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Finite time stability and sliding mode control for uncertain variable fractional order nonlinear systems

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Abstract

This paper deals with the finite time stability and control for a class of uncertain variable fractional order nonlinear systems. The variable fractional Lyapunov direct method is developed to provide the basis for the stability proof of the system considered. The sliding mode control method is applied for robust control of uncertain variable fractional order systems; furthermore, the chattering phenomenon is avoided. And the finite time stability of the systems under control law is proved based on the proposed stability criterion. Finally, numerical simulations are proposed and the efficiency of the controller is verified.

Keywords: Uncertain variable fractional order system; Finite time stability; Robust control; Sliding mode control method

1 Introduction

In recent years, fractional calculus (FC) has been used to describe the natural behavior in many research fields [1–9]. And the fractional order differential models can better describe some complex dynamic phenomena in many practical engineering problems [10–16]. Thus, the theory of FC has been developed rapidly. For example, a definition of variable fractional order operator (VFO) was proposed in order to describe the complex phenomena of the mechanical modeling [17]. FC has a vast of applications in areas of physics and engineering, and it has been applied to chaos control of the fractional order(FO) dynamical systems [18]. And it has become a hot spot in research for the theory analysis and application of control in FO dynamical systems [19–21], and there exist many control methods to deal with the control problem of the chaotic systems, such as adaptive control, backstepping method, feedback control method, and H_∞ approach [22–25]. Monje et al. [22] detailed fractional order systems and controls by use of fractional calculus in the description and modeling of systems and in a range of control design and practical applications. Aguilar et al. [25] investigated the chaos control for a class of variable-order fractional chaotic systems using robust control strategy. Moreover, sliding mode control (SMC) [26–28], which has the advantage of better transient performance, easy realization, rapid response, and insensitivity to external disturbances and so on, is frequently employed, see [29–37]. Pisano et al. [29] applied the sliding mode control approaches to stabilize a class of linear uncertain fractional order dynamics and presented two sliding mode control schemes. Jakovljevic et al. [30] dealt with applications of sliding mode based fractional control techniques to address tracking and stabilization control tasks for some classes of nonlinear uncertain fractional order systems.

Aghababa [38] introduced a suitable robust SMC law to realize control in a given finite time for integer-order nonautonomous chaotic systems. By use of the SMC approach, a feedback control has been designed to guarantee asymptotical stability of the chaotic systems in [39]. Combined with a global SMC, Saleh et al. [40] presented a novel adaptive stabilization technique for disturbed chaotic flow. Motivated by the research in SMC of the constant FO chaotic system, many researchers have exploited the VFO operators to investigate the dynamical and control problems [17, 41–44]. However, from the mathematical analysis point, it has not been resolved for the problem of the theory to prove the stability of the controller. To the authors' knowledge, due to the complexity of VFO systems, the results are rare on this topic. In addition, many studies have been devoted to the simulation of the fraction order system in recent years, and a large number of methods have emerged and the theories have gradually improved [45–50].

In the present paper, the finite time control is discussed for VFO chaotic systems in the presence of uncertainties and external disturbances. The Lyapunov direct method is extended to the VFO form, and a finite time stability theorem is proposed. Based on the stable results, a VFO sliding mode manifold is designed. And in order to guarantee the finite time reach of the system state trajectories to the above sliding mode manifold, a SMC law is designed in the VFO form. In this paper, our main contribution is to realize the stabilization of variable fractional order uncertain systems in finite time by the SMC approach. Moreover, the theoretical proof is given by use of the variable fractional order Lyapunov theorem. Lastly, simulation results are proposed to display the effectiveness and usefulness of the theoretical analysis.

The organization of this article is presented as follows. The basic definitions of VFO calculus and the basic description of the system are given in Sect. 2. Section 3 is devoted to obtaining the stability of VFO differential system in a finite time and provide the design strategy of the VFOSMC. Section 4 provides the numerical simulations for the viability of the theoretical results.

2 Preliminaries

The following definitions of VFO operators are adopted in this article.

Definition 2.1 ([51]) When the order $q(t)$ depends on time t, there is an obvious way for accounting for the variation:

$$I_t^{q(t)}x(t) = \frac{1}{\Gamma(q(t))}\int_{t_0}^{t}(t-s)^{q(t)-1}x(s)\,ds, \quad 0 < q(t) < 1, \tag{1}$$

provided the integration is defined on $t \in [t_0, T]$, and $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 ([51]) The definition of VFO derivatives is as follows:

$$ {}^{C}D_{t}^{q(t)}x(t) = \frac{1}{\Gamma(1-q(t))}\int_{t_0}^{t}(t-s)^{-q(t)}x'(s)\,ds, \quad 0 < q(t) < 1, \tag{2}$$

provided the integration is defined on $t \in [t_0, T]$, and $\Gamma(\cdot)$ is the gamma function.

When $q(t)$ is a constant, Definitions 2.1 and 2.1 are reduced to the Caputo constant fractional order operators.

The n-dimensional uncertain VFO nonlinear dynamical system is described by the following equations:

$$\begin{cases} {}_{t_0}^{C}D_{t}^{q(t)}x_1(t) = f_1(t,X) + \Delta f_1(t,X) + d_1(t) + u_1(t), \\ {}_{t_0}^{C}D_{t}^{q(t)}x_2(t) = f_2(t,X) + \Delta f_2(t,X) + d_2(t) + u_2(t), \\ \dots \\ {}_{t_0}^{C}D_{t}^{q(t)}x_n(t) = f_n(t,X) + \Delta f_n(t,X) + d_n(t) + u_n(t), \end{cases} \tag{3}$$

where $0 < q_1 \leq q(t) \leq q_2 < 1$, q_1, q_2 are finite constant. $X(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T \in R^n$ is the states vector, $f_i(t,X) \in R$, $i = 1, 2, \dots, n$, denotes a nonlinear function of X and t, $u_i(t) \in R$, $i = 1, 2, \dots, n$, is the control input. $d_i(t) \in R$ represents an unknown model uncertainty, and $\Delta f_i(t,X) \in R$ represents the external disturbances of the system for $i = 1, 2, \dots, n$, which is required to satisfy the following assumptions.

Assumption 1 Suppose that the unknown model uncertainty $\Delta f_i(t,X)$ for $i = 1, 2, \dots, n$ is differentiable and satisfies

$$ \left| D^{q(t)} \Delta f_i(t,X) \right| \leq M_i^{\Delta f}, $$

where $M_i^{\Delta f} > 0$ is a constant which is known for $i = 1, 2, \dots, n$.

Assumption 2 Suppose that the external disturbance $d_i(t)$ is differentiable for $i = 1, 2, \dots, n$,

$$ \left| D^{q(t)} d_i(t) \right| \leq M_i^{d}, $$

where $M_i^{d} > 0$ is a constant which is known for $i = 1, 2, \dots, n$.

Remark 1 From the applied points of view, the uncertain terms and external disturbances are always bounded, a designed control input always has a finite magnitude. Thus, the above assumption is realistic and not restricting.

3 Main results

3.1 Stability analysis of the VFO system

This part is to derive some criteria of stability for the VFO differential systems.

Definition 3.1 The constant x_0 is an equilibrium point of the VFO system

$$\begin{cases} {}_{t_0}^{C}D_t^{q(t)}x(t) = f(t,x(t)), & t \in (t_0, T], \\ x(t_0) = x_0, \end{cases} \tag{4}$$

where $0 < q_1 \leq q(t,x(t)) \leq q_2 < 1$ if $f(t,x_0) = 0$.

The following theorem is an extended Lyapunov direct method into a VFO form, which provides the basis of the asymptotic stability analysis for the VFO system.

Theorem 3.2 *Suppose that* $x = 0$ *is an equilibrium point of VFO system* (4) *and* $D \subset R^n$ *is a domain containing the origin. If there exists a continuously differential function* $V(t,x(t)) : [0,\infty) \times D \to R$ *such that, for arbitrary positive constants* α_1, α_2, α_3, a, b, *the following inequality holds*:

$$\begin{cases} \alpha_1\|x\|^a \leq V(t,x(t)) \leq \alpha_2\|x\|^{ab}, \\ D^{q(t)}V(t,x(t)) \leq -\alpha_3\|x\|^{ab}, \end{cases} \tag{5}$$

where $x \in D$, $0 < q_1 \leq q(t) \leq q_2 < 1$, $t \in [0,\infty)$.

Then the equilibrium point of system (4) *is asymptotically stable*.

Proof Divide the interval $[0,\infty)$ into the subintervals $[t_k, t_{k+1}]$, $k = 1,2,\ldots,n,\ldots,$ which requires $\lim_{k\to\infty} t_k = \infty$. Denote $\chi_k = t_{k+1} - t_k$ with $\inf_k \chi_k > 0$ and $0 < \sup_k \chi_k < 1$ for $k = 1,2,\ldots,n,\ldots,$ then the following inequality is obtained:

$$\chi_k^{-q(t)} \leq \begin{cases} (\frac{1}{\chi_k})^{q_2}, & 0 < \chi_k < 1, \\ (\frac{1}{\chi_k})^{q_1}, & 1 \leq \chi_k, \end{cases} \tag{6}$$

with $\widehat{\chi_k} = max\{(\frac{1}{\chi_k})^{q_2}, (\frac{1}{\chi_k})^{q_1}\}$. According to the property of $\Gamma(t)$ on $(0,1]$, we have $\Gamma(1-q_1) \leq \Gamma(1-q(t)) \leq \Gamma(1-q_2)$, which gives that, for $t \in [t_k, t_{k+1})$, $k = 1,\ldots,n,\ldots,$ together with Definition 2.1 and Definition 2.2

$$\begin{aligned} {}_{t_k}^{C}D_t^{q(t)}V\bigl(t,x(t)\bigr) &= \int_{t_k}^{t} \frac{(t-s)^{-q(t)}}{\Gamma(1-q(t))} V'\bigl(s,x(s)\bigr)\,ds \\ &\geq \frac{1}{\Gamma(1-q_1)} \int_{t_k}^{t} (t-s)^{-q(t)} V'\bigl(s,x(s)\bigr)\,ds \\ &\geq \frac{\widehat{\chi_k}}{\Gamma(1-q_1)} \int_{t_k}^{t} \left(\frac{t-s}{\chi_k}\right)^{-q_2} V'\bigl(s,x(s)\bigr)\,ds \\ &= \frac{1}{H_k}\,{}_{t_k}^{C}D_t^{q_2}V\bigl(t,x(t)\bigr), \end{aligned} \tag{7}$$

where $H_k = \frac{\Gamma(1-q_1)}{\chi_k^{q_2}\widehat{\chi_k}\Gamma(1-q_2)} > 0$. Thus, we obtain

$${}_{t_k}^{C}D_t^{q_2}V\bigl(t,x(t)\bigr) \leq H_k\,{}_{t_k}^{C}D_t^{q(t)}V\bigl(t,x(t)\bigr), \quad t \in [t_k, t_{k+1}), k = 1,\ldots,n,$$

then we get

$$ {}_0^C D_t^{q_2} V\big(t,x(t)\big) \le H {}_0^C D_t^{q(t)} V\big(t,x(t)\big) \quad \text{for } t \in [0,\infty), \tag{8} $$

where $H = H_i$ for $t \in [t_i, t_{i+1})$. From inequalities (5) and (8), it is obtained that

$$ \begin{aligned} {}_0^C D_t^{q_2} V\big(t,x(t)\big) &\le H {}_0^C D_t^{q(t)} V\big(t,x(t)\big) \\ &\le -H\alpha_3\alpha_2^{-1} V\big(t,x(t)\big). \end{aligned} \tag{9} $$

Then a nonnegative function $\overline{G(t)}$ exists such that

$$ {}_0^C D_t^{q_2} V\big(t,x(t)\big) + \overline{G(t)} = -\alpha_3\alpha_2^{-1} H V\big(t,x(t)\big). \tag{10} $$

Applying the Laplace transform to (10) with $V(0) = V(0,x(0))$, we derive that

$$ s^{q_2} V(s) - V(0)s^{q_2-1} + \overline{G(s)} = -\alpha_3\alpha_2^{-1} H V(s), $$

where $V(s) = L\{V(t,x(t))\}$ and $G(s) = L\{D(t)\}$. After some manipulations, it is obtained that

$$ V(s) = \frac{V(0)s^{q_2-1} - G(s)}{s^{q_2} + H\alpha_3\alpha_2^{-1}}. $$

By the inverse Laplace transform, one can get

$$ \begin{aligned} V(t) &= V(0)E_{q_2}\big(-\alpha_3\alpha_2^{-1} H t^{q_2}\big) - \int_0^t (t-s)^{q_2-1} E_{q_2,q_2}\big[-H\alpha_3\alpha_2^{-1}(t-s)^{q_2}\big] G(s)\, ds \\ &\le V(0)E_{q_2}\big(-\alpha_3\alpha_2^{-1} H t^{q_2}\big). \end{aligned} $$

In terms of inequality (5), it implies that

$$ \|x\| \le \big[V(0)\alpha_1^{-1} E_{q_2}\big(-H\alpha_3\alpha_2^{-1} t^{q_2}\big)\big]^{\frac{1}{a}}. $$

Thus, the proof is completed. □

The coming definition and theorems are concerned with finite time stability of the systems.

Definition 3.3 Assume that D is some open connected set, $W(Y,t)$ is a function of variables Y, t. Then a function $Y(t)$, $t_0 \le t < T$, $T > t_0$ is called a solution of the differential inequality

$$ {}_{t_0}^C D^q Y(t) \le W\big(Y(t), t\big) \tag{11} $$

on $[t_0, T)$ if $Y(t)$, and its fractional order derivative satisfies inequality (11) on $[t_0, T)$.

Theorem 3.4 *Suppose that $W(X,t)$ is continuous on $D \in R^2$, which is an open connected set, and $X(t)$ is a solution of the following initial value problem*:

$$ {}_{t_0}^{C}D^q X(t) = \lambda X(t), X(t_0) = x_0, $$

on $[t_0, T]$, where λ is a known constant. If $Y(t)$ is a solution of inequality (11) *on $[t_0, T]$ with $Y(t_0) \leq X(t_0)$, then $Y(t) \leq X(t)$ for $t_0 \leq t \leq T$.*

Proof Set $P(t) = Y(t) - X(t)$, taking the FO derivative on time yields

$$ \begin{aligned} {}_{t_0}^{C}D^q P(t) &= {}_{t_0}^{C}D^q \bigl(Y(t) - X(t)\bigr) \\ &= {}_{t_0}^{C}D^q Y(t) - {}_{t_0}^{C}D^q X(t) \\ &\leq \lambda Y(t) - \lambda X(t) \\ &= \lambda P(t), \end{aligned} $$

which combined with $Y(t_0) \leq X(t_0)$ gives $P(t_0) \leq 0$. The following is to validate the inequality holds:

$$ P(t) \leq 0, \quad \forall t \in [t_0, T). \tag{12} $$

By the contradiction method, if there exist $t_1, t_2 \in (t_0, T)$, $t_1 < t_2$ satisfying

$$ \begin{cases} P(t) < 0, & t \in (t_0, t_1), \\ P(t) > 0, & t \in (t_1, t_2], \\ P(t_1) = 0, & t = t_1. \end{cases} $$

Suppose that $\lambda > 0$, applying the fractional operator I^q to the following inequality:

$$ {}_{t_0}^{C}D^q P(t) \leq \lambda P(t), $$

then it is obtained that

$$ P(t_1) - P(t_0) \leq \frac{1}{\Gamma(q)} \int_{t_0}^{t_1} (t_1 - s)^{q-1} \lambda P(s)\, ds. $$

Since $P(t) < 0$ for $t \in (t_0, t_1)$, then $-P(t_0) < 0$, which is a contradiction. Assume $\lambda < 0$, following a similar approach, we have

$$ P(t_2) - P(t_1) \leq \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} \lambda P(s)\, ds < 0, $$

then $P(t_2) < 0$, which means that inequality (12) holds. □

Theorem 3.5 *Assume that $V(t)$ is a continuous and positive definite function which satisfies*

$$ {}_{t_0}^{C}D^{q_2} V(t) \leq -\alpha V(t) \tag{13} $$

for $t \geq t_0$, where α is a positive constant. Then the following inequality can be got:

$$V(t) \leq V(t_0)E_{q_2}\left(-\alpha t^{q_2}\right) \tag{14}$$

for $t_0 \leq t < t^*$ with $t^* = (\frac{\Gamma(q_2+1)}{\alpha})^{q_2^{-1}}$, where $E_{q_2}(t)$ is a Mittag-Leffler function which is denoted by $E_{q_2}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(q_2 k+1)}$. Moreover, $V(t_0) \geq 0$ with any given t_0, and $V(t) = 0$ for $t \geq t^*$.

Proof Consider the initial value problem of the FO differential systems:

$${}_{t_0}^{C}D^{q_2}X(t) = -\alpha X(t), \quad X(t_0) = V(t_0).$$

Then this initial value problem has a unique solution as follows:

$$X(t) = X(t_0)E_{q_2}\left(-\alpha t^{q_2}\right) \quad \text{for } t_0 \leq t.$$

Therefore, on the basis of Theorem 3.4, one obtains that

$$V(t) \leq X(t) = V(t_0)E_{q_2}\left(-\alpha t^{q_2}\right) \quad \text{for } t_0 \leq t < t^*,$$

where $t^* = (\frac{\Gamma(q_2+1)}{\alpha})^{q_2^{-1}}$ and $V(t) = 0$ for $\forall t \geq t^*$. □

3.2 Finite time control of the VFO system by SMC approach

For VFO differential system (3), the VFO sliding mode is proposed as follows:

$$s_i(t) = {}_{0}^{C}D^{q(t)}x_i + \beta_i x_i + \bar{\beta}_i sgn(x_i)|x_i|^{q(t)}, \tag{15}$$

where $\beta_i > 0$, $\bar{\beta}_i > 0$ for $i = 1, 2, \dots, n$.

Remark 2 The representation of the sliding mode is related to the variable fractional order operator, and it is focused on the VFO systems. When the VFO parameter is a constant, it can be used to deal with the constant FO systems.

If the states of the system reach the sliding mode surface, then it is obtained that

$$s_i(t) = 0, \quad i = 1, 2, \dots, n.$$

Thus,

$${}_{0}^{C}D^{q(t)}x_i = -\beta_i x_i - \bar{\beta}_i sgn(x_i)|x_i|^{q(t)}. \tag{16}$$

Lemma 3.6 *Assume that $x(t)$ is a continuous differential function, then the following inequality holds for any time instant $t \geq 0$:*

$$\frac{1}{2}{}_{0}^{C}D_t^{q(t)}x^2(t) \leq x(t){}_{0}^{C}D_t^{q(t)}x(t), \quad 0 < q(t) < 1.$$

Theorem 3.7 *Consider the VFO sliding mode dynamics* (16), *then its state trajectories converge to zero asymptotically in a finite time*.

Proof Choose the Lyapunov functional as

$$V_1(t) = \sum_{i=1}^{n} x_i^2(t), \tag{17}$$

and apply VFO derivative $q(t)$ to Lyapunov function (17) with respect to time. We obtain that along with Lemma 3.6 by subsisting equation (16) of ${}_0^C D_t^{q(t)} x_i(t)$,

$$\begin{aligned}
{}_0^C D_t^{q(t)} V_1(t) &= \sum_{i=1}^{n} {}_0^C D_t^{q(t)} x_i^2(t) \\
&\leq \sum_{i=1}^{n} x_i(t) {}_0^C D_t^{q(t)} x_i(t) \\
&= \sum_{i=1}^{n} x_i(t) \bigl[-\beta_i x_i(t) - \bar{\beta}_i sgn(x_i)|x_i|^{q(t)}\bigr] \\
&\leq -a V_1(t),
\end{aligned}$$

where $a = \min\{\beta_i, \bar{\beta}_i\}$. Then, according to Theorem 3.4 and Theorem 3.5, the state variables x_i, $i = 1, 2, \dots, n$, asymptotically tend to zero in a finite time.

The following step is to design a robust sliding control law based on the sliding mode approach. Then the state trajectories of the VFO system are forced to the sliding mode surface in a finite time. Subsequently, the control law is given as follows:

$$u_i(t) = u_{eq}^i + u_{sw}^i \tag{18}$$

with

$$\begin{cases}
u_{eq}^i = -f_i(t, X) - \bar{\beta}_i sgn(x_i)|x_i|^{q(t)} - \beta_i x_i, \\
{}_0^C D^{q(t)} u_{sw}^i = -(M_i^{\Delta f} + M_i^d) sgn(s_i) - \xi_1^i s_i - \xi_2^i sgn(s_i),
\end{cases} \tag{19}$$

where u_{eq}^i is the equivalent control, and u_{sw}^i is the reaching law with $\bar{\beta}_i > 0$, $\beta_i > 0$, $\xi_1^i > 0$, $\xi_2^i > 0$, $i = 1, 2, \dots, n$. □

The next theorem ensures that system trajectories (3) converge to the sliding mode surface under the controller.

Theorem 3.8 *Consider VFO system* (3). *If controller* (18) *is applied to system* (3) *with* $\bar{\beta}_i > 0$, $\beta_i > 0$, $\xi_1^i > 0$, $\xi_2^i > 0$, *then the states of system* (3) *are driven to reach to the sliding mode surface* (15) *asymptotically from the initial conditions in the finite time and stay on it forever*.

Proof The Lyapunov function is defined as

$$V_2(s_i) = \frac{1}{2}\sum_{i=1}^{n} s_i^2(t). \tag{20}$$

Calculating the variable fractional order derivative $q(t)$ for $V_2(s_i)$ with respect to time, it is obtained from Lemma 3.6 that

$${}_0^C D_t^{q(t)} V_2(s_i) \le \sum_{i=1}^{n} s_i {}_0^C D_t^{q(t)} s_i. \tag{21}$$

According to the sliding mode (15), it is rewritten under controller (18) as follows:

$$\begin{aligned}
{}_0^C D_t^{q(t)} V_2(s_i) &\le \sum_{i=1}^{n} s_i\big[{}_0^C D_t^{q(t)}\big({}_0^C D_t^{q(t)} x_i + \beta_i x_i + \bar{\beta}_i sgn(x_i)|x_i|^{q(t)}\big)\big] \\
&= \sum_{i=1}^{n} s_i\big[{}_0^C D_t^{q(t)}\big(f_i(t,X) + \Delta f_i(t,X) + d_i(t) + u_i(t) + \beta_i x_i + \bar{\beta}_i sgn(x_i)|x_i|^{q(t)}\big)\big] \\
&= \sum_{i=1}^{n} s_i\big[{}_0^C D_t^{q(t)}\big(f_i(t,X) + \Delta f_i(t,X) + d_i(t) - f_i(t,X) - \beta_i x_i - \bar{\beta}_i sgn(x_i)|x_i|^{q(t)} \\
&\quad + u_{sw}^i + \beta_i x_i + \bar{\beta}_i sgn(x_i)|x_i|^{q(t)}\big)\big] \\
&= \sum_{i=1}^{n} s_i\big[{}_0^C D_t^{q(t)} \Delta f_i(t,X) + {}_0^C D_t^{q(t)} d_i(t) + {}_0^C D_t^{q(t)} u_{sw}^i\big] \\
&\le \sum_{i=1}^{n}\big[M_i^{\Delta}|s_i| + M_i^d|s_i| - M_i^{\Delta}|s_i| - M_i^d|s_i| - \xi_1^i s_i^2 - \xi_2^i |s_i|\big] \\
&= -\sum_{i=1}^{n} \xi_1^i s_i^2 - \sum_{i=1}^{n} \xi_2^i |s_i| \\
&\le -\xi V_2(s_i),
\end{aligned}$$

where $\xi = \min\{\xi_1^1, \xi_1^2, \dots, \xi_1^n\}$ is a constant.

From Theorem 3.4 and Theorem 3.5, we obtain that the state trajectories of VFO system (3) will be driven to $s_i(t) = 0$, $i = 1, 2, \dots, n$, as $t \to \infty$ in a finite time and stay on it forever. Combined with Theorem 3.7 and Theorem 3.8, the trajectories will converge to zero asymptotically in a finite time. □

4 Numerical simulation

The simulation results are presented to validate our theoretical results in this section.

4.1 Control of the VFO brushless motor system by SMC

The VFO brushless motor system is stated as follows:

$$\begin{cases}
{}_0^C D^{q(t)} x_1 = -0.875 x_1 + x_2 x_3 + \Delta f_1(X,t) + d_1(t) + u_1(t), \\
{}_0^C D^{q(t)} x_2 = -x_2 + 55 x_3 - x_1 x_3 + \Delta f_2(X,t) + d_2(t) + u_2(t), \\
{}_0^C D^{q(t)} x_3 = 4(x_2 - x_3) + \Delta f_3(X,t) + d_3(t) + u_3(t),
\end{cases} \tag{22}$$

where $\Delta f_i(X,t)$, $i = 1,2,3$, and $d_i(t)$, $i = 1,2,3$, denote the perturbation and uncertainty terms of the system, respectively,

$$\begin{cases} \Delta f_1(X,t) + d_1(t) = -0.15 \cdot sin(2t) + 0.2 \cdot cos(3t)x_1, \\ \Delta f_2(X,t) + d_2(t) = 0.2 \cdot sin(3t) + 0.25 \cdot sin(4t)x_2, \\ \Delta f_3(X,t) + d_3(t) = -0.25 \cdot cos(4t) + 0.3 \cdot sin(2t)x_3. \end{cases} \tag{23}$$

Under the initial conditions

$$x_1(0) = 10, \qquad x_2(0) = -5, \qquad x_3(0) = 5,$$

system (22) is chaotic behavior with the following VFO:

$$q(t) = \begin{cases} 0.96 + 0.002t/T, & t \in [0,T], \\ 0.96, & t > T. \end{cases}$$

The chaotic trajectories of uncontrolled system (22) are illustrated in Fig. 1. From the figure, we can find that the system presents chaotic behavior under the initial value condition.

According to the sliding mode surface (15), the following sliding mode surfaces in this simulation are utilized:

$$s_i(t) = {}_0^C D^{q(t)} x_i + \beta_i x_i + \bar{\beta}_i sgn(x_i)|x_i|^{q(t)}, \quad i = 1,2,3. \tag{24}$$

Subsequently, the controller is designed according to (18) in order to stabilize the chaotic system

$$\begin{cases} u_1(t) = 0.875x_1 - x_2x_3 - \bar{\beta} sgn(x_1)|x_1|^{q(t)} - \beta x_1 + u_{sw}^1, \\ {}_0^C D^{q(t)} u_{sw}^1 = -(M_1^{\Delta f} + M_1^d) sgn(s_1) - \xi_1^{(1)} s_1 - \xi_2^{(1)} sgn(s_1), \\ u_2(t) = x_2 - 55x_3 + x_1x_3 - \bar{\beta} sgn(x_2)|x_2|^{q(t)} - \beta x_2 + u_{sw}^2, \\ {}_0^C D^{q(t)} u_{sw}^2 = -(M_2^{\Delta f} + M_2^d) sgn(s_2) - \xi_1^{(2)} s_2 - \xi_2^{(2)} sgn(s_2), \\ u_3(t) = -4(x_2 - x_3) - \bar{\beta} sgn(x_3)|x_3|^{q(t)} - \beta x_3 + u_{sw}^3, \\ {}_0^C D^{q(t)} u_{sw}^3 = -(M_3^{\Delta f} + M_3^d) sgn(s_3) - \xi_1^{(3)} s_3 - \xi_2^{(3)} sgn(s_3), \end{cases} \tag{25}$$

the constant parameters are

$$\begin{cases} \beta_1 = 8, \quad \bar{\beta}_1 = 6, \quad \beta_2 = 4, \quad \bar{\beta}_2 = 5, \quad \beta_3 = 3, \quad \bar{\beta}_3 = 2, \\ M_i^{\Delta f} = M_i^d = 0.05, \quad \xi_1^{(i)} = 0.5, \quad \xi_2^{(i)} = 0.4, \quad i = 1,2,3. \end{cases} \tag{26}$$

Then the trajectories of the system under the controller are depicted in Fig. 2, which shows that the trajectories of the system can be stabilized to the origin. The time responses are showed in Fig. 3 and Fig. 4 for the control inputs (25) and the sliding mode surfaces (24), which show that the time responses have been driven to the origin in a finite time. It is concluded that the state variables converge to the origin in a finite time. Moreover, the chaotic behavior is suppressed.

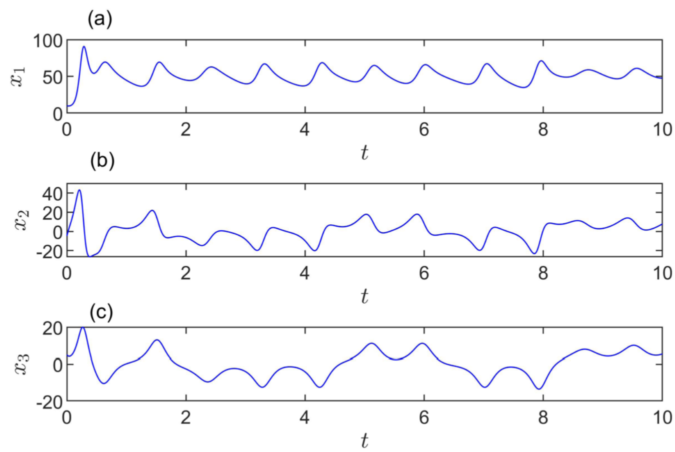

Figure 1 The chaotic trajectories of uncontrolled system (22)

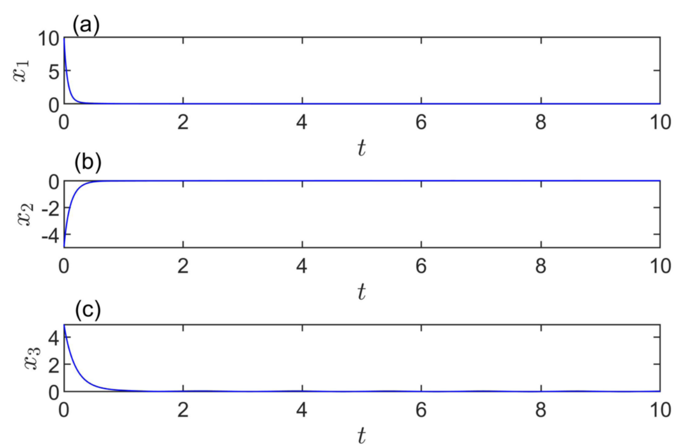

Figure 2 The state trajectories of system (22) under controller (25)

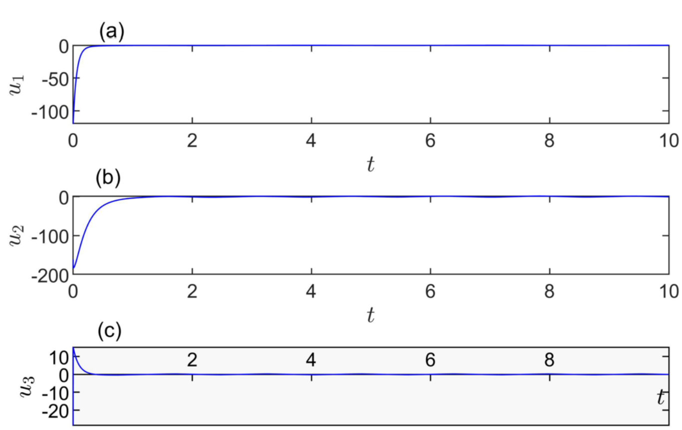

Figure 3 The history of control input (25)

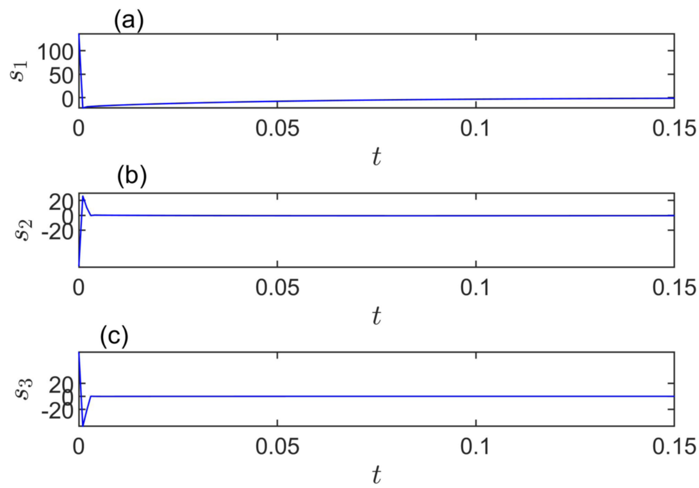

Figure 4 The time response of the sliding mode surface (24)

4.2 Control of the VFO electrostatic transducer by SMC approach

Consider the VFO uncertain nonlinear system

$$\begin{cases} {}_0^C D^{q(t)} x_1 = x_2 + u_1(t), \\ {}_0^C D^{q(t)} x_2 = x_3 + 0.15cos(2t) + u_2(t), \\ {}_0^C D^{q(t)} x_3 = -(6.8 - 0.2sin(t))x_1 - 3.92x_2 - x_3 + (1 + 0.3 * cos(0.5t))x_1^2(t) \\ \qquad\qquad + 1.2 * cos(3t) + u_3(t), \end{cases} \tag{27}$$

with the model uncertainty terms of the system as follows:

$$\begin{cases} \Delta f_1(X,t) + d_1(t) = 0, \\ \Delta f_2(X,t) + d_2(t) = 0.15cos(2t), \\ \Delta f_3(X,t) + d_3(t) = 1.2cos(3t). \end{cases} \tag{28}$$

By (15), the sliding mode is designed as

$$s_i(t) = D^{q(t)} x_i + \beta_i x_i + \bar{\beta}_i sgn(x_i)|x_i|^{q(t)}, \quad i = 1, 2, 3, \tag{29}$$

in terms of (18), the controller is proposed as

$$\begin{cases} u_1(t) = -x_2 - \bar{\beta} sgn(x_1)|x_1|^{q(t)} - \beta x_1 + u_1^{sw}, \\ {}_0^C D^{q(t)} u_1^{sw} = -(M_1^{\Delta f} + M_1^d) sgn(s_1) - \xi_1^{(1)} s_1 - \xi_2^{(1)} sgn(s_1), \\ u_2(t) = -x_3 - \bar{\beta} sgn(x_2)|x_2|^{q(t)} - \beta x_2 + u_2^{sw}, \\ {}_0^C D^{q(t)} u_2^{sw} = -(M_2^{\Delta f} + M_2^d) sgn(s_2) - \xi_1^{(2)} s_2 - \xi_2^{(2)} sgn(s_2), \\ u_3(t) = (6.8 - 0.2sin(t))x_1 + 3.92x_2 + x_3 - (1 + 0.3cos(0.5t))x_1^2 - \bar{\beta} sgn(x_3)|x_3|^{q(t)} \\ \qquad - \beta x_3 + u_{sw}^3, \\ {}_0^C D^{q(t)} u_{sw}^3 = -(M_3^{\Delta f} + M_3^d) sgn(s_3) - \xi_1^{(3)} s_3 - \xi_2^{(3)} sgn(s_3), \end{cases} \tag{30}$$

the parameters satisfy

$$\beta_i = \bar{\beta}_i = 2, \qquad M_i^{\Delta f} = M_i^d = 1, \qquad \xi_1^{(i)} = 4, \qquad \xi_2^{(i)} = 5, \quad i = 1,2,3,4. \tag{31}$$

Under the initial value

$$x_1(0) = 1, \qquad x_2(0) = 1, \qquad x_3(0) = 1,$$

system (27) shows chaotic phenomena, which is indicated by Fig. 5. By using the proposed SMC (30), the state trajectories of system (27) are described by Fig. 6. From the figure, we can see that the system converges to zero quickly. Figure 7 and Fig. 8 demonstrate that the sliding mode surface (29) responses converge to zero and controller (30) can stabilize system (27) effectively in a finite time. Therefore, the control inputs give a good performance in practice.

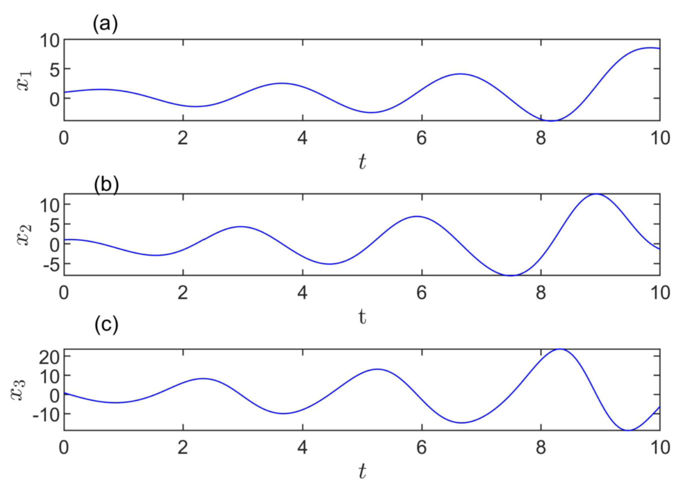

Figure 5 The chaotic trajectories of uncontrolled system (27)

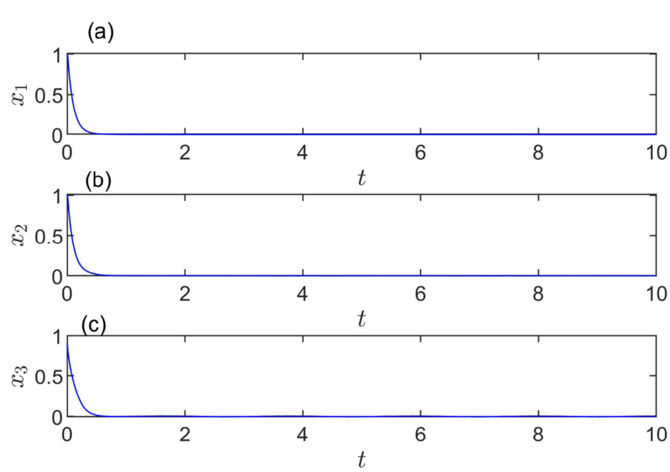

Figure 6 The state trajectories of the system under controller (30)

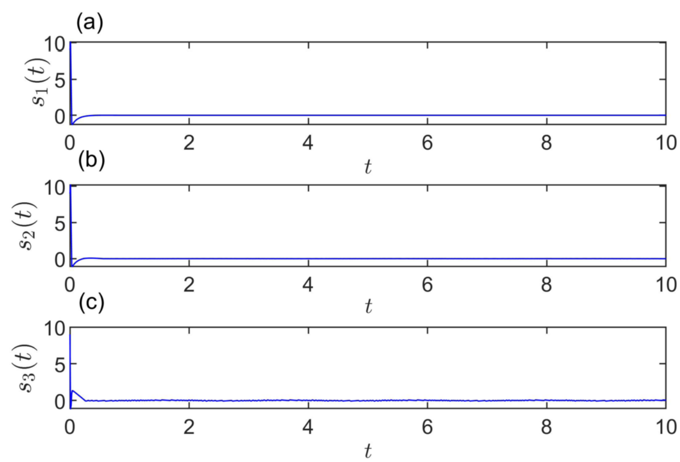

Figure 7 The time response of the sliding mode surface (29)

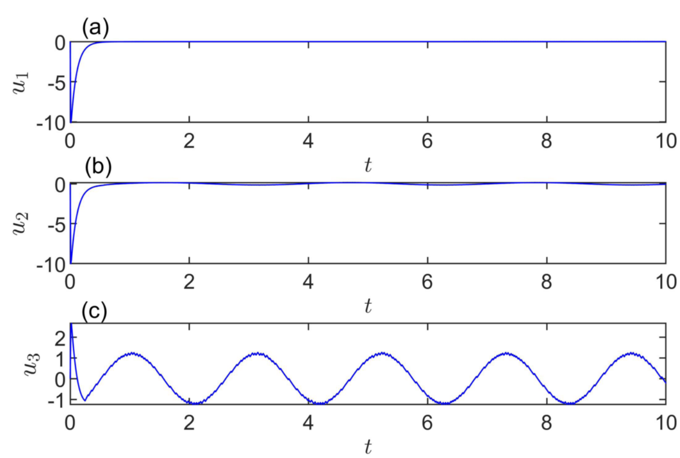

Figure 8 The history of the control input (30)

5 Conclusion

This paper investigates a control problem of the VFO nonlinear system. A robust controller is proposed to stabilize the system in the present with uncertainty and external disturbance. By applying the sliding mode control to the system, a VFO derivative sliding mode surface is designed. And then, a control law has been designed which is free of chattering signal for a kind of VFO system. According to the proposed stability criteria, the finite time stability of the controlled systems has been proved. Lastly, numerical results are provided to illustrate the validity and efficiency of the proposed FO controllers. Furthermore, the proposed results motivate the development of theoretical and practical tools for implementing the proposed controllers to the fractional model.

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Availability of data and materials
This paper is a theoretical work, and it is not based on any data.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have worked in research in an equal way to obtain the results of this paper. All authors read and approved the final manuscript.

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