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On innovations of n-dimensional integral-type inequality on time scales

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Abstract

Integral-type inequalities and dynamic equations have an important place in time scales. In this paper, we present some innovations of n-dimensional Minkowski's integral-type inequality on time scales via $\lozenge_\alpha$-integral.

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1 Introduction

For more than a quarter century, the theory of time scales, whose founder was German mathematician Stefan Hilger [1], played an important role in differential calculus, difference calculus, and quantum calculus. Later, this theory was quickly developed by many mathematicians, who added many innovations to the literature by using integral-type inequalities and dynamic equations on time scales [2–12]. Wong et al. [6, 7] expressed some integral equations on time scales. Ozkan et al. [10] demonstrated extensions of some integral inequalities on time scales. Yang [13] obtained a extension of $\lozenge_\alpha$-integral Hölder's inequality. Georgiev et al. [14] demonstrated two dimensional integral inequalities on time scales. Anderson [15, 16] demonstrated some dynamic integral inequalities in two independent variables on time scales. Tuna and Kutukcu [17] reached some general conclusions about Hardy's integral inequalities by using Hölder inequalities with delta integral on time scales. Chen [18] demonstrated some generalizations of the Minkowski's integral inequality. Akın [19] showed new properties of fractional inequalities by using fractional maximal integral operators and synchronous functions on time scales. Recently, numerous applications have had an accelerating effect on the development of mathematical inequalities. These applications also attracted the attention of researchers from various disciplines, for example, quantum mechanics, phsical problems, wave equations, heat transfer, and economic problems [20–24].

The organization of this paper is as follows. In Sect. 2, we give necessary definitions, lemmas, and theorems. In Sect. 3, we demonstrate some innovations of n-dimensional Minkowski's integral-type inequalities on time scales via $\lozenge_\alpha$-integral.

2 Mathematical background

Let us give basic information about time scales in general. For more detailed information, we refer the readers to [1–38].

The time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. $[a,b]$ is an arbitrary interval on time scale $\mathbb{T}$, and by $[a,b]_{\mathbb{T}}$ we denote $[a,b]\cap\mathbb{T}$.

Definition 2.1 ([31]) The mappings $\sigma,\rho:\mathbb{T}\to\mathbb{T}$ defined by $\sigma(t)=\inf\{s\in\mathbb{T}:s>t\}$, $\rho(t)=\sup\{s\in\mathbb{T}:s>t\}$ for $t\in\mathbb{T}$. Respectively, $\sigma(t)$ is the forward jump operator and $\rho(t)$ is the backward jump operator.

If $\sigma(t)>t$, then t is right-scattered, and if $\sigma(t)=t$, then t is called right-dense. If $\rho(t)<t$, then t is left-scattered, and if $\rho(t)=t$, then t is called left-dense.

Definition 2.2 ([31]) The mappings $\mu,\vartheta:\mathbb{T}\to\mathbb{R}^+$ defined as $\mu(t)=\sigma(t)-t$, $\vartheta(t)=t-\rho(t)$ are called graininess mappings.

If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k=\mathbb{T}-\{m\}$. Otherwise, $\mathbb{T}^k=\mathbb{T}$. Briefly,

$$\mathbb{T}^k=\begin{cases}\mathbb{T}\setminus(\rho\sup\mathbb{T},\sup\mathbb{T}] & \text{if }\sup\mathbb{T}<\infty,\\ \mathbb{T} & \text{if }\sup\mathbb{T}=\infty.\end{cases}$$

Similarly,

$$\mathbb{T}_k=\begin{cases}\mathbb{T}\setminus[\inf\mathbb{T},\sigma(\inf\mathbb{T})], & |\inf\mathbb{T}|<\infty,\\ \mathbb{T}, & \inf\mathbb{T}=-\infty.\end{cases}$$

Let $h:\mathbb{T}\to\mathbb{R}$, and let $t\in\mathbb{T}^k$ $(t\neq\min\mathbb{T})$. If h is Δ-differentiable at point t, then h is continuous at point t, and if h is left continuous at point t and t is right-scattered, then h is Δ-differentiable at point t, and

$$h^{\Delta}(t)=\frac{h^{\sigma}(t)-h(t)}{\mu(t)}.$$

Let t be right-dense. If h is Δ-differentiable at point t and $\lim_{s\to t}\frac{h(t)-h(s)}{t-s}$, then

$$h^{\Delta}(t)=\lim_{s\to t}\frac{h(t)-h(s)}{t-s},$$

and if h is Δ-differentiable at point t, then $h^{\sigma}(t)=h(t)+\mu(t)h^{\Delta}(t)$.

Remark 2.3 If $\mathbb{T}=\mathbb{R}$, then $h^{\Delta}(t)=h'(t)$, and if $\mathbb{T}=\mathbb{Z}$, then $h^{\Delta}(t)$ reduces to $\Delta h(t)$.

Let $n\in N$. For $i=\{1,2,\dots,n\}$, we denote by $\mathbb{T}_i$ a time scale.

Definition 2.4 ([14]) The set $\aleph^n=\mathbb{T}_1\times\mathbb{T}_2\times\cdots\times\mathbb{T}_n=\{t=(t_1,t_2,\dots,t_n):t_i\in\mathbb{T}_i,i=1,2,\dots,n\}$ is called an n-dimensional time scale.

Definition 2.5 ([14]) For $a = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ and $b = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n$, we write $a \geq b$ whenever $a_i \geq b_i$ for all $i = 1, 2, \dots, n$.

Definition 2.6 ([25]) If $H : \mathbb{T} \to \mathbb{R}$ is a Δ-antiderivative of $h : \mathbb{T} \to \mathbb{R}$, then $H^{\Delta} = h(t)$ for all $t \in \mathbb{T}$, and we define the Δ-integral of h by

$$\int_s^t h(\tau)\Delta\tau = H(t) - H(s)$$

for $s, t \in \mathbb{T}$.

Proposition 2.7 ([25]) *Let $u, v, \alpha, t \in \mathbb{T}$ and $c \in \mathbb{R}$. If $f(\gamma)$ and $g(\gamma)$ are Δ-integrable functions on $[u, v]_{\mathbb{T}}$, then the following statements are valid*:

(i) $\int_u^t [f(\gamma) + g(\gamma)]\Delta\gamma = \int_u^t f(\gamma)\Delta\gamma + \int_u^t g(\gamma)\Delta\gamma$,
(ii) $\int_u^t cf(\gamma)\Delta\gamma = c\int_u^t f(\gamma)\Delta\gamma$,
(iii) $\int_u^t f(\gamma)\Delta\gamma = -\int_t^u f(\gamma)\Delta\gamma$,
(iv) $\int_u^t f(\gamma)\Delta\gamma = \int_u^v f(\gamma)\Delta\gamma + \int_v^t f(\gamma)\Delta\gamma$,
(v) $\int_u^u f(\gamma)\Delta\gamma = 0$.

Lemma 2.8 ([25]) *Let $u, v, \alpha, t \in \mathbb{T}$ with $u < v$. Suppose that $h(\gamma)$, $g(\gamma)$ are Δ-integrable functions on $[u, v]_{\mathbb{T}}$. Then we have*

(a) *If $h(\gamma) \geq 0$ for all $\gamma \in [u, v]_{\mathbb{T}}$, then $\int_u^v h(\gamma)\Delta\gamma \geq 0$,*
(b) *If $h(\gamma) \leq g(\gamma)$ for all $\gamma \in [u, v]_{\mathbb{T}}$, then $\int_u^v h(\gamma)\Delta\gamma \leq \int_u^v g(\gamma)\Delta\gamma$,*
(c) *If $h(\gamma) \geq 0$ for all $\gamma \in [u, v]_{\mathbb{T}}$, then $h(\gamma) = 0$ iff $\int_u^v h(\gamma)\Delta\gamma = 0$.*

Definition 2.9 ([25]) Let $h : \mathbb{T}_k \to \mathbb{R}$ be ∇-differentiable at $t \in \mathbb{T}_k$. If $\varepsilon > 0$, then there exists a neighborhood V of t such that

$$\left|h\big(\rho(t)\big) - h(s) - h^{\nabla}(t)\big(\rho(t) - s\big)\right| \leq \varepsilon\left|\rho(t) - s\right|$$

for all $s \in V$.

Definition 2.10 ([25]) Let $H : \mathbb{T} \to \mathbb{R}$ be a ∇-antiderivative of $h : \mathbb{T} \to \mathbb{R}$. Then we define

$$\int_s^t h(\tau)\nabla\tau = H(t) - H(s)$$

for all $s, t \in \mathbb{T}$.

Let $f(t)$ be differentiable on $\mathbb{T}$ for all $t \in \mathbb{T}$. Then we define $f^{\diamondsuit_\alpha}(t)$ by

$$f^{\diamondsuit_\alpha}(t) = \alpha f^{\Delta}(t) + (1 - \alpha)f^{\nabla}(t)$$

for $0 \leq \alpha \leq 1$.

Proposition 2.11 ([25]) *If $f, h : \mathbb{T} \to \mathbb{R}$ are $\diamondsuit_\alpha$-differentiable for all $\alpha, t \in \mathbb{T}$, then*

(i) *$f + h : \mathbb{T} \to \mathbb{R}$ is $\diamondsuit_\alpha$-differentiable for $t \in \mathbb{T}$ with*

$$(f + h)^{\diamondsuit_\alpha}(t) = f^{\diamondsuit_\alpha}(t) + h^{\diamondsuit_\alpha}(t).$$

(ii) *For* $k \in \mathbb{R}$, $kf : \mathbb{T} \to \mathbb{R}$ *is* $\Diamond_\alpha$*-differentiable for all* $\alpha, t \in \mathbb{T}$ *with*

$$(kf)^{\Diamond_\alpha}(t) = kf^{\Diamond_\alpha}(t).$$

(iii) $fh : \mathbb{T} \to \mathbb{R}$ *is* $\Diamond_\alpha$*-differentiable for all* $\alpha, t \in \mathbb{T}$ *with*

$$(fh)^{\Diamond_\alpha}(t) = f^{\Diamond_\alpha}(t)h(t) + \alpha f^{\sigma}(t)h^{\Delta}(t) + (1-\alpha)f^{\rho}(t)h^{\nabla}(t).$$

Definition 2.12 ([25]) For $\alpha, b, t \in \mathbb{T}$ and $f : \mathbb{T} \to \mathbb{R}$, we have

$$\int_b^t f(\delta)\Diamond_\alpha\delta = \alpha\int_b^t f(\delta)\Delta\delta + (1-\alpha)\int_b^t f(\delta)\nabla\delta$$

for $0 \le \alpha \le 1$.

Definition 2.13 ([31, p. 6]) If $f \in C_{rd}(\mathbb{T}, \mathbb{R})$ and $t \in \mathbb{T}^k$, then we have

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t).$$

Theorem 2.14 ([31, Theorem 1.1.2]) *If* f *is* Δ*-integrable on* $[a,b]$, *then* $|f|$ *is* Δ*-integrable on* $[a,b]$, *and* $|\int_a^b f(\gamma)\Delta\gamma| \le \int_a^b |f(\gamma)|\Delta\gamma$.

Theorem 2.15 ([38]) *If two functions* $g, h : I \to \mathbb{R}$ *are* Δ*-integrable on* $I = [a,b] \in \mathbb{T}$ *with* $0 < l \le g^p, h^p \le L < \infty$. *If* $p > 1$, *then we have*

$$\left(\int_a^b \left|g(\gamma)\right|^p\Delta\gamma\right)^{\frac{1}{p}} + \left(\int_a^b \left|h(\gamma)\right|^p\Delta\gamma\right)^{\frac{1}{p}} \le 2\left(\frac{L}{l}\right)^{\frac{1}{p}}\left(\int_a^b \left|g(\gamma)+h(\gamma)\right|^p\Delta\gamma\right)^{\frac{1}{p}}.$$

3 Main results

In this section, we state and prove our main results.

Theorem 3.1 *If two mappings* $g, h : I \to \mathbb{R}$ *are* $\Diamond_\alpha$*-integrable on* $I = [a,b] \in \mathbb{T}$ *with* $0 < l \le g^p, h^p \le L < \infty$, $p > 1$, *then we have*

$$\left(\int_a^b \left|g(\gamma)\right|^p\Diamond_\alpha\gamma\right)^{\frac{1}{p}} + \left(\int_a^b \left|h(\gamma)\right|^p\Diamond_\alpha\gamma\right)^{\frac{1}{p}}$$
$$\le 2\left(\frac{p}{(p-1)}\frac{L}{l}\right)^{\frac{1}{p}}\left(\int_a^b \left|g(\gamma)+h(\gamma)\right|^p\Diamond_\alpha\gamma\right)^{\frac{1}{p}}. \tag{1}$$

Proof We know that (Theorem 3.1, [38]) if $0 < l \le g^p \le L < \infty$, then

$$l^{\frac{1}{p}} \le g \le L^{\frac{1}{p}}. \tag{2}$$

Similarly, if $0 < l \le h^p \le L < \infty$, then

$$l^{\frac{1}{p}} \le h \le L^{\frac{1}{p}}. \tag{3}$$

Multiplying both sides of inequalities (2) and (3) by $(\int_a^b |g(\gamma)|^p \Diamond_\alpha \gamma)^{\frac{1}{p}}$ and $(\int_a^b |h(\gamma)|^p \Diamond_\alpha \gamma)^{\frac{1}{p}}$, respectively, we have

$$\begin{aligned}(p-1)^{\frac{1}{p}} l^{\frac{1}{p}} \left(\int_a^b \left|g(\gamma)\right|^p \Diamond_\alpha \gamma\right)^{\frac{1}{p}} &\le L^{\frac{1}{p}} \left(\int_a^b \left|g(\gamma)\right|^p \Diamond_\alpha \gamma\right)^{\frac{1}{p}} \\ &\le p^{\frac{1}{p}} L^{\frac{1}{p}} \left(\int_a^b \left|g(\gamma)+h(\gamma)\right|^p \Diamond_\alpha \gamma\right)^{\frac{1}{p}} \end{aligned} \tag{4}$$

and

$$\begin{aligned}(p-1)^{\frac{1}{p}} l^{\frac{1}{p}} \left(\int_a^b \left|h(\gamma)\right|^p \Diamond_\alpha \gamma\right)^{\frac{1}{p}} &\le L^{\frac{1}{p}} \left(\int_a^b \left|h(\gamma)\right|^p \Diamond_\alpha \gamma\right)^{\frac{1}{p}} \\ &\le p^{\frac{1}{p}} L^{\frac{1}{p}} \left(\int_a^b \left|g(\gamma)+h(\gamma)\right|^p \Diamond_\alpha \gamma\right)^{\frac{1}{p}}. \end{aligned} \tag{5}$$

Now, if we add inequalities (4) and (5) to each other, then we have

$$\begin{aligned}&(p-1)^{\frac{1}{p}} l^{\frac{1}{p}} \left[\left(\int_a^b \left|g(\gamma)\right|^p \Diamond_\alpha \gamma\right)^{\frac{1}{p}} + \left(\int_a^b \left|g(\gamma)\right|^p \Diamond_\alpha \gamma\right)^{\frac{1}{p}}\right] \\ &\quad \le 2p^{\frac{1}{p}} L^{\frac{1}{p}} \left(\int_a^b \left|g(\gamma)+h(\gamma)\right|^p \Diamond_\alpha \gamma\right)^{\frac{1}{p}}. \end{aligned} \tag{6}$$

Thus we have proved inequality (1). □

Theorem 3.2 *Let $g,h : [a_1,b_1] \times [a_1,b_1] \times \cdots \times [a_i,b_i] \to \mathbb{R}$ be rd-continuous for $I = [a_i,b_i] \in \mathbb{T}_i$, $1 \le i \le n$. If $p > 1$, then we have*

$$\begin{aligned}&\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_i}^{b_i} \left|g(\gamma_1,\gamma_2,\dots,\gamma_i)+h(\gamma_1,\gamma_2,\dots,\gamma_i)\right|^p \Diamond_\alpha \gamma_i\right)^{\frac{1}{p}} \\ &\quad \le \frac{p}{(p-1)} 2^{1-\frac{1}{p}} \left[\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_i}^{b_i} \left|g(\gamma_1,\gamma_2,\dots,\gamma_i)\right|^p \Diamond_\alpha \gamma_i\right)^{\frac{1}{p}} \right. \\ &\qquad \left. + \left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_i}^{b_i} \left|h(\gamma_1,\gamma_2,\dots,\gamma_i)\right|^p \Diamond_\alpha \gamma_i\right)^{\frac{1}{p}}\right] \end{aligned} \tag{7}$$

for $1 \le i \le n$.

Proof Let $g,h : [a_1,b_1] \times [a_1,b_1] \times \cdots \times [a_i,b_i] \to \mathbb{R}$ be *rd*-continuous and finite for $[a_i,b_i] \in \mathbb{T}_i \subset \mathbb{R}$. We know that [36]

$$\left|g(\gamma,\theta)+h(\gamma,\theta)\right| \le \left|g(\gamma,\theta)\right| + \left|h(\gamma,\theta)\right|. \tag{8}$$

Taking the pth power of both sides, we obtain

$$\left|g(\gamma_1,\gamma_2,\dots,\gamma_i)+h(\gamma_1,\gamma_2,\dots,\gamma_i)\right|^p \le 2^{p-1}\left[\left|g(\gamma_1,\gamma_2,\dots,\gamma_i)\right|^p + \left|h(\gamma_1,\gamma_2,\dots,\gamma_i)\right|^p\right] \tag{9}$$

for $1 \le i \le n$.

Taking the two-dimensional $\diamondsuit_\alpha$-integral over $[a_i, b_i]$ of both sides of inequality (9) and the power of order $\frac{1}{p}$, we obtain

$$\begin{aligned}
&\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_i}^{b_i}\left|g(\gamma_1,\gamma_2,\ldots,\gamma_i)+h(\gamma_1,\gamma_2,\ldots,\gamma_i)\right|^p\diamondsuit_\alpha\gamma_i\right)^{\frac{1}{p}}\\
&\quad\le\frac{p}{(p-1)}2^{1-\frac{1}{p}}\left[\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_i}^{b_i}\left|g(\gamma_1,\gamma_2,\ldots,\gamma_i)\right|^p\diamondsuit_\alpha\gamma_i\right)^{\frac{1}{p}}\right.\\
&\qquad\left.+\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_i}^{b_i}\left|h(\gamma_1,\gamma_2,\ldots,\gamma_i)\right|^p\diamondsuit_\alpha\gamma_i\right)^{\frac{1}{p}}\right].
\end{aligned}$$

Thus the proof of Theorem 3.2 is completed. □

Theorem 3.3 *Let $M((\gamma_1,\gamma_2,\ldots,\gamma_i),(\theta_1,\theta_2,\ldots,\theta_i))$, $g(\gamma_1,\gamma_2,\ldots,\gamma_i)$, $h(\theta_1,\theta_2,\ldots,\theta_i)$, $\Phi(\gamma_1,\gamma_2,\ldots,\gamma_i)$, $\Psi(\theta_1,\theta_2,\ldots,\theta_i)$ be nonnegative functions, and let $g,h:[a_1,b_1]\times[a_1,b_1]\times\cdots\times[a_i,b_i]\to\mathbb{R}$ be rd-continuous and finite for $[a_i,b_i]\in\mathbb{T}_i\subset\mathbb{R}$. Let*

$$\begin{aligned}
H(\theta_1,\theta_2,\ldots,\theta_i)&=\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_i}^{b_i}\frac{M((\gamma_1,\gamma_2,\ldots,\gamma_i),(\theta_1,\theta_2,\ldots,\theta_i))}{\Psi(\theta_1,\theta_2,\ldots,\theta_i)^p}\diamondsuit_\alpha\gamma_i,\\
K(\gamma_1,\gamma_2,\ldots,\gamma_i)&=\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_i}^{b_i}\frac{M((\gamma_1,\gamma_2,\ldots,\gamma_i),(\theta_1,\theta_2,\ldots,\theta_i))}{\Phi(\gamma_1,\gamma_2,\ldots,\gamma_i)^p}\diamondsuit_\alpha\theta_i
\end{aligned}$$

for $[a_i,b_i]\in\mathbb{T}_i\subset\mathbb{R}$ and $1\le i\le n$. If $p>1$, then we have the following inequalities:

$$\begin{aligned}
&\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_i}^{b_i}M\big((\gamma_1,\gamma_2,\ldots,\gamma_i),(\theta_1,\theta_2,\ldots,\theta_i)\big)\right.\\
&\qquad\times\left|g(\gamma_1,\gamma_2,\ldots,\gamma_i)+h(\theta_1,\theta_2,\ldots,\theta_i)\right|^p\diamondsuit_\alpha\gamma_i\\
&\qquad+\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_i}^{b_i}M\big((\gamma_1,\gamma_2,\ldots,\gamma_i),(\theta_1,\theta_2,\ldots,\theta_i)\big)\\
&\qquad\left.\times\left|g(\gamma_1,\gamma_2,\ldots,\gamma_i)+h(\theta_1,\theta_2,\ldots,\theta_i)\right|^p\diamondsuit_\alpha\theta_i\right)^{\frac{1}{p}}\\
&\quad\le\frac{p}{(p-1)}2^{1-\frac{1}{p}}\left[\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_i}^{b_i}\Phi(\gamma_1,\gamma_2,\ldots,\gamma_i)^pK(\gamma_1,\gamma_2,\ldots,\gamma_i)\right.\right.\\
&\qquad\left.\times\left|g(\gamma_1,\gamma_2,\ldots,\gamma_i)\right|^p\diamondsuit_\alpha\gamma_i\right)^{\frac{1}{p}}\\
&\qquad+\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_i}^{b_i}\Psi(\theta_1,\theta_2,\ldots,\theta_i)^pH(\theta_1,\theta_2,\ldots,\theta_i)\right.\\
&\qquad\left.\left.\times\left|h(\theta_1,\theta_2,\ldots,\theta_i)\right|^p\diamondsuit_\alpha\theta_i\right)^{\frac{1}{p}}\right],
\end{aligned}\tag{10}$$

$$\begin{aligned}
&\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_i}^{b_i}H(\theta_1,\theta_2,\ldots,\theta_i)^{1-p}\Psi(\theta_1,\theta_2,\ldots,\theta_i)^{p(1-p)}\\
&\qquad\times\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_i}^{b_i}M\big((\gamma_1,\gamma_2,\ldots,\gamma_i),(\theta_1,\theta_2,\ldots,\theta_i)\big)g(\gamma_1,\gamma_2,\ldots,\gamma_i)\diamondsuit_\alpha\gamma_i\right)^p\diamondsuit_\alpha\theta_i
\end{aligned}$$

$$\leq \frac{p}{(p-1)} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_i}^{b_i} \Phi(\gamma_1, \gamma_2, \ldots, \gamma_i)^p K(\gamma_1, \gamma_2, \ldots, \gamma_i) \times \left|g(\gamma_1, \gamma_2, \ldots, \gamma_i)\right|^p \lozenge_\alpha \gamma_i, \tag{11}$$

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_i}^{b_i} K(\gamma_1, \gamma_2, \ldots, \gamma_i)^{1-p} \Phi(\gamma_1, \gamma_2, \ldots, \gamma_i)^{p(1-p)} \times \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_i}^{b_i} M\bigl((\gamma_1, \gamma_2, \ldots, \gamma_i), (\theta_1, \theta_2, \ldots, \theta_i)\bigr) \times \left|g(\theta_1, \theta_2, \ldots, \theta_i)\right| \lozenge_\alpha \theta_i\right)^p \lozenge_\alpha \gamma_i$$
$$\leq \frac{p}{(p-1)} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_i}^{b_i} \Psi(\theta_1, \theta_2, \ldots, \theta_i)^p H(\theta_1, \theta_2, \ldots, \theta_i) \times \left|g(\theta_1, \theta_2, \ldots, \theta_i)\right|^p \lozenge_\alpha \theta_i. \tag{12}$$

Proof First, let us prove inequality (10).

Let $g, h : [a_1, b_1] \times [a_1, b_1] \times \cdots \times [a_i, b_i] \to \mathbb{R}$ be rd-continuous and finite for $[a_i, b_i] \in \mathbb{T}_i \subset \mathbb{R}$, and let $M((\gamma_1, \gamma_2, \ldots, \gamma_i), (\theta_1, \theta_2, \ldots, \theta_i))$ be nonnegative kernel functions. Let us consider the equation

$$\left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_i}^{b_i} M\bigl((\gamma_1, \gamma_2, \ldots, \gamma_i), (\theta_1, \theta_2, \ldots, \theta_i)\bigr) \times \left|g(\gamma_1, \gamma_2, \ldots, \gamma_i) + h(\theta_1, \theta_2, \ldots, \theta_i)\right|^p \lozenge_\alpha \gamma_i + \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_i}^{b_i} M\bigl((\gamma_1, \gamma_2, \ldots, \gamma_i), (\theta_1, \theta_2, \ldots, \theta_i)\bigr) \times \left|g(\gamma_1, \gamma_2, \ldots, \gamma_i) + h(\theta_1, \theta_2, \ldots, \theta_i)\right|^p \lozenge_\alpha \theta_i\right)^{\frac{1}{p}}$$
$$= \frac{p}{(p-1)} \left(\left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_i}^{b_i} M\bigl((\gamma_1, \gamma_2, \ldots, \gamma_i), (\theta_1, \theta_2, \ldots, \theta_i)\bigr) \times \left|\frac{g(\gamma_1, \gamma_2, \ldots, \gamma_i)\Phi(\gamma_1, \gamma_2, \ldots, \gamma_i)}{\Phi(\gamma_1, \gamma_2, \ldots, \gamma_i)} + \frac{h(\gamma_1, \gamma_2, \ldots, \gamma_i)\Psi(\theta_1, \theta_2, \ldots, \theta_i)}{\Psi(\theta_1, \theta_2, \ldots, \theta_i)}\right|^p \lozenge_\alpha \gamma_i\right) + \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_i}^{b_i} M\bigl((\gamma_1, \gamma_2, \ldots, \gamma_i), (\theta_1, \theta_2, \ldots, \theta_i)\bigr) \times \left|\frac{g(\gamma_1, \gamma_2, \ldots, \gamma_i)\Phi(\gamma_1, \gamma_2, \ldots, \gamma_i)}{\Phi(\gamma_1, \gamma_2, \ldots, \gamma_i)} + \frac{h(\gamma_1, \gamma_2, \ldots, \gamma_i)\Psi(\theta_1, \theta_2, \ldots, \theta_i)}{\Psi(\theta_1, \theta_2, \ldots, \theta_i)}\right|^p \lozenge_\alpha \theta_i\right)\right)^{\frac{1}{p}}. \tag{13}$$

Applying the Minkowski inequality to the right side of inequality (13), by Theorem 3.2 we complete the proof of inequality (10).

Now let us prove inequality (11). Let us consider the equation

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_i}^{b_i} H(\theta_1, \theta_2, \ldots, \theta_i)^{1-p} \Psi(\theta_1, \theta_2, \ldots, \theta_i)^{p(1-p)} \times \left(\int_{a_i}^{b_i} M\bigl((\gamma_1, \gamma_2, \ldots, \gamma_i), (\theta_1, \theta_2, \ldots, \theta_i)\bigr)\left|g(\gamma_1, \gamma_2, \ldots, \gamma_i)\right| \lozenge_\alpha \gamma_i\right)^p \lozenge_\alpha \theta_i$$

$$= \frac{p}{(p-1)} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a}^{b} M\big((\gamma_1, \gamma_2, \ldots, \gamma_i), (\theta_1, \theta_2, \ldots, \theta_i)\big)^{1-p}$$
$$\times \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_i}^{b_i} M\big((\gamma_1, \gamma_2, \ldots, \gamma_i), \theta\big) \big| g(\gamma_1, \gamma_2, \ldots, \gamma_i) \big| \Diamond_\alpha \gamma_i \right)^p \Diamond_\alpha \theta_i. \tag{14}$$

Applying the Hölder inequality to the right side of equality (14), we obtain

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_i}^{b_i} H(\theta_1, \theta_2, \ldots, \theta_i)^{1-p} \Psi(\theta_1, \theta_2, \ldots, \theta_i)^{p(1-p)}$$
$$\times \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_i}^{b_i} M\big((\gamma_1, \gamma_2, \ldots, \gamma_i), (\theta_1, \theta_2, \ldots, \theta_i)\big) \big| g(\gamma_1, \gamma_2, \ldots, \gamma_i) \big| \Diamond_\alpha \gamma_i \right)^p \Diamond_\alpha \theta_i$$
$$\leq \frac{p}{(p-1)} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_i}^{b_i} M\big((\gamma_1, \gamma_2, \ldots, \gamma_i), (\theta_1, \theta_2, \ldots, \theta_i)\big)$$
$$\times \left[\left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_i}^{b_i} \big| g(\gamma_1, \gamma_2, \ldots, \gamma_i) \big|^p \Diamond_\alpha \gamma_i \right)^{\frac{1}{p}} \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_i}^{b_i} \Diamond_\alpha \gamma_i \right)^{\frac{1}{q}} \right] \Diamond_\alpha \theta_i$$
$$\leq \frac{p}{(p-1)} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_i}^{b_i} M\big((\gamma_1, \gamma_2, \ldots, \gamma_i), (\theta_1, \theta_2, \ldots, \theta_i)\big)$$
$$\times \left[\left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_i}^{b_i} \big| g(\gamma_1, \gamma_2, \ldots, \gamma_i) \big|^p \Diamond_\alpha \gamma_i \right)^{\frac{1}{p}} \right] \Diamond_\alpha \theta_i$$
$$= \frac{p}{(p-1)} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_i}^{b_i} \Phi(\gamma_1, \gamma_2, \ldots, \gamma_i)^p K(\gamma_1, \gamma_2, \ldots, \gamma_i) \big| g(\gamma_1, \gamma_2, \ldots, \gamma_i) \big|^p \Diamond_\alpha \gamma_i.$$

Thus we have completed the proof of Theorem 3.3. The reader can see the proof of inequality (12), similar to the proof of inequality (11). □

Remark 3.4 Let $f(\theta_1, \theta_2, \ldots, \theta_i)$, $g(\gamma_1, \gamma_2, \ldots, \gamma_i)$, $h(\theta_1, \theta_2, \ldots, \theta_i)$, $\Phi(\gamma_1, \gamma_2, \ldots, \gamma_i)$, $\Psi(\theta_1, \theta_2, \ldots, \theta_i)$ be nonnegative functions. If we put

$$M\big((\gamma_1, \gamma_2, \ldots, \gamma_i), (\theta_1, \theta_2, \ldots, \theta_i)\big) = \begin{cases} f(\theta_1, \theta_2, \ldots, \theta_i), & (\gamma_1, \gamma_2, \ldots, \gamma_i) \leq (\theta_1, \theta_2, \ldots, \theta_i), \\ 0, & (\gamma_1, \gamma_2, \ldots, \gamma_i) > (\theta_1, \theta_2, \ldots, \theta_i), \end{cases}$$

and

$$M\big((\gamma_1, \gamma_2, \ldots, \gamma_i), (\theta_1, \theta_2, \ldots, \theta_i)\big) = \begin{cases} f(\theta_1, \theta_2, \ldots, \theta_i), & (\gamma_1, \gamma_2, \ldots, \gamma_i) > (\theta_1, \theta_2, \ldots, \theta_i), \\ 0, & (\gamma_1, \gamma_2, \ldots, \gamma_i) \leq (\theta_1, \theta_2, \ldots, \theta_i), \end{cases}$$

then the inequalities in Theorem 3.3 are provided.

4 Conclusion

Recently, the concept of inequalities in time scales has gained an important place in the scientific literature. Mathematicians have emphasized many aspects of integral inequalities. For example, transformations, inverse conversions, extensions, and so on. However, we found that little work has been done on multidimensional inequalities in time scales. In this paper, we proved some innovations of n-dimensional Minkowski's $\Diamond_\alpha$-integral inequality in time scales. We think that our method is applicable to different integral-type inequalities as well. As a result, we predict that new versions of known inequalities may be obtained.

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