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Certain fractional conformable inequalities for the weighted and the extended Chebyshev functionals

Asifa Tassaddiq^{1*} , Gauhar Rahman², Kottakkaran Sooppy Nisar³ and Muhammad Samraiz⁴

*Correspondence:

a.tassaddiq@mu.edu.sa

¹Department of Basic Sciences and Humanities, College of Computer and Information Sciences, Majmaah University, Al Majmaah, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

The main aim of this present paper is to establish fractional conformable inequalities for the weighted and extended Chebyshev functionals. We present some special cases of our main result in terms of the Riemann–Liouville fractional integral operator and classical inequalities.

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1 Introduction

Fractional calculus is the study of integrals and derivatives of arbitrary order which was a natural outgrowth of conventional definitions of calculus integral and derivative. In all areas of sciences, especially in mathematics, fractional calculus is a developing field with deep applications, though the idea was introduced more than three hundred years ago. Many theories of mathematics applicable to the study of fractional calculus were emerging at the end of the 19th century.

Fractional integral has been widely studied in the literature. The idea has been defined by many mathematicians with slightly different formulas, for example, Riemann–Liouville, Weyl, Erdélyi–Kober, Hadamard integral, and Liouville and Katugampola fractional integrals [21, 25, 26, 28, 34]. In the last few years, Khalil et al. [27] and Adeljawad [1] established a new class of fractional derivatives and integrals, called fractional conformable derivatives and integrals. Jarad et al. [23] introduced the fractional conformable integral operators. Based on that notion, one obtains generalizations of the Hadamard, Hermite–Hadamard, Opial, Grüss, Ostrowski, and Chebyshev inequalities, among others [2, 9, 14, 22, 35, 36, 39]). Furthermore, Set et al. [40–44] have contributed significant investigations in this direction. To study the further recent analysis for such a type of inequalities, the interested reader is referred to [5, 18, 24, 32, 33].

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In [7], the Chebyshev functional for two integrable functions f and g on $[a, b]$ is defined as

$$\mathcal{T}(f, g) = \frac{1}{b-a} \int_a^b f(\tau)g(\tau) d\tau - \frac{1}{b-a} \left(\int_a^b f(\tau) d\tau \right) \frac{1}{b-a} \left(\int_a^b g(\tau) d\tau \right). \tag{1}$$

In [3, 4, 15, 17], the applications and several inequalities related to (1) are found. In ([10], also see [7]), the Chebyshev functional is defined by

$$\mathcal{T}(f, g, h) = \int_a^b h(\tau) d\tau \int_a^b h(\tau)f(\tau)g(\tau) d\tau - \int_a^b h(\tau)f(\tau) d\tau \int_a^b h(\tau)g(\tau) d\tau, \tag{2}$$

where f and g are integrable on $[a, b]$ and h is a positive and integrable function on $[a, b]$. Applications of the functional defined in (2) are found in probability and statistical problems. Further applications in differential and integral equations are found in [6, 16, 31]. Elezovic et al. [19] defined

$$\begin{aligned} |\mathcal{T}(f, g, h)| &\leq \frac{1}{2} \left(\int_a^b \int_a^b h(\theta)h(\vartheta) |\theta - \vartheta|^{\frac{1}{p'} + \frac{1}{q'}} \left| \int_{\vartheta}^{\theta} |f'(\tau)|^p d\tau \right|^{\frac{r}{p}} d\theta d\vartheta \right)^{\frac{1}{r}} \\ &\quad \times \left(\int_a^b \int_a^b h(\theta)h(\vartheta) |\theta - \vartheta|^{\frac{1}{p'} + \frac{1}{q'}} \left| \int_{\vartheta}^{\theta} |g'(\tau)|^q d\tau \right|^{\frac{r}{q}} d\theta d\vartheta \right)^{\frac{1}{r}} \\ &\leq \frac{1}{2} \|f'\| \|g'\| \left(\int_a^b \int_a^b h(\theta)h(\vartheta) |\theta - \vartheta|^{\frac{1}{p'} + \frac{1}{q'}} d\theta d\vartheta \right), \end{aligned} \tag{3}$$

where $f' \in L^p([a, b])$, $g' \in L^q([a, b])$, $p, q, r > 1$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{r} + \frac{1}{r'} = 1$. In [13], the authors defined the following fractional integral inequality for Chebyshev functionals:

$$\begin{aligned} &2|\mathcal{I}^\alpha h(\tau)\mathcal{I}^\alpha hf(\tau) - \mathcal{I}^\alpha hf(\tau)\mathcal{I}^\alpha hg(\tau)| \\ &\leq \frac{\|f'\|_p \|g'\|_q}{\Gamma^2(\alpha)} \int_0^\tau \int_0^\tau (\tau - \theta)^{\alpha-1} (\tau - \vartheta)^{\alpha-1} |\theta - \vartheta| h(\theta)h(\vartheta) d\theta d\vartheta, \end{aligned} \tag{4}$$

where $f' \in L^p([0, \infty[)$, $g' \in L^q([0, \infty[)$, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Let us consider the extended Chebyshev functional [8, 30]

$$\begin{aligned} \tilde{\mathcal{T}}(f, g, h, h') &= \int_a^b h'(\tau) d\tau \int_a^b h(\tau)f(\tau)g(\tau) d\tau + \int_a^b h(\tau) d\tau \int_a^b h'(\tau)f(\tau)g(\tau) d\tau \\ &\quad - \int_a^b h(\tau)f(\tau) d\tau \int_a^b h'(\tau)g(\tau) d\tau \\ &\quad - \int_a^b h'(\tau)f(\tau) d\tau \int_a^b h(\tau)g(\tau) d\tau. \end{aligned} \tag{5}$$

In [4, 11, 29], various researchers have addressed the functionals (2) and (5). Recently Rahman et al. [38] defined fractional conformable inequalities for Chebyshev functionals (1) and (2). The present paper aims to develop certain fractional conformable inequalities for the Chebyshev functionals (2) and (5). Also, we will discuss some particular cases of our main result.

2 Preliminaries

In this section, we present the following well-known definitions from [20, 23].

Definition 2.1 The Riemann–Liouville fractional integral $\mathcal{J}_{a^+}^\alpha$ and $\mathcal{J}_{b^-}^\alpha$ of order $\alpha > 0$, for a continuous function $f \in [a, b]$, is defined by

$$\mathcal{J}_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau - t)^{\alpha-1} f(t) dt, \quad a < \tau \leq b, \tag{6}$$

where Γ is the gamma function; for further details as regards gamma and related functions, see [45].

Definition 2.2 The fractional conformable integral ${}^\beta \mathcal{J}_0^\alpha$ of order $\beta > 0$, for a continuous function is defined by

$${}^\beta \mathcal{J}_0^\alpha f(\tau) = \frac{1}{\Gamma(\beta)} \int_a^\tau \left(\frac{\tau^\alpha - t^\alpha}{\alpha} \right)^{\beta-1} \frac{f(t)}{t^{1-\alpha}} dt; \quad 0 < \tau \leq b. \tag{7}$$

Clearly one can get ${}^0 \mathcal{J}_0^\alpha f(\tau) = f(\tau)$ and

$${}^\beta \mathcal{J}_0^\alpha {}^\gamma \mathcal{J}_0^\alpha f(\tau) = {}^{\beta+\gamma} \mathcal{J}_0^\alpha f(\tau) = {}^\gamma \mathcal{J}_0^\alpha {}^\beta \mathcal{J}_0^\alpha f(\tau); \quad \beta, \gamma > 0.$$

In [23, 35, 37, 38, 40, 43], one has studied fractional conformable integral operators and has established certain inequalities by employing the said fractional integral operators.

Remark 1 If we consider $\alpha = 1$, then (7) will lead to the fractional integral in (6).

3 Main results

In this section, we establish certain fractional conformable inequalities for the weighted and the extended Chebyshev functionals.

Theorem 3.1 *Let f and g be two differentiable functions on $[0, \infty)$ and let h be positive and integrable function on $[0, \infty)$. If $f' \in L^p([0, \infty[)$, $g' \in L^q([0, \infty[)$, $p, q, r > 1$ with $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{r} + \frac{1}{r'} = 1$, then the following inequality holds for all $\tau > 0, \alpha, \beta > 0$:*

$$\begin{aligned} & 2 \left| {}^\beta \mathcal{J}^\alpha h(\tau) {}^\beta \mathcal{J}^\alpha hfg(\tau) - {}^\beta \mathcal{J}^\alpha hf(\tau) {}^\beta \mathcal{J}^\alpha hg(\tau) \right| \\ & \leq \left(\frac{\|f'\|_p^r}{\Gamma(\beta)} \int_0^\tau \int_0^\tau \left(\frac{\tau^\alpha - \theta^\alpha}{\alpha} \right)^{\beta-1} \left(\frac{\tau^\alpha - \vartheta^\alpha}{\alpha} \right)^{\beta-1} \right. \\ & \quad \left. \times \theta^{\alpha-1} \vartheta^{\alpha-1} h(\theta)h(\vartheta) |\theta - \vartheta|^{\frac{1}{p'} + \frac{1}{q'}} d\theta d\vartheta \right)^{\frac{1}{r}} \\ & \quad \times \left(\frac{\|g'\|_q^{r'}}{\Gamma(\beta)} \int_0^\tau \int_0^\tau \left(\frac{\tau^\alpha - \theta^\alpha}{\alpha} \right)^{\beta-1} \left(\frac{\tau^\alpha - \vartheta^\alpha}{\alpha} \right)^{\beta-1} \right. \\ & \quad \left. \times \theta^{\alpha-1} \vartheta^{\alpha-1} h(\theta)h(\vartheta) |\theta - \vartheta|^{\frac{1}{p'} + \frac{1}{q'}} d\theta d\vartheta \right)^{\frac{1}{r'}} \end{aligned}$$

$$\begin{aligned} &\leq \frac{\|f'\|_\alpha \|g'\|_q}{\Gamma^2(\beta)} \left(\int_0^\tau \int_0^\tau \left(\frac{\tau^\alpha - \theta^\alpha}{\alpha}\right)^{\beta-1} \left(\frac{\tau^\alpha - \vartheta^\alpha}{\alpha}\right)^{\beta-1} \right. \\ &\quad \left. \times \theta^{\alpha-1} \vartheta^{\alpha-1} h(\theta) h(\vartheta) |\theta - \vartheta|^{\frac{1}{p} + \frac{1}{q}} d\theta d\vartheta \right). \end{aligned} \tag{8}$$

Proof Let us define

$$H(\theta, \vartheta) = (f(\theta) - f(\vartheta))(g(\theta) - g(\vartheta)); \quad \theta, \vartheta \in (0, \tau). \tag{9}$$

Multiplying (9) by $\frac{1}{\Gamma(\beta)} \left(\frac{\tau^\alpha - \theta^\alpha}{\alpha}\right)^{\beta-1} \theta^{\alpha-1} h(\theta)$ and then integrating with respect to θ over $(0, \tau)$, we have

$$\begin{aligned} &\frac{1}{\Gamma(\beta)} \int_0^\tau \left(\frac{\tau^\alpha - \theta^\alpha}{\alpha}\right)^{\beta-1} \theta^{\alpha-1} h(\theta) H(\theta, \vartheta) d\theta \\ &= {}^\beta \mathcal{J}^\alpha hfg(\tau) - g(\vartheta) {}^\beta \mathcal{J}^\alpha hf(\tau) - f(\theta) {}^\beta \mathcal{J}^\alpha hg(\tau) + f(\vartheta)g(\vartheta) {}^\beta \mathcal{J}^\alpha h(\tau). \end{aligned} \tag{10}$$

Again, multiplying (10) by $\frac{1}{\Gamma(\beta)} \left(\frac{\tau^\alpha - \vartheta^\alpha}{\alpha}\right)^{\beta-1} \vartheta^{\alpha-1} h(\vartheta)$ and then integrating with respect to ν over $(0, \tau)$, we have

$$\begin{aligned} &\frac{1}{\Gamma^2(\beta)} \int_0^\tau \int_0^\tau \left(\frac{\tau^\alpha - \theta^\alpha}{\alpha}\right)^{\beta-1} \theta^{\alpha-1} \left(\frac{\tau^\alpha - \vartheta^\alpha}{\alpha}\right)^{\beta-1} \vartheta^{\alpha-1} h(\theta) h(\vartheta) H(\theta, \vartheta) d\theta d\vartheta \\ &= 2({}^\beta \mathcal{J}^\alpha h(\tau) {}^\beta \mathcal{J}^\alpha hfg(\tau) - {}^\beta \mathcal{J}^\alpha hf(\tau) {}^\beta \mathcal{J}^\alpha hg(\tau)). \end{aligned} \tag{11}$$

Also, on the other hand, we have

$$H(\theta, \vartheta) = \int_\vartheta^\theta \int_\vartheta^\theta f'(x)g'(y) dx dy. \tag{12}$$

By employing the Hölder inequality, we have

$$|f(\theta) - f(\vartheta)| \leq |\theta - \vartheta|^{\frac{1}{p}} \left| \int_\vartheta^\theta |f'(x)|^p dx \right|^{\frac{1}{p}} \tag{13}$$

and

$$|g(\theta) - g(\vartheta)| \leq |\theta - \vartheta|^{\frac{1}{q}} \left| \int_\vartheta^\theta |g'(y)|^q dy \right|^{\frac{1}{q}}. \tag{14}$$

Then H becomes

$$|H(\theta, \vartheta)| \leq |\theta - \vartheta|^{\frac{1}{p} + \frac{1}{q}} \left| \int_\vartheta^\theta |f'(x)|^p dx \right|^{\frac{1}{p}} \left| \int_\vartheta^\theta |g'(\tau)|^q dy \right|^{\frac{1}{q}}. \tag{15}$$

Therefore, from (11) and (15), we can write

$$\begin{aligned} &2|{}^\beta \mathcal{J}^\alpha h(\tau) {}^\beta \mathcal{J}^\alpha hfg(\tau) - {}^\beta \mathcal{J}^\alpha hf(\tau) {}^\beta \mathcal{J}^\alpha hg(\tau)| \\ &= \frac{1}{\Gamma^2(\beta)} \int_0^\tau \int_0^\tau \left(\frac{\tau^\alpha - \theta^\alpha}{\alpha}\right)^{\beta-1} \theta^{\alpha-1} \left(\frac{\tau^\alpha - \vartheta^\alpha}{\alpha}\right)^{\beta-1} \vartheta^{\alpha-1} h(\theta) h(\vartheta) |H(\theta, \vartheta)| d\theta d\vartheta \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{\Gamma^2(\beta)} \int_0^\tau \int_0^\tau \left(\frac{\tau^\alpha - \theta^\alpha}{\alpha}\right)^{\beta-1} \theta^{\alpha-1} \left(\frac{\tau^\alpha - \vartheta^\alpha}{\alpha}\right)^{\beta-1} \vartheta^{\alpha-1} h(\theta)h(\vartheta) \\ &\quad \times |\theta - \vartheta|^{\frac{1}{p'} + \frac{1}{q'}} \left| \int_\vartheta^\theta |f'(x)|^p dx \right|^{\frac{1}{p}} \left| \int_\vartheta^\theta |g'(\tau)|^q dt \right|^{\frac{1}{q}} d\theta d\vartheta. \end{aligned} \tag{16}$$

Now, by using the Hölder inequality for the double integral, we have

$$\begin{aligned} &2|\beta \mathfrak{J}^\alpha h(\tau)^\beta \mathfrak{J}^\alpha hfg(\tau) - \beta \mathfrak{J}^\alpha hf(\tau)^\beta \mathfrak{J}^\alpha hg(\tau)| \\ &\leq \frac{1}{\Gamma^2(\beta)} \left(\int_0^\tau \int_0^\tau \left(\frac{\tau^\alpha - \theta^\alpha}{\alpha}\right)^{\beta-1} \theta^{\alpha-1} \left(\frac{\tau^\alpha - \vartheta^\alpha}{\alpha}\right)^{\beta-1} \right. \\ &\quad \times \vartheta^{\alpha-1} h(\theta)h(\vartheta) |\theta - \vartheta|^{\frac{1}{p'} + \frac{1}{q'}} \left| \int_\vartheta^\theta |f'(x)|^p dx \right|^{\frac{r}{p}} d\theta d\vartheta \Big)^{\frac{1}{r}} \\ &\quad \times \left(\int_0^\tau \int_0^\tau \left(\frac{\tau^\alpha - \theta^\alpha}{\alpha}\right)^{\beta-1} \theta^{\alpha-1} \left(\frac{\tau^\alpha - \vartheta^\alpha}{\alpha}\right)^{\beta-1} \right. \\ &\quad \times \vartheta^{\alpha-1} h(\theta)h(\vartheta) |\theta - \vartheta|^{\frac{1}{p'} + \frac{1}{q'}} \left| \int_\vartheta^\theta |g'(x)|^q dx \right|^{\frac{r'}{q}} d\theta d\vartheta \Big)^{\frac{1}{r'}}. \end{aligned} \tag{17}$$

Now, using the following properties:

$$\left| \int_\vartheta^\theta |f'(x)|^p dx \right| \leq \|f'\|_p^p, \quad \left| \int_\vartheta^\theta |g'(y)|^q dy \right| \leq \|g'\|_q^q, \tag{18}$$

(17) can be written as

$$\begin{aligned} &2|\beta \mathfrak{J}^\alpha h(\tau)^\beta \mathfrak{J}^\alpha hfg(\tau) - \beta \mathfrak{J}^\alpha hf(\tau)^\beta \mathfrak{J}^\alpha hg(\tau)| \\ &\leq \left(\frac{\|f'\|_p^r}{\Gamma^r(\beta)} \int_0^\tau \int_0^\tau \left(\frac{\tau^\alpha - \theta^\alpha}{\alpha}\right)^{\beta-1} \theta^{\alpha-1} \left(\frac{\tau^\alpha - \vartheta^\alpha}{\alpha}\right)^{\beta-1} \right. \\ &\quad \times \vartheta^{\alpha-1} h(\theta)h(\vartheta) |\theta - \vartheta|^{\frac{1}{p'} + \frac{1}{q'}} d\theta d\vartheta \Big)^{\frac{1}{r}} \\ &\quad \times \left(\frac{\|g'\|_q^{r'}}{\Gamma^{r'}(\beta)} \int_0^\tau \int_0^\tau \left(\frac{\tau^\alpha - \theta^\alpha}{\alpha}\right)^{\beta-1} \theta^{\alpha-1} \left(\frac{\tau^\alpha - \vartheta^\alpha}{\alpha}\right)^{\beta-1} \right. \\ &\quad \times \vartheta^{\alpha-1} h(\theta)h(\vartheta) |\theta - \vartheta|^{\frac{1}{p'} + \frac{1}{q'}} d\theta d\vartheta \Big)^{\frac{1}{r'}}. \end{aligned} \tag{19}$$

Therefore,

$$\begin{aligned} &2|\beta \mathfrak{J}^\alpha h(\tau)^\beta \mathfrak{J}^\alpha hfg(\tau) - \beta \mathfrak{J}^\alpha hf(\tau)^\beta \mathfrak{J}^\alpha hg(\tau)| \\ &\leq \frac{\|f'\|_p^r \|g'\|_q^{r'}}{\Gamma^2(\beta)} \left(\int_0^\tau \int_0^\tau \left(\frac{\tau^\alpha - \theta^\alpha}{\alpha}\right)^{\beta-1} \left(\frac{\tau^\alpha - \vartheta^\alpha}{\alpha}\right)^{\beta-1} \right. \\ &\quad \times \theta^{\alpha-1} \vartheta^{\alpha-1} h(\theta)h(\vartheta) |\theta - \vartheta|^{\frac{1}{p'} + \frac{1}{q'}} d\theta d\vartheta \Big), \end{aligned}$$

which gives the required proof. □

By considering $\alpha = 1$ in Theorem 3.1, we get the following well-known result of Dahmani et al. [12].

Corollary 1 *Let f and g be two differentiable functions on $[0, \infty)$ and let h be positive and integrable function on $[0, \infty)$. If $f' \in L^p([0, \infty[)$, $g' \in L^q([0, \infty[)$, $p, q, r > 1$ with $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{r} + \frac{1}{r'} = 1$, then the following inequality holds for all $\tau > 0, \beta > 0$:*

$$\begin{aligned}
 & 2|\mathfrak{J}^\beta h(\tau)\mathfrak{J}^\beta hfg(\tau) - \mathfrak{J}^\beta hf(\tau)\mathfrak{J}^\beta hg(\tau)| \\
 & \leq \left(\frac{\|f'\|_p^r}{\Gamma(\beta)} \int_0^\tau \int_0^\tau (\tau - \theta)^{\beta-1} (\tau - \vartheta)^{\beta-1} h(\theta)h(\vartheta)|\theta - \vartheta|^{\frac{1}{p'} + \frac{1}{q'}} d\theta d\vartheta \right)^{\frac{1}{r}} \\
 & \quad \times \left(\frac{\|g'\|_q^{r'}}{\Gamma(\beta)} \int_0^\tau \int_0^\tau (\tau - \theta)^{\beta-1} (\tau - \vartheta)^{\beta-1} h(\theta)h(\vartheta)|\theta - \vartheta|^{\frac{1}{p'} + \frac{1}{q'}} d\theta d\vartheta \right)^{\frac{1}{r'}} \\
 & \leq \frac{\|f'\|_p^r \|g'\|_q^{r'}}{\Gamma^2(\beta)} \left(\int_0^\tau \int_0^\tau (\tau - \theta)^{\beta-1} (\tau - \vartheta)^{\beta-1} h(\theta)h(\vartheta)|\theta - \vartheta|^{\frac{1}{p'} + \frac{1}{q'}} d\theta d\vartheta \right). \tag{20}
 \end{aligned}$$

Remark 2 Similarly, by considering $\alpha = \beta = 1$ in Theorem 3.1, we get the inequality (3).

Theorem 3.2 *Let f and g be two differentiable functions on $[0, \infty)$ and let h and h' be positive and integrable functions on $[0, \infty)$. If $f' \in L^p([0, \infty[)$, $g' \in L^q([0, \infty[)$, $p, q, r > 1$ with $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{r} + \frac{1}{r'} = 1$, then the following inequality holds for all $\tau > 0, \alpha, \beta > 0$:*

$$\begin{aligned}
 & |\beta \mathfrak{J}^\alpha h'(\tau)^\beta \mathfrak{J}^\alpha hfg(\tau) + \beta \mathfrak{J}^\alpha h(\tau)^\beta \mathfrak{J}^\alpha h'fg(\tau) - \beta \mathfrak{J}^\alpha hf(\tau)^\beta \mathfrak{J}^\alpha h'g(\tau) \\
 & \quad - \beta \mathfrak{J}^\alpha hf(\tau)^\beta \mathfrak{J}^\alpha hg(\tau)| \\
 & \leq \frac{\|f'\|_p^r \|g'\|_q^{r'}}{\Gamma^2(\beta)} \int_0^\tau \int_0^\tau \left(\frac{\tau^\alpha - \theta^\alpha}{\alpha} \right)^{\beta-1} \left(\frac{\tau^\alpha - \vartheta^\alpha}{\alpha} \right)^{\beta-1} \\
 & \quad \times \theta^{\alpha-1} \vartheta^{\alpha-1} h(\theta)h'(\vartheta)|\theta - \vartheta|^{\frac{1}{p'} + \frac{1}{q'}} d\theta d\vartheta. \tag{21}
 \end{aligned}$$

Proof Multiplying (10) by $\frac{1}{\Gamma(\beta)} \left(\frac{\tau^\alpha - \vartheta^\alpha}{\alpha} \right)^{\mu-1} \vartheta^{\alpha-1} h(\vartheta)$ and then integrating with respect to ϑ over $(0, \tau)$, we have

$$\begin{aligned}
 & \frac{1}{\Gamma^2(\beta)} \int_0^\tau \int_0^\tau \left(\frac{\tau^\alpha - \theta^\alpha}{\alpha} \right)^{\beta-1} \theta^{\alpha-1} \left(\frac{\tau^\alpha - \vartheta^\alpha}{\alpha} \right)^{\mu-1} \vartheta^{\alpha-1} h(\theta)h(\vartheta)H(\theta, \vartheta) d\theta d\vartheta \\
 & = \beta \mathfrak{J}^\alpha h'(\tau)^\beta \mathfrak{J}^\alpha hfg(\tau) + \beta \mathfrak{J}^\alpha h(\tau)^\beta \mathfrak{J}^\alpha h'fg(\tau) \\
 & \quad - \beta \mathfrak{J}^\alpha hf(\tau)^\beta \mathfrak{J}^\alpha h'g(\tau) - \beta \mathfrak{J}^\alpha hf(\tau)^\beta \mathfrak{J}^\alpha hg(\tau). \tag{22}
 \end{aligned}$$

Using (15) in (22), we obtain

$$\begin{aligned}
 & |\beta \mathfrak{J}^\alpha h'(\tau)^\beta \mathfrak{J}^\alpha hfg(\tau) + \beta \mathfrak{J}^\alpha h(\tau)^\beta \mathfrak{J}^\alpha h'fg(\tau) - \beta \mathfrak{J}^\alpha hf(\tau)^\beta \mathfrak{J}^\alpha h'g(\tau) \\
 & \quad - \beta \mathfrak{J}^\alpha hf(\tau)^\beta \mathfrak{J}^\alpha hg(\tau)| \\
 & = \frac{1}{\Gamma(\beta)\Gamma(\mu)} \int_0^\tau \int_0^\tau \left(\frac{\tau^\alpha - \theta^\alpha}{\alpha} \right)^{\beta-1} \vartheta^{\alpha-1} \left(\frac{\tau^\alpha - \vartheta^\alpha}{\alpha} \right)^{\beta-1} \vartheta^{\alpha-1} h(\theta)h'|H(\theta, \vartheta)| d\theta d\vartheta
 \end{aligned}$$

$$\begin{aligned} &\leq \frac{\|f'\|_\alpha \|g'\|_\alpha}{\Gamma^2(\beta)} \int_0^\tau \int_0^\tau \left(\frac{\tau - \theta^\alpha}{\alpha}\right)^{\beta-1} \theta^{\alpha-1} \left(\frac{\tau - \vartheta^\alpha}{\alpha}\right)^{\mu-1} \vartheta^{\alpha-1} |\theta - \vartheta|^{\frac{1}{p'} + \frac{1}{q'}} \\ &\quad \times \left| \int_\theta^\vartheta |f'(x)|^p dx \right|^{\frac{1}{p}} \left| \int_\vartheta^\theta |g'(\tau)|^q dt \right|^{\frac{1}{q}} h(\theta)h'(\vartheta) d\theta d\vartheta. \end{aligned} \tag{23}$$

Applying the similar procedure of Theorem 3.1, we obtain the desired proof. □

If we consider $\alpha = 1$ in Theorem 3.2, then we get the following well-known result [12].

Corollary 2 *Let f and g be two differentiable functions on $[0, \infty)$ and let h and h' be positive and integrable functions on $[0, \infty)$. If $f' \in L^p([0, \infty[), g' \in L^q([0, \infty[), p, q, r > 1$ with $\frac{1}{p} + \frac{1}{p'} = 1, \frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{r} + \frac{1}{r'} = 1$, then the following inequality holds for all $\tau > 0, \beta > 0$:*

$$\begin{aligned} &|\mathfrak{J}^\beta h'(\tau)\mathfrak{J}^\beta hfg(\tau) + \mathfrak{J}^\beta h(\tau)\mathfrak{J}^\beta h'fg(\tau) - \mathfrak{J}^\beta hf(\tau)\mathfrak{J}^\beta h'g(\tau) - \mathfrak{J}^\beta h'f(\tau)\mathfrak{J}^\beta hg(\tau)| \\ &\leq \frac{\|f'\|_\alpha \|g'\|_q}{\Gamma^2(\beta)} \int_0^\tau \int_0^\tau (\tau - \theta)^{\beta-1} (\tau - \vartheta)^{\beta-1} \\ &\quad \times \theta^{\alpha-1} \vartheta^{\alpha-1} h(\theta)h'(\vartheta) |\theta - \vartheta|^{\frac{1}{p'} + \frac{1}{q'}} d\theta d\vartheta. \end{aligned} \tag{24}$$

Remark 3 If we let $\beta = \alpha = 1$ in Theorem 3.2, then we get the inequality (4).

4 Concluding remarks

In this paper, we established certain fractional conformable inequalities related to the weighted and the extended Chebyshev functionals. The inequalities obtained in the present paper are more general than the existing classical inequalities cited therein. This work will reduce to the inequalities some Riemann–Liouville integral inequalities by taking $\alpha = 1$, which have been presented earlier by [12]. Also, one can get the classical results by taking $\alpha = \beta = 1$.

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Author details

¹Department of Basic Sciences and Humanities, College of Computer and Information Sciences, Majmaah University, Al Majmaah, Saudi Arabia. ²Department of Mathematics, Shaheed Benazir Bhutto University, Sheringal, Pakistan. ³Department of Mathematics, College of Arts and Sciences, Prince Sattam Bin Abdulaziz University, Wadi Aldawasir, Saudi Arabia. ⁴Department of Mathematics, University of Sargodha, Sargodha, Pakistan.

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