation*

$$x'(t) + \gamma Z_0(t) x\bigl(\sigma(t)\bigr) = 0 \tag{E_0}$$

are oscillatory for some $\gamma \in (0,1)$ and that (3.2) holds. Then the positive solution class $\mathcal{N}_0 = \varnothing$ for Eq. (1.1). In addition, if

$$\frac{1}{e} \geq \int_t^{\sigma(t)} \gamma Z_0(s)\,ds$$

for all large enough t, then for Eq. (E_0) there exists an eventually non-oscillatory solution (Kneser solution).

Proof The second part of the theorem can be omitted due to the proof proposed by Kusano and Fukagai [16]. We have the following proof to the first part of this theorem.

By the principle of the reduction to absurdity, we can suppose on the contrary that for Eq. (1.1) there exists $x(t) > 0$, which belongs to $\mathcal{N}_0$. Integrating Eq. (1.1) $n-1$ times from t to ∞, one gets

$$0 \leq \frac{x(\sigma(t))}{(n-2)!} \int_t^\infty (s-t)^{n-2} q(s)\,ds + x'(t). \tag{4.2}$$

In addition, we can discover that $x(t)$ fulfills the inequality

$$x'(t) + Q_0(t)x\bigl(\sigma(t)\bigr) \geq 0. \tag{4.3}$$

However, we can check that Theorem 1 of [16] can guarantee that for Eq. (E_0) there exists a positive solution. This contradiction ends the proof of Theorem 5. □

Thanks to the above theorems, we can easily derive the following oscillation theorem for Eq. (1.1).

Theorem 6 *Suppose that all solutions of the first-order differential equations* (E_0) *and* (E_k) *are oscillatory for some* $\gamma \in (0,1)$ *and that* (3.2) *holds, then all solutions of Eq.* (1.1) *are oscillatory.*

We get an effortlessly confirmable theorem for the oscillatory properties of the investigated trinomial differential equations by applying some sufficient conditions to the oscillation of the first-order advanced equations.

Theorem 7 *Suppose that*

$$\liminf_{t\to\infty} \int_t^{\sigma(t)} Z_i(s)\,ds > \frac{1}{e}, \tag{C_i}$$

where $i = k$, $k = 0,1,2,3,\ldots,n-1$ *and that* (3.2) *holds. Then Eq.* (1.1) *is oscillatory. In addition, if*

$$\frac{1}{e} \geq \int_t^{\sigma(t)} Z_i(s)\,ds$$

for all large enough t, *then for Eq.* (E_k) *there exists an eventually non-oscillatory solution.*

Proof The second part of the theorem can be omitted due to the proof proposed by Kusano and Fukagai [16]. We have the following proof to the first part of this theorem.

By the principle of the reduction to absurdity, we can suppose on the contrary that for Eq. (1.1) there exists $x(t)$, which is an eventually positive solution. Theorem 1 guarantees that $x(t) \in \mathcal{N}_k$. It follows from (C_k) that there is some $\gamma \in (0,1)$ which can guarantee that

$$\frac{1}{e} < \liminf_{t\to\infty} \int_t^{\sigma(t)} \gamma Z_i(s)\,ds,$$

which by Theorem 2.4.1 of [24] implies that all solutions of Eq. (E_k) are oscillatory, which based on Theorem 4 guarantees $\mathcal{N}_k = \varnothing$. This contradiction ends the proof of Theorem 7. □

Example 1 Based on Example 3.5 in [11], we investigate the following linear fourth-order trinomial ADE:

$$x^{(4)}(t) + \frac{a}{t^3}x'(t) + \frac{b}{t^4}x(\lambda t) = 0, \quad b > 0, \lambda > 1, a \in \left(0, \frac{2}{3\sqrt{3}}\right). \tag{E_1}$$

For the investigated Eq. (E_1), the corresponding Eq. (3.1) has the following form:

$$y'''(t) + \frac{a}{t^3}y(t) = 0$$

with a solution $y(t) = t^{-0.1} > 0$. By direct calculation, we can effortlessly see that $0 < a = 0.231 < \frac{2\sqrt{3}}{9} \approx 0.3849$. Via a direct computation with Eq. (E_1) we see that

$$Z_1(t) = \left(\frac{ab}{36} + \frac{b}{6}\right)\frac{1}{t},$$
$$Z_3(t) = \left(\frac{2a}{\lambda^3} + b + \frac{a}{\lambda^3}\ln\lambda\right)\frac{1}{t} - \left(at_1 + \frac{bt_1}{2}\right)\frac{1}{t^2},$$

where t_1 is large enough. Thanks to Theorem 7, all solutions of Eq. (E_1) are oscillatory under the following conditions:

$$\left(\frac{ab}{36} + \frac{b}{6}\right)\ln\lambda > \frac{1}{e},$$
$$\left(\frac{2a}{\lambda^3} + b + \frac{a}{\lambda^3}\ln\lambda\right)\ln\lambda > \frac{1}{e}.$$

For example, with $\lambda = e$ it happens provided that $b > 2.1254$.

Example 2 We study the fifth-order ADE of Euler type

$$x^{(5)}(t) + \frac{a}{t^4}x'(t) + \frac{b}{t^5}x(\lambda t) = 0, \quad b > 0, \lambda > 1, a \in \left(0, \frac{9}{16}\right). \tag{E_2}$$

For the investigated Eq. (E_2), the corresponding Eq. (3.1) has the form

$$y^{(4)}(t) + \frac{a}{t^4}y(t) = 0$$

with an eventually positive solution $y(t) = t^{0.1}$, where $0 < a = 0.4959 < \frac{9}{16} = 0.5625$. It is seen via a direct computation with Eq. (E_2) that

$$Z_0(t) = \frac{b}{24}\cdot\frac{1}{t},$$
$$Z_2(t) = Z_4(t) = \left(\frac{ab}{576} + \frac{b}{24}\right)\frac{1}{t}.$$

Based on Theorem 7, all solutions of Eq. (E_2) are oscillatory under the following conditions:

$$\frac{b}{24}\cdot\ln\lambda > \frac{1}{e},$$
$$\left(\frac{ab}{576}+\frac{b}{24}\right)\ln\lambda > \frac{1}{e}.$$

For e.g. $\lambda = e$ it happens provided that $b > 8.8291$.

5 Integral criteria of Eq. (1.1)

We will propose some new integral conditions for oscillatory properties to investigate advanced differential equations in this section which will provide a more precise result than the comparison criteria. At first, we need to use the preliminary results in Sect. 3.

Theorem 8 *Suppose that* (3.2) *and*

$$\begin{aligned}\limsup_{t\to\infty}\Bigg(&\frac{1}{\sigma(t)}\int_{t_*}^{\sigma(t)}(n-2)s^{n-1}p(s)\,ds+\sigma(t)\int_{\sigma(t)}^{\infty}(n-2)s^{n-3}p(s)\,ds\\&+\frac{1}{\sigma(t)}\int_{t_*}^{t}s\sigma(s)^{n-1}q(s)\,ds+\int_{t}^{\sigma(t)}s\sigma(s)^{n-2}q(s)\,ds\\&+\sigma(t)\int_{\sigma(t)}^{\infty}\sigma(s)^{n-2}q(s)\,ds\Bigg)>(n-1)!\end{aligned}\tag{5.1}$$

hold. Then $\mathcal{N}_{n-1}$ is an empty set for Eq. (1.1).

Proof By the principle of the reduction to absurdity, we suppose on the contrary that Eq. (1.1) has a positive solution $x(t)$ and this solution belongs to the positive solution class $\mathcal{N}_{n-1}$. Applying the estimation of the inequality offered by Theorem 3 which is a previous result proposed by Kiguradze [19], we have

$$\begin{aligned}(n-1)!\frac{x(t)}{t^{n-2}}&>(n-2)!\frac{x'(t)}{t^{n-3}}>\cdots>2\frac{x^{(n-3)}(t)}{t}\geq x^{(n-2)}(t)\\&\geq\int_{t_1}^{t}(n-2)\frac{(s-t_1)p(s)}{s}x(s)\,ds+(t-t_1)\int_{t}^{\infty}(n-2)\frac{x(s)}{s}p(s)\,ds\\&\quad+\int_{t_1}^{t}(s-t_1)q(s)x\big(\sigma(s)\big)\,ds+(t-t_1)\int_{t}^{\infty}q(s)x\big(\sigma(s)\big)\,ds\\&\geq\gamma\bigg(\int_{t_1}^{t}(n-2)p(s)x(s)\,ds+t\int_{t}^{\infty}\frac{(n-2)p(s)}{s}x(s)\,ds\\&\quad+t\int_{t}^{\infty}q(s)x\big(\sigma(s)\big)\,ds+\int_{t_1}^{t}sq(s)x\big(\sigma(s)\big)\,ds\bigg)\end{aligned}$$

for some $\gamma\in(0,1)$. Thus,

$$\begin{aligned}\frac{(n-1)!}{\gamma}\geq{}&\frac{1}{\sigma(t)}\int_{t_*}^{\sigma(t)}(n-2)s^{n-1}p(s)\,ds+\sigma(t)\int_{\sigma(t)}^{\infty}(n-2)s^{n-3}p(s)\,ds\\&+\frac{1}{\sigma(t)}\int_{t_*}^{t}s\sigma(s)^{n-1}q(s)\,ds+\int_{t}^{\sigma(t)}s\sigma(s)^{n-2}q(s)\,ds+\sigma(t)\int_{\sigma(t)}^{\infty}\sigma(s)^{n-2}q(s)\,ds,\end{aligned}$$

from which one can conclude to a contradiction with the hypotheses of Theorem 8. This ends the proof. □

Theorem 9 *Assume that* (3.2) *and*

$$\limsup_{t\to\infty}\left(\frac{1}{\sigma(t)}\int_{t_*}^{t}s^{n-1}\sigma(s)\big[P_k(s)+q(s)\big]\,ds+\int_{t}^{\sigma(t)}s^{n-1}\big[P_k(s)+q(s)\big]\,ds\right.$$
$$\left.+(n-1)\sigma(t)\int_{\sigma(t)}^{\infty}\big(s-\sigma(t)\big)^{n-2}\big[P_k(s)+q(s)\big]\,ds\right)>(n-1)!$$

hold. Then the positive solution class $\mathcal{N}_k$ is empty for Eq. (1.1).

Proof By the principle of the reduction to absurdity, we suppose on the contrary that Eq. (1.1) has a positive solution $x(t)$ and this solution belongs to the positive solution class $\mathcal{N}_k$. After that, we integrate (4.2) from t_1 to t to get

$$x(t)\geq\int_{t_1}^{t}\int_{u}^{\infty}x\big(\sigma(s)\big)Q_k(s)\frac{(s-u)^{n-2}}{(n-2)!}\,ds\,du,\qquad Q_k(t)=P_k(t)+q(t),\tag{5.2}$$

or

$$\begin{aligned}x(t)&\geq\int_{t_1}^{t}\int_{u}^{\infty}x\big(\sigma(s)\big)Q_k(s)\frac{(s-u)^{n-2}}{(n-2)!}\,ds\,du\\&=\int_{t_1}^{\sigma(t)}x\big(\sigma(s)\big)Q_k(s)\int_{t_1}^{s}\frac{(s-u)^{n-2}}{(n-2)!}\,du\,ds\\&\quad+\int_{\sigma(t)}^{\infty}x\big(\sigma(s)\big)Q_k(s)\int_{t_1}^{\sigma(t)}\frac{(s-u)^{n-2}}{(n-2)!}\,du\,ds\\&=\int_{t_1}^{\sigma(t)}x\big(\sigma(s)\big)Q_k(s)\frac{(s-t_1)^{n-1}}{(n-1)!}\,ds\\&\quad+\int_{\sigma(t)}^{\infty}x\big(\sigma(s)\big)Q_k(s)\frac{(s-t_1)^{n-1}-(s-\sigma(t))^{n-1}}{(n-2)!}\,ds.\end{aligned}$$

Denoting $\mathcal{A}=s-t_1$, $\mathcal{B}=s-\sigma(t)$ and applying the Lagrangian middle-value theorem yields $\mathcal{A}^{n-1}-\mathcal{B}^{n-1}\geq(n-1)(\mathcal{A}-\mathcal{B})\mathcal{B}^{n-2}$ for $\mathcal{A}\geq\mathcal{B}\geq 0$. Therefore, the last inequality leads to

$$\begin{aligned}x\big(\sigma(t)\big)&\geq\frac{\gamma}{(n-1)!}\int_{t_*}^{\sigma(t)}x\big(\sigma(s)\big)Q_k(s)s^{n-1}\,ds\\&\quad+\frac{\gamma\sigma(t)}{(n-2)!}\int_{t}^{\infty}x\big(\sigma(s)\big)Q_k(s)\big(s-\sigma(t)\big)^{n-2}\,ds\\&\geq\frac{\gamma x(\sigma(t))}{(n-1)!}\left(\frac{1}{\sigma(t)}\int_{t_*}^{t}s^{n-1}\sigma(s)Q_k(s)\,ds+\int_{t}^{\sigma(t)}s^{n-1}Q_k(s)\,ds\right.\\&\quad\left.+(n-1)\sigma(t)\int_{\sigma(t)}^{\infty}\big(s-\sigma(t)\big)^{n-2}Q_k(s)\,ds\right)\end{aligned}$$

for any $\gamma\in(0,1)$, from which one concludes to a contradiction with the hypotheses of this theorem. This ends the proof. □

Theorem 10 *Assume that* (3.2) *and*

$$\limsup_{t\to\infty}\left(\frac{(n-1)!x(\sigma(t))}{x(t_1)}+\frac{1}{x(t_1)}\int_{t_1}^{\sigma(t)}x\bigl(\sigma(s)\bigr)q(s)(s-t_1)^{n-1}\,ds\right.$$
$$\left.+\frac{n-1}{x(t_1)}\int_{\sigma(t)}^{\infty}x\bigl(\sigma(s)\bigr)q(s)(s-t_1)^{n-1}\,ds\right)$$
$$<(n-1)!$$

hold. Then the positive solution class $\mathcal{N}_0=\varnothing$ *for Eq.* (1.1).

Proof For the sake of proof by contradiction, we suppose that for Eq. (1.1) there exists an eventually positive solution $x(t)$ and this solution belongs to the positive solution class $\mathcal{N}_0$. Next, we integrate (4.2) from t_1 to t to obtain

$$x(t_1)-x(t)\le\int_{t_1}^{t}\int_{u}^{\infty}x\bigl(\sigma(s)\bigr)q(s)\frac{(s-u)^{n-2}}{(n-2)!}\,ds\,du \tag{5.3}$$

or

$$x(t_1)-x\bigl(\sigma(t)\bigr)\le\int_{t_1}^{\sigma(t)}x\bigl(\sigma(s)\bigr)q(s)\frac{(s-t_1)^{n-1}}{(n-1)!}\,ds$$
$$+\int_{\sigma(t)}^{\infty}x\bigl(\sigma(s)\bigr)q(s)\frac{(s-t_1)^{n-1}-(s-\sigma(t))^{n-1}}{(n-1)!}\,ds.$$

Denoting $\mathcal{A}=s-t_1$, $\mathcal{B}=s-\sigma(t)$ and applying the Lagrangian middle-value theorem yield $\mathcal{A}^{n-1}-\mathcal{B}^{n-1}\le(n-1)(\mathcal{A}-\mathcal{B})\mathcal{B}^{n-2}$ for $\mathcal{A}\ge\mathcal{B}\ge0$, the above estimate guarantees

$$x(t_1)-x\bigl(\sigma(t)\bigr)\le\int_{t_1}^{\sigma(t)}x\bigl(\sigma(s)\bigr)q(s)\frac{(s-t_1)^{n-1}}{(n-1)!}\,ds$$
$$+\int_{\sigma(t)}^{\infty}x\bigl(\sigma(s)\bigr)q(s)\frac{(\sigma(t)-t_1)(s-t_1)^{n-1}}{(n-1)!}\,ds.$$

It is obvious that

$$\limsup_{t\to\infty}\left(\frac{(n-1)!x(\sigma(t))}{x(t_1)}+\frac{1}{x(t_1)}\int_{t_1}^{\sigma(t)}x\bigl(\sigma(s)\bigr)q(s)(s-t_1)^{n-1}\,ds\right.$$
$$\left.+\frac{n-1}{x(t_1)}\int_{\sigma(t)}^{\infty}x\bigl(\sigma(s)\bigr)q(s)(s-t_1)^{n-1}\,ds\right)$$
$$\ge(n-1)!,$$

for any $\gamma\in(0,1)$, from which one can conclude to a contradiction with the hypotheses of this theorem. This ends the proof. □

According to the above theorems, we can deduce the sufficient conditions to guarantee the oscillatory behavior of Eq. (1.1).

Theorem 11 *If all hypotheses of Theorems* 8–10 *are satisfied, then we can conclude that all solutions of the nth-order trinomial differential equation Eq.* (1.1) *are oscillatory.*

Example 3 We discuss once more the fourth-order trinomial ADE

$$x^{(4)}(t) + \frac{a}{t^3}x'(t) + \frac{b}{t^4}x(\lambda t) = 0, \quad b > 0, \lambda > 1, a \in \left(0, \frac{2}{3\sqrt{3}}\right).$$

Based on the conditions of Theorems 8 and 9, we have

$$4a + 2b\lambda^2 + b\lambda^2 \ln\lambda > 3!,$$
$$(2 + \ln\lambda)\left(\frac{ab}{6} + b\right) > 3!,$$

which guarantee that the positive solution classes $\mathcal{N}_k$ and $\mathcal{N}_{n-1}$ are empty and all solutions of Eq. (E_1) are oscillatory. For instance, for $\lambda = e$ it happens provided that $b > 1.9259$.

By the same principle, we can also apply integral criteria to study the fifth-order trinomial ADE of Euler type. Hence, it is omitted.

6 Conclusion

In this article, two methods, comparison criteria and integral criteria, have been applied to obtain some sufficient conditions of asymptotic and oscillatory behavior of a higher-order advanced trinomial differential equation under the substantial difficulty derived from the middle positive or negative term $p(t)x'(t)$. In 2015, Džurina, Baculíková and Jadlovská [11] have obtained the oscillatory behavior of the following equation:

$$x^{(4)}(t) + p(t)x'(t) + q(t)x\big(\tau(t)\big) = 0. \tag{6.1}$$

The previous fourth-order differential equation can be generalized to higher order by the two general types

$$x^{(n)}(t) + p(t)x'(t) + q(t)x\big(\tau(t)\big) = 0 \tag{6.2}$$

or

$$x^{(n+3)}(t) + p(t)x^{(n)}(t) + q(t)x\big(\tau(t)\big) = 0. \tag{6.3}$$

The second equation has been addressed by Liang [27] in 2014. However, so far no researcher addressed the investigation of the first type advanced equations in the theory of oscillatory and asymptotic behaviors. Therefore, this gap in the theory of oscillation has been filled.

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