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An algorithm for approximating a common solution of some nonlinear problems in Banach spaces with an application

Abdulmalik U. Bello[1,2*] and Monday O. Nnakwe[1,3]

*Correspondence:
uabdulmalik@aust.edu.ng
[1] African University of Science and Technology, Km 10 Airport Road, Abuja, Nigeria
[2] Federal University Dutsinma, Katsina, Nigeria
Full list of author information is available at the end of the article

Abstract

In this paper, we construct a new Halpern-type subgradient extragradient iterative algorithm. The sequence generated by this algorithm converges strongly to a common solution of a variational inequality, an equilibrium problem, and a J-fixed point of a continuous J-pseudo-contractive map in a uniformly smooth and two-uniformly convex real Banach space. Also, the theorem is applied to approximate a common solution of a variational inequality, an equilibrium problem, and a convex minimization problem. Moreover, a numerical example is given to illustrate the implementability of our algorithm. Finally, the theorem proved complements, improves, and unifies some related recent results in the literature.

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Keywords: J-pseudo-contractive maps; J-fixed points; Variational inequality; Equilibrium problems

1 Introduction

Let $\mathcal{Q}^*$ be the dual space of a real normed linear space $\mathcal{Q}$ and $\mathcal{D}$ be a nonempty, closed, and convex subset of $\mathcal{Q}$. In this paper, we study the classical variational inequality of Fichera [1] and Stampacchia [2], the equilibrium problem of Blum and Otelli [3], and some fixed point problems.

Let $A : \mathcal{D} \to \mathcal{Q}^*$ be a given map. A *variational inequality problem* is the following:

$$\text{find} \quad x \in \mathcal{D} \quad \text{such that} \quad \langle y - x, Ax \rangle \geq 0 \quad \text{for all } y \in \mathcal{D}. \tag{1.1}$$

Variational inequality was first developed to solve equilibrium problems, precisely the Signorini problem posed by Antonio [4] in the year 1959. This problem was later solved by Fichera [1] in the year 1963. In 1964, Stampacchia [2] studied the regularity problem for partial differential equations and thereby coined the name "variational inequality", stating nothing but the principle of complementary virtual work in its inequality form. Variational inequality has been found to have numerous applications in many areas of sciences; see, for example, [5–7].

For earlier and more recent results on the existence of solutions and iterative methods for solving variational inequalities; see, for example, [8–19].

Let $\Theta : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ be a bifunctional. An *equilibrium problem* is to find

$$x \in \mathcal{D} \quad \text{such that} \quad \Theta(x,y) \geq 0 \quad \text{for all } y \in \mathcal{D}. \tag{1.2}$$

Numerous problems in physics, optimization, and economics reduce to a problem of finding solutions of inequality (1.2). Some methods have been proposed to solve equilibrium problems in Hilbert spaces and more general Banach spaces; see, for example, [20–25].

We remark that the following conditions will be needed in solving the equilibrium problem (1.2):

(A_1) $\Theta(x,x) = 0$ for all $x \in \mathcal{D}$;

(A_2) Θ is monotone, i.e., $\Theta(x,y) + \Theta(y,x) \leq 0$ for all $x,y \in \mathcal{D}$;

(A_3) $\limsup_{t\downarrow 0} \Theta(x + t(z-x), y) \leq \Theta(x,y)$ for all $x,y,z \in \mathcal{D}$;

(A_4) for all $x \in \mathcal{D}$, $\Theta(x,\cdot)$ is convex and lower semi-continuous.

A map $A : \mathcal{Q} \to \mathcal{Q}$ is called *accretive* if, for each $x,y \in \mathcal{Q}$, there exists $j(x-y) \in J(x-y)$ such that

$$\bigl\langle Ax - Ay, j(x-y)\bigr\rangle \geq 0, \tag{1.3}$$

where $J : \mathcal{Q} \to 2^{\mathcal{Q}^*}$ is the normalized duality map. In Hilbert spaces, accretive maps are called monotone.

Accretive maps were introduced independently in the year 1967 by Browder [9] and Kato [26]. Interest in this class of maps stems mainly from their firm connection with the existence theory for nonlinear equations of evolution in Banach spaces, i.e., equations of the form

$$\begin{cases} x'(s) + Ax(s) = v(x(s)), & s \geq 0, \\ x(0) = x_0. \end{cases} \tag{1.4}$$

At equilibrium state, and setting $v \equiv 0$ in equation (1.4), we obtain the following equation:

$$Ax = 0. \tag{1.5}$$

In many cases, where the map A is accretive, solutions of equation (1.5) represent the equilibrium state of the system described by equation (1.4).

For solving equation (1.5), Browder [9] in the year 1967 introduced a self-map $S := I - A$ on a real Banach space, which he called a *pseudo-contractive map.* Approximating zeros of accretive maps is equivalent to approximating fixed points of pseudo-contractive maps, assuming existence of such zeros. For earlier and more recent results on the approximation of fixed points of pseudo-contractive maps, the reader may consult any of the following: [27–36].

A map $A : \mathcal{Q} \to \mathcal{Q}^*$ is called *monotone* if, for each $x,y \in \mathcal{Q}$, the following inequality holds:

$$\langle x - y, Ax - Ay\rangle \geq 0.$$

The sub-differential of a convex and proper function h, defined on a real Banach space and denoted by ∂h, is a monotone map, and for each x in the domain of h, $0 \in \partial h(x)$ if and only if x minimizes the function h; see, for example, [37]. In particular, setting the sub-differential equivalently as A, we have $0 \in Ax$, which reduces to $Ax = 0$ for the case where $\partial h \equiv A$ is single-valued. Therefore, approximating zeros of such monotone maps is equivalent to finding a minimizer of some convex function.

It is obvious that the fixed point technique introduced by Browder in the year 1967 for approximating zeros of accretive maps is not applicable in this case, where A from a real Banach space to its dual space is monotone.

Hence, there is the need to develop techniques for approximating zeros of monotone maps.

To approximate zeros of monotone maps, Zegeye [38] in the year 2008 introduced a map $S := J - A$ from a real Banach space to its dual space. He called the map *semi-pseudo mapping*, where $A : \mathcal{Q} \to 2^{\mathcal{Q}^*}$ is a monotone map. Also, in the year 2016, Chidume and Idu [39] studied this class of maps and called it *J-pseudo-contractive*.

An element x in $\mathcal{D}$ is called a *semi-fixed point* (*J-fixed point*) of S from $\mathcal{D}$ to $\mathcal{Q}^*$ if

$$Sx = Jx, \tag{1.6}$$

where J is the normalized duality map and single-valued in this case; see, for example, [38, 39].

Approximating zeros of these monotone maps is equivalent to approximating J-fixed points of J-pseudo-contractive maps, assuming the existence of such zeros, which is also equivalent to finding a minimizer of some convex functions.

We remark that in real Hilbert spaces, and also smooth and strictly convex real Banach spaces, the notion of J-fixed points coincides with the classical definition of fixed points. However, if the space is not strictly convex, J may fail to be one-to-one. Thus, the inverse of J may not exist. For more recent works on J-fixed points, see, for example, [40–48].

In 2014, Zegeye and Shazard [49] studied the problem of finding a common solution in the set of fixed points of a Lipschitz pseudo-contractive map S and solution sets of a variational inequality for a γ-inverse strongly monotone map A in a real Hilbert space H by considering the following iterative algorithm:

$$\begin{cases} x_o \in C \subset H, \\ y_n = (1-\beta_n)x_n + \beta_n S x_n, \\ x_{n+1} = P_C[(1-\alpha_n)(\delta_n x_n + \theta_n S y_n + \gamma_n P_C[I - \gamma_n A]x_n)], \end{cases} \tag{1.7}$$

where P_C is the metric projection from H onto C and $\{\delta_n\}$, $\{\gamma_n\}$, $\{\theta_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $]0,1[$ satisfying appropriate conditions. They proved that the sequence generated by the algorithm converges strongly to an element in the solution set of the problem.

Also, in the year 2015, Alghamdi *et al.* [17] studied a Halpern-type extragradient method, to approximate a common solution of a variational inequality and a fixed point problem of a continuous pseudo-contractive map in a real Hilbert space. They proved the following theorem.

Theorem 1.1 *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $S : C \to C$ be a continuous pseudo-contractive map. Let $A : C \to H$ be an L-Lipschitz* *monotone map. Assume that* $\mathcal{F} := F(S) \cap VI(A,C) \neq \emptyset$, *where* $F(S)$ *is a fixed point set of* S *and* $VI(A,C)$ *is the solution set of a variational inequality. Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_o, \quad u \in C, \\ z_n = P_C[x_n - \gamma_n A x_n], \\ x_{n+1} := \alpha_n u + (1-\alpha_n)(a_n x_n + b_n K^S_{r_n} x_n + c_n P_C[x_n - \gamma_n A z_n]), \end{cases}$$

where $\{\gamma_n\} \subset [a,b[\subset]0,\frac{1}{L}[$, *and* $\{a_n\},\{b_n\},\{c_n\} \subset (a,b) \subset]0,1[$, $\{\alpha_n\} \subset]0,c[\subset]0,1[$ *satisfying the following conditions*: (*i*) $a_n + b_n + c_n = 1$, (*ii*) $\lim \alpha_n = 0$, $\sum \alpha_n = \infty$, *and* $K^S_{r_n}$ *is a resolvent map for* S *from* H *unto* C. *Then* $\{x_n\}$ *converges strongly to the point* x^* *of* $\mathcal{F}$ *nearest to* u.

Motivated by the results in [17, 39, 49], we present in this paper a Halpern-type subgradient-extragradient algorithm for which the sequence generated by the algorithm converges strongly to a common solution of a *variational inequality, an equilibrium problem, and J-fixed points of a continuous J-pseudo-contractive map* in a uniformly smooth and two-uniformly convex real Banach space. Also, the theorem is applied to approximate a common solution of a variational inequality, an equilibrium problem, and a convex minimization problem. Moreover, a numerical example is given to illustrate the implementability of our algorithm. Finally, the theorem proved complements, improves, and unifies some related recent results in the literature.

2 Preliminaries

Let $\mathcal{Q}^*$ be the dual space of a real normed linear space $\mathcal{Q}$ and $\mathcal{D}$ be a nonempty, closed, and convex subset of $\mathcal{Q}$. We denote $x_n \rightharpoonup x^*$ and $x_n \to x^*$ to indicate that the sequence $\{x_n\}$ converges *weakly* to x^* and converges *strongly* to x^*, respectively. Also, $VI(A,\mathcal{D})$, $EP(\Theta)$, and $F_J(S)$ denote the set of solutions of variational inequalities, the set of solutions of equilibrium problems, and the set of J-fixed points of S, respectively.

Let $\mathcal{Q}$ be a smooth real normed linear space and $\phi : \mathcal{Q} \times \mathcal{Q} \to \mathbb{R}$ be a map defined by

$$\phi(x,y) := \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \text{for all } x,y \in \mathcal{Q}, \tag{2.1}$$

where $J : \mathcal{Q} \to 2^{\mathcal{Q}^*}$ is the normalized duality map defined by

$$J(x) := \{x^* \in \mathcal{Q}^* : \langle x, x^*\rangle = \|x\|\|x^*\|, \|x\| = \|x^*\|\}.$$

The map ϕ was introduced by Alber [7] and has been studied extensively by a host of other authors.

For any $x,y,z \in \mathcal{Q}$ and $\alpha \in]0,1[$, the following properties are true:

(P_1) $(\|x\| - \|y\|)^2 \le \phi(x,y) \le (\|x\| + \|y\|)^2$,

(P_2) $\phi(x, J^{-1}(\alpha Jy + (1-\alpha)Jz)) \le \alpha\phi(x,y) + (1-\alpha)\phi(x,z)$,

(P_3) $\phi(x,z) = \phi(x,y) + \phi(y,z) + 2\langle y - x, Jz - Jy\rangle$,

(P_4) $\phi(x,y) \le \|x\|\|Jx - Jy\| + \|y\|\|x - y\|$.

Definition 2.1 ([7]) Let $\mathcal{Q}$ be a smooth, strictly convex, and reflexive real Banach space, and let $\mathcal{D}$ be a nonempty, closed, and convex subset of $\mathcal{Q}$. The map $\Pi_{\mathcal{D}} : \mathcal{Q} \to \mathcal{D}$ defined by $\tilde{x} = \Pi_{\mathcal{D}}(x) \in \mathcal{D}$ such that $\phi(\tilde{x}, x) = \inf_{y\in\mathcal{D}} \phi(y, x)$ is called the *generalized projection* of $\mathcal{Q}$ onto $\mathcal{D}$.

Definition 2.2 ([9]) A map $S : \mathcal{Q} \to \mathcal{Q}$ is called a *pseudo-contractive map* if, for all $x, y \in \mathcal{Q}$, there exists $j(x - y)$ in $J(x - y)$ such that

$$\bigl\langle Sx - Sy, j(x - y)\bigr\rangle \le \|x - y\|^2,$$

where J is the normalized duality map on $\mathcal{Q}$.

Definition 2.3 ([39]) A map $S : \mathcal{Q} \to \mathcal{Q}^*$ is called a *J-pseudo-contractive map* if, for all $x, y \in \mathcal{Q}$, the following inequality holds:

$$\langle x - y, Sx - Sy\rangle \le \langle x - y, Jx - Jy\rangle.$$

In real Hilbert spaces, the of notion pseudo-contractive maps coincides with that of J-pseudo-contractive maps.

Definition 2.4 ([37]) The sub-differential of a convex function h is a map $\partial h : \mathcal{Q} \to 2^{\mathcal{Q}^*}$, defined by

$$\partial h(x) = \bigl\{x^* \in \mathcal{Q}^* : h(y) - h(x) \ge \bigl\langle y - x, x^*\bigr\rangle, \forall y \in \mathcal{Q}\bigr\}.$$

Lemma 2.5 ([7]) *Let $\mathcal{D}$ be a nonempty, closed, and convex subset of a smooth, strictly convex, and reflexive real Banach space $\mathcal{Q}$. Then*:

1. *If $x \in \mathcal{Q}$ and $y \in \mathcal{D}$, then $\tilde{x} = \Pi_{\mathcal{D}}x$ if and only if $\langle \tilde{x} - y, Jx - J\tilde{x}\rangle \ge 0$ for all $y \in \mathcal{D}$, where $\Pi_{\mathcal{D}}$ is a generalized projection of $\mathcal{Q}$ onto $\mathcal{D}$ in Definition 2.1*;
2. $\phi(y, \tilde{x}) + \phi(\tilde{x}, x) \le \phi(y, x)$ for all $x \in \mathcal{Q}$, $y \in \mathcal{D}$.

Lemma 2.6 ([50]) *Let $\mathcal{Q}$ be a two-uniformly convex and smooth real Banach space. Then there exists a positive constant c such that*

$$\phi(x, y) \ge c\|x - y\|^2, \quad \forall x, y \in \mathcal{Q}.$$

Remark 1 Without loss of generality, we may assume $c \in\,]0, 1[$.

Lemma 2.7 ([51]) *Let $\{x_n\}$ and $\{y_n\}$ be two sequences of a uniformly convex and smooth real Banach space. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Remark 2 Using Condition (P_4), the converse of Lemma 2.7 is also true whenever $\{x_n\}$ and $\{y_n\}$ are bounded.

Lemma 2.8 ([52]) *Let $\mathcal{D}$ be a nonempty, closed, and convex subset of a reflexive space $\mathcal{Q}$ and A be a monotone and hemi-continuous map from $\mathcal{D}$ into $\mathcal{Q}^*$. Let $\mathcal{R} \subset \mathcal{Q} \times \mathcal{Q}^*$ be a* *map defined by*

$$\mathcal{R}x = \begin{cases} Ax + N_{\mathcal{D}}(x), & \text{if } x \in \mathcal{D}, \\ \emptyset, & \text{if } x \notin \mathcal{D}, \end{cases}$$

where $N_{\mathcal{D}}(x)$ *is defined as follows*: $N_{\mathcal{D}}(x) = \{w^* \in \mathcal{Q}^* : \langle x - z, w^* \rangle \geq 0, \forall z \in \mathcal{D}\}$. *Then* $\mathcal{R}$ *is maximal monotone and* $\mathcal{R}^{-1}(0) = VI(A, \mathcal{D})$.

Lemma 2.9 ([53]) *Let* $\{a_n\}$ *be a sequence of nonnegative numbers satisfying the condition*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \beta_n, \quad n \geq 0,$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences of real numbers such that* (i) $\{\alpha_n\} \subset [0,1]$ *and* $\sum \alpha_n = \infty$; (ii) $\limsup \beta_n \leq 0$. *Then* $\lim a_n = 0$.

Lemma 2.10 ([54]) *Let* $\{a_n\}$ *be a sequence of real numbers such that there exists a subsequence* $\{a_{m_j}\}$ *of* $\{a_n\}$ *such that* $a_{m_j} < a_{m_j+1}$ *for all* $j \in \mathbb{N}$. *Then there exists a nondecreasing sequence* $\{n_k\}$ *of* $\mathbb{N}$ *such that* $\lim_{k\to\infty} n_k = \infty$, *and the following properties are satisfied by all* (*sufficiently large*) *number* $k \in \mathbb{N}$:

$$a_{n_k} \leq a_{n_k+1} \quad \text{and} \quad a_k \leq a_{n_k+1}.$$

In fact, n_k *is the largest number* n *in the set* $\{1, \dots, k\}$ *such that* $a_n < a_{n+1}$ *holds*.

Lemma 2.11 ([55]) *Let* $\mathcal{Q}^*$ *be the dual space of a reflexive, strictly convex, and smooth Banach space* $\mathcal{Q}$. *Then*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^* \rangle \leq V(x, x^* + y^*) \quad \text{for all } x \in \mathcal{Q} \text{ and } x^*, y^* \in \mathcal{Q}^*.$$

Lemma 2.12 ([47]) *Let* $\mathcal{Q}^*$ *be the dual space of a uniformly smooth and strictly convex real Banach space* $\mathcal{Q}$. *Let* $\mathcal{D}$ *be a nonempty, closed, and convex subset of* $\mathcal{Q}$ *and* $S : \mathcal{D} \to \mathcal{Q}^*$ *be a continuous* J*-pseudo-contractive map. Let* $r > 0$ *and* $x \in \mathcal{Q}$. *Then the following conditions hold*:

1. *There exists* $z \in \mathcal{D}$ *such that* $\langle w - z, Sz \rangle - \frac{1}{r}\langle w - z, (1+r)Jz - Jx \rangle \leq 0$, $\forall w \in \mathcal{D}$.
2. *Define a map* $T_r^S : \mathcal{Q} \to \mathcal{D}$ *by*

$$T_r^S(x) := \left\{ z \in \mathcal{D} : \langle w - z, Sz \rangle - \frac{1}{r}\langle w - z, (1+r)Jz - Jx \rangle \leq 0, \forall w \in \mathcal{D} \right\}, \quad x \in \mathcal{Q}.$$

Then:

(a) T_r^S *is single-valued*;

(b) T_r^S *is a firmly nonexpansive-type map, i.e.*,

$$\langle T_r^S x - T_r^S y, JT_r^S x - JT_r^S y \rangle \leq \langle T_r^S x - T_r^S y, Jx - Jy \rangle, \quad \forall x, y \in \mathcal{Q};$$

(c) $F(T_r^S) = F_J(S)$, *where* $F(T_r^S)$ *denotes the fixed point set of the map* T_r^S;

(d) $F_J(S)$ *is closed and convex*;

(e) $\phi(q, T_r^S x) + \phi(T_r^S x, x) \leq \phi(q, x)$, $\forall q \in F(T_r^S)$, $x \in \mathcal{Q}$.

3 Main results

In the sequel, $c \in]0,1[$ is the constant appearing in Lemma 2.6.

Theorem 3.1 *Let $\mathcal{Q}^*$ be the dual space of a uniformly smooth and two-uniformly convex real Banach space $\mathcal{Q}$. Let $\mathcal{D}$ be a nonempty, closed, and convex subset of $\mathcal{Q}$. Let $A : \mathcal{D} \to \mathcal{Q}^*$ be a monotone and L-Lipschitz map, $\Theta : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ be a bi-functional satisfying conditions A_1 to A_4, and $S : \mathcal{D} \to \mathcal{Q}^*$ be a continuous J-pseudo-contractive map with $\Omega := F_J(S) \cap VI(A,\mathcal{D}) \cap EP(\Theta) \neq \emptyset$. Let the sequence $\{x_n\}$ be generated by*

$$\begin{cases} x_0 \in \mathcal{D}, \\ z_n = \Pi_{\mathcal{D}} J^{-1}(Jx_n - \tau A x_n), \\ T_n = \{w \in \mathcal{Q} : \langle w - z_n, Jx_n - \tau A x_n - Jz_n \rangle \leq 0\}, \\ x_{n+1} = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)[\beta J v_n + (1-\beta) J w_n]), \end{cases} \tag{3.1}$$

where $v_n = K^{\Theta}_{r_n} T^S_{r_n} x_n$ with $K^{\Theta}_{r_n}$ and $T^S_{r_n}$ as the resolvent maps for Θ and S, respectively, $\{r_n\} \subset [a,\infty[$ for some $a > 0$, $w_n = \Pi_{T_n} J^{-1}(Jx_n - \tau A z_n)$, $\tau \in]0,1[$ with $\tau < \frac{c}{L}$ and $\{\alpha_n\} \subset]0,1[$ with $\lim \alpha_n = 0$ and $\sum \alpha_n = \infty$. Then the sequence $\{x_n\}$ converges strongly to a point $\Pi_\Omega x_0$.

Proof We divide the proof into two steps.

Step 1. We show that $\{x_n\}$ is bounded.

Let $p \in \Omega$. Setting $t_n := J^{-1}(Jx_n - \tau A z_n)$ and $y_n = T^S_{r_n} x_n$, then $v_n = K^{\Theta}_{r_n} y_n$.

Applying the definition of A, Lemma 2.5, and property P_3, we compute as follows:

$$\begin{aligned} \phi(p,w_n) &\leq \phi(p,t_n) - \phi(w_n,t_n) \\ &= \|p\|^2 - 2\langle p, Jx_n - \tau A z_n\rangle - \|w_n\|^2 + 2\langle w_n, Jx_n - \tau A z_n\rangle \\ &= \phi(p,x_n) - \phi(w_n,x_n) + 2\tau\langle p - w_n, A z_n\rangle \\ &= \phi(p,x_n) - \phi(w_n,x_n) + 2\tau\big[\langle p - z_n, Az_n - Ap\rangle + \langle p - z_n, Ap\rangle + \langle z_n - w_n, Az_n\rangle\big] \\ &\leq \phi(p,x_n) - \phi(w_n,x_n) + 2\tau\langle z_n - w_n, Az_n\rangle \\ &= \phi(p,x_n) - \big[\phi(w_n,z_n) + \phi(z_n,x_n) + 2\langle z_n - w_n, Jx_n - Jz_n\rangle\big] + 2\tau\langle z_n - w_n, Az_n\rangle \\ &= \phi(p,x_n) - \phi(w_n,z_n) - \phi(z_n,x_n) + 2\langle w_n - z_n, Jx_n - \tau A z_n - Jz_n\rangle. \end{aligned} \tag{3.2}$$

Since $w_n \in T_n$, we have that $\langle w_n - z_n, Jx_n - \tau A x_n - Jz_n\rangle \leq 0$. Thus,

$$\begin{aligned} \langle w_n - z_n, Jx_n - \tau A z_n - Jz_n\rangle &= \langle w_n - z_n, Jx_n - \tau A x_n - Jz_n\rangle + \tau\langle w_n - z_n, Ax_n - Az_n\rangle \\ &\leq \tau\langle w_n - z_n, Ax_n - Az_n\rangle. \end{aligned} \tag{3.3}$$

Substituting inequality (3.3) in inequality (3.2) and using the Lipschitz condition of A and Lemma 2.6, we have

$$\begin{aligned} \phi(p,w_n) &\leq \phi(p,x_n) - \phi(w_n,z_n) - \phi(z_n,x_n) + 2\tau\langle w_n - z_n, Ax_n - Az_n\rangle \\ &\leq \phi(p,x_n) - \phi(w_n,z_n) - \phi(z_n,x_n) + 2\tau L\|w_n - z_n\|\|x_n - z_n\| \end{aligned}$$

$$\begin{aligned} &\leq \phi(p,x_n) - \phi(w_n,z_n) - \phi(z_n,x_n) + \frac{\tau L}{c}\big(\phi(w_n,z_n) + \phi(z_n,x_n)\big) \\ &= \phi(p,x_n) - \left(1 - \frac{\tau L}{c}\right)\big(\phi(w_n,z_n) + \phi(z_n,x_n)\big). \end{aligned} \tag{3.4}$$

Also, by a result of Blum and Otelli [3], and also by Lemma 2.12(2(e)), we have

$$\begin{aligned} \phi(p,v_n) &\leq \phi(p,y_n) - \phi(v_n,y_n) \\ &\leq \phi(p,x_n) - \phi(y_n,x_n) - \phi(v_n,y_n). \end{aligned} \tag{3.5}$$

Now, using the recursion formula (3.1), property P_2, inequalities (3.4) and (3.5), we have

$$\begin{aligned} \phi(p,x_{n+1}) &= \phi\big(p,J^{-1}\big(\alpha_n Jx_0 + (1-\alpha_n)\big[\beta Jv_n + (1-\beta)Jw_n\big]\big)\big) \\ &\leq \alpha_n\phi(p,x_0) + (1-\alpha_n)\big[\beta\phi(p,v_n) + (1-\beta)\phi(p,w_n)\big] \\ &\leq \alpha_n\phi(p,x_0) + (1-\alpha_n)\beta\big[\phi(p,x_n) - \phi(y_n,x_n) - \phi(v_n,y_n)\big] \\ &\quad + (1-\alpha_n)(1-\beta)\left[\phi(p,x_n) - \left(1 - \frac{\tau L}{c}\right)\big(\phi(w_n,z_n) + \phi(z_n,x_n)\big)\right] \\ &\leq \alpha_n\phi(p,x_0) + (1-\alpha_n)\phi(p,x_n) - (1-\alpha_n)\beta\big[\phi(y_n,x_n) + \phi(v_n,y_n)\big] \\ &\quad - (1-\alpha_n)(1-\beta)\left(1 - \frac{\tau L}{c}\right)\big[\phi(w_n,z_n) + \phi(z_n,x_n)\big] \qquad (3.6) \\ &\leq \alpha_n\phi(p,x_0) + (1-\alpha_n)\phi(p,x_n) \\ &\leq \max\big\{\phi(p,x_0),\phi(p,x_n)\big\}. \qquad (3.7) \end{aligned}$$

Hence, by induction, we have that $\phi(p,x_n) \leq \phi(p,x_0)$, $\forall n \geq 0$, which implies that $\{\phi(p,x_n)\}$ is bounded. By property P_1, $\{x_n\}$ is bounded. Consequently, $\{w_n\}$, $\{v_n\}$, $\{y_n\}$, and $\{z_n\}$ are bounded.

Step 2. We show that $\{x_n\}$ converges strongly to a point $u := \Pi_\Omega x_0$.

Two cases arise.

Case 1. There exists $N_0 \in \mathbb{N}$ such that $\phi(u,x_n) \geq \phi(u,x_{n+1})$, $\forall n \geq N_0$.

This implies that $\lim_{n\to\infty}\phi(u,x_n)$ exists.

Claim 1. $\lim_{n\to\infty}\|y_n - x_n\| = \lim_{n\to\infty}\|v_n - y_n\| = \lim_{n\to\infty}\|w_n - z_n\| = \lim_{n\to\infty}\|z_n - x_n\| = \lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Setting $\sigma := (1-\beta)(1-\alpha_n)(1-\frac{\tau L}{c}) > 0$ and $\xi := (1-\alpha_n)\beta > 0$, from inequality (3.6), we have

$$\phi(y_n,x_n) + \phi(v_n,y_n) \leq \xi^{-1}\big(\phi(u,x_n) - \phi(u,x_{n+1}) + \alpha_n\phi(u,x_0)\big), \tag{3.8}$$

$$\phi(w_n,z_n) + \phi(z_n,x_n) \leq \sigma^{-1}\big(\phi(u,x_n) - \phi(u,x_{n+1}) + \alpha_n\phi(u,x_0)\big). \tag{3.9}$$

Using the condition on $\{\alpha_n\}$ and taking limit on both sides of inequalities (3.8) and (3.9), we have

$$\lim_{n\to\infty}\phi(y_n,x_n) = \lim_{n\to\infty}\phi(v_n,y_n) = \lim_{n\to\infty}\phi(w_n,z_n) = \lim_{n\to\infty}\phi(z_n,x_n) = 0. \tag{3.10}$$

By Lemma 2.7 and equation (3.10), we obtain that

$$\lim_{n\to\infty}\|y_n - x_n\| = \lim_{n\to\infty}\|v_n - y_n\| = \lim_{n\to\infty}\|w_n - z_n\| = \lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{3.11}$$

Combining equation (3.11), recursion formula (3.1) and applying the triangle inequality, we have

$$\lim_{n\to\infty}\|v_n - x_n\| = \lim_{n\to\infty}\|w_n - x_n\| = \lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.12}$$

Claim 2. We show that $\Omega_w(x_n) \subset \Omega$; $\Omega_\omega(x_n)$ is the set of weak sub-sequential limits of $\{x_n\}$.

First, we show that $\Omega_w(x_n) \subset VI(A,\mathcal{D})$.

Let $x^* \in \Omega_\omega(x_n)$ and $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup x^*$ as $j \to \infty$.

Applying the weak convergence of $\{x_n\}$ and equation (3.11), we have that $y_{n_j} \rightharpoonup x^*$ as $j \to \infty$.

Let

$$\mathcal{R}x = \begin{cases} Ax + N_{\mathcal{D}}(x) & \text{if } x \in \mathcal{D}, \\ \emptyset & \text{if } x \notin \mathcal{D}, \end{cases}$$

be as defined in Lemma 2.8. Then $\mathcal{R}$ is maximal monotone, and $0 \in \mathcal{R}x \iff x \in VI(A,\mathcal{D})$. It is known that if $\mathcal{R}$ is maximal monotone, then given $(x,v^*) \in \mathcal{Q} \times \mathcal{Q}^*$ such that if $\langle x - y, v^* - y^*\rangle \geq 0$, $\forall (y,y^*) \in G(\mathcal{R})$, where $G(\mathcal{R})$ denotes the graph of $\mathcal{R}$, one has $v^* \in \mathcal{R}x$.

Claim. $(x^*,0) \in G(\mathcal{R})$.

Let $(x,z^*) \in G(\mathcal{R})$. It suffices to show that $\langle x - x^*, z^*\rangle \geq 0$.

Now, $(x,z^*) \in G(\mathcal{R}) \implies z^* \in \mathcal{R}x = Ax + N_{\mathcal{D}}(x), \Rightarrow z^* - Ax \in N_{\mathcal{D}}(x)$.

This implies that $\langle x - t, z^* - Ax\rangle \geq 0$, $\forall t \in \mathcal{D}$. In particular, $\langle x - z_n, z^* - Ax\rangle \geq 0$.

But $z_n = \Pi_{\mathcal{D}} J^{-1}(Jx_n - \tau Ax_n)$, $\forall n \geq 0$, and $x \in \mathcal{D}$. By the characterization of the generalized projection, we have

$$\langle z_n - x, Jx_n - \tau Ax_n - Jz_n\rangle \geq 0.$$

This implies that

$$\left\langle x - z_n, \frac{Jz_n - Jx_n}{\tau} + Ax_n \right\rangle \geq 0, \quad \forall n \geq 0. \tag{3.13}$$

Using inequality (3.13) and the fact that $\langle x - z_n, z^* - Ax\rangle \geq 0$, we get that

$$\begin{aligned} \langle x - z_{n_j}, z^*\rangle &\geq \langle x - z_{n_j}, Ax\rangle \geq \langle x - z_{n_j}, Ax\rangle - \left\langle x - z_{n_j}, \frac{Jz_{n_j} - Jx_{n_j}}{\tau} + Ax_{n_j}\right\rangle \\ &= \langle x - z_{n_j}, Az_{n_j} - Ax_{n_j}\rangle - \left\langle x - z_{n_j}, \frac{Jz_{n_j} - Jx_{n_j}}{\tau}\right\rangle. \end{aligned}$$

Applying the monotonicity condition on A, equation (3.11), and the uniform continuity of J on bounded subset sets of $\mathcal{Q}$, we have

$$\langle x - x^*, z^*\rangle \geq 0,$$

which implies that $\Omega_w(x_n) \subset VI(A,\mathcal{D})$.

Next, we show that $\Omega_w(x_n) \subset F_J(S)$.

Since $\lim_{n\to\infty} \|y_n - x_n\| = 0$ and J is uniformly continuous on bounded set, and also $\{r_n\} \subset [a,\infty[$ by assumption, we get that $\lim_{n\to\infty} \frac{\|Jy_n - Jx_n\|}{r_n} = 0$. But $y_n = T^S_{r_n} x_n$. By Lemma 2.12(2), we have

$$\langle y - y_n, Sy_n\rangle - \frac{1}{r_n}\big\langle y - y_n, (1+r_n)Jy_n - Jx_n\big\rangle \leq 0, \quad \forall y \in \mathcal{D}. \tag{3.14}$$

Let $\alpha \in]0,1]$ and $y \in \mathcal{D}$. Then $y_\alpha = \alpha y + (1-\alpha)x^* \in \mathcal{D}$. By inequality (3.14), Definition 2.3, and for some constant $M_0 > 0$, we get that

$$\begin{aligned}
\langle y_{n_j} - y_\alpha, Sy_\alpha\rangle &\geq \langle y_{n_j} - y_\alpha, Sy_\alpha\rangle + \langle y_\alpha - y_{n_j}, Sy_{n_j}\rangle - \frac{1}{r_{n_j}}\big\langle y_\alpha - y_{n_j}, (1+r_{n_j})Jy_{n_j} - Jx_{n_j}\big\rangle \\
&= \langle y_{n_j} - y_\alpha, Sy_\alpha - Sy_{n_j}\rangle - \frac{1}{r_{n_j}}\big\langle y_\alpha - y_{n_j}, (1+r_{n_j})Jy_{n_j} - Jx_{n_j}\big\rangle \\
&\geq \langle y_{n_j} - y_\alpha, Jy_\alpha - Jy_{n_j}\rangle - \frac{1}{r_{n_j}}\big\langle y_\alpha - y_{n_j}, (1+r_{n_j})Jy_{n_j} - Jx_{n_j}\big\rangle \\
&\geq \langle y_{n_j} - y_\alpha, Jy_\alpha\rangle - M_0\frac{\|Jy_{n_j} - Jx_{n_j}\|}{r_{n_j}}.
\end{aligned} \tag{3.15}$$

Taking limit on both sides of inequality (3.15), we have

$$\big\langle x^* - y_\alpha, Sy_\alpha\big\rangle \geq \big\langle x^* - y_\alpha, Jy_\alpha\big\rangle. \tag{3.16}$$

From inequality (3.16), we have

$$\big\langle x^* - y, S\big(x^* + \alpha\big(y - x^*\big)\big)\big\rangle \geq \big\langle x^* - y, J\big(x^* + \alpha\big(y - x^*\big)\big)\big\rangle. \tag{3.17}$$

Using the fact that S is continuous and J is uniformly continuous on bounded subsets of $\mathcal{Q}$, letting $\alpha \downarrow 0$, we get from inequality (3.17) that

$$\big\langle x^* - y, Sx^*\big\rangle \geq \big\langle x^* - y, Jx^*\big\rangle, \quad \forall y \in \mathcal{D} \quad\Longleftrightarrow\quad 0 \geq \big\langle x^* - y, Jx^* - Sx^*\big\rangle, \quad \forall y \in \mathcal{D}.$$

Set $y := J^{-1}(Sx^*)$. Since $\mathcal{Q}^*$ is strictly convex and J^{-1} is monotone, we get that

$$\big\langle x^* - J^{-1}\big(Sx^*\big), Jx^* - Sx^*\big\rangle = 0, \tag{3.18}$$

which implies that $Sx^* = Jx^*$. Thus, $x^* \in F_J(S)$, which implies that $\Omega_w(x_n) \subset F_J(S)$.

Finally, we show that $\Omega_w(x_n) \subset EP(\Theta)$.

Since $\lim_{n\to\infty} \|v_n - y_n\| = 0$ and J is uniformly continuous on bounded sets, and also $\{r_n\} \subset [a,\infty[$ by assumption, we get that $\lim_{n\to\infty} \frac{\|Jv_n - Jy_n\|}{r_n} = 0$. But $v_n = K^\Theta_{r_n} y_n$. By a result of Blum and Otelli [3], we have

$$\Theta(v_n, y) + \frac{1}{r_n}\langle y - v_n, Jv_n - Jy_n\rangle \geq 0, \quad \forall y \in \mathcal{D}. \tag{3.19}$$

By A_2, we have that $\frac{1}{r_{n_j}}\langle y - v_{n_j}, Jv_{n_j} - Jy_{n_j}\rangle \geq \Theta(y, v_{n_j})$. Since $y \mapsto \Theta(v, y)$ is convex and lower semi-continuous, we obtain from the above inequality that $0 \geq \Theta(y, x^*)$, $\forall y \in \mathcal{D}$. For $\alpha \in$

$]0,1]$ and $y \in \mathcal{D}$, letting $y_\alpha = \alpha y + (1-\alpha)x^*$, then $y_\alpha \in \mathcal{D}$, since $\mathcal{D}$ is closed and convex. Hence,

$$0 \geq \Theta\big(y_\alpha, x^*\big), \quad \forall y \in \mathcal{D}.$$

By A_1 and A_4, we have

$$\begin{aligned} 0 &= \Theta(y_\alpha, y_\alpha) \leq \alpha\Theta(y_\alpha, y) + (1-\alpha)\Theta\big(y_\alpha, x^*\big) \leq \alpha\Theta(y_\alpha, y) \\ &\leq \Theta\big(x^* + \alpha\big(y - x^*\big), y\big). \end{aligned} \tag{3.20}$$

Letting $\alpha \downarrow 0$, by A_3, we obtain that $\Theta(x^*, y) \geq 0$. Hence, $\Omega_w(x_n) \subset EP(\Theta)$. Using this and the fact that $\Omega_w(x_n) \subset VI(A, \mathcal{D})$ and $\Omega_w(x_n) \subset F_J(S)$, we conclude that

$$x^* \in \Omega := F_J(S) \cap VI(A, \mathcal{D}) \cap EP(\Theta).$$

Claim 3. We show that $\{x_n\}$ converges strongly to the point $u := \Pi_\Omega x_0$.

Since $\{x_n\}$ is bounded, then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup w$ and

$$\limsup_{n\to\infty}\langle x_n - u, Jx_0 - Ju\rangle = \lim_{j\to\infty}\langle x_{n_j} - u, Jx_0 - Ju\rangle = \langle w - u, Jx_0 - Ju\rangle. \tag{3.21}$$

Now, applying Lemma 2.11, inequalities (3.4), (3.5), equation (3.12), and some $M_0 > 0$, we have

$$\begin{aligned} \phi(u, x_{n+1}) &= V\big(u, \alpha_n Jx_0 + (1-\alpha_n)\big[\beta Jv_n + (1-\beta)Jw_n\big]\big) \\ &\leq V\big(u, \alpha_n Ju + (1-\alpha_n)\big[\beta Jv_n + (1-\beta)Jw_n\big]\big) + 2\alpha_n\langle x_{n+1} - u, Jx_0 - Ju\rangle \\ &\leq (1-\alpha_n)V\big(u, \beta Jv_n + (1-\beta)Jw_n\big) + 2\alpha_n\langle x_{n+1} - u, Jx_0 - Ju\rangle \\ &\leq (1-\alpha_n)\big[\beta V(u, Jv_n) + (1-\beta)V(u, Jw_n)\big] + 2\alpha_n\langle x_{n+1} - u, Jx_0 - Ju\rangle \\ &= (1-\alpha_n)\big[\beta\phi(u, v_n) + (1-\beta)\phi(u, w_n)\big] + 2\alpha_n\langle x_{n+1} - u, Jx_0 - Ju\rangle \\ &\leq (1-\alpha_n)\phi(u, x_n) + 2\alpha_n\big(\langle x_n - u, Jx_0 - Ju\rangle + \|x_{n+1} - x_n\|M_0\big). \end{aligned} \tag{3.22}$$

By inequality (3.21), Lemmas 2.5 and 2.9, it follows from inequality (3.22) that $\lim_{n\to\infty}\phi(u, x_n) = 0$. Hence, by Lemma 2.7, we get that $\lim_{n\to\infty}\|x_n - u\| = 0$.

Case 2. There exists a subsequence $\{x_{m_j}\} \subset \{x_n\}$ such that $\phi(u, x_{m_j+1}) > \phi(u, x_{m_j})$ for all $j \in \mathbb{N}$, $u \in \Omega$. By Lemma 2.10, there exists a nondecreasing sequence $\{n_i\} \subset \mathbb{N}$ such that $\lim_{i\to\infty} n_i = \infty$ and the following inequalities hold:

$$\phi(u, x_{n_i}) \leq \phi(u, x_{n_i+1}) \quad \text{and} \quad \phi(u, x_i) \leq \phi(u, x_{n_i+1}) \quad \text{for all } i \in \mathbb{N}.$$

Now, from inequality (3.6), we have

$$\phi(u, x_{n_i}) \leq \phi(u, x_{n_i+1})$$

$$\leq \alpha_n\phi(u,x_0) + (1-\alpha_n)\phi(u,x_{n_i}) - (1-\alpha_{n_i})\beta\big[\phi(y_{n_i},x_{n_i}) + \phi(v_{n_i},y_{n_i})\big]$$
$$- (1-\alpha_{n_i})(1-\beta)\left(1-\frac{\tau L}{c}\right)\big[\phi(w_{n_i},z_{n_i}) + \phi(z_{n_i},x_{n_i})\big]. \tag{3.23}$$

From inequality (3.23), with $\sigma = (1-\beta)(1-\alpha_{n_i})(1-\frac{\tau L}{c}) > 0$ and $\xi = (1-\alpha_{n_i})\beta > 0$, we have

$$\begin{aligned} &\phi(y_{n_i},x_{n_i}) + \phi(v_{n_i},y_{n_i}) \leq \xi^{-1}\alpha_{n_i}\phi(u,x_0), \\ &\phi(w_{n_i},z_{n_i}) + \phi(z_{n_i},x_{n_i}) \leq \sigma^{-1}\alpha_{n_i}\phi(u,x_0). \end{aligned} \tag{3.24}$$

Since $\alpha_{n_i} \to 0$ as $i \to \infty$, we get that

$$\lim_{i\to\infty}\phi(y_{n_i},x_{n_i}) = \lim_{i\to\infty}\phi(v_{n_i},y_{n_i}) = \lim_{i\to\infty}\phi(w_{n_i},z_{n_i}) = \lim_{i\to\infty}\phi(z_{n_i},x_{n_i}) = 0.$$

Using similar arguments as in Case 1 above, we have the fact that

(1) $\lim_{i\to\infty}\|y_{n_i}-x_{n_i}\| = \lim_{i\to\infty}\|v_{n_i}-y_{n_i}\| = \lim_{i\to\infty}\|w_{n_i}-z_{n_i}\| = \lim_{i\to\infty}\|z_{n_i}-x_{n_i}\| = \lim_{i\to\infty}\|x_{n_i+1}-x_{n_i}\| = 0$;

(2) $\Omega_w(x_{n_i}) \subset \Omega := F_J(S) \cap VI(A,\mathcal{D}) \cap EP(\Theta)$.

Next, we show that $\{x_i\}$ converges strongly to a point $u := \Pi_\Omega x_0$.Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{i_j}} \rightharpoonup z$ as $j \to \infty$ and

$$\limsup_{i\to\infty}\langle x_{n_i}-u, Jx_0-Ju\rangle = \lim_{j\to\infty}\langle x_{n_{i_j}}-u, Jx_0-Ju\rangle = \langle z-u, Jx_0-Ju\rangle. \tag{3.25}$$

From inequality (3.22) and Lemma 2.10, we get that

$$\begin{aligned} \phi(u,x_{n_i+1}) &\leq (1-\alpha_{n_i})\phi(u,x_{n_i}) + 2\alpha_{n_i}\big(\langle x_{n_i}-u, Jx_0-Ju\rangle + \|x_{n_i+1}-x_{n_i}\|M_0\big) \\ &\leq (1-\alpha_{n_i})\phi(u,x_{n_i+1}) + 2\alpha_{n_i}\big(\langle x_{n_i}-u, Jx_0-Ju\rangle + \|x_{n_i+1}-x_{n_i}\|M_0\big). \end{aligned}$$

Since $\alpha_{n_i} > 0$, for all $i \geq 1$, we get that

$$\phi(u,x_i) \leq \phi(u,x_{n_i+1}) \leq 2\langle x_{n_i}-u, Jx_0-Ju\rangle + \|x_{n_i+1}-x_{n_i}\|M_0.$$

By Lemma 2.5 and the fact that $\lim_{i\to\infty}\|x_{n_i+1}-x_{n_i}\| = 0$, we have

$$\limsup_{i\to\infty}\phi(u,x_i) \leq \limsup_{i\to\infty} 2\langle x_{n_i}-u, Jx_0-Ju\rangle + 2M_0\limsup_{i\to\infty}\|x_{n_i+1}-x_{n_i}\|,$$

which implies that $\limsup_{i\to\infty}\phi(u,x_i) \leq 0$. By Lemma 2.7, we conclude that $x_i \to u$, as $i \to \infty$. □

Corollary 3.2 *Let $\mathcal{Q}^*$ be the dual space of a uniformly smooth and two-uniformly convex real Banach space $\mathcal{Q}$. Let $\mathcal{D}$ be a nonempty, closed, and convex subset of $\mathcal{Q}$. Let $A : \mathcal{D} \to \mathcal{Q}^*$ be a monotone and L-Lipschitz map, $\Theta : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ be a bi-functional satisfying conditions A_1 to A_4, with $K^\Theta_{r_n}$ as the resolvent map of Θ. Let $B : \mathcal{D} \to \mathcal{Q}^*$ be a continuous monotone map with $\Omega := B^{-1}(0) \cap VI(A,\mathcal{D}) \cap EP(\Theta) \neq \emptyset$ and $\{x_n\}$ be a sequence generated by algorithm (3.1). Assume $\tau \in]0,1[$ with $\tau < \frac{c}{L}$, $\{r_n\} \subset [a,\infty[$ for some $a > 0$ and $\{\alpha_n\} \subset]0,1[$ with $\lim \alpha_n = 0$ and $\sum \alpha_n = \infty$. Then the sequence $\{x_n\}$ converges strongly to a point $\Pi_\Omega x_0$.*

Proof Set $S := J - B$. Then we have that S is a continuous J-pseudo-contractive map with $\Omega := F_J(S) \cap VI(A,\mathcal{D}) \cap EP(\Theta) = B^{-1}(0) \cap VI(A,\mathcal{D}) \cap EP(\Theta)$. Hence, the result follows from Theorem 3.1. □

4 Application to convex optimization problem

In this section, we apply our theorem to finding a minimizer of a convex function defined on a real Banach space.

Theorem 4.1 *Let $\mathcal{Q}^*$ be the dual space of a uniformly smooth and two-uniformly convex real Banach space $\mathcal{Q}$. Let $\mathcal{D}$ be a nonempty, closed, and convex subset of $\mathcal{Q}$. Let $A : \mathcal{D} \to \mathcal{Q}^*$ be a monotone and L-Lipschitz map, $\Theta : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ be a bi-functional satisfying conditions A_1 to A_4, with $K^{\Theta}_{r_n}$ as the resolvent map of Θ. Let $h : \mathcal{D} \to \mathcal{Q}$ be a Fréchet differentiable and convex functions and $dh : \mathcal{D} \to \mathcal{Q}^*$ be a monotone and continuous map with $\Omega := dh^{-1}(0) \cap VI(A,\mathcal{D}) \cap EP(\Theta) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by algorithm (3.1). Assume $\tau \in\,]0,1[$ with $\tau < \frac{c}{L}$, $\{r_n\} \subset [a,\infty[$ for some $a > 0$ and $\{\alpha_n\} \subset\,]0,1[$ with $\lim \alpha_n = 0$ and $\sum \alpha_n = \infty$. Then the sequence $\{x_n\}$ converges strongly to a point $\Pi_{\Omega} x_0$.*

Proof Setting $dh = B$ in Corollary 3.2, then $J - dh$ is a continuous J-pseudo-contractive map. Furthermore, we get that $dh^{-1}(0) \cap VI(A,\mathcal{D}) \cap EP(\Theta) = \arg\inf_{y\in\mathcal{D}} h(y) \cap VI(A,\mathcal{D}) \cap EP(\Theta)$. Therefore, the result follows from Corollary 3.2. □

5 Numerical experiment

Here, we present an example to confirm the implementability of our algorithm (3.1).

Example 1 Let $\mathcal{Q} = L_p^{\mathbb{R}}([0,1])$, $1 < p \leq 2$.Then $\mathcal{Q}^* = L_q^{\mathbb{R}}([0,1])$, $\frac{1}{p} + \frac{1}{q} = 1$. Let $\mathcal{D} := \overline{B_p(0,1)} \subset \mathcal{Q}$,

$$\|x\|_{\mathcal{L}_p} := \left(\int_0^1 |x(t)|^p\,dt\right)^{\frac{1}{p}} \quad \text{and}$$
$$T = \left\{w \in \mathcal{Q} : \int_0^1 \big([w-z](t)[Jx - \tau Ax - Jz](t)\big)\,dt \leq 0\right\},$$

where $A : \mathcal{D} \to \mathcal{Q}^*$ is a map defined by

$$(Ax)(t) = Jx(t) \quad \text{for all } t \in [0,1].$$

Clearly, A monotone and L-Lipschitz and $VI(A,\mathcal{D}) = \{0\}$.

Let $B : \mathcal{D} \to \mathcal{Q}^*$ be a map defined by

$$(Bx)(t) = (1+t)Jx(t) \quad \text{for all } t \in [0,1].$$

Clearly, B is monotone and continuous. Define $S := J - B$. Therefore, S is a continuous J-pseudo-contractive map with $F_J(S) = \{0\}$. Furthermore, from Lemma 2.12, we have

$$T_r^S(x) := \left\{z \in \mathcal{D} : \langle w - z, Sz\rangle - \frac{1}{r}\langle w - z, (1+r)Jz - Jx\rangle \leq 0, \forall w \in \mathcal{D}\right\}, \quad x \in \mathcal{Q}.$$

Therefore, for any $x \in \mathcal{Q}$ and for some $z \in \mathcal{Q}$, we have

$$\left(T_r^S x\right)(t) = J^{-1}\left(\frac{Jx(t)}{1+tr}\right), \quad x \in \mathcal{Q}.$$

Let $\Theta : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ be a map defined by $\Theta(x,y) = \langle y - x, Jx\rangle$, $\forall y \in \mathcal{D}$.

Clearly, Θ satisfies conditions A_1 to A_4 and $0 \in EP(\Theta)$. Moreover, from a result of Blum and Otelli, we have

$$K_r^{\Theta}(x) := \left\{z \in \Theta(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \forall y \in \mathcal{D}\right\}.$$

Thus, for any $x \in \mathcal{Q}$ and for some $z \in \mathcal{D}$, we have that

$$\left\langle y, Jz(r+1) - Jx\right\rangle \geq \left\langle z, Jz(r+1) - Jx\right\rangle, \quad \forall y \in \mathcal{D}.$$

Hence,

$$\left(K_r^{\Theta}(x)\right)(t) = \frac{x(t)}{(r+1)} \quad \text{for all } t \in [0,1].$$

Therefore, $\Omega := F_J(S) \cap VI(A,\mathcal{D}) \cap EP(\Theta) = \{0\}$.

Let $P_T : \mathbb{R} \to T$ and $P_{\mathcal{D}} : \mathbb{R} \to \mathcal{D}$ be maps defined by

$$P_T(u) := \begin{cases} u - \max\{0, \frac{\langle a, u-z\rangle}{\|a\|^2}\}a, & \text{if } a \neq 0, \\ u, & \text{if } a = 0, \end{cases}$$

$$P_{\mathcal{D}}(x) := \begin{cases} x, & \text{if } x \in \mathcal{D}, \\ x_0 + \frac{r}{\|x - x_0\|}(x - x_0), & \text{if } x \notin \mathcal{D}, \end{cases}$$

where $u := Jx - \tau A(z)$ and $a := Jx - \tau A(x) - Jz$.

For implementation, we choose $p = 2$, $\beta = \frac{1}{2}$, $\tau = 0.000868$, $\alpha_n = \frac{1}{(n+5)}$, and $r_n = 10$, $\forall n \geq 0$.

Then we compute the $(n+1)$th iteration as follows:

$$\begin{cases} x_0(t) = e^{-t} \quad \text{or} \quad x_0(t) = \sin(t), \\ z_n(t) = \begin{cases} 0.999132 x_n(t), & \text{if } \|x_n\| \leq 1, \\ 0.999132 \frac{x_n(t)}{\|x_n\|}, & \text{if } \|x_n\| > 1, \end{cases} \\ \text{setting:} \\ a_n(t) = 0.999132 x_n(t) - z_n(t), \\ w_n(t) = x_n(t) - 0.000868 z_n(t), \\ w_n(t) - z_n(t) = x_n(t) - 1.000868 z_n(t), \\ v_n(t) = \frac{x_n(t)}{22(10t+1)} + \frac{1}{2} \cdot \begin{cases} w_n(t) - \max\{0, \frac{\int_0^1 a_n(t)(w_n(t) - z_n(t))\,dt}{\|a_n\|^2}\} \cdot a_n(t), & \text{if } a_n(t) \neq 0, \\ w_n(t), & \text{if } a_n(t) = 0, \end{cases} \\ x_{n+1}(t) = \frac{1}{n+5} x_0(t) + (1 - \frac{1}{n+1}) v_n(t). \end{cases}$$

Remark 3 Theorem 3.1 extends and improves the results in [17, 49] in the following ways:

(*a*) Theorem 3.1 extends the results in [17, 49] from a real Hilbert space to a uniformly smooth and two-uniformly convex real Banach space.

(*b*) In Theorem 3.1, a continuous J-pseudo-contractive map was studied, which contains the class of Lipschitz pseudo-contractive maps studied in [49].

(*c*) The theorem in [17, 49] did not study equilibrium problems, whereas in Theorem 3.1 it was studied.

(*d*) Finally, a subgradient-extragradient algorithm has an advantage in computing over the extragradient method proposed in [12] (see also [13]).

6 Conclusion

In this paper, we constructed a new Halpern-type subgradient-extragradient iterative algorithm whose sequence approximates a common solution of some nonlinear problems in Banach spaces. Also, the theorem is applied to approximate a common solution of a variational inequality, an equilibrium problem, and a convex minimization problem. Moreover, the theorem proved is applicable in L_p (l_p or $W_p^m(\Omega)$ spaces, $1 < p \leq 2$, where $W_p^m(\Omega)$ denotes the usual Sobolev space. The analytical representations of the duality map in these spaces where $p^{-1} + q^{-1} = 1$ is given in Theorem 3.1 of [55], page 36. Finally, a numerical example is given to illustrate the implementability of our algorithm.

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All authors contributed equally. All authors read and approved the final manuscript.

Author details
[1] African University of Science and Technology, Km 10 Airport Road, Abuja, Nigeria. [2] Federal University Dutsinma, Katsina, Nigeria. [3] Department of Mathematics and Statistics, Auburn University, Alabama, AL 36849, USA.

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References

1. Fichera, G.: Problemi elastostatici con vincoli unilaterali; il problema di Signorini con ambigue condizioni al contorno (Elastostatic problems with unilateral constraints; the Signorini's problem with ambiguous boundary conditions). Mem. Accad. Naz. Lincei **7**, 91–140 (1964)
2. Stampacchia, G.: Formes bilinéaires coercitives sur le ensembles convexes. C. R. Acad. Sci. Paris **258**, 4413–4416 (1964)
3. Blum, E., Oettli, W.: From optimization and variational inequalities to equilibrium problems. Math. Stud. **63**, 123–145 (1994)
4. Antonio, S.: Questioni di elasticità nonlinearizzata e semilinearizzata. Rend. Mat. **18**, 1–45 (1959)
5. Ceng, L.C., Cho, S.Y., Qin, X., Yao, J.C.: A general system of variational inequalities with nonlinear mappings in Banach spaces. J. Nonlinear Convex Anal. **20**, 395–410 (2019)
6. Kinderlehrer, D., Stampaccia, G.: An Iteration to Variational Inequalities and Their Applications. Academic Press, New York (1990)
7. Alber, Ya.: Metric and generalized projection operators in Banach spaces: properties and applications. In: Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, pp. 15–50. Dekker, New York (1996)
8. Ceng, L.C., Petrusel, A., Yao, J.C.: On Mann viscosity subgradient extragradient algorithms for fixed point problems of finitely many strict pseudocontractions and variational inequalities. Mathematics **7**, Article ID 925 (2019) 14 pp.

9. Browder, F.E.: Nonlinear mappings of nonexpansive and accretive-type in Banach spaces. Bull. Am. Math. Soc. **73**, 875–882 (1967)
10. Chidume, C.E., Nnyaba, U.V., Romanus, O.M., Ezea, C.G.: Convergence theorems for strictly *J*-pseudocontractions with application to zeros of Gamma-inverse strongly monotone maps. Panam. Math. J. **26**, 57–76 (2016)
11. Yao, Y., Marino, G., Muglia, L.: A modified Korpelevich's method convergent to the minimum-norm solution of a variational inequality. Optimization **63**, 559–569 (2014)
12. Korpelevič, G.M.: An extragradient method for finding saddle points and for other problems. Èkon. Mat. Metody **12**, 747–756 (1967) (in Russian)
13. Censor, Y., Gibali, A., Reich, S.: The subgradient extragradient method for solving variational inequalities in Hilbert space. J. Optim. Theory Appl. **148**, 318–335 (2011)
14. Kraikaew, R., Saejung, S.: Strong convergence of the Halpern subgradient extragradient method for solving variational inequalities in Hilbert spaces. J. Optim. Theory Appl. **163**, 399–412 (2014)
15. Ceng, L.C., Petrusel, A., Yao, J.C., Yao, Y.: Systems of variational inequalities with hierarchical variational inequality constraints for Lipschitzian pseudocontractions. Fixed Point Theory **20**, 113–133 (2019)
16. Gibali, A., Reich, S., Zalas, R.: Outer approximation methods for solving variational inequalities in Hilbert space. Optimization **66**, 417–437 (2017)
17. Alghamdi, M.A., Shahzad, N., Zegeye, H.: A scheme for a solution of a variational inequality for a monotone mapping and a fixed point of a pseudocontractive mapping. J. Inequal. Appl. **2015**, 292 (2015). https://doi.org/10.1186/s13660-015-0804-3
18. Censor, Y., Gibali, A., Reich, S.: Strong convergence of subgradient extragradient methods for the variational inequality problem in Hilbert space. Optim. Methods Softw. **26**, 827–845 (2011)
19. Chidume, C.E., Nnakwe, M.O.: Iterative algorithms for split variational inequalities and generalized split feasibility problems, with applications. J. Nonlinear Var. Anal. **3**(2), 127–140 (2019)
20. Ezeora, J.N.: Convergence theorem for generalized mixed equilibrium problems and common fixed point problems for a family of multivalued mappings. Int. J. Anal. Appl. **10**(1), 48–57 (2016)
21. Van Hieu, D.: Halpern subgradient extragradient method extended to equilibrium problems. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **111**, 823840 (2017). https://doi.org/10.1007/s13398-016-0328-9
22. Kumam, P.: A hybrid approximation method for equilibrium and fixed point problems for a monotone mapping and a nonexpansive. Nonlinear Anal. Hybrid Syst. **2**, 1245–1255 (2008)
23. Moudafi, A.: Weak convergence theorems for nonexpansive mappings and equilibrium problems. J. Nonlinear Convex Anal. **9**, 37–43 (2008)
24. Qin, X., Cho, Y.J., Kang, S.M.: Convergence theorems of common elements for equilibrium problems and fixed point problem in Banach spaces. J. Comput. Appl. Math. **225**, 20–30 (2009)
25. Takahashi, W., Zembayashi, K.: Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces. Nonlinear Anal. **70**, 45–57 (2009)
26. Kato, T.: Nonlinear semigroups and evolution equations. J. Math. Soc. Jpn. **19**, 511–520 (1967)
27. Chidume, C.E., Bello, A.U., Usman, B.: Iterative algorithms for zeros of strongly monotone Lipschitz maps in classical Banach spaces. SpringerPlus **4**(1), 9
28. Chidume, C.E., Bello, A.U., Oyindo, M.O.: Convergence theorem for a countable family of multi-valued strictly pseudo-contractive mappings in Hilbert spaces. Int. J. Math. Anal. **9**(27), 1331–1340 (2015)
29. Chidume, C.E., Ndambomve, P., Bello, A.U., Okpala, M.E.: The multiple-sets split equality fixed point problem for countable families of multi-valued demi-contractive mappings. Int. J. Math. Anal. **9**(10), 453–469 (2015)
30. Chidume, C.E., Ndambomve, P., Bello, A.U., Okpala, M.E.: Strong convergence theorem for fixed points of nearly uniformly L-Lipschitzian asymptotically generalized Φ-hemicontractive mappings. Int. J. Math. Anal. **9**(52), 2555–2569 (2015)
31. Bruck, R.E.: A strongly convergent iterative method for the solution of $0 \in Ux$ for a maximal monotone operator U in Hilbert space. J. Math. Anal. Appl. **48**, 114–126 (1974)
32. Reich, S.: Extension problems for accretive sets in Banach spaces. J. Funct. Anal. **26**, 378–395 (1977)
33. Schu, J.: Iterative construction of fixed points of asymptotically nonexpansive mappings. J. Math. Anal. Appl. **158**, 407–413 (1991)
34. Kirk, W.A.: On local expansions and accretive mappings. Int. J. Math. Math. Sci. **6**, 419–429 (1983)
35. Chidume, C.E., Mutangadura, S.A.: An example on the Mann iteration method for Lipschitz pseudocontrations. Proc. Am. Math. Soc. **129**(8), 2359–2363 (2001)
36. Nnakwe, M.O., Ifebude, B.C.: A common fixed point of an infinite family of pseudo-contractive maps. Thai J. Math. **18**(3), 1387–1400 (2020)
37. Rockarfellar, R.T.: Convex Analysis. Princeton University Press, Princeton (1970). p. 242. ISBN 0-691-08069-0
38. Zegeye, H.: Strong convergence theorems for maximal monotone mappings in Banach spaces. J. Math. Anal. Appl. **343**, 663–671 (2008)
39. Chidume, C.E., Idu, K.O.: Approximation of zeros of bounded maximal monotone maps, solutions of Hammerstein integral equations and convex minimization problems. Fixed Point Theory Appl. **2016**, 97 (2016). https://doi.org/10.1186/s13663-016-0582-8
40. Liu, B.: Fixed point of strong duality pseudocontractive mappings and applications. Abstr. Appl. Anal. **2012**, Article ID 623625 (2012). https://doi.org/10.1155/2012/623625
41. Su, Y., Xu, H.K.: A duality fixed point theorem and applications. Fixed Point Theory **13**(1), 259–265 (2012)
42. Cheng, Q., Su, Y., Zhang, J.: Duality fixed point and zero point theorems and applications. Abstr. Appl. Anal. **2012**, Article ID 391301 (2012). https://doi.org/10.1155/2012/391301
43. Chidume, C.E., Otubo, E.E., Ezea, C.G.: Strong convergence theorem for a common fixed point of an infinite family of *J*-nonexpansive maps with applications. Aust. J. Math. Anal. Appl. **13**(1), 15, 1–13 (2016)
44. Chidume, C.E., Nnakwe, M.O.: A new Halpern-type algorithm for a generalized mixed equilibrium problem and a countable family of generalized-J-nonexpansive maps, with applications. Carpath. J. Math. **34**(2), 191–198 (2018)
45. Chidume, C.E., Nnakwe, M.O., Otubo, E.E.: A new iterative algorithm for a generalized mixed equilibrium problem and a countable family of nonexpansive-type maps, with applications. Fixed Point Theory **21**(1), 109–124 (2020)

46. Chidume, C.E., Nnakwe, M.O.: A strong convergence theorem for an inertial algorithm for a countable family of generalized nonexpansive maps. Fixed Point Theory **21**(2), 441–452 (2020)
47. Nnakwe, M.O.: An algorithm for approximating a common solution of variational inequality and convex minimization problems. Optimization. https://doi.org/10.1080/02331934.2020.1777995
48. Chidume, C.E., Adamu, A., Chinwendu, L.O.: Strong convergence theorem for some nonexpansive mappings in certain Banach spaces. Thai J. Math. **18**(3), 1537–1548 (2020)
49. Zegeye, H., Shahzad, H.: Solutions of variational inequality problems in the set of fixed points of pseudocontractive mappings. Carpath. J. Math. **30**, 257–265 (2014)
50. Xu, H.K.: Inequalities in Banach spaces with applications. Nonlinear Anal. **16**(12), 1127–1138 (1991)
51. Kamimura, S., Takahashi, W.: Strong convergence of a proximal-type algorithm in a Banach space. SIAM J. Optim. **13**, 938–945 (2002)
52. Rockafellar, R.T.: On the maximality of sums of nonlinear monotone operators. Trans. Am. Math. Soc. **149**, 75–88 (1970)
53. Xu, H.K.: Iterative algorithms for nonlinear operators. J. Lond. Math. Soc. **66**(2), 240–256 (2002)
54. Mainge, P.E.: The viscosity approximation process for quasi-nonexpansive mappings in Hilbert spaces. Comput. Math. Appl. **59**, 74–79 (2010)
55. Alber, Y., Ryazantseva, I.: Nonlinear Ill Posed Problems of Monotone Type. Springer, London (2006)