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On the ω-multiple Charlier polynomials

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Abstract

The main aim of this paper is to define and investigate more general multiple Charlier polynomials on the linear lattice $\omega\mathbb{N} = \{0, \omega, 2\omega, \ldots\}$, $\omega \in \mathbb{R}$. We call these polynomials ω-multiple Charlier polynomials. Some of their properties, such as the raising operator, the Rodrigues formula, an explicit representation and a generating function are obtained. Also an $(r+1)$th order difference equation is given. As an example we consider the case $\omega = \frac{3}{2}$ and define $\frac{3}{2}$-multiple Charlier polynomials. It is also mentioned that, in the case $\omega = 1$, the obtained results coincide with the existing results of multiple Charlier polynomials.

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1 Introduction

In [3] the authors introduced the δ_ω-Appell polynomial sets which are defined by

$$\delta_\omega P_{n+1}(x) = (n+1)P_n(x), \quad n \geq 0,$$

where

$$\delta_\omega\big(f(x)\big) := \frac{\Delta_\omega(f(x))}{\omega} := \frac{f(x+\omega) - f(x)}{\omega}, \quad \omega \neq 0.$$

They proved an equivalent definition in terms of the generating function:

$$A(t)(1+\omega t)^{\frac{x}{\omega}} = \sum_{n=0}^{\infty} \frac{P_n(\omega; x)}{n!} t^n,$$

where

$$A(t) = \sum_{k=0}^{\infty} a_k t^k, \quad a_0 \neq 0.$$

It has also been shown in [3] that among all the δ_ω- Appell polynomials, d-orthogonal polynomial sets should have the generating function of the form

$$G(x,t)=\exp\bigl(H_d(t)\bigr)(1+\omega t)^{\frac{x}{\omega}}=\sum_{n=0}^{\infty}\frac{P_n(x)}{n!}t^n,$$

where H_d is a polynomial of degree d. In the special case

$$H_d(t)=-at,\quad a\neq 0,$$

we have the polynomials generated as follows:

$$\exp(-at)(1+\omega t)^{\frac{x}{\omega}}=\sum_{n=0}^{\infty}\frac{P_n(x)}{n!}t^n,\quad a\neq 0.$$

These polynomials can be called ω-Charlier polynomials, since the case $\omega=1$ gives the usual Charlier polynomials.

On the other hand, in a recent paper, the multiple Δ_ω-Appell polynomials were defined [8] by the generating function

$$\begin{aligned}&A(t_1,t_2,\dots,t_n)\bigl(1+\omega(t_1+t_2+\cdots+t_r)\bigr)^{\frac{x}{\omega}}\\&\quad=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}P_{\vec{n}}(x)\frac{t_1^{n_1}t_2^{n_2}\cdots t_r^{n_r}}{n_1!n_2!\cdots n_r!}.\end{aligned}\tag{1}$$

Inspired by these observations, in this paper we aim to introduce the ω-multiple Charlier polynomials starting from the multiple orthogonality relations with respect to the weight function of the form

$$w_i(x)=\frac{a_i^x}{\Gamma_\omega(x+\omega)},\quad x\in\mathbb{R}^+,i=1,\dots r,$$

and investigate certain of their properties such as raising operator, Rodrigues formula, explicit representation and generating function. We also obtain an $(r+1)$th order difference equation and give some special examples for certain choices of ω. So it can be easily observed from the generating function of ω-multiple Charlier polynomials that these polynomials are examples of Δ_ω-multiple Appell polynomials.

We will start by recalling some basic knowledge about the discrete orthogonal and discrete multiple orthogonal polynomials.

The nth degree monic orthogonal polynomial p_n is defined by

$$\int p_n(x)x^k\,d\mu(x)=0,\quad k=0,1,2,\dots,n-1,$$

where μ is a positive measure on the real line. In general, in the case of discrete orthogonal polynomials, the term x^k is replaced by $(-x)_k$, since $\Delta(-x)_k=-k(-x)_{k-1}$, where

$$(a)_k=a(a+1)\dots(a+k-1)$$

is the Pochhammer symbol and

$$\Delta f(x) = f(x+1) - f(x),$$

is the forward difference operator.

The classical orthogonal polynomials (on a linear lattice) of a discrete variable are the Hahn, Meixner, Kravchuk and Charlier polynomials. The main concern of this paper is the Charlier polynomials.

The orthogonality measure (Poisson distribution) for Charlier polynomials is

$$\mu = \sum_{k=0}^{\infty} \frac{a^k}{k!} \delta_k,$$

with $k \in \mathbb{N}$ ($\mathbb{N} := \{0, 1, 2, \ldots\}$) and $a > 0$.

The type *II* multiple orthogonal polynomials $p_{\vec{n}}$ of degree $\leq |\vec{n}| := n_1 + \cdots + n_r$ ($r \geq 2$) with respect to r non-negative measures $\mu_1, \ldots, \mu_r$ on $\mathbb{R}$, are defined by

$$\int_{I_i} p_{\vec{n}}(x) x^k \, d\mu_i(x) = 0, \quad k = 0, 1, \ldots, n_i - 1 \ (i = 1, \ldots, r). \tag{2}$$

Here

$$\operatorname{supp}(\mu_i) = \bigl\{ x \in \mathbb{R} : \mu_i\bigl((x - \epsilon, x + \epsilon)\bigr) > 0 \text{ for all } \epsilon > 0 \bigr\}$$

and I_i ($i = 1, 2, \ldots, r$) is the smallest interval containing $\operatorname{supp}(\mu_i)$. Conditions (2) give $|\vec{n}|$ linear equations for the $|\vec{n}| + 1$ unknown coefficients of $p_{\vec{n}}$. If $p_{\vec{n}}$ is unique (up to a multiplicative factor) and has degree $|\vec{n}|$, then $\vec{n}$ is said to be normal. In general, the monic polynomials are considered.

In the case where we have r non-negative discrete measures on $\mathbb{R}$:

$$\mu_i = \sum_{m=0}^{N_i} \rho_{i,m} \delta_{x_{i,m}}, \quad \rho_{i,m} > 0, x_{i,m} \in \mathbb{R}, N_i \in \mathbb{N} \cup \{\infty\}, i = 1, \ldots, r,$$

where all $x_{i,m}$ are different for each $m = 0, 1, \ldots, N_i$ ($i = 1, 2, \ldots, r$), we have the discrete multiple orthogonal polynomials (on the linear lattice), and the above orthogonality conditions can be written as

$$\sum_{j=0}^{\infty} p_{\vec{n}}(j)(-j)_k \rho_{i,j} = 0, \quad k = 0, 1, \ldots, n_i - 1, i = 1, \ldots, r, \tag{3}$$

where $p_{\vec{n}}$ is a polynomial of degree $\leq |\vec{n}|$.

In this paper, we pay attention to the *AT* system of r non-negative discrete measures; we recall its definition.

Definition 1.1 ([1]) An *AT* system of r non-negative discrete measures is a system of measures

$$\mu_i = \sum_{m=0}^{N} \rho_{i,m} \delta_{x_m}, \quad \rho_{i,m} > 0, x_m \in \mathbb{R}, N \in \mathbb{N} \cup \{+\infty\}, i = 1, \ldots, r,$$

where $\operatorname{supp}(\mu_i)$ $(i = 1, \ldots, r)$ is the closure of x_m and the orthogonality intervals (2) are the same, namely I. It is also assumed that there exist r continuous functions $w_1, \ldots, w_r$ on I with $w_i(x_m) = \rho_{i,m}$ $(m = 1, \ldots, N, i = 1, \ldots, r)$ such that the $|\vec{n}|$ functions

$$\{w_1, xw_1, \ldots, x^{n_1-1}w_1, w_2, xw_2, \ldots, x^{n_2-1}w_2, \ldots, w_r, xw_r, \ldots, x^{n_r-1}w_r\},$$

form a Chebyshev system on I for each multi-index $|\vec{n}| < N + 1$. This means that all the linear combinations of the form

$$\sum_{i=1}^{r} Q_{n_i-1} w_i(x),$$

where Q_{n_i-1} is a polynomial of degree $\leq n_i - 1$, has at most $|\vec{n}| - 1$ zeros on I.

Remark 1.1 If we have r continuous functions $w_1, \ldots, w_r$ on I with $w_i(x_m) = \rho_{i,m}$, then the orthogonality conditions (3) can be written as

$$\sum_{j=0}^{\infty} p_{\vec{n}}(j)(-j)_k w_i(j) = 0, \quad k = 0, 1, \ldots, n_i - 1, i = 1, \ldots, r.$$

As is pointed out in [1], in an *AT* system every discrete multiple orthogonal polynomials of type *II* corresponding to the multi-index $\vec{n}$ has exact degree $|\vec{n}|$, and every multi-index $\vec{n}$ with $|\vec{n}| < N + 1$ is normal.

Recently, some discrete multiple orthogonal polynomials and their structural properties have been studied in [1]. Difference equations for discrete classical multiple orthogonal polynomials have been studied in [5]. In [7], the ratio asymptotics and the zeros of multiple Charlier polynomials have been investigated. Nearest neighbor recurrence relations for multiple orthogonal polynomials were investigated in [10]. The $(r + 1)$th order difference equations for the multiple Charlier and Meixner polynomials have been studied in [9]. Furthermore, in [2], the q-Charlier multiple orthogonal polynomials and some of their structural properties were studied.

The main aim of this paper is to extend the idea of discrete multiple orthogonality to more general linear lattice $\omega\mathbb{N} = \{0, \omega, 2\omega, \ldots\}$ for the ω-multiple Charlier polynomials. We note that in a recent paper this type of discrete orthogonality is used to define ω-multiple Meixner polynomials [11].

We organize the paper as follows: In Sect. 2, we define ω-multiple Charlier polynomials and obtain a raising operator and the Rodrigues formula for them. In Sect. 3, the explicit representation and generating function are given for the ω-multiple Charlier polynomials. In Sect. 4, recurrence relations are given. In Sect. 5, we obtain $(r + 1)$th order difference equations satisfied by ω-multiple Charlier polynomials. In Sect. 6, as an illustrative example, we consider the case $\omega = \frac{3}{2}$ and exhibit our main results for this particular case. In the last section, it is shown that the special cases of the results obtained in Sects. 2, 3, 4 and 5 coincide with the corresponding results for multiple Charlier polynomials obtained in the earlier papers. Some concluding remarks are also stated.

2 Discrete ω-multiple Charlier orthogonal polynomials

In this section, we define ω-multiple Charlier polynomials. We present a raising operator and the Rodrigues formula for them. We start by defining the discrete multiple orthogonality on the linear lattice $\omega\mathbb{N} = \{0, \omega, 2\omega, \ldots\}$ $(\omega > 0)$ and call them ω-multiple orthogonal polynomials.

Definition 2.1 The ω-multiple orthogonal polynomials are defined as

$$\sum_{k=0}^{\infty} p_{\overrightarrow{n}}(\omega k)(-\omega k)_{j,\omega} w_i(\omega k) = 0, \quad j = 0, \ldots, n_i - 1, i = 1, 2, \ldots, r,$$

where ω is a fixed positive real number, $\overrightarrow{n} = (n_1, \ldots, n_r)$ and $p_{\overrightarrow{n}}$ is a polynomial of degree $|\overrightarrow{n}|$ and

$$\begin{aligned}(-\omega k)_{j,\omega} &= (-\omega k)(-\omega k + \omega) \ldots \bigl(-\omega k + \omega(j-1)\bigr) \\ &= \omega^j (-k)_j.\end{aligned}$$

Now we choose the orthogonality measures as

$$\mu_i = \sum_{k=0}^{+\infty} \frac{a_i^{\omega k}}{\Gamma_\omega(\omega k + \omega)} \delta_{\omega k}, \quad a_i > 0, i = 1, \ldots, r,$$

where $a_1, \ldots, a_r$ are different parameters and

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}}\, dt, \quad x > 0$$

is the k-gamma function [6].

For each measure the weights form an extended Poisson distribution on $\omega\mathbb{N}$ ($\omega\mathbb{N} = \{0, \omega, 2\omega, \ldots\}$). It is easily seen from Example 2.1 in [1] that these r measures form a *Chebyshev* system on $\mathbb{R}^+$ for every $\overrightarrow{n} = (n_1, \ldots, n_r) \in \omega\mathbb{N}^r$ since the weight functions,

$$w_i(x) = \frac{a_i^x}{\Gamma_\omega(x + \omega)}, \quad x \in \mathbb{R}^+, i = 1, \ldots, r,$$

are continuous and they have no zeros on $\mathbb{R}^+$. So every multi-index is normal and the monic solution is unique.

The corresponding multiple orthogonality conditions are given on $\omega\mathbb{N}$ as

$$\sum_{k=0}^{\infty} C_{\overrightarrow{n}}^{\overrightarrow{a}}(\omega k)(-\omega k)_{j,\omega} \frac{a_i^{\omega k}}{\Gamma_\omega(\omega k + \omega)} = 0, \quad j = 0, \ldots, n_i - 1, i = 1, 2, \ldots, r, \tag{4}$$

where $\overrightarrow{n} = (n_1, \ldots, n_r)$ and $\overrightarrow{a} = (a_1, \ldots, a_r)$. We represent these polynomials by $C_{\overrightarrow{n}}^{\overrightarrow{a}}$ and call them ω-multiple Charlier orthogonal polynomials.

Theorem 2.2 *The raising relation for the ω-multiple Charlier polynomials is given as*

$$\frac{a_i^{\omega}}{w_i(x)}\nabla_{\omega}\big[w_i(x)C_{\vec{n}}^{\vec{a}}(x)\big] = -C_{\vec{n}+\vec{e_i}}^{\vec{a}}(x), \quad i = 1,\dots,r. \tag{5}$$

where $\nabla_{\omega}f(x) = f(x) - f(x-\omega)$ *and* $\vec{e}_i = (0,\dots,0,1,\dots,0)$.

Proof Applying the product rule $\nabla_{\omega}[f(x)g(x)] = f(x)\nabla_{\omega}g(x) + g(x-\omega)\nabla_{\omega}f(x)$, we have

$$\nabla_{\omega}\big[w_i(x)C_{\vec{n}}^{\vec{a}}(x)\big] = w_i(x)\nabla_{\omega}C_{\vec{n}}^{\vec{a}}(x) + C_{\vec{n}}^{\vec{a}}(x-\omega)\nabla_{\omega}w_i(x). \tag{6}$$

Since $\nabla_{\omega}w_i(x) = w_i(x)[1 - \frac{x}{a_i^{\omega}}]$, we get by using (6)

$$\begin{aligned}\nabla_{\omega}\big[w_i(x)C_{\vec{n}}^{\vec{a}}(x)\big] &= w_i(x)\left[\nabla_{\omega}C_{\vec{n}}^{\vec{a}}(x) + C_{\vec{n}}^{\vec{a}}(x-\omega)\left[1 - \frac{x}{a_i^{\omega}}\right]\right] \\ &= -\frac{w_i(x)}{a_i^{\omega}}P_{\vec{n}+\vec{e_i}}^{\vec{\alpha}}(x).\end{aligned} \tag{7}$$

Hence

$$\sum_{x=0}^{\infty}(-x)_{j,\omega}\nabla_{\omega}\big[w_i(x)C_{\vec{n}}^{\vec{a}}(x)\big] = -\frac{1}{a_i^{\omega}}\sum_{x=0}^{\infty}w_i(x)(-x)_{j,\omega}P_{\vec{n}+\vec{e}_i}^{\vec{a}}(x).$$

Applying the ω-summation by parts formula, which is

$$\sum_{x=0}^{\infty}\Delta_{\omega}\big[f(\omega x)\big]g(\omega x) = -\sum_{x=0}^{\infty}\nabla_{\omega}\big[g(\omega x)\big]f(\omega x), \quad g(-\omega) = 0,$$

we get

$$\sum_{x=0}^{\infty}(-\omega x)_{j,\omega}\nabla_{\omega}\big[w_i(\omega x)C_{\vec{n}}^{\vec{a}}(\omega x)\big] = -\sum_{x=0}^{\infty}\Delta_{\omega}\big[(-\omega x)_{j,\omega}\big]w_i(\omega x)C_{\vec{n}}^{\vec{a}}(\omega x).$$

Since $\Delta_{\omega}(-\omega x)_{j,\omega} = -\omega j(-\omega x)_{j-1,\omega}$, we have

$$\sum_{x=0}^{\infty}\omega j(-\omega x)_{j-1,\omega}w_i(\omega x)C_{\vec{n}}^{\vec{a}}(\omega x) = -\frac{1}{a_i^{\omega}}\sum_{x=0}^{\infty}w_i(\omega x)(-\omega x)_{j,\omega}P_{\vec{n}+\vec{e_i}}^{\vec{a}}(\omega x).$$

Then, for $j = 0,\dots,n$, the summation on the left will be zero from the ω-multiple orthogonality conditions. Hence

$$-\frac{1}{a_i^{\omega}}\sum_{x=0}^{\infty}w_i(\omega x)(-\omega x)_{j,\omega}P_{\vec{n}+\vec{e_i}}^{\vec{a}}(\omega x) = 0. \tag{8}$$

By the uniqueness of the ω-multiple orthogonal polynomials, we have

$$P_{\vec{n}+\vec{e_i}}^{\vec{a}}(x) = C_{\vec{n}+\vec{e_i}}^{\vec{a}}(x).$$

Considering the above equality in (7), the proof is completed. □

Theorem 2.3 *The Rodrigues formula for the ω-multiple Charlier polynomials is given by*

$$C_{\vec{n}}^{\vec{a}}(x)=\left[\prod_{j=1}^{r}\left(-a_j^{\omega}\right)^{n_j}\right]\Gamma_\omega(x+\omega)\left[\prod_{i=1}^{r}\left(\frac{1}{a_i^x}\nabla_\omega^{n_i}\left(a_i^x\right)\right)\right]\left(\frac{1}{\Gamma_\omega(x+\omega)}\right). \tag{9}$$

Proof We will give the proof for the case $r=2$. The proof of the general case is similar. Repeatedly using the raising operators, we find, since $C_{0,0}^{a_1,a_2}(x)=1$, that

$$\begin{aligned}
C_{n_1,n_2}^{a_1,a_2}(x)&=\frac{(-a_1^{\omega})^{n_1}(-a_2^{\omega})^{n_2}}{a_1^x a_2^x}\Gamma_\omega(x+\omega)\nabla_\omega^{n_1}\left[\left(a_1^x\right)\nabla_\omega^{n_2}\left[\left(a_2^x\right)\frac{1}{\Gamma_\omega(x+\omega)}\right]\right]\\
&=\left[\prod_{j=1}^{2}\left(-a_j^{\omega}\right)^{n_j}\right]\Gamma_\omega(x+\omega)\left[\prod_{i=1}^{2}\left(\frac{1}{a_i^x}\nabla_\omega^{n_i}\left(a_i^x\right)\right)\right]\left(\frac{1}{\Gamma_\omega(x+\omega)}\right).
\end{aligned}$$

Hence, we get (9) for $r=2$. □

3 Explicit representation and generating function

In this section, we use the Rodrigues type formula (9) to give the explicit representation of the multiple ω-Charlier polynomials. Furthermore, we obtain the generating function for these polynomials.

Theorem 3.1 *The explicit representation for the ω-multiple Charlier polynomials is given by*

$$\begin{aligned}
C_{\vec{n}}^{\vec{a}}(x)&=\left(-a_1^{\omega}\right)^{n_1}\left(-a_2^{\omega}\right)^{n_2}\ldots\left(-a_r^{\omega}\right)^{n_r}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\cdots\sum_{k_r=0}^{n_r}\frac{(-n_1)_{k_1}(-n_2)_{k_2}\ldots(-n_r)_{k_r}}{k_1!k_2!\ldots k_r!}\\
&\quad\times\left(-\frac{x}{\omega}\right)_{k_1+k_2+\cdots+k_r}\left(\left(-\frac{1}{a_1}\right)^{\omega}\omega\right)^{k_1}\cdots\left(\left(-\frac{1}{a_r}\right)^{\omega}\omega\right)^{k_r}.
\end{aligned}\tag{10}$$

Proof We will give the proof for $r=2$. The general case (10) can be proved in a similar manner. Using (9) for $r=2$, we write

$$C_{n_1,n_2}^{a_1,a_2}(x)=\left(-a_1^{\omega}\right)^{n_1}\left(-a_2^{\omega}\right)^{n_2}\Gamma_\omega(x+\omega)\frac{1}{a_1^x}\nabla_\omega^{n_1}\left(a_1^x\right)\left(\frac{1}{a_2^x}\nabla_\omega^{n_2}\frac{a_2^x}{\Gamma_\omega(x+\omega)}\right).$$

Since $\nabla_\omega^n f(x)=\sum_{i=1}^{n}(-1)^i\binom{n}{i}f(x-i\omega)$, we have

$$\begin{aligned}
C_{n_1,n_2}^{a_1,a_2}(x)&=\left(-a_1^{\omega}\right)^{n_1}\left(-a_2^{\omega}\right)^{n_2}\Gamma_\omega(x+\omega)\sum_{k=0}^{n_2}\frac{1}{a_2^x}\binom{n_2}{k}(-1)^k a_2^{x-k\omega}\\
&\quad\times\left(\frac{1}{a_1^x}\nabla_\omega^{n_1}\left(\frac{a_1^x}{\Gamma_\omega(x+\omega-k\omega)}\right)\right)\\
&=\left(-a_1^{\omega}\right)^{n_1}\left(-a_2^{\omega}\right)^{n_2}\Gamma_\omega(x+\omega)\sum_{k=0}^{n_2}\binom{n_2}{k}a_2^{-k\omega}\sum_{m=0}^{n_1}\binom{n_1}{m}(-1)^{k+m}\\
&\quad\times\frac{a_1^{-m\omega}}{\Gamma_\omega(x+\omega-k\omega-m\omega)}
\end{aligned}$$

$$
\begin{aligned}
&= \left(-a_1^{\omega}\right)^{n_1}\left(-a_2^{\omega}\right)^{n_2}\sum_{m=0}^{n_1}\sum_{k=0}^{n_2}(-n_1)_m(-n_2)_k\frac{a_1^{-m\omega}}{m!}\frac{a_2^{-k\omega}}{k!}\\
&\quad\times\frac{\Gamma_\omega(x+\omega)}{\Gamma_\omega(x+\omega-k\omega-m\omega)} \qquad (11)\\
&= \left(-a_1^{\omega}\right)^{n_1}\left(-a_2^{\omega}\right)^{n_2}\sum_{m=0}^{n_1}\sum_{k=0}^{n_2}\frac{(-n_1)_m(-n_2)_k(-\frac{x}{\omega})_{k+m}}{m!k!}\\
&\quad\times\left(\left(-\frac{1}{a_1}\right)^{\omega}\omega\right)^{m}\left(\left(-\frac{1}{a_2}\right)^{\omega}\omega\right)^{k}.
\end{aligned}
$$

Whence the result. □

Corollary 3.2 *Equation* (11) *can be written as*

$$
C_{n_1,n_2}^{a_1,a_2}(x) = \left(-a_1^{\omega}\right)^{n_1}\left(-a_2^{\omega}\right)^{n_2}\lim_{\gamma\to+\infty}F_2\left(-\frac{x}{\omega},-n_1,-n_2;\gamma,\gamma;\left(-\frac{1}{a_1}\right)^{\omega}\gamma\omega,\left(-\frac{1}{a_2}\right)^{\omega}\gamma\omega\right)
$$

where

$$
F_2(\alpha,\beta,\beta';\gamma,\gamma';x,y) = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m(\gamma')_n m!n!}x^m y^n
$$

is the second Appell hypergeometric functions of two variables [4].

Theorem 3.3 *The ω-multiple Charlier polynomials have the following generating function*

$$
\begin{aligned}
&\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}C_{\vec{n}}^{\vec{a}}(x)\frac{t_1^{n_1}t_2^{n_2}\ldots t_r^{n_r}}{n_1!n_2!\ldots n_r!}\\
&= (1+\omega t_1+\omega t_2+\cdots+\omega t_r)^{\frac{x}{\omega}}\exp\left(-a_1^{\omega}t_1-\cdots-a_r^{\omega}t_r\right) \qquad (12)\\
&\left(\sum_{i=1}^{r}|t_i|<\omega^{-r}\right).
\end{aligned}
$$

Proof Using the explicit form of the polynomials given in Theorem 3.1, we can write

$$
\begin{aligned}
&\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}C_{\vec{n}}^{\vec{a}}(x)\frac{t_1^{n_1}t_2^{n_2}\ldots t_r^{n_r}}{n_1!n_2!\ldots n_r!}\\
&= \sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}\sum_{k_1=0}^{n_1}\cdots\sum_{k_r=0}^{n_r}\frac{(-a_1^{\omega})^{n_1}\ldots(-a_r^{\omega})^{n_r}(-n_1)_{k_1}\ldots(-n_r)_{k_r}}{k_1!k_2!\ldots k_r!}\left(-\frac{x}{\omega}\right)_{|\vec{k}|}\\
&\quad\times\left(\left(-\frac{1}{a_1}\right)^{\omega}\omega\right)^{k_1}\left(\left(-\frac{1}{a_2}\right)^{\omega}\omega\right)^{k_2}\cdots\left(\left(-\frac{1}{a_r}\right)^{\omega}\omega\right)^{k_r}\frac{t_1^{n_1}t_2^{n_2}\ldots t_r^{n_r}}{n_1!n_2!\ldots n_r!}.
\end{aligned}
$$

Using the Cauchy product of the series, we get for $\sum_{i=1}^{r}|t_i| < \omega^{-r}$

$$\begin{aligned}
&\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_r=0}^{\infty} C_{\vec{n}}^{\vec{a}}(x)\frac{t_1^{n_1}t_2^{n_2}\dots t_r^{n_r}}{n_1!n_2!\dots n_r!} \\
&\quad= \sum_{k_1=0}^{\infty}\cdots\sum_{k_r=0}^{\infty}\frac{(-\omega t_1)^{k_1}\dots(-\omega t_r)^{k_r}}{k_1!\dots k_r!}\left(-\frac{x}{\omega}\right)_{|\vec{k}|}\sum_{l_1=0}^{\infty}\cdots\sum_{l_r=0}^{\infty}\frac{(-a_1^{\omega}t_1)^{l_1}\dots(-a_r^{\omega}t_r)^{l_r}}{l_1!\dots l_r!} \\
&\quad= (1+\omega t_1+\omega t_2+\cdots+\omega t_r)^{\frac{x}{\omega}}\exp\bigl(-a_1^{\omega}t_1-\cdots-a_r^{\omega}t_r\bigr).
\end{aligned}$$

Whence the result. □

Remark 3.1 It can be easily seen from (1) and (12) that ω-multiple Charlier polynomials are an example of the Δ_ω-multiple Appell polynomials.

4 Recurrence relations

The main aim of this section is to obtain some recurrence relations for ω-multiple Charlier polynomials. Throughout this section, we concentrate on the case $r = 2$, since the proof techniques for the general r will be similar.

Proposition 4.1 *Let $G(x,t_1,t_2) = (1+\omega t_1+\omega t_2)^{\frac{x}{\omega}}e^{-(a_1^{\omega}t_1+a_2^{\omega}t_2)}$. We have the properties*

$$\frac{\partial}{\partial t_1}G(x,t_1,t_2) - \frac{\partial}{\partial t_2}G(x,t_1,t_2) = \bigl(a_2^{\omega}-a_1^{\omega}\bigr)G(x,t_1,t_2) \tag{13}$$

and

$$(1+\omega t_1+\omega t_2)\frac{\partial}{\partial t_1}G(x,t_1,t_2) = \bigl(x-a_1^{\omega}(1+\omega t_1+\omega t_2)\bigr)G(x,t_1,t_2). \tag{14}$$

Proof The proofs can be given by elementary calculations. □

Theorem 4.2 *The recurrence relations*

$$\bigl(a_2^{\omega}-a_1^{\omega}\bigr)C_{n_1,n_2}^{a_1,a_2}(x) = C_{n_1+1,n_2}^{a_1,a_2}(x) - C_{n_1,n_2+1}^{a_1,a_2}(x), \tag{15}$$

$$\begin{aligned}
xC_{n_1,n_2}^{a_1,a_2}(x) &= C_{n_1+1,n_2}^{a_1,a_2}(x) + \bigl(a_1^{\omega}+\omega n_1+\omega n_2\bigr)C_{n_1,n_2}^{a_1,a_2}(x) \\
&\quad+ \bigl(\omega a_1^{\omega}n_1+\omega a_2^{\omega}n_2\bigr)C_{n_1,n_2-1}^{a_1,a_2}(x) + n_1a_1^{\omega}\omega\bigl(a_1^{\omega}-a_2^{\omega}\bigr)C_{n_1-1,n_2-1}^{a_1,a_2}(x),
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
xC_{n_1,n_2}^{a_1,a_2}(x) &= C_{n_1+1,n_2}^{a_1,a_2}(x) + \bigl(a_1^{\omega}+\omega n_1+\omega n_2\bigr)C_{n_1,n_2}^{a_1,a_2}(x) \\
&\quad+ \omega a_2^{\omega}n_2C_{n_1,n_2-1}^{a_1,a_2}(x) + \omega a_1^{\omega}n_1C_{n_1-1,n_2}^{a_1,a_2}(x),
\end{aligned} \tag{17}$$

hold for the ω-multiple Charlier polynomials.

Proof Using (13), we get

$$\begin{aligned}&\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left[\left(a_2^{\omega}-a_1^{\omega}\right)C_{n_1,n_2}^{a_1,a_2}(x)\right]\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}\\&\quad=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left[C_{n_1+1,n_2}^{a_1,a_2}(x)-C_{n_1,n_2+1}^{a_1,a_2}(x)\right]\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}.\end{aligned}$$

Comparing the coefficients of $\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}$, (15) follows.

The left hand side of (14) can be written as

$$\begin{aligned}&(1+\omega t_1+\omega t_2)\frac{\partial}{\partial t_1}G(x,t_1,t_2)\\&\quad=(1+\omega t_1+\omega t_2)\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}C_{n_1,n_2}^{a_1,a_2}(x)\frac{t_1^{n_1-1}}{(n_1-1)!}\frac{t_2^{n_2}}{n_2!}\\&\quad=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left[C_{n_1+1,n_2}^{a_1,a_2}(x)+\omega n_1C_{n_1,n_2}^{a_1,a_2}(x)+\omega n_2C_{n_1+1,n_2-1}^{a_1,a_2}(x)\right]\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}.\end{aligned}\tag{18}$$

The right hand side of (14) will be

$$\begin{aligned}&\left(x-a_1^{\omega}(1+\omega t_1+\omega t_2)\right)G(x,t_1,t_2)\\&\quad=\left(x-a_1^{\omega}(1+\omega t_1+\omega t_2)\right)\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}C_{n_1,n_2}^{a_1,a_2}(x)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}\\&\quad=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left[\left(x-a_1^{\omega}\right)C_{n_1,n_2}^{a_1,a_2}(x)-a_1^{\omega}\omega n_1C_{n_1-1,n_2}^{a_1,a_2}(x)-a_1^{\omega}\omega n_2C_{n_1,n_2-1}^{a_1,a_2}(x)\right]\\&\qquad\times\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}.\end{aligned}\tag{19}$$

Combining (18) and (19), we get

$$\begin{aligned}&C_{n_1+1,n_2}^{a_1,a_2}(x)+\omega n_1C_{n_1,n_2}^{a_1,a_2}(x)+\omega n_2C_{n_1+1,n_2-1}^{a_1,a_2}(x)\\&\quad=\left(x-a_1^{\omega}\right)C_{n_1,n_2}^{a_1,a_2}(x)-a_1^{\omega}\omega n_1C_{n_1-1,n_2}^{a_1,a_2}(x)-a_1^{\omega}\omega n_2C_{n_1,n_2-1}^{a_1,a_2}(x).\end{aligned}\tag{20}$$

From (15), replacing n_2 by n_2-1 and n_1 by n_1-1, we have

$$C_{n_1+1,n_2-1}^{a_1,a_2}(x)=C_{n_1,n_2}^{a_1,a_2}(x)+\left(a_2^{\omega}-a_1^{\omega}\right)C_{n_1,n_2-1}^{a_1,a_2}(x)\tag{21}$$

and

$$C_{n_1-1,n_2}^{a_1,a_2}(x)=C_{n_1,n_2-1}^{a_1,a_2}(x)-\left(a_2^{\omega}-a_1^{\omega}\right)C_{n_1-1,n_2-1}^{a_1,a_2}(x),\tag{22}$$

respectively.

Using (21) and (22), we get (16).

Using (22), we have

$$\left(a_1^{\omega} - a_2^{\omega}\right) C_{n_1-1,n_2-1}^{a_1,a_2}(x) = C_{n_1-1,n_2}^{a_1,a_2}(x) - C_{n_1,n_2-1}^{a_1,a_2}(x). \tag{23}$$

Comparing (23) and (16), we get (17). □

5 Difference equations for ω-multiple Charlier polynomials

In this section, we obtain the $(r+1)$th difference equation for ω-multiple Charlier polynomials. As a corollary, we give the third order difference equation for the case $r = 2$. We start with the following theorem which will be needed for the main result.

Theorem 5.1 *The raising operator can be rewritten as*

$$L_{a_i}\left[C_{\vec{n}}^{\vec{a}}(x)\right] = -C_{\vec{n}+\vec{e_i}}^{\vec{a}}(x), \quad i = 1,2,\ldots,r, \tag{24}$$

where $\overrightarrow{e_i} = (0,\ldots,0,1,\ldots,0)$ *and* $L_{a_i}[\cdot]$ *is defined by*

$$L_{a_i}[y] = x\nabla_{\omega} y + \left(a_i^{\omega} - x\right)y.$$

Proof From the raising relation (5), we have

$$a_i^{\omega}\nabla_{\omega}\left[w_i(x)C_{\vec{n}}^{\vec{a}}(x)\right] = -w_i(x)C_{\vec{n}+\vec{e_i}}^{\vec{a}}(x).$$

Applying the ω-product rule, we can write

$$a_i^{\omega}\left[C_{\vec{n}}^{\vec{a}}(x)\nabla_{\omega} w_i(x) + w_i(x-\omega)\nabla_{\omega} C_{\vec{n}}^{\vec{a}}(x)\right] = -w_i(x)C_{\vec{n}+\vec{e_i}}^{\vec{a}}(x). \tag{25}$$

Since $\nabla_{\omega} w_i(x) = w_i(x)[1 - \frac{x}{a_{i^{\omega}}}]$, we get by using (25)

$$a_i^{\omega}\left[C_{\vec{n}}^{\vec{a}}(x)w_i(x)\left[1 - \frac{x}{a_i^{\omega}}\right] + \frac{a_i^{x-\omega}}{\Gamma_{\omega}(x)}\nabla_{\omega} C_{\vec{n}}^{\vec{a}}(x)\right] = -w_i(x)C_{\vec{n}+\vec{e_i}}^{\vec{a}}(x).$$

Hence

$$x\nabla_{\omega} C_{\vec{n}}^{\vec{a}}(x) + \left(a_i^{\omega} - x\right)C_{\vec{n}}^{\vec{a}}(x) = -C_{\vec{n}+\vec{e_i}}^{\vec{a}}(x),$$

and therefore

$$L_{a_i}\left[C_{\vec{n}}^{\vec{a}}(x)\right] = x\nabla_{\omega} C_{\vec{n}}^{\vec{a}}(x) + \left(a_i^{\omega} - x\right)C_{\vec{n}}^{\vec{a}}(x),$$

where

$$L_{a_i}\left[C_{\vec{n}}^{\vec{a}}(x)\right] = -C_{\vec{n}+\vec{e_i}}^{\vec{a}}(x).$$

This completes the proof. □

Theorem 5.2 *The lowering operator of the polynomials is determined from the following relation*:

$$\Delta_\omega C_{\overrightarrow{n}}^{\overrightarrow{a}}(x) = \sum_{i=1}^{r} \omega n_i C_{\overrightarrow{n}-\overrightarrow{e_i}}^{\overrightarrow{a}}(x), \tag{26}$$

where $\overrightarrow{e_i} = (0,\dots,1,\dots,0)$.

Proof Applying Δ_ω on both sides of (12), we get

$$\begin{aligned}
&\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}\Delta_\omega C_{\overrightarrow{n}}^{\overrightarrow{a}}(x)\frac{t_1^{n_1}t_2^{n_2}\dots t_r^{n_r}}{n_1!n_2!\dots n_r!} \\
&\quad = \Delta_\omega\bigl[(1+\omega t_1+\omega t_2+\cdots+\omega t_r)^{\frac{x}{\omega}}\exp\bigl(-a_1^\omega t_1-\cdots-a_r^\omega t_r\bigr)\bigr] \\
&\quad = \exp\bigl(-a_1^\omega t_1-\cdots-a_r^\omega t_r\bigr)\Delta_\omega\bigl[(1+\omega t_1+\omega t_2+\cdots+\omega t_r)^{\frac{x}{\omega}}\bigr] \\
&\quad = \sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}\bigl(\omega n_1 C_{n_1-1,\dots,n_r}^{\overrightarrow{a}}(x)+\cdots+\omega n_r C_{n_1,\dots,n_r-1}^{\overrightarrow{a}}(x)\bigr)\frac{t_1^{n_1}t_2^{n_2}\dots t_r^{n_r}}{n_1!n_2!\dots n_r!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t_1^{n_1}t_2^{n_2}\dots t_r^{n_r}}{n_1!n_2!\dots n_r!}$, we get the result. □

Corollary 5.3 *In particular, if* $r=2$,

$$\Delta_\omega C_{n_1,n_2}^{a_1,a_2}(x) = \omega n_1 C_{n_1-1,n_2}^{a_1,a_2} + \omega n_2 C_{n_1,n_2-1}^{a_1,a_2}(x).$$

Theorem 5.4 *The ω-multiple Charlier polynomial* $\{C_{\overrightarrow{n}}^{\overrightarrow{a}}(x)\}_{|n|=0}^{\infty}$ *satisfies the following* $(r+1)$ *order difference equation*:

$$L_{a_1}L_{a_2}\cdots L_{a_r}\bigl[\Delta_\omega C_{\overrightarrow{n}}^{\overrightarrow{a}}(x)\bigr] + \sum_{i=1}^{r}\omega n_i L_{a_1}L_{a_2}\dots L_{a_i-1}L_{a_i+1}\dots L_{a_r}\bigl[C_{\overrightarrow{n}}^{\overrightarrow{a}}(x)\bigr] = 0,$$

where $L_{a_i}[\cdot]$ *is the raising operator* $(i=1,\dots,r)$ *given in Theorem* 5.1..

Proof Applying $L_{a_1}\dots L_{a_r}$ to both sides of (26), we get

$$L_{a_1}\dots L_{a_r}\bigl[\Delta_\omega C_{\overrightarrow{n}}^{\overrightarrow{a}}(x)\bigr] = \sum_{i=1}^{r}\omega n_i L_{a_1}\dots L_{a_r} C_{\overrightarrow{n}-\overrightarrow{e_i}}^{\overrightarrow{a}}(x).$$

Since $L_{a_j}L_{a_k}(y) = L_{a_k}L_{a_j}(y)$ for $a_j, a_k \in \mathbb{R}$, we obtain for $i=1,2,\dots,r$

$$\begin{aligned}
L_{a_1}\dots L_{a_r} &= L_{a_1}\dots L_{a_{i-1}}L_{a_i}L_{a_{i+1}}L_{a_{i+2}}\dots L_{a_r} \\
&= L_{a_1}\dots L_{a_{i-1}}L_{a_{i+1}}L_{a_i}L_{a_{i+2}}\dots L_{a_r} \\
&\ \ \vdots \\
&= L_{a_1}\dots L_{a_{i-1}}L_{a_{i+1}}\dots L_{a_r}L_{a_i}.
\end{aligned}$$

Hence

$$L_{a_1}\dots L_{a_r}\big[\Delta_\omega C_{\overrightarrow{n}}^{\overrightarrow{a}}(x)\big]=\sum_{i=1}^{r}\omega n_i L_{a_1}\dots L_{a_{i-1}}L_{a_{i+1}}\dots L_{a_r}L_{a_i}\big[C_{\overrightarrow{n}-\overrightarrow{e_i}}^{\overrightarrow{a}}(x)\big].$$

Using (24) with $\overrightarrow{n}$ replaced by $\overrightarrow{n}-\overrightarrow{e_i}$, we get the result. □

Corollary 5.5 *The ω-multiple Charlier polynomial $\{C_{n_1,n_2}^{a_1,a_2}(x)\}_{n_1+n_2=0}^{\infty}$ satisfies the following difference equation*:

$$\begin{aligned}&x(x-\omega)\Delta_\omega\nabla_\omega^2 y+x\big(2\omega+a_1^\omega+a_2^\omega-2x\big)\Delta_\omega\nabla_\omega y+\big[\big(a_1^\omega-x\big)\big(a_2^\omega-x\big)-x\omega\big]\Delta_\omega y\\&\quad+(\omega n_1+\omega n_2)x\nabla_\omega y+\big(n_1\big(a_2^\omega-x\big)+n_2\big(a_1^\omega-x\big)\big)\omega y=0.\end{aligned}\tag{27}$$

6 Special cases of the ω-multiple Charlier polynomials

In this section, as an illustrative example of our new definition and its main results, we consider the case $\omega=\frac{3}{2}$ and define $\frac{3}{2}$-multiple Charlier polynomials. The corresponding consequences of our main results for $\frac{3}{2}$-multiple Charlier polynomials are also given.

Taking the weight function as

$$w_i(x)=\frac{a_i^x}{x(\frac{3}{2})^{\frac{2x-3}{3}}\Gamma(\frac{2x}{3})},$$

we can define the $\frac{3}{2}$-multiple Charlier polynomial by the following orthogonality conditions:

$$\sum_{k=0}^{\infty}C_{\overrightarrow{n}}^{\overrightarrow{a}}\left(\frac{3k}{2}\right)\left(\frac{3}{2}\right)^j(-k)_j\frac{a_i^{\frac{3k}{2}}}{(\frac{3k}{2})\Gamma_{\frac{3}{2}}(\frac{3k}{2})}=0,\quad j=0,\dots,n_i-1,i=1,\dots r.$$

Their explicit representation can be written from Theorem 3.1 as

$$\begin{aligned}C_{\overrightarrow{n}}^{\overrightarrow{a}}(x)&=(-a_1)^{\frac{3n_1}{2}}\dots(-a_r)^{\frac{3n_r}{2}}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\dots\sum_{k_r=0}^{n_r}\frac{(-n_1)_{k_1}\dots(-n_r)_{k_r}}{k_1!\dots k_r!}\\&\quad\times\left(-\frac{2x}{3}\right)_{k_1+\dots+k_r}\left(-\frac{1}{a_1}\right)^{\frac{3k_1}{2}}\dots\left(-\frac{1}{a_r}\right)^{\frac{3k_r}{2}}\left(\frac{3}{2}\right)^{k_1+k_2+\dots+k_r}.\end{aligned}$$

The generating function of the $\frac{3}{2}$-multiple Charlier polynomials is written from Theorem 3.3 as

$$\begin{aligned}&\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\dots\sum_{n_r=0}^{\infty}C_{\overrightarrow{n}}^{\overrightarrow{a}}(x)\frac{t_1^{n_1}t_2^{n_2}\dots t_r^{n_r}}{n_1!n_2!\dots n_r!}\\&\quad=\left(1+\frac{3}{2}(t_1+t_2+\dots+t_r)\right)^{\frac{2x}{3}}\exp\big(-a_1^{\frac{3}{2}}t_1-a_2^{\frac{3}{2}}t_2-\dots-a_r^{\frac{3}{2}}t_r\big)\\&\quad\left(\sum_{i=1}^{r}|t_i|<\left(\frac{2}{3}\right)^r\right).\end{aligned}$$

Their recurrence relations can be written from Theorem 4.2 as

$$\big(a_2^{\frac{3}{2}} - a_1^{\frac{3}{2}}\big)C_{n_1,n_2}^{a_1,a_2}(x) = C_{n_1+1,n_2}^{a_1,a_2}(x) - C_{n_1,n_2+1}^{a_1,a_2}(x),$$

$$\begin{aligned} xC_{n_1,n_2}^{a_1,a_2}(x) &= C_{n_1+1,n_2}^{a_1,a_2}(x) + \left(a_1^{\frac{3}{2}} + \frac{3}{2}(n_1+n_2)\right)C_{n_1,n_2}^{a_1,a_2}(x) \\ &\quad + \left(\frac{3}{2}\big(a_1^{\frac{3}{2}}n_1 + a_2^{\frac{3}{2}}n_2\big)\right)C_{n_1,n_2-1}^{a_1,a_2}(x) + \frac{3}{2}n_1a_1^{\frac{3}{2}}\big(a_1^{\frac{3}{2}} - a_2^{\frac{3}{2}}\big)C_{n_1-1,n_2-1}^{a_1,a_2}(x), \end{aligned}$$

and

$$\begin{aligned} xC_{n_1,n_2}^{a_1,a_2}(x) &= C_{n_1+1,n_2}^{a_1,a_2}(x) + \left(a_1^{\frac{3}{2}} + \frac{3}{2}(n_1+n_2)\right)C_{n_1,n_2}^{a_1,a_2}(x) \\ &\quad + \frac{3}{2}a_2^{\frac{3}{2}}n_2C_{n_1,n_2-1}^{a_1,a_2}(x) + \frac{3}{2}a_1^{\frac{3}{2}}n_1C_{n_1-1,n_2}^{a_1,a_2}(x). \end{aligned}$$

The difference equation of the $\frac{3}{2}$-multiple Charlier polynomials for the case $r = 2$ is

$$\begin{aligned} &x\left(x - \frac{3}{2}\right)\Delta_{\frac{3}{2}}\nabla_{\frac{3}{2}}^2 y + x\big(3 + a_1^{\frac{3}{2}} + a_2^{\frac{3}{2}} - 2x\big)\Delta_{\frac{3}{2}}\nabla_{\frac{3}{2}}y + \left[\big(a_1^{\frac{3}{2}} - x\big)\big(a_2^{\frac{3}{2}} - x\big) - \frac{3x}{2}\right]\Delta_{\frac{3}{2}}y \\ &\quad + \frac{3}{2}(n_1+n_2)x\nabla_{\frac{3}{2}}y + \big(n_1\big(a_2^{\frac{3}{2}} - x\big) + n_2\big(a_1^{\frac{3}{2}} - x\big)\big)\frac{3y}{2} = 0. \end{aligned}$$

7 Concluding remarks and observations

The multiple Charlier polynomials $C_{\vec{n}}^{\vec{a}}$ were introduced in [1]. The raising operators and Rodrigues formula were obtained. The explicit representation, recurrence relation and generating function were investigated in [1] and [7]. Also an $(r + 1)$th order difference equation was investigated in [5].

In this paper, we define the ω-multiple Charlier polynomials by the orthogonality condition (4). We obtain the raising relation, the Rodrigues formula, an explicit representation, a recurrence relation and a generating function. Also an $(r+1)$th order difference equation was obtained. All our results coincide in the case $\omega = 1$ with the corresponding versions of the multiple Charlier polynomials. For instance, this is so in the case $\omega = 1$.

The raising relation (5) coincides with the raising operators given in ([1], pp. 30). That is

$$\frac{a_i}{w_i(x)}\nabla\big(w_i(x)C_{\vec{n}}^{\vec{a}}(x)\big) = -C_{\vec{n}+\vec{e_i}}(x), \quad i = 1,\dots,r,$$

where

$$w_i(x) = \frac{a_i^x}{\Gamma(x+1)}, \quad x \in \mathbb{R}^+, i = 1,\dots,r.$$

The Rodrigues formula (9) coincides with the Rodrigues formula given in ([1], pp. 31). That is,

$$C_{\vec{n}}^{\vec{a}}(x) = \left[\prod_{j=1}^{r}(-a_j)^{n_j}\right]\Gamma(x+1)\left[\prod_{i=1}^{r}\left(\frac{1}{a_i^x}\nabla^{n_i}a_i^x\right)\right]\left(\frac{1}{\Gamma(x+1)}\right).$$

The explicit representation (10) coincides with the explicit representation given in ([7], pp. 824). That is,

$$C_{\vec{n}}^{\vec{a}}(x) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} (-n_1)_{k_1} \cdots (-n_r)_{k_r} (-x)_{k_1+\cdots+k_r} \frac{(-a_1)^{n_1-k_1} \ldots (-a_r)^{n_r-k_r}}{k_1! \ldots k_r!}.$$

The recurrence relation ($r = 2$) (16) coincides with the recurrence relation ($r = 2$) given in ([1], pp. 32). That is,

$$\begin{aligned} xC_{n_1,n_2}^{a_1,a_2}(x) &= C_{n_1+1,n_2}^{a_1,a_2}(x) + (a_1 + n_1 + n_2) C_{n_1,n_2}^{a_1,a_2}(x) \\ &\quad + (a_1 n_1 + a_2 n_2) C_{n_1,n_2-1}^{a_1,a_2}(x) + n_1 a_1 (a_1 - a_2) C_{n_1-1,n_2-1}^{a_1,a_2}. \end{aligned}$$

The generating function (12) coincides with the generating function given in ([7], pp. 825). That is,

$$\begin{aligned} &\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} C_{\vec{n}}^{\vec{a}}(x) \frac{t_1^{n_1} t_2^{n_2} \ldots t_r^{n_r}}{n_1! n_2! \ldots n_r!} \\ &\quad = (1 + t_1 + t_2 + \cdots + t_r)^x \exp(-a_1 t_1 - a_2 t_2 - \cdots - a_r t_r) \\ &\qquad \left(\sum_{j=1}^{r} |t_j| < 1 \right). \end{aligned}$$

The third order difference equation (27) coincides with the third order difference equation in ([5], pp. 137). That is,

$$\begin{aligned} &x(x-1)\Delta\nabla^2 y + x(2 + a_1 + a_2 - 2x)\Delta\nabla y + \big[(a_1 - x)(a_2 - x) - x\big]\Delta y \\ &\quad + (n_1 + n_2)x\nabla y + \big[n_1(a_2 - x) + n_2(a_1 - x)\big]y = 0. \end{aligned}$$

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References

1. Arvesu, J., Coussement, J., Van Assche, W.: Some discrete multiple orthogonal polynomials. J. Comput. Appl. Math. **153**, 19–45 (2003)
2. Arvesu, J., Ramirez-Aberasturis, A.M.: On the q-Charlier multiple orthogonal polynomials. SIGMA **11**, 026 (2015)
3. Cheikh, B.Y., Zaghouani, A.: Some discrete d-orthogonal polynomial sets. J. Comput. Appl. Math. **156**, 253–263 (2003)
4. Erdelyi, A.: Higher Transcendental Functions, Vol. I. McGraw-Hill, New York (1953)
5. Lee, D.W.: Difference equations for discrete classical multiple orthogonal polynomials. J. Approx. Theory **150**, 132–152 (2008)
6. Mubeen, S., Rehman, A.: A note on k-gamma function and Pochhammer k-symbol. J. Inform. Math. Sci. **6**, 93–107 (2014)
7. Ndayiragije, F., Van Assche, W.: Asymptotics for the ratio and the zeros of multiple Charlier polynomials. J. Approx. Theory **164**, 823–840 (2012)
8. Sadjang, N.P., Mboutngam, S.: On multiple Δ_ω- Appell polynomials. arXiv:1806.00032v1 [math.CA]
9. Van Assche, W.: Difference equations for multiple Charlier and Meixner polynomials. In: Proceedings of the Sixth International Conference on Difference Equations, Augsburg, Germany, pp. 549–557 (2001)
10. Van Assche, W.: Nearest neighbor recurrence relations for multiple orthogonal polynomials. J. Approx. Theory **163**, 1427–1448 (2011)
11. Zorlu, O.S., Elidemir, İ.: On the ω-multiple Meixner polynomials of the first kind. J. Inequal. Appl. **2020**, 167 (2020)