

RESEARCH

Open Access



# Modification of certain fractional integral inequalities for convex functions

Pshtiwan Othman Mohammed<sup>1</sup> and Thabet Abdeljawad<sup>2,3,4\*</sup>

\*Correspondence:

[tabeljawad@psu.edu.sa](mailto:tabeljawad@psu.edu.sa)

<sup>2</sup>Department of Mathematics and General Sciences, Prince Sultan University, Riyadh, Saudi Arabia

<sup>3</sup>Department of Medical Research, China Medical University, Taichung, Taiwan

Full list of author information is available at the end of the article

## Abstract

We consider the modified Hermite–Hadamard inequality and related results on integral inequalities, in the context of fractional calculus using the Riemann–Liouville fractional integrals. Our results generalize and modify some existing results. Finally, some applications to special means of real numbers are given. Moreover, some error estimates for the midpoint formula are pointed out.

**MSC:** 26D07; 26D15; 26D10; 26A33

**Keywords:** Riemann–Liouville fractional integral; Convex function; Hermite–Hadamard inequality; Special means; Midpoint formula

## 1 Introduction

The generalization of certain integral inequalities to the fractional scope, in both continuous and discrete versions, have attracted many researchers in the recent few years and before [1, 19, 20]. In this article, our work is devoted to Hadamard–Hermite type for convex functions in the framework of Riemann–Liouville fractional type integrals.

A function  $g : \mathcal{I} \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is said to be convex on the interval  $\mathcal{I}$ , if the inequality

$$g(\ell x + (1 - \ell)y) \leq \ell g(x) + (1 - \ell)g(y) \quad (1)$$

holds for all  $x, y \in \mathcal{I}$  and  $\ell \in [0, 1]$ . We say that  $g$  is concave if  $-g$  is convex.

For convex functions (1), many equalities and inequalities have been established by many authors; such as the Hardy type inequality [3], Ostrowski type inequality [7], Olsen type inequality [8], Gagliardo–Nirenberg type inequality [22], midpoint type inequality [10] and trapezoidal type inequality [14]. But the most important inequality is the Hermite–Hadamard type inequality [6], which is defined by

$$g\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u} \int_u^v g(x) dx \leq \frac{g(u)+g(v)}{2}, \quad (2)$$

where  $g : \mathcal{I} \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is assumed to be a convex function on  $\mathcal{I}$  where  $u, v \in \mathcal{I}$  with  $u < v$ .

A number of mathematicians in the field of applied and pure mathematics have devoted their efforts to generalizing, refining, finding counterparts of, and extending the Hermite–

© The Author(s) 2020. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Hadamard inequality (2) for different classes of convex functions and mappings. For more recent results obtained in view of inequality (2), we refer the reader to [2, 4, 6, 13, 16, 18].

In [21], Sarikaya et al. obtained the Hermite–Hadamard inequalities in fractional integral form:

$$g\left(\frac{u+v}{2}\right) \leq \frac{\Gamma(\vartheta+1)}{2(v-u)^\vartheta} [\mathfrak{J}_{u^+}^\vartheta g(v) + \mathfrak{J}_{v^-}^\vartheta g(u)] \leq \frac{g(u)+g(v)}{2}, \tag{3}$$

where  $g : [u, v] \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is assumed to be a positive convex function on  $[u, v]$ ,  $g \in L_1[u, v]$  with  $u < v$ , and  $\mathfrak{J}_{u^+}^\vartheta$  and  $\mathfrak{J}_{v^-}^\vartheta$  are the left-sided and right-sided Riemann–Liouville fractional integrals of order  $\vartheta > 0$ , which, respectively, are defined by [9]

$$\begin{aligned} \mathfrak{J}_{u^+}^\vartheta g(x) &= \frac{1}{\Gamma(\vartheta)} \int_u^x (x-\ell)^{\vartheta-1} g(\ell) d\ell, \quad x > u, \\ \mathfrak{J}_{v^-}^\vartheta g(x) &= \frac{1}{\Gamma(\vartheta)} \int_x^v (\ell-x)^{\vartheta-1} g(\ell) d\ell, \quad x < v. \end{aligned}$$

It is clear that inequality (3) is a generalization of Hermite–Hadamard inequality (2). If we take  $\vartheta = 1$  in (3) we obtain (2). Many inequalities have been established in view of inequality (3); for more details see [5, 10, 11, 14, 15, 21, 23].

Recently, in [12], Mehrez and Agarwal obtained a new modification of the Hermite–Hadamard inequality (2); this is given by

$$g\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u} \int_u^v g(x) dx \leq \frac{g(\frac{3v-u}{2}) + 2g(\frac{u+v}{2}) + g(\frac{3u-v}{2})}{4}. \tag{4}$$

Furthermore, Mehrez and Agarwal obtained many inequalities in view of inequalities (4); for which we refer the reader to their interesting paper [12].

The aim of this paper is to establish new inequalities of Hermite–Hadamard type for convex functions via Riemann–Liouville fractional integrals.

### 2 Preliminary lemmas

In order to obtain our main results, we need some qualities which are stated in the following lemmas.

**Lemma 1** ([23]) *Let  $g : [u, v] \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $(u, v)$  with  $u < v$ . If  $g' \in L_1[u, v]$ , then we have*

$$\begin{aligned} &\frac{\Gamma(\vartheta+1)}{2(v-u)^\vartheta} [\mathfrak{J}_{u^+}^\vartheta g(v) + \mathfrak{J}_{v^-}^\vartheta g(u)] - g\left(\frac{u+v}{2}\right) \\ &= \frac{v-u}{2} \left[ \int_0^1 \kappa g'(\ell u + (1-\ell)v) d\ell - \int_0^1 [(1-\ell)^\vartheta - \ell^\vartheta] g'(\ell u + (1-\ell)v) d\ell \right], \tag{5} \end{aligned}$$

where

$$\kappa = \begin{cases} 1 & 0 \leq \ell < \frac{1}{2}, \\ -1 & \frac{1}{2} \leq \ell < 1. \end{cases}$$

**Lemma 2** ([5]) *Let  $g : \mathcal{I} \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a twice differentiable function on  $\mathcal{I}^\circ$  (the interior of  $\mathcal{I}$ ). Assume that  $u, v \in \mathcal{I}^\circ$  with  $u < v$ . If  $g'' \in L_1[u, v]$ , then for  $\vartheta > 0$  we have*

$$\begin{aligned} & \frac{g(u) + g(v)}{2} - \frac{\Gamma(\vartheta + 1)}{2(v - u)^\vartheta} [\mathfrak{J}_{u^+}^\vartheta g(v) + \mathfrak{J}_{v^-}^\vartheta g(u)] \\ &= \frac{(v - u)^2}{2(\vartheta + 1)} \int_0^1 \ell(1 - \ell)^\vartheta [g''(\ell u + (1 - \ell)v) + g''((1 - \ell)u + \ell v)] d\ell. \end{aligned} \tag{6}$$

**Lemma 3** *Let  $g : \mathcal{I} \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable function on  $\mathcal{I}^\circ$  and  $g \in L_1[u, v]$ . If  $g$  is a convex function on  $[u, v]$ , then for  $\vartheta > 0$  we have*

$$g\left(\frac{u + v}{2}\right) \leq \frac{\Gamma(\vartheta + 1)}{2(v - u)^\vartheta} [\mathfrak{J}_{u^+}^\vartheta g(v) + \mathfrak{J}_{v^-}^\vartheta g(u)] \leq \frac{g(\frac{3v-u}{2}) + 2g(\frac{u+v}{2}) + g(\frac{3u-v}{2})}{4} \tag{7}$$

and

$$\left| \frac{\Gamma(\vartheta + 1)}{2(v - u)^\vartheta} [\mathfrak{J}_{u^+}^\vartheta g(v) + \mathfrak{J}_{v^-}^\vartheta g(u)] - \frac{1}{2}g\left(\frac{u + v}{2}\right) \right| \leq \frac{g(\frac{3v-u}{2}) + g(\frac{3u-v}{2})}{4}. \tag{8}$$

*Proof* From the definition of Riemann–Liouville fractional integral, we have

$$\begin{aligned} & \frac{\Gamma(\vartheta + 1)}{2(v - u)^\vartheta} [\mathfrak{J}_{u^+}^\vartheta g(v) + \mathfrak{J}_{v^-}^\vartheta g(u)] \\ &= \frac{\vartheta}{2(v - u)^\vartheta} \left[ \int_u^v (v - x)^{\vartheta-1} g(x) dx + \int_u^v (x - u)^{\vartheta-1} g(x) dx \right]. \end{aligned}$$

By using the change of the variable  $x = \frac{3}{4}\ell + \frac{u+v}{4}$  for  $\ell \in [\frac{3u-v}{3}, \frac{3v-u}{3}]$ , we obtain

$$\begin{aligned} & \frac{\Gamma(\vartheta + 1)}{2(v - u)^\vartheta} [\mathfrak{J}_{u^+}^\vartheta g(v) + \mathfrak{J}_{v^-}^\vartheta g(u)] \\ &= \frac{3\vartheta}{8(v - u)^\vartheta} \left[ \int_{\frac{3u-v}{3}}^{\frac{3v-u}{3}} \left(\frac{3v - u}{4} - \frac{3}{4}\ell\right)^{\vartheta-1} g\left(\frac{3}{4}\ell + \frac{u + v}{4}\right) d\ell \right. \\ & \quad \left. + \int_{\frac{3u-v}{3}}^{\frac{3v-u}{3}} \left(\frac{3}{4}\ell - \frac{3u - v}{4}\right)^{\vartheta-1} g\left(\frac{3}{4}\ell + \frac{u + v}{4}\right) d\ell \right]. \end{aligned} \tag{9}$$

Since  $g$  is convex on  $[u, v]$ , we have

$$g\left(\frac{3}{4}\ell + \frac{u + v}{4}\right) = g\left(\frac{\frac{3}{2}\ell + \frac{u+v}{2}}{2}\right) \leq \frac{1}{2}g\left(\frac{3}{2}\ell\right) + \frac{1}{2}g\left(\frac{u + v}{2}\right).$$

It follows from this and (9) that

$$\begin{aligned} & \frac{\Gamma(\vartheta + 1)}{2(v - u)^\vartheta} [\mathfrak{J}_{u^+}^\vartheta g(v) + \mathfrak{J}_{v^-}^\vartheta g(u)] \\ & \leq \frac{3\vartheta}{16(v - u)^\vartheta} \left[ \int_{\frac{3u-v}{3}}^{\frac{3v-u}{3}} \left(\frac{3v - u}{4} - \frac{3}{4}\ell\right)^{\vartheta-1} \left\{ g\left(\frac{3}{2}\ell\right) + g\left(\frac{u + v}{2}\right) \right\} d\ell \right. \\ & \quad \left. + \int_{\frac{3u-v}{3}}^{\frac{3v-u}{3}} \left(\frac{3}{4}\ell - \frac{3u - v}{4}\right)^{\vartheta-1} \left\{ g\left(\frac{3}{2}\ell\right) + g\left(\frac{u + v}{2}\right) \right\} d\ell \right] \end{aligned}$$

$$\begin{aligned}
 &= \frac{g\left(\frac{u+v}{2}\right)}{2} + \frac{3\vartheta}{16(v-u)^\vartheta} \left[ \int_{\frac{3u-v}{3}}^{\frac{3v-u}{3}} \left(\frac{3v-u}{4} - \frac{3}{4}\ell\right)^{\vartheta-1} g\left(\frac{3}{2}\ell\right) d\ell \right. \\
 &\quad \left. + \int_{\frac{3u-v}{3}}^{\frac{3v-u}{3}} \left(\frac{3}{4}\ell - \frac{3u-v}{4}\right)^{\vartheta-1} g\left(\frac{3}{2}\ell\right) d\ell \right].
 \end{aligned}$$

Again, by using the change of the variable  $z = \frac{3}{2}\ell$  for  $\ell \in [\frac{3u-v}{2}, \frac{3v-u}{2}]$ , we obtain

$$\begin{aligned}
 &\frac{\Gamma(\vartheta+1)}{2(v-u)^\vartheta} [\mathfrak{J}_{u^+}^\vartheta g(v) + \mathfrak{J}_{v^-}^\vartheta g(u)] \\
 &\leq \frac{g\left(\frac{u+v}{2}\right)}{2} + \frac{\vartheta}{2^{\vartheta+2}(v-u)^\vartheta} \\
 &\quad \times \left[ \int_{\frac{3u-v}{2}}^{\frac{3v-u}{2}} \left(\frac{3u-v}{2} - z\right)^{\vartheta-1} g(z) dz + \int_{\frac{3u-v}{2}}^{\frac{3v-u}{2}} \left(z - \frac{3v-u}{2}\right)^{\vartheta-1} g(z) dz \right] \\
 &= \frac{g\left(\frac{u+v}{2}\right)}{2} + \frac{\Gamma(\vartheta+1)}{2^{\vartheta+2}(v-u)^\vartheta} \left[ \mathfrak{J}_{\left(\frac{3u-v}{2}\right)^+}^\vartheta g\left(\frac{3v-u}{2}\right) + \mathfrak{J}_{\left(\frac{3v-u}{2}\right)^-}^\vartheta g\left(\frac{3u-v}{2}\right) \right]. \tag{10}
 \end{aligned}$$

Replace  $u$  by  $\frac{3u-v}{2}$  and  $v$  by  $\frac{3v-u}{2}$  in the right-hand side of inequality (3) and multiply both sides by  $\frac{1}{2}$ , we get

$$\frac{\Gamma(\vartheta+1)}{2^{\vartheta+2}(v-u)^\vartheta} \left[ \mathfrak{J}_{\left(\frac{3u-v}{2}\right)^+}^\vartheta g\left(\frac{3v-u}{2}\right) + \mathfrak{J}_{\left(\frac{3v-u}{2}\right)^-}^\vartheta g\left(\frac{3u-v}{2}\right) \right] \leq \frac{g\left(\frac{3v-u}{2}\right) + g\left(\frac{3u-v}{2}\right)}{4}. \tag{11}$$

From (10) and (11), we obtain the desired inequality (7) and from (7) we can easily obtain the inequality (8). These complete the proof of Lemma 3.  $\square$

*Remark 1* If we use  $\vartheta = 1$  in Lemma 3, then Lemma 3 reduces to Lemma 3 in [12]. In particular, inequalities (7) reduces to the inequalities (4).

### 3 Hermite–Hadamard type inequalities

Our main results start from the following theorem.

**Theorem 1** Let  $g : \mathcal{I}^o \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $\mathcal{I}^o$ ,  $u, v \in \mathcal{I}^o$  with  $u < v$ . Let  $g' \in L_1[\frac{3u-v}{2}, \frac{3v-u}{2}]$  and  $g' : [\frac{3u-v}{2}, \frac{3v-u}{2}] \rightarrow \mathbb{R}$  be a continuous function on  $[\frac{3u-v}{2}, \frac{3v-u}{2}]$ . If  $|g'|^q, q \geq 1$  is a convex function on  $[\frac{3u-v}{2}, \frac{3v-u}{2}]$ , then

$$\begin{aligned}
 &\left| \frac{\Gamma(\vartheta+1)}{2(v-u)^\vartheta} [\mathfrak{J}_{u^+}^\vartheta g(v) + \mathfrak{J}_{v^-}^\vartheta g(u)] - g\left(\frac{u+v}{2}\right) \right| \\
 &\leq \frac{v-u}{2} \left( \frac{2^\vartheta-1}{2^\vartheta(\vartheta+1)} + \frac{1}{2} \right) \left( \left| g'\left(\frac{3u-v}{2}\right) \right|^q + \left| g'\left(\frac{3v-u}{2}\right) \right|^q \right)^{\frac{1}{q}}. \tag{12}
 \end{aligned}$$

*Proof* First we prove the theorem for  $q = 1$ . By changing the variables  $u \rightarrow \frac{3u-v}{2}$  and  $v \rightarrow \frac{3v-u}{2}$  in Lemma 1, we get

$$\begin{aligned}
 &\frac{\Gamma(\vartheta+1)}{2^{\vartheta+1}(v-u)^\vartheta} \left[ \mathfrak{J}_{\left(\frac{3u-v}{2}\right)^+}^\vartheta g\left(\frac{3v-u}{2}\right) + \mathfrak{J}_{\left(\frac{3v-u}{2}\right)^-}^\vartheta g\left(\frac{3u-v}{2}\right) \right] \\
 &= g\left(\frac{u+v}{2}\right) + (v-u) \left[ \int_0^1 \kappa g'\left(\frac{3v-u}{2} + 2(v-u)\ell\right) d\ell \right.
 \end{aligned}$$

$$\begin{aligned}
 & - \int_0^1 [(1-\ell)^\vartheta - \ell^\vartheta] g' \left( \frac{3v-u}{2} + 2(v-u)\ell \right) d\ell \\
 & = g \left( \frac{u+v}{2} \right) + (v-u) \left[ \int_0^{\frac{1}{2}} g' \left( \frac{3v-u}{2} + 2(v-u)\ell \right) d\ell \right. \\
 & \quad - \int_{\frac{1}{2}}^1 g' \left( \frac{3v-u}{2} + 2(v-u)\ell \right) d\ell \\
 & \quad \left. + \int_0^1 [\ell^\vartheta - (1-\ell)^\vartheta] g' \left( \frac{3v-u}{2} + 2(v-u)\ell \right) d\ell \right]. \tag{13}
 \end{aligned}$$

From (10) and (13), we find

$$\begin{aligned}
 & \left| \frac{\Gamma(\vartheta+1)}{2(v-u)^\vartheta} [\mathfrak{J}_{u^+}^\vartheta g(v) + \mathfrak{J}_{v^-}^\vartheta g(u)] - g \left( \frac{u+v}{2} \right) \right| \\
 & \leq \frac{v-u}{2} \left[ \int_0^{\frac{1}{2}} [(1-\ell)^\vartheta - \ell^\vartheta + 1] \left| g' \left( \frac{3v-u}{2} + 2(v-u)\ell \right) \right| d\ell \right. \\
 & \quad \left. + \int_{\frac{1}{2}}^1 [\ell^\vartheta - (1-\ell)^\vartheta + 1] \left| g' \left( \frac{3v-u}{2} + 2(v-u)\ell \right) \right| d\ell \right]. \tag{14}
 \end{aligned}$$

Using the convexity of  $|g'|$  on  $[\frac{3u-v}{2}, \frac{3v-u}{2}]$ , we obtain

$$\begin{aligned}
 & \int_0^{\frac{1}{2}} [(1-\ell)^\vartheta - \ell^\vartheta + 1] \left| g' \left( \frac{3v-u}{2} + 2(v-u)\ell \right) \right| d\ell \\
 & = \int_0^{\frac{1}{2}} [(1-\ell)^\vartheta - \ell^\vartheta + 1] \left| g' \left( \frac{3u-v}{2}\ell + \frac{3v-u}{2}(1-\ell) \right) \right| d\ell \\
 & \leq \int_0^{\frac{1}{2}} [(1-\ell)^\vartheta - \ell^\vartheta + 1] \left\{ \ell \left| g' \left( \frac{3u-v}{2} \right) \right| + (1-\ell) \left| g' \left( \frac{3v-u}{2} \right) \right| \right\} d\ell \\
 & = \left( \frac{1}{8} + \frac{1}{(\vartheta+1)(\vartheta+2)} - \frac{1}{(\vartheta+1)2^{\vartheta+1}} \right) \left| g' \left( \frac{3u-v}{2} \right) \right| \\
 & \quad + \left( \frac{3}{8} + \frac{1}{\vartheta+2} - \frac{1}{(\vartheta+1)2^{\vartheta+1}} \right) \left| g' \left( \frac{3v-u}{2} \right) \right|. \tag{15}
 \end{aligned}$$

Analogously, we obtain

$$\begin{aligned}
 & \int_{\frac{1}{2}}^1 [\ell^\vartheta - (1-\ell)^\vartheta + 1] \left| g' \left( \frac{3v-u}{2} + 2(v-u)\ell \right) \right| d\ell \\
 & \leq \left( \frac{3}{8} + \frac{1}{\vartheta+2} - \frac{1}{(\vartheta+1)2^{\vartheta+1}} \right) \left| g' \left( \frac{3u-v}{2} \right) \right| \\
 & \quad + \left( \frac{1}{8} + \frac{1}{(\vartheta+1)(\vartheta+2)} - \frac{1}{(\vartheta+1)2^{\vartheta+1}} \right) \left| g' \left( \frac{3v-u}{2} \right) \right|. \tag{16}
 \end{aligned}$$

Using (15) and (16) in (14), we get inequality (12) for  $q = 1$ .

For  $q > 1$  we use the Hölder inequality and the convexity of  $|g'|^q$  on  $[\frac{3u-v}{2}, \frac{3v-u}{2}]$  to obtain

$$\begin{aligned}
 & \int_0^{\frac{1}{2}} [(1-\ell)^\vartheta - \ell^\vartheta + 1] \left| g' \left( \frac{3v-u}{2} + 2(v-u)\ell \right) \right| d\ell \\
 & \leq \left( \int_0^{\frac{1}{2}} [(1-\ell)^\vartheta - \ell^\vartheta + 1] d\ell \right)^{1-\frac{1}{q}} \\
 & \quad \times \left( \int_0^{\frac{1}{2}} [(1-\ell)^\vartheta - \ell^\vartheta + 1] \left| g' \left( \frac{3u-v}{2}\ell + \frac{3v-u}{2}(1-\ell) \right) \right|^q d\ell \right)^{\frac{1}{q}} \\
 & \leq \left( \int_0^{\frac{1}{2}} [(1-\ell)^\vartheta - \ell^\vartheta + 1] d\ell \right)^{1-\frac{1}{q}} \\
 & \quad \times \left( \int_0^{\frac{1}{2}} [(1-\ell)^\vartheta - \ell^\vartheta + 1] \left\{ \ell \left| g' \left( \frac{3u-v}{2} \right) \right|^q + (1-\ell) \left| g' \left( \frac{3v-u}{2} \right) \right|^q \right\} d\ell \right)^{\frac{1}{q}} \\
 & = \left( \frac{1}{2} + \frac{1}{\vartheta+1} \left( 1 - \frac{1}{2^\vartheta} \right) \right)^{1-\frac{1}{q}} \left( \left[ \frac{1}{8} + \frac{1}{(\vartheta+1)(\vartheta+2)} - \frac{1}{(\vartheta+1)2^{\vartheta+1}} \right] \left| g' \left( \frac{3u-v}{2} \right) \right|^q \right. \\
 & \quad \left. + \left[ \frac{3}{8} + \frac{1}{\vartheta+2} - \frac{1}{(\vartheta+1)2^{\vartheta+1}} \right] \left| g' \left( \frac{3v-u}{2} \right) \right|^q \right)^{\frac{1}{q}}. \tag{17}
 \end{aligned}$$

Analogously,

$$\begin{aligned}
 & \int_{\frac{1}{2}}^1 [\ell^\vartheta - (1-\ell)^\vartheta + 1] \left| g' \left( \frac{3v-u}{2} + 2(v-u)\ell \right) \right| d\ell \\
 & \leq \left( \frac{1}{2} + \frac{1}{\vartheta+1} \left( 1 - \frac{1}{2^\vartheta} \right) \right)^{1-\frac{1}{q}} \left( \left[ \frac{3}{8} + \frac{1}{\vartheta+2} - \frac{1}{(\vartheta+1)2^{\vartheta+1}} \right] \left| g' \left( \frac{3u-v}{2} \right) \right|^q \right. \\
 & \quad \left. + \left[ \frac{1}{8} + \frac{1}{(\vartheta+1)(\vartheta+2)} - \frac{1}{(\vartheta+1)2^{\vartheta+1}} \right] \left| g' \left( \frac{3v-u}{2} \right) \right|^q \right)^{\frac{1}{q}}. \tag{18}
 \end{aligned}$$

Using (17) and (18) in (14), we get

$$\begin{aligned}
 & \left| \frac{\Gamma(\vartheta+1)}{2(v-u)^\vartheta} [\mathfrak{J}_{u^+}^\vartheta g(v) + \mathfrak{J}_{v^-}^\vartheta g(u)] - g\left(\frac{u+v}{2}\right) \right| \\
 & \leq \frac{v-u}{2} \left( \frac{1}{2} + \frac{1}{\vartheta+1} \left( 1 - \frac{1}{2^\vartheta} \right) \right)^{1-\frac{1}{q}} \{ (c_1 + d_1)^{\frac{1}{q}} + (c_2 + d_2)^{\frac{1}{q}} \} \tag{19}
 \end{aligned}$$

where

$$\begin{aligned}
 c_1 &= \left[ \frac{1}{8} + \frac{1}{(\vartheta+1)(\vartheta+2)} - \frac{1}{(\vartheta+1)2^{\vartheta+1}} \right] \left| g' \left( \frac{3u-v}{2} \right) \right|^q, \\
 c_2 &= \left[ \frac{3}{8} + \frac{1}{\vartheta+2} - \frac{1}{(\vartheta+1)2^{\vartheta+1}} \right] \left| g' \left( \frac{3u-v}{2} \right) \right|^q, \\
 d_1 &= \left[ \frac{3}{8} + \frac{1}{\vartheta+2} - \frac{1}{(\vartheta+1)2^{\vartheta+1}} \right] \left| g' \left( \frac{3v-u}{2} \right) \right|^q, \\
 d_2 &= \left[ \frac{1}{8} + \frac{1}{(\vartheta+1)(\vartheta+2)} - \frac{1}{(\vartheta+1)2^{\vartheta+1}} \right] \left| g' \left( \frac{3v-u}{2} \right) \right|^q.
 \end{aligned}$$

Applying the formula

$$\sum_{k=1}^n (c_k + d_k)^m \leq \sum_{k=1}^n c_k^m + \sum_{k=1}^n d_k^m, \quad 0 \leq m < 1,$$

for (19) and then using the fact that  $|x_1^r + x_2^r| \leq |x_1 + x_2|^r$ ,  $x_1, x_2, r \in [0, 1]$ , we obtain the inequality (12). This completes the proof of Theorem 1.  $\square$

**Corollary 1** *With similar assumptions to Theorem 1, if  $\vartheta = 1$ , then*

$$\left| \frac{1}{v-u} \int_u^v g(x) dx - g\left(\frac{u+v}{2}\right) \right| \leq \frac{3(v-u)}{8} \left( \left| g'\left(\frac{3u-v}{2}\right) \right|^q + \left| g'\left(\frac{3v-u}{2}\right) \right|^q \right)^{\frac{1}{q}}, \quad (20)$$

which is obtained by Mehrez and Agarwal in [12, Theorem 1].

*Remark 2* In [23], the following inequality has been established:

$$\left| \frac{1}{v-u} \int_u^v g(x) dx - g\left(\frac{u+v}{2}\right) \right| \leq \frac{3(v-u)}{8} (|g'(u)| + |g'(v)|). \quad (21)$$

We show an analytical and numerical comparison between the left-hand side of inequalities (20) and (21).

1. Let  $q = 1$  and  $u, v \in \mathbb{R}$  with  $u < v$ . Then:
  - (a) If the function  $|g'|$  is increasing on  $[\frac{3u-v}{2}, \frac{3v-u}{2}]$ . Since  $\frac{3u-v}{2} < u < v < \frac{3v-u}{2}$ , we obtain

$$g'\left(\frac{3u-v}{2}\right) < g'(u) \quad \text{and} \quad g'(v) < g'\left(\frac{3v-u}{2}\right);$$

or if the function  $|g'|$  is decreasing on  $[\frac{3u-v}{2}, \frac{3v-u}{2}]$ , we obtain

$$g'(u) < g'\left(\frac{3u-v}{2}\right) \quad \text{and} \quad g'\left(\frac{3v-u}{2}\right) < g'(v).$$

In those cases, comparison does not occur analytically between inequalities (20) and (21).

- (b) If the function  $|g'|$  is increasing on  $[\frac{3u-v}{2}, u]$ , and decreasing on  $[v, \frac{3v-u}{2}]$ , then we have

$$g'\left(\frac{3u-v}{2}\right) < g'(u) \quad \text{and} \quad g'\left(\frac{3v-u}{2}\right) < g'(v).$$

This tells us the right-hand side of inequality (20) is better than the right-hand side of inequality (21).

- (c) If the function  $|g'|$  is decreasing on  $[\frac{3u-v}{2}, u]$ , and increasing on  $[v, \frac{3v-u}{2}]$ , then we conclude that the right-hand side of inequality (21) is better than the right-hand side of inequality (20).

- Suppose that  $\mathbf{m}$  and  $\mathbf{n}$  represent the right-hand side of inequalities (20) and (21), respectively. Let  $[u, v] = [-1, -\frac{1}{2}]$  and  $g(x) = e^x$ , then we obtain  $\mathbf{m} = 0.199744$  and  $\mathbf{n} = 0.167444$  when  $q = 1$  and  $\mathbf{m} = 0.155592$  when  $q = 2$ . Then we conclude that the right-hand side of inequality (20) is worse than the right-hand side of inequalities (21) when  $q = 1$ , but better when  $q = 2$ .

**Theorem 2** Let  $g : \mathcal{I}^o \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $\mathcal{I}^o$ ,  $u, v \in \mathcal{I}^o$  with  $u < v$ . Let  $g' \in L_1[\frac{3u-v}{2}, \frac{3v-u}{2}]$  and  $g' : [\frac{3u-v}{2}, \frac{3v-u}{2}] \rightarrow \mathbb{R}$  be a continuous function on  $[\frac{3u-v}{2}, \frac{3v-u}{2}]$ . If  $|g'|^q, q > 1$  is a convex function on  $[\frac{3u-v}{2}, \frac{3v-u}{2}]$ , then

$$\begin{aligned} & \left| \frac{\Gamma(\vartheta + 1)}{2(v-u)^\vartheta} [\mathfrak{J}_{u^+}^\vartheta g(v) + \mathfrak{J}_{v^-}^\vartheta g(u)] - g\left(\frac{u+v}{2}\right) \right| \\ & \leq \frac{v-u}{2} \left( \frac{2^{\vartheta p} - 1}{2^{\vartheta p}(\vartheta p + 1)} + \frac{1}{2} \right)^{\frac{1}{p}} \\ & \quad \times \left\{ \left( \frac{|g'(\frac{3u-v}{2})|^q + 3|g'(\frac{3v-u}{2})|^q}{8} \right)^{\frac{1}{q}} + \left( \frac{3|g'(\frac{3u-v}{2})|^q + |g'(\frac{3v-u}{2})|^q}{8} \right)^{\frac{1}{q}} \right\} \\ & \leq \frac{v-u}{2} \left( \frac{2^{\vartheta p} - 1}{2^{\vartheta p}(\vartheta p + 1)} + \frac{1}{2} \right)^{\frac{1}{p}} \left( \frac{|g'(\frac{3u-v}{2})|^q + |g'(\frac{3v-u}{2})|^q}{2} \right)^{\frac{1}{q}}, \end{aligned} \tag{22}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ .

*Proof* Applying Hölder’s inequality and the convexity of  $|g'|^q, q > 1$  on  $[\frac{3u-v}{2}, \frac{3v-u}{2}]$ , we obtain

$$\begin{aligned} & \int_0^{\frac{1}{2}} [(1-\ell)^\vartheta - \ell^\vartheta + 1] \left| g' \left( \frac{3u-v}{2}\ell + \frac{3v-u}{2}(1-\ell) \right) \right| d\ell \\ & \leq \left( \int_0^{\frac{1}{2}} [(1-\ell)^\vartheta - \ell^\vartheta + 1]^p d\ell \right)^{\frac{1}{p}} \\ & \quad \times \left( \int_0^{\frac{1}{2}} \left| g' \left( \frac{3u-v}{2}\ell + \frac{3v-u}{2}(1-\ell) \right) \right|^q d\ell \right)^{\frac{1}{q}} \\ & \leq \left( \int_0^{\frac{1}{2}} [(1-\ell)^\vartheta - \ell^\vartheta + 1]^p d\ell \right)^{\frac{1}{p}} \\ & \quad \times \left( \int_0^{\frac{1}{2}} \left\{ \ell \left| g' \left( \frac{3u-v}{2} \right) \right|^q + (1-\ell) \left| g' \left( \frac{3v-u}{2} \right) \right|^q \right\} d\ell \right)^{\frac{1}{q}}. \end{aligned} \tag{23}$$

Since  $(H_1 - H_2)^q \leq H_1^q - H_2^q$  for each  $H_1, H_2 > 0$  and  $q > 1$ , (23) becomes

$$\begin{aligned} & \int_0^{\frac{1}{2}} [(1-\ell)^\vartheta - \ell^\vartheta + 1] \left| g' \left( \frac{3v-u}{2} + 2(v-u)\ell \right) \right| d\ell \\ & \leq \left( \int_0^{\frac{1}{2}} [1 + (1-\ell)^{\vartheta p} - \ell^{\vartheta p}] d\ell \right)^{\frac{1}{p}} \end{aligned}$$

$$\begin{aligned} & \times \left( \left| g' \left( \frac{3u-v}{2} \right) \right|^q \int_0^{\frac{1}{2}} \ell \, d\ell + \left| g' \left( \frac{3v-u}{2} \right) \right|^q \int_0^{\frac{1}{2}} (1-\ell) \, d\ell \right)^{\frac{1}{q}} \\ & = \left( \frac{2^{\vartheta p} - 1}{2^{\vartheta p}(\vartheta p + 1)} + \frac{1}{2} \right)^{\frac{1}{p}} \left( \frac{|g'(\frac{3u-v}{2})|^q + 3|g'(\frac{3v-u}{2})|^q}{8} \right)^{\frac{1}{q}}. \end{aligned} \tag{24}$$

In a similar manner, we get

$$\begin{aligned} & \int_{\frac{1}{2}}^1 [\ell^{\vartheta} - (1-\ell)^{\vartheta} + 1] \left| g' \left( \frac{3u-v}{2} \ell + \frac{3v-u}{2} (1-\ell) \right) \right| d\ell \\ & \leq \left( \frac{2^{\vartheta p} - 1}{2^{\vartheta p}(\vartheta p + 1)} + \frac{1}{2} \right)^{\frac{1}{p}} \left( \frac{3|g'(\frac{3u-v}{2})|^q + |g'(\frac{3v-u}{2})|^q}{8} \right)^{\frac{1}{q}}. \end{aligned} \tag{25}$$

Using (24) and (25) in (14), we get

$$\begin{aligned} & \left| \frac{\Gamma(\vartheta + 1)}{2(v-u)^{\vartheta}} [\mathfrak{J}_{u^+}^{\vartheta} g(v) + \mathfrak{J}_{v^-}^{\vartheta} g(u)] - g\left(\frac{u+v}{2}\right) \right| \\ & \leq \frac{v-u}{2} \left( \frac{2^{\vartheta p} - 1}{2^{\vartheta p}(\vartheta p + 1)} + \frac{1}{2} \right)^{\frac{1}{p}} \\ & \quad \times \left\{ \left( \frac{|g'(\frac{3u-v}{2})|^q + 3|g'(\frac{3v-u}{2})|^q}{8} \right)^{\frac{1}{q}} + \left( \frac{3|g'(\frac{3u-v}{2})|^q + |g'(\frac{3v-u}{2})|^q}{8} \right)^{\frac{1}{q}} \right\}, \end{aligned}$$

which proves the first inequality of (22). Applying the fact

$$|c_1^m + c_2^m| \leq |c_1 + c_2|^m, \quad 0 \leq m < 1, \quad c_1, c_2 \in [0, 1]$$

to the last inequality, we get the second inequality of (22). This completes the proof.  $\square$

Collecting both of Theorems 1 and 2 we obtain the following corollary.

**Corollary 2** *Let  $\frac{1}{p} + \frac{1}{q} = 1$ , then from Theorems 1 and 2, we have*

$$\begin{aligned} & \left| \frac{\Gamma(\vartheta + 1)}{2(v-u)^{\vartheta}} [\mathfrak{J}_{u^+}^{\vartheta} g(v) + \mathfrak{J}_{v^-}^{\vartheta} g(u)] - g\left(\frac{u+v}{2}\right) \right| \\ & \leq \frac{v-u}{2} \left( \left| g' \left( \frac{3u-v}{2} \right) \right|^q + \left| g' \left( \frac{3v-u}{2} \right) \right|^q \right)^{\frac{1}{q}} \min\{\gamma_1, \gamma_2\}, \end{aligned}$$

where  $\gamma_1 = \left( \frac{2^{\vartheta} - 1}{2^{\vartheta}(\vartheta + 1)} + \frac{1}{2} \right)$  and  $\gamma_2 = \left( \frac{2^{\vartheta p} - 1}{2^{\vartheta p + \frac{p}{q}}(\vartheta p + 1)} + \frac{1}{2^{\frac{p}{q} + 1}} \right)^{\frac{1}{p}}$ .

**Corollary 3** *With similar assumptions to Theorem 2 if  $\vartheta = 1$ , we have*

$$\begin{aligned} & \left| \frac{1}{v-u} \int_u^v g(x) \, dx - g\left(\frac{u+v}{2}\right) \right| \\ & \leq (v-u) \left( \frac{1}{2^{p+1}(p+1)} \right)^{\frac{1}{p}} \left( \frac{|g'(\frac{3u-v}{2})|^q + |g'(\frac{3v-u}{2})|^q}{2} \right)^{\frac{1}{q}}, \end{aligned}$$

which is obtained by Mehrez and Agarwal in [12, Theorem 2].

**Theorem 3** Let  $g : \mathcal{I}^o \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $\mathcal{I}^o$ ,  $u, v \in \mathcal{I}^o$  with  $u < v$ . Let  $g'' : [\frac{3u-v}{2}, \frac{3v-u}{2}] \rightarrow \mathbb{R}$  be a continuous function on  $[\frac{3u-v}{2}, \frac{3v-u}{2}]$ . If  $|g''|^q, q \geq 1$  is a convex function on  $[\frac{3u-v}{2}, \frac{3v-u}{2}]$ , then

$$\begin{aligned} & \left| \frac{\Gamma(\vartheta + 1)}{2(v-u)^\vartheta} [\mathfrak{J}_{u^+}^\vartheta g(v) + \mathfrak{J}_{v^-}^\vartheta g(u)] - \frac{g(\frac{3v-u}{2}) + 2g(\frac{u+v}{2}) + g(\frac{3u-v}{2})}{4} \right| \\ & \leq \frac{\vartheta(v-u)^2}{2(\vartheta+1)(\vartheta+2)} \left\{ \left( \frac{2\vartheta+4}{3\vartheta+9} \left| g''\left(\frac{3u-v}{2}\right) \right|^q + \frac{\vartheta+5}{3\vartheta+9} \left| g''\left(\frac{3v-u}{2}\right) \right|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left( \frac{\vartheta+5}{3\vartheta+9} \left| g''\left(\frac{3u-v}{2}\right) \right|^q + \frac{2\vartheta+4}{3\vartheta+9} \left| g''\left(\frac{3v-u}{2}\right) \right|^q \right)^{\frac{1}{q}} \right\}. \end{aligned} \tag{26}$$

*Proof* From Lemma 2 we have

$$\begin{aligned} & \frac{\Gamma(\vartheta + 1)}{2^{\vartheta+1}(v-u)^\vartheta} \left[ \mathfrak{J}_{\left(\frac{3u-v}{2}\right)^+}^\vartheta g\left(\frac{3v-u}{2}\right) + \mathfrak{J}_{\left(\frac{3v-u}{2}\right)^-}^\vartheta g\left(\frac{3u-v}{2}\right) \right] \\ & = \frac{g\left(\frac{3u-v}{2}\right) + g\left(\frac{3v-u}{2}\right)}{2} \\ & \quad - \frac{2(v-u)^2}{\vartheta+1} \int_0^1 \ell(1-\ell^\vartheta) \left[ g''\left(\ell\left(\frac{3u-v}{2}\right) + (1-\ell)\left(\frac{3v-u}{2}\right)\right) \right. \\ & \quad \left. + g''\left((1-\ell)\left(\frac{3u-v}{2}\right) + \ell\left(\frac{3v-u}{2}\right)\right) \right] d\ell. \end{aligned} \tag{27}$$

From (27) and (10), we have

$$\begin{aligned} & \left| \frac{\Gamma(\vartheta + 1)}{2(v-u)^\vartheta} [\mathfrak{J}_{u^+}^\vartheta g(v) + \mathfrak{J}_{v^-}^\vartheta g(u)] - \frac{g(\frac{3v-u}{2}) + 2g(\frac{u+v}{2}) + g(\frac{3u-v}{2})}{4} \right| \\ & \leq \frac{(v-u)^2}{\vartheta+1} \int_0^1 \ell(1-\ell^\vartheta) \\ & \quad \times \left[ \left| g''\left(\ell\left(\frac{3u-v}{2}\right) + (1-\ell)\left(\frac{3v-u}{2}\right)\right) \right| \right. \\ & \quad \left. + \left| g''\left((1-\ell)\left(\frac{3u-v}{2}\right) + \ell\left(\frac{3v-u}{2}\right)\right) \right| \right] d\ell. \end{aligned} \tag{28}$$

Using the convexity of  $|g''|^q, q > 1$  on  $[\frac{3u-v}{2}, \frac{3v-u}{2}]$  and Hölder’s inequality, we have

$$\begin{aligned} & \int_0^1 \ell(1-\ell^\vartheta) \left[ \left| g''\left(\ell\left(\frac{3u-v}{2}\right) + (1-\ell)\left(\frac{3v-u}{2}\right)\right) \right| \right. \\ & \quad \left. + \left| g''\left((1-\ell)\left(\frac{3u-v}{2}\right) + \ell\left(\frac{3v-u}{2}\right)\right) \right| \right] d\ell \\ & \leq \left( \int_0^1 \ell(1-\ell^\vartheta) d\ell \right)^{1-\frac{1}{q}} \\ & \quad \times \left[ \left( \int_0^1 \ell(1-\ell^\vartheta) \left\{ \ell \left| g''\left(\left(\frac{3u-v}{2}\right)\right|^q + (1-\ell) \left| g''\left(\frac{3v-u}{2}\right) \right|^q \right\} d\ell \right)^{\frac{1}{q}} \right. \right. \end{aligned}$$

$$\begin{aligned}
 & + \left( \int_0^1 \ell(1-\ell^\vartheta) \left\{ (1-\ell) \left| g'' \left( \left( \frac{3u-v}{2} \right)^q + \ell \left| g'' \left( \frac{3v-u}{2} \right) \right|^q \right\} d\ell \right)^{\frac{1}{q}} \right] \\
 & = \frac{\vartheta}{2(\vartheta+2)} \left\{ \left( \frac{2\vartheta+4}{3\vartheta+9} \left| g'' \left( \frac{3u-v}{2} \right) \right|^q + \frac{\vartheta+5}{3\vartheta+9} \left| g'' \left( \frac{3v-u}{2} \right) \right|^q \right)^{\frac{1}{q}} \right. \\
 & \quad \left. + \left( \frac{\vartheta+5}{3\vartheta+9} \left| g'' \left( \frac{3u-v}{2} \right) \right|^q + \frac{2\vartheta+4}{3\vartheta+9} \left| g'' \left( \frac{3v-u}{2} \right) \right|^q \right)^{\frac{1}{q}} \right\}. \tag{29}
 \end{aligned}$$

Using (29) in (28) we get (26) for  $q > 1$ .

Now by using the convexity of  $|g''|$ , we find

$$\begin{aligned}
 & \int_0^1 \ell(1-\ell^\vartheta) \left[ \left| g'' \left( \ell \left( \frac{3u-v}{2} \right) + (1-\ell) \left( \frac{3v-u}{2} \right) \right) \right| \right. \\
 & \quad \left. + \left| g'' \left( (1-\ell) \left( \frac{3u-v}{2} \right) + \ell \left( \frac{3v-u}{2} \right) \right) \right| \right] d\ell \\
 & \leq \left( \int_0^1 \ell(1-\ell^\vartheta) \left\{ \ell \left| g'' \left( \left( \frac{3u-v}{2} \right) \right) \right| + (1-\ell) \left| g'' \left( \frac{3v-u}{2} \right) \right| \right\} d\ell \right) \\
 & \quad + \left( \int_0^1 \ell(1-\ell^\vartheta) \left\{ (1-\ell) \left| g'' \left( \left( \frac{3u-v}{2} \right) \right) \right| + \ell \left| g'' \left( \frac{3v-u}{2} \right) \right| \right\} d\ell \right) \\
 & = \left[ \frac{\vartheta}{3(\vartheta+3)} \left| g'' \left( \frac{3u-v}{2} \right) \right| + \frac{\vartheta(\vartheta+5)}{6(\vartheta+2)(\vartheta+3)} \left| g'' \left( \frac{3v-u}{2} \right) \right| \right. \\
 & \quad \left. + \frac{\vartheta}{3(\vartheta+3)} \left| g'' \left( \frac{3v-u}{2} \right) \right| + \frac{\vartheta(\vartheta+5)}{6(\vartheta+2)(\vartheta+3)} \left| g'' \left( \frac{3u-v}{2} \right) \right| \right] \\
 & = \frac{\vartheta}{2(\vartheta+2)} \left( \left| g'' \left( \frac{3u-v}{2} \right) \right| + \left| g'' \left( \frac{3v-u}{2} \right) \right| \right). \tag{30}
 \end{aligned}$$

Substituting (30) into (28) we deduce that the inequality (26) holds true for  $q = 1$ . Hence the proof of Theorem 3 is completed.  $\square$

**Corollary 4** *With similar assumptions to Theorem 3 if  $\vartheta = 1$ , we have*

$$\begin{aligned}
 & \left| \frac{1}{v-u} \int_u^v g(x) dx - \frac{g\left(\frac{3v-u}{2}\right) + 2g\left(\frac{u+v}{2}\right) + g\left(\frac{3u-v}{2}\right)}{4} \right| \\
 & \leq \frac{(v-u)^2}{6} \left( \frac{\left| g''\left(\frac{3u-v}{2}\right) \right|^q + \left| g''\left(\frac{3v-u}{2}\right) \right|^q}{2} \right)^{\frac{1}{q}}. \tag{31}
 \end{aligned}$$

*Remark 3* In [12, Theorem 3], Mehrez and Agarwal obtained the following inequality:

$$\begin{aligned}
 & \left| \frac{1}{v-u} \int_u^v g(x) dx - \frac{g\left(\frac{3v-u}{2}\right) + 2g\left(\frac{u+v}{2}\right) + g\left(\frac{3u-v}{2}\right)}{4} \right| \\
 & \leq \frac{(v-u)^2}{3} \left( \frac{\left| g''\left(\frac{3u-v}{2}\right) \right|^q + \left| g''\left(\frac{3v-u}{2}\right) \right|^q}{2} \right)^{\frac{1}{q}}. \tag{32}
 \end{aligned}$$

The right-hand side of (31) confirms the modification of our work compared with (32).

*Remark 4* If  $g''(x)$  is bounded on the interval  $[\frac{3u-v}{2}, \frac{3v-u}{2}]$ , then Theorem 3 reduces to

$$\left| \frac{\Gamma(\vartheta + 1)}{2(v-u)^\vartheta} [\mathfrak{J}_{u^+}^\vartheta g(v) + \mathfrak{J}_{v^-}^\vartheta g(u)] - \frac{g(\frac{3v-u}{2}) + 2g(\frac{u+v}{2}) + g(\frac{3u-v}{2})}{4} \right| \leq \frac{M\vartheta(v-u)^2}{(\vartheta + 1)(\vartheta + 2)}$$

for some  $M \in \mathbb{R}$ .

**Theorem 4** Let  $g : \mathcal{I}^o \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $\mathcal{I}^o$ ,  $u, v \in \mathcal{I}^o$  with  $u < v$ . Let  $g'' : [\frac{3u-v}{2}, \frac{3v-u}{2}] \rightarrow \mathbb{R}$  be a continuous function on  $[\frac{3u-v}{2}, \frac{3v-u}{2}]$ . If  $|g''|^q, q > 1$  is a convex function on  $[\frac{3u-v}{2}, \frac{3v-u}{2}]$ , then

$$\begin{aligned} & \left| \frac{\Gamma(\vartheta + 1)}{2(v-u)^\vartheta} [\mathfrak{J}_{u^+}^\vartheta g(v) + \mathfrak{J}_{v^-}^\vartheta g(u)] - \frac{g(\frac{3v-u}{2}) + 2g(\frac{u+v}{2}) + g(\frac{3u-v}{2})}{4} \right| \\ & \leq \frac{2(v-u)^2}{\vartheta + 1} \beta^{\frac{1}{p}} (p + 1, \vartheta p + 1) \left( \frac{|g''(\frac{3u-v}{2})|^q + |g''(\frac{3v-u}{2})|^q}{2} \right)^{\frac{1}{q}}, \end{aligned} \tag{33}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ .

*Proof* From inequality (28) and the Hölder inequality, we have

$$\begin{aligned} & \left| \frac{\Gamma(\vartheta + 1)}{2(v-u)^\vartheta} [\mathfrak{J}_{u^+}^\vartheta g(v) + \mathfrak{J}_{v^-}^\vartheta g(u)] - \frac{g(\frac{3v-u}{2}) + 2g(\frac{u+v}{2}) + g(\frac{3u-v}{2})}{4} \right| \\ & \leq \frac{(v-u)^2}{\vartheta + 1} \left( \int_0^1 \ell^p (1 - \ell^\vartheta)^p d\ell \right)^{\frac{1}{p}} \\ & \quad \times \left[ \left( \int_0^1 \left| g'' \left( \ell \left( \frac{3u-v}{2} \right) + (1-\ell) \left( \frac{3v-u}{2} \right) \right) \right|^q d\ell \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left( \int_0^1 \left| g'' \left( \ell \left( \frac{3v-u}{2} \right) + (1-\ell) \left( \frac{3u-v}{2} \right) \right) \right|^q d\ell \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Using the fact that  $|g''|^q, q > 1$  is convex on  $[\frac{3u-v}{2}, \frac{3v-u}{2}]$ , we have

$$\begin{aligned} & \left| \frac{\Gamma(\vartheta + 1)}{2(v-u)^\vartheta} [\mathfrak{J}_{u^+}^\vartheta g(v) + \mathfrak{J}_{v^-}^\vartheta g(u)] - \frac{g(\frac{3v-u}{2}) + 2g(\frac{u+v}{2}) + g(\frac{3u-v}{2})}{4} \right| \\ & \leq \frac{(v-u)^2}{\vartheta + 1} \left( \int_0^1 \ell^p (1 - \ell^\vartheta)^p d\ell \right)^{\frac{1}{p}} \\ & \quad \times \left[ \left( \int_0^1 \left\{ \ell \left| g'' \left( \left( \frac{3u-v}{2} \right) + (1-\ell) \left( \frac{3v-u}{2} \right) \right) \right|^q d\ell \right\}^{\frac{1}{q}} \right. \right. \\ & \quad \left. \left. + \left( \int_0^1 \left\{ (1-\ell) \left| g'' \left( \left( \frac{3u-v}{2} \right) + \ell \left( \frac{3v-u}{2} \right) \right) \right|^q d\ell \right\}^{\frac{1}{q}} \right) \right] \\ & = \frac{2(v-u)^2}{\vartheta + 1} \beta^{\frac{1}{p}} (p + 1, \vartheta p + 1) \left( \frac{|g''(\frac{3u-v}{2})|^q + |g''(\frac{3v-u}{2})|^q}{2} \right)^{\frac{1}{q}}. \end{aligned} \tag{34}$$

Observe that  $\ell^p (1 - \ell^\vartheta)^p \leq \ell^p ([1 - \ell]^\vartheta)^p = \ell^p (1 - \ell)^{\vartheta p}$ . So, (34) completes the proof of Theorem 4. □

**Corollary 5** *With similar assumptions to Theorem 4 if  $\vartheta = 1$ , we have*

$$\begin{aligned} & \left| \frac{1}{v-u} \int_u^v g(x) dx - \frac{g(\frac{3v-u}{2}) + 2g(\frac{u+v}{2}) + g(\frac{3u-v}{2})}{4} \right| \\ & \leq \frac{(v-u)^2}{4} \left( \frac{\sqrt{\pi} \Gamma(p+1)}{2\Gamma(p+\frac{3}{2})} \right)^{\frac{1}{p}} \left( \frac{|g''(\frac{3u-v}{2})|^q + |g''(\frac{3v-u}{2})|^q}{2} \right)^{\frac{1}{q}}. \end{aligned} \tag{35}$$

*Proof* The proof of this corollary follows from the facts that

$$\beta(p+1, +1) = \frac{1}{2^{2p+1}} \frac{\sqrt{\pi} \Gamma(p+1)}{\Gamma(p+\frac{3}{2})}. \quad \square$$

*Remark 5* The right-hand side of inequality (35) confirms the modification of our work compared with the right-hand side of inequality (3.24) in [12, Theorem 4].

*Remark 6* If  $g''(x)$  is bounded on the interval  $[\frac{3u-v}{2}, \frac{3v-u}{2}]$ , then Theorem 4 reduces to

$$\begin{aligned} & \left| \frac{\Gamma(\vartheta+1)}{2(v-u)^\vartheta} [\mathfrak{J}_{u^+}^\vartheta g(v) + \mathfrak{J}_{v^-}^\vartheta g(u)] - \frac{g(\frac{3v-u}{2}) + 2g(\frac{u+v}{2}) + g(\frac{3u-v}{2})}{4} \right| \\ & \leq \beta^{\frac{1}{p}}(p+1, \vartheta p+1) \frac{2M(v-u)^2}{\vartheta+1} \end{aligned}$$

for  $\frac{1}{p} = 1 - \frac{1}{q}$  and for some  $M \in \mathbb{R}$ .

**Theorem 5** *With similar assumptions to Theorem 4, we have*

$$\begin{aligned} & \left| \frac{\Gamma(\vartheta+1)}{2(v-u)^\vartheta} [\mathfrak{J}_{u^+}^\vartheta g(v) + \mathfrak{J}_{v^-}^\vartheta g(u)] - \frac{g(\frac{3v-u}{2}) + 2g(\frac{u+v}{2}) + g(\frac{3u-v}{2})}{4} \right| \\ & \leq \frac{(v-u)^2}{\vartheta+1} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left( \frac{1}{(\vartheta q+1)(\vartheta q+2)} \right)^{\frac{1}{q}} \\ & \quad \times \left[ \left( \left| g''\left(\frac{3u-v}{2}\right) \right|^q + (\vartheta q+1) \left| g''\left(\frac{3v-u}{2}\right) \right|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left( (\vartheta q+1) \left| g''\left(\frac{3u-v}{2}\right) \right|^q + \left| g''\left(\frac{3v-u}{2}\right) \right|^q \right)^{\frac{1}{q}} \right]. \end{aligned} \tag{36}$$

*Proof* By using the Hölder inequality and the convexity of  $|g''|^q, q > 1$  on  $[\frac{3u-v}{2}, \frac{3v-u}{2}]$ , we have

$$\begin{aligned} & \int_0^1 \ell(1-\ell)^\vartheta \left[ \left| g''\left(\ell\left(\frac{3u-v}{2}\right) + (1-\ell)\left(\frac{3v-u}{2}\right)\right) \right| \right. \\ & \quad \left. + \left| g''\left((1-\ell)\left(\frac{3u-v}{2}\right) + \ell\left(\frac{3v-u}{2}\right)\right) \right| \right] d\ell \\ & \leq \left( \int_0^1 \ell^p d\ell \right)^{\frac{1}{p}} \\ & \quad \times \left[ \left( \int_0^1 (1-\ell)^\vartheta d\ell \right)^q \left\{ \ell \left| g''\left(\left(\frac{3u-v}{2}\right)\right) \right|^q + (1-\ell) \left| g''\left(\left(\frac{3v-u}{2}\right)\right) \right|^q \right\} d\ell \right]^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned}
 & + \left( \int_0^1 (1 - \ell^\vartheta)^q \left\{ (1 - \ell) \left| g'' \left( \left( \frac{3u - v}{2} \right) \right|^q + \ell \left| g'' \left( \frac{3v - u}{2} \right) \right|^q \right\} d\ell \right)^{\frac{1}{q}} \\
 & = \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left[ \left( \frac{1}{(\vartheta q + 1)(\vartheta q + 2)} \left| g'' \left( \frac{3u - v}{2} \right) \right|^q + \frac{1}{\vartheta q + 2} \left| g'' \left( \frac{3v - u}{2} \right) \right|^q \right)^{\frac{1}{q}} \right. \\
 & \quad \left. + \left( \frac{1}{\vartheta q + 2} \left| g'' \left( \frac{3u - v}{2} \right) \right|^q + \frac{1}{(\vartheta q + 1)(\vartheta q + 2)} \left| g'' \left( \frac{3v - u}{2} \right) \right|^q \right)^{\frac{1}{q}} \right]. \tag{37}
 \end{aligned}$$

Using (37) in (28), we obtain the required inequality (36). □

**Corollary 6** *With similar assumptions to Theorem 5 if  $\vartheta = 1$ , we have*

$$\begin{aligned}
 & \left| \frac{1}{v - u} \int_u^v g(x) dx - \frac{g\left(\frac{3v-u}{2}\right) + 2g\left(\frac{u+v}{2}\right) + g\left(\frac{3u-v}{2}\right)}{4} \right| \\
 & \leq (v - u)^2 \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left( \frac{1}{(q+1)(q+2)} \right)^{\frac{1}{q}} \\
 & \quad \times \left( \left| g'' \left( \frac{3u - v}{2} \right) \right|^q + (q + 1) \left| g'' \left( \frac{3v - u}{2} \right) \right|^q \right)^{\frac{1}{q}}. \tag{38}
 \end{aligned}$$

*Remark 7* The right-hand side of inequality (38) confirms the modification of our work compared with the right-hand side of inequality (3.27) in [12, Theorem 5].

**Theorem 6** *With similar assumptions to Theorem 3, we have*

$$\begin{aligned}
 & \left| \frac{\Gamma(\vartheta + 1)}{2(v - u)^\vartheta} [\mathfrak{J}_{u^+}^\vartheta g(v) + \mathfrak{J}_{v^-}^\vartheta g(u)] - \frac{g\left(\frac{3v-u}{2}\right) + 2g\left(\frac{u+v}{2}\right) + g\left(\frac{3u-v}{2}\right)}{4} \right| \\
 & \leq \frac{(v - u)^2}{2(\vartheta + 1)} \left( \frac{2}{(\vartheta q + 1)(\vartheta q + 2)(\vartheta q + 3)} \right)^{\frac{1}{q}} \\
 & \quad \times \left[ \left( 2 \left| g'' \left( \frac{3u - v}{2} \right) \right|^q + (\vartheta q + 1) \left| g'' \left( \frac{3v - u}{2} \right) \right|^q \right)^{\frac{1}{q}} \right. \\
 & \quad \left. + \left( (\vartheta q + 1) \left| g'' \left( \frac{3u - v}{2} \right) \right|^q + 2 \left| g'' \left( \frac{3v - u}{2} \right) \right|^q \right)^{\frac{1}{q}} \right]. \tag{39}
 \end{aligned}$$

*Proof* Let  $q > 1$ , then, by using the Hölder inequality and the convexity of  $|g''|^q$  on  $[\frac{3u-v}{2}, \frac{3v-u}{2}]$ , we have

$$\begin{aligned}
 & \int_0^1 \ell(1 - \ell^\vartheta) \left[ \left| g'' \left( \ell \left( \frac{3u - v}{2} \right) + (1 - \ell) \left( \frac{3v - u}{2} \right) \right) \right| \right. \\
 & \quad \left. + \left| g'' \left( (1 - \ell) \left( \frac{3u - v}{2} \right) + \ell \left( \frac{3v - u}{2} \right) \right) \right| \right] d\ell \leq \left( \int_0^1 \ell d\ell \right)^{1 - \frac{1}{q}} \\
 & \quad \times \left[ \left( \int_0^1 \ell(1 - \ell^\vartheta)^q \left\{ \ell \left| g'' \left( \left( \frac{3u - v}{2} \right) \right|^q + (1 - \ell) \left| g'' \left( \frac{3v - u}{2} \right) \right|^q \right\} d\ell \right)^{\frac{1}{q}} \right. \right. \\
 & \quad \left. \left. + \left( \int_0^1 \ell(1 - \ell^\vartheta)^q \left\{ (1 - \ell) \left| g'' \left( \left( \frac{3u - v}{2} \right) \right|^q + \ell \left| g'' \left( \frac{3v - u}{2} \right) \right|^q \right\} d\ell \right)^{\frac{1}{q}} \right] \right)
 \end{aligned}$$

$$\begin{aligned}
 &\leq \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left[ \left( \beta(3, \vartheta q + 1) \left| g''\left(\frac{3u-v}{2}\right) \right|^q + \beta(2, \vartheta q + 2) \left| g''\left(\frac{3v-u}{2}\right) \right|^q \right)^{\frac{1}{q}} \right. \\
 &\quad \left. + \left( \beta(2, \vartheta q + 2) \left| g''\left(\frac{3u-v}{2}\right) \right|^q + \beta(3, \vartheta q + 1) \left| g''\left(\frac{3v-u}{2}\right) \right|^q \right)^{\frac{1}{q}} \right] \\
 &= \frac{1}{2} \left( \frac{2}{(\vartheta q + 1)(\vartheta q + 2)(\vartheta q + 3)} \right)^{\frac{1}{q}} \left[ \left( 2 \left| g''\left(\frac{3u-v}{2}\right) \right|^q + (\vartheta q + 1) \left| g''\left(\frac{3v-u}{2}\right) \right|^q \right)^{\frac{1}{q}} \right. \\
 &\quad \left. + \left( (\vartheta q + 1) \left| g''\left(\frac{3u-v}{2}\right) \right|^q + 2 \left| g''\left(\frac{3v-u}{2}\right) \right|^q \right)^{\frac{1}{q}} \right]. \tag{40}
 \end{aligned}$$

Using (40) in (28) we obtain the inequality (39) for  $q > 1$ .

Now, using the convexity of  $|g''|$  and the properties of the modulus, we find

$$\begin{aligned}
 &\int_0^1 \ell(1-\ell^\vartheta) \left[ \left| g''\left(\ell\left(\frac{3u-v}{2}\right) + (1-\ell)\left(\frac{3v-u}{2}\right)\right) \right| \right. \\
 &\quad \left. + \left| g''\left((1-\ell)\left(\frac{3u-v}{2}\right) + \ell\left(\frac{3v-u}{2}\right)\right) \right| \right] d\ell \\
 &= \int_0^1 \ell^\vartheta \left[ \ell(1-\ell^\vartheta) \left| g''\left(\ell\left(\frac{3u-v}{2}\right) + (1-\ell)\left(\frac{3v-u}{2}\right)\right) \right| \right. \\
 &\quad \left. + \ell(1-\ell^\vartheta) \left| g''\left((1-\ell)\left(\frac{3u-v}{2}\right) + \ell\left(\frac{3v-u}{2}\right)\right) \right| \right] d\ell \\
 &\leq \left( \int_0^1 d\ell \right) \left[ \left( \int_0^1 \ell(1-\ell)^\vartheta \left\{ \ell \left| g''\left(\left(\frac{3u-v}{2}\right)\right) \right| + (1-\ell) \left| g''\left(\frac{3v-u}{2}\right) \right| \right\} d\ell \right) \right. \\
 &\quad \left. + \left( \int_0^1 \ell(1-\ell)^\vartheta \left\{ (1-\ell) \left| g''\left(\left(\frac{3u-v}{2}\right)\right) \right| + \ell \left| g''\left(\frac{3v-u}{2}\right) \right| \right\} d\ell \right) \right] \\
 &= \left( \frac{2}{(\vartheta + 1)(\vartheta + 2)} \right) \left( \left| g''\left(\frac{3u-v}{2}\right) \right| + \left| g''\left(\frac{3v-u}{2}\right) \right| \right). \tag{41}
 \end{aligned}$$

Substituting (41) into (28) we deduce that the inequality (39) holds true for  $q = 1$ . Thus (40), (41) and (28) complete the proof of Theorem 6. □

**Corollary 7** *With similar assumptions to Theorem 5 if  $\vartheta = 1$ , we have*

$$\begin{aligned}
 &\left| \frac{1}{v-u} \int_u^v g(x) dx - \frac{g\left(\frac{3v-u}{2}\right) + 2g\left(\frac{u+v}{2}\right) + g\left(\frac{3u-v}{2}\right)}{4} \right| \\
 &\leq \frac{(v-u)^2}{2} \left( \frac{2}{(q+1)(q+2)(q+3)} \right)^{\frac{1}{q}} \\
 &\quad \times \left( 2 \left| g''\left(\frac{3u-v}{2}\right) \right|^q + (q+1) \left| g''\left(\frac{3v-u}{2}\right) \right|^q \right)^{\frac{1}{q}}. \tag{42}
 \end{aligned}$$

*Remark 8* The right-hand side of inequality (42) confirms the modification of our work compared with the right-hand side of inequality (3.29) in [12, Theorem 6].

Collecting Theorems 3–6 we obtain the following corollary.

**Corollary 8** *From Theorems 3–6 we deduce that*

$$\begin{aligned} & \left| \frac{\Gamma(\vartheta + 1)}{2(\nu - u)^\vartheta} [\mathfrak{J}_{u^+}^\vartheta g(\nu) + \mathfrak{J}_{\nu^-}^\vartheta g(u)] - \frac{g(\frac{3\nu-u}{2}) + 2g(\frac{u+\nu}{2}) + g(\frac{3u-\nu}{2})}{4} \right| \\ & \leq \frac{(\nu - u)^2}{2(\vartheta + 1)} \min\{\gamma_3, \gamma_4, \gamma_5, \gamma_6\}, \end{aligned}$$

where

$$\begin{aligned} \gamma_3 &= \frac{\vartheta}{(\vartheta + 2)} \left\{ \left( \frac{2\vartheta + 4}{3\vartheta + 9} \left| g''\left(\frac{3u - \nu}{2}\right) \right|^q + \frac{\vartheta + 5}{3\vartheta + 9} \left| g''\left(\frac{3\nu - u}{2}\right) \right|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left( \frac{\vartheta + 5}{3\vartheta + 9} \left| g''\left(\frac{3u - \nu}{2}\right) \right|^q + \frac{2\vartheta + 4}{3\vartheta + 9} \left| g''\left(\frac{3\nu - u}{2}\right) \right|^q \right)^{\frac{1}{q}} \right\}, \\ \gamma_4 &= \beta^{\frac{1}{p}}(p + 1, \vartheta p + 1) \left( \frac{|g''(\frac{3u-\nu}{2})|^q + |g''(\frac{3\nu-u}{2})|^q}{2} \right)^{\frac{1}{q}}, \\ \gamma_5 &= \frac{1}{2} \left( \frac{1}{p + 1} \right)^{\frac{1}{p}} \left( \frac{1}{(\vartheta q + 1)(\vartheta q + 2)} \right)^{\frac{1}{q}} \\ & \quad \times \left[ \left( \left| g''\left(\frac{3u - \nu}{2}\right) \right|^q + (\vartheta q + 1) \left| g''\left(\frac{3\nu - u}{2}\right) \right|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left( (\vartheta q + 1) \left| g''\left(\frac{3u - \nu}{2}\right) \right|^q + \left| g''\left(\frac{3\nu - u}{2}\right) \right|^q \right)^{\frac{1}{q}} \right], \\ \gamma_6 &= \left( \frac{2}{(\vartheta q + 1)(\vartheta q + 2)(\vartheta q + 3)} \right)^{\frac{1}{q}} \\ & \quad \times \left[ \left( 2 \left| g''\left(\frac{3u - \nu}{2}\right) \right|^q + (\vartheta q + 1) \left| g''\left(\frac{3\nu - u}{2}\right) \right|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left( (\vartheta q + 1) \left| g''\left(\frac{3u - \nu}{2}\right) \right|^q + 2 \left| g''\left(\frac{3\nu - u}{2}\right) \right|^q \right)^{\frac{1}{q}} \right], \end{aligned}$$

for  $q > 1$ .

A few results for concave functions will be extended in the following theorems.

**Theorem 7** *Let  $g : \mathcal{I}^o \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $\mathcal{I}^o$ ,  $u, \nu \in \mathcal{I}^o$  with  $u < \nu$ . Let  $g'' : [\frac{3u-\nu}{2}, \frac{3\nu-u}{2}] \rightarrow \mathbb{R}$  be a continuous function on  $[\frac{3u-\nu}{2}, \frac{3\nu-u}{2}]$ . If  $|g''|^q$  is concave on  $[\frac{3u-\nu}{2}, \frac{3\nu-u}{2}]$ , then*

$$\begin{aligned} & \left| \frac{\Gamma(\vartheta + 1)}{2(\nu - u)^\vartheta} [\mathfrak{J}_{u^+}^\vartheta g(\nu) + \mathfrak{J}_{\nu^-}^\vartheta g(u)] - \frac{g(\frac{3\nu-u}{2}) + 2g(\frac{u+\nu}{2}) + g(\frac{3u-\nu}{2})}{4} \right| \\ & \leq \frac{2(\nu - u)^2}{\vartheta + 1} \beta^{\frac{1}{p}}(p + 1, \vartheta p + 1) \left| g''\left(\frac{u + \nu}{2}\right) \right|, \end{aligned} \tag{43}$$

where  $q = \frac{p}{p-1}$  such that  $p \in \mathbb{R}, p > 1$ .

*Proof* Applying Hölder’s inequality to (28), we get

$$\begin{aligned}
 & \left| \frac{\Gamma(\vartheta + 1)}{2(\nu - u)^\vartheta} [\mathfrak{J}_{u^+}^\vartheta g(\nu) + \mathfrak{J}_{\nu^-}^\vartheta g(u)] - \frac{g(\frac{3\nu-u}{2}) + 2g(\frac{u+\nu}{2}) + g(\frac{3u-\nu}{2})}{4} \right| \\
 & \leq \left( \frac{(\nu - u)^2}{\vartheta + 1} \int_0^1 \ell^\vartheta (1 - \ell^\vartheta)^p d\ell \right)^{\frac{1}{p}} \\
 & \quad \times \left\{ \left[ \int_0^1 \left| g'' \left( \ell \left( \frac{3u - \nu}{2} \right) + (1 - \ell) \left( \frac{3\nu - u}{2} \right) \right) \right| d\ell \right]^{\frac{1}{q}} \right. \\
 & \quad \left. + \left[ \int_0^1 \left| g'' \left( \ell \left( \frac{3\nu - u}{2} \right) + (1 - \ell) \left( \frac{3u - \nu}{2} \right) \right) \right| d\ell \right]^{\frac{1}{q}} \right\}. \tag{44}
 \end{aligned}$$

By using the concavity of  $|g''|^q$  on  $[\frac{3u-\nu}{2}, \frac{3\nu-u}{2}]$  and the integral Jensen’s inequality, we get

$$\begin{aligned}
 & \int_0^1 \left| g'' \left( \ell \left( \frac{3u - \nu}{2} \right) + (1 - \ell) \left( \frac{3\nu - u}{2} \right) \right) \right| d\ell \\
 & \leq \left( \int_0^1 \ell^0 d\ell \right) \left| g'' \left( \frac{\int_0^1 \ell^0 (\ell(\frac{3u-\nu}{2}) + (1-\ell)(\frac{3\nu-u}{2})) d\ell}{\int_0^1 \ell^0 d\ell} \right) \right|^q = \left| g'' \left( \frac{u + \nu}{2} \right) \right|^q \tag{45}
 \end{aligned}$$

and analogously

$$\int_0^1 \left| g'' \left( \ell \left( \frac{3\nu - u}{2} \right) + (1 - \ell) \left( \frac{3u - \nu}{2} \right) \right) \right| d\ell \leq \left| g'' \left( \frac{u + \nu}{2} \right) \right|^q. \tag{46}$$

Thus substituting the obtained results of (45) and (46) in (44), we get (43) as desired.  $\square$

**Theorem 8** Let  $g : \mathcal{I}^o \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $\mathcal{I}^o$ ,  $u, \nu \in \mathcal{I}^o$  with  $u < \nu$ . Let  $g'' : [\frac{3u-\nu}{2}, \frac{3\nu-u}{2}] \rightarrow \mathbb{R}$  be a continuous function on  $[\frac{3u-\nu}{2}, \frac{3\nu-u}{2}]$ . Assume that  $\frac{1}{p} + \frac{1}{q} = 1$  with  $p \geq 1$  such that  $|g''|^q$  is concave on  $[\frac{3u-\nu}{2}, \frac{3\nu-u}{2}]$ . Then

$$\begin{aligned}
 & \left| \frac{\Gamma(\vartheta + 1)}{2(\nu - u)^\vartheta} [\mathfrak{J}_{u^+}^\vartheta g(\nu) + \mathfrak{J}_{\nu^-}^\vartheta g(u)] - \frac{g(\frac{3\nu-u}{2}) + 2g(\frac{u+\nu}{2}) + g(\frac{3u-\nu}{2})}{4} \right| \\
 & \leq \frac{\vartheta(\nu - u)^2}{2(\vartheta + 1)(\vartheta + 2)} \\
 & \quad \times \left( \left| g'' \left( \frac{5\vartheta + 7}{6(\vartheta + 3)}u + \frac{\vartheta + 11}{6(\vartheta + 3)}\nu \right) \right| + \left| g'' \left( \frac{\vartheta + 11}{6(\vartheta + 3)}u + \frac{5\vartheta + 7}{6(\vartheta + 3)}\nu \right) \right| \right). \tag{47}
 \end{aligned}$$

*Proof* From the concavity of  $|g''|^q$  and the power-mean inequality, we have

$$\begin{aligned}
 & |g''(\ell x + (1 - \ell)y)|^q > \ell |g''(x)|^q + (1 - \ell) |g''(y)|^q \\
 & \geq (\ell |g''(x)| + (1 - \ell) |g''(y)|)^q
 \end{aligned}$$

for all  $x, y \in [\frac{3u-\nu}{2}, \frac{3\nu-u}{2}]$  and  $\ell \in [0, 1]$ . This also gives

$$|g''(\ell x + (1 - \ell)y)| \geq \ell |g''(x)| + (1 - \ell) |g''(y)|$$

this means that  $|g''|$  is also concave. Again by the Jensen integral inequality, we obtain

$$\begin{aligned} & \int_0^1 \left| g'' \left( \ell \left( \frac{3u-v}{2} \right) + (1-\ell) \left( \frac{3v-u}{2} \right) \right) \right| d\ell \\ & \leq \left( \int_0^1 \ell(1-\ell^\vartheta) d\ell \right) \left| g'' \left( \frac{\int_0^1 \ell(1-\ell^\vartheta) \left( \ell \left( \frac{3u-v}{2} \right) + (1-\ell) \left( \frac{3v-u}{2} \right) \right) d\ell}{\int_0^1 \ell(1-\ell^\vartheta) d\ell} \right) \right|^q \\ & = \frac{\vartheta}{2(\vartheta+2)} \left| g'' \left( \frac{5\vartheta+7}{6(\vartheta+3)} u + \frac{\vartheta+11}{6(\vartheta+3)} v \right) \right|^q, \end{aligned} \tag{48}$$

and analogously

$$\int_0^1 \left| g'' \left( \ell \left( \frac{3v-u}{2} \right) + (1-\ell) \left( \frac{3u-v}{2} \right) \right) \right| d\ell \leq \left| g'' \left( \frac{\vartheta+11}{6(\vartheta+3)} u + \frac{5\vartheta+7}{6(\vartheta+3)} v \right) \right|^q. \tag{49}$$

Thus substituting the obtained results of (48) and (49) in (44), we get (47) as desired.  $\square$

### 4 Applications

In this section some applications are presented to demonstrate the usefulness of our obtained results in the previous sections.

#### 4.1 Applications to special means

Let  $u$  and  $v$  are two arbitrary positive real numbers such that  $u \neq v$ , we consider the following special means [17].

(i) The arithmetic mean:

$$A = A(u, v) = \frac{u+v}{2}.$$

(ii) The inverse arithmetic mean:

$$H = H(u, v) = \frac{2}{\frac{1}{u} + \frac{1}{v}}, \quad u, v \neq 0.$$

(iii) The geometric mean:

$$G = G(u, v) = \sqrt{uv}.$$

(iv) The logarithmic mean:

$$L(u, v) = \frac{v-u}{\log(v) - \log(u)}, \quad u \neq v.$$

(v) The generalized logarithmic mean:

$$L_n(u, v) = \left[ \frac{v^{n+1} - u^{n+1}}{(v-u)(n+1)} \right]^{\frac{1}{n}}, \quad n \in \mathbb{Z} \setminus \{-1, 0\}.$$

**Proposition 1** *Let  $u, v \in \mathbb{R}$  with  $0 < u < v$  and  $n \in \mathbb{Z}/\{0, -1\}$ , then we have*

$$\begin{aligned}
 & |L_n^n(u, v) - A^n(u, v)| \\
 & \leq \min\{\delta_1, \delta_2\} \frac{|n|(v-u)}{2^{\frac{1}{p}}} \left[ A \left( \left| \left( \frac{3u-v}{2} \right) \right|^{(n-1)q}, \left| \left( \frac{3v-u}{2} \right) \right|^{(n-1)q} \right) \right]^{\frac{1}{q}}, \tag{50}
 \end{aligned}$$

where  $p = \frac{q}{q-1}$ .

*Proof* The proof of this proposition follows from Corollary 2 with  $\vartheta = 1$  and  $g(x) = x^n$ .  $\square$

**Proposition 2** *Let  $u, v \in \mathbb{R}$  with  $0 < u < v$  and  $n \in \mathbb{Z}/\{0, -1\}$ , then we have*

$$\begin{aligned}
 & |G^{-1}(u, v) - A^{-1}(u, v)| \\
 & \leq \min\{\delta_1, \delta_2\} \frac{v-u}{2^{1-\frac{1}{q}}} \left[ A \left( \left| \left( \frac{3u-v}{2} \right) \right|^{-2q}, \left| \left( \frac{3v-u}{2} \right) \right|^{-2q} \right) \right]^{\frac{1}{q}} \tag{51}
 \end{aligned}$$

for  $q \geq 1$ .

*Proof* The assertion follows from Corollary 2 with  $\vartheta = 1$  and  $g(x) = \frac{1}{x}$ .  $\square$

**Proposition 3** *Let  $|n| \geq 3$  and  $u, v \in \mathbb{R}$  with  $0 < u < v$ , then*

$$\begin{aligned}
 & |L_n^n(v^{-1}, u^{-1}) - H^{-n}(v, u)| \\
 & \leq \min\{\delta_1, \delta_2\} \frac{|n|(v^{-1} - u^{-1})}{2^{\frac{1}{p}}} \left[ H \left( \left| \left( \frac{3u-v}{2} \right) \right|^{(n-1)q}, \left| \left( \frac{3v-u}{2} \right) \right|^{(n-1)q} \right) \right]^{\frac{-1}{q}} \tag{52}
 \end{aligned}$$

and

$$\begin{aligned}
 & |L^{-1}(v^{-1}, u^{-1}) - H(v, u)| \\
 & \leq \min\{\delta_1, \delta_2\} \frac{v^{-1} - u^{-1}}{2^{1-\frac{1}{q}}} \left[ A \left( \left| \left( \frac{3u-v}{2} \right) \right|^{-2q}, \left| \left( \frac{3v-u}{2} \right) \right|^{-2q} \right) \right]^{\frac{1}{q}} \tag{53}
 \end{aligned}$$

for  $q \geq 1$ .

*Proof* Observe that  $A^{-1}(u^{-1}, v^{-1}) = H(u, v) = \frac{2}{\frac{1}{u} + \frac{1}{v}}$ . So, making the change of variables  $u \rightarrow v^{-1}$  and  $v \rightarrow u^{-1}$  in the inequalities (50) and (51), we can deduce the desired inequalities (52) and (53), respectively.  $\square$

**Proposition 4** *Let  $u, v \in \mathbb{R}$  with  $0 < u < v$  and  $n \in \mathbb{Z}/\{0, -1\}$ , then we have*

$$|G^{-2}(u, v) - A^{-2}(u, v)| \leq \min\{\delta_1, \delta_2\} \frac{v-u}{2^{\frac{2}{p}-1}} \left[ A \left( \left| \left( \frac{3u-v}{2} \right) \right|^{-3q}, \left| \left( \frac{3v-u}{2} \right) \right|^{-3q} \right) \right]^{\frac{1}{q}},$$

where  $p = \frac{q}{q-1}$ .

*Proof* The proof of this proposition follows from Corollary 2 with  $\vartheta = 1$  and  $g(x) = \frac{1}{x^2}$ .  $\square$

### 4.2 The midpoint formula

Let  $d$  be a partition of the interval  $[u, v]$  such that  $u = x_0 < x_1 < x_2 < \dots < x_{m-1} < x_m = v$ . Consider the quadrature formula [17]

$$\int_u^v g(x) dx = T(g, d) + \mathcal{E}(g, d), \tag{54}$$

where  $\mathcal{E}(g, d)$  represents the associated approximation error and

$$T(g, d) = \sum_{k=0}^{m-1} g\left(\frac{x_k + x_{k+1}}{2}\right)(x_{k+1} - x_k)$$

is the midpoint version.

**Proposition 5** *Let  $g : \mathcal{I}^o \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $\mathcal{I}^o$ ,  $u, v \in \mathcal{I}^o$  with  $u < v$ . Let  $g' \in L_1[\frac{3u-v}{2}, \frac{3v-u}{2}]$  and  $g' : [\frac{3u-v}{2}, \frac{3v-u}{2}] \rightarrow \mathbb{R}$  be a continuous function on  $[\frac{3u-v}{2}, \frac{3v-u}{2}]$ . If  $|g'|^q, q \geq 1$  is a convex function, then, for every partition  $d$  of  $[\frac{3u-v}{2}, \frac{3v-u}{2}]$  in (54), we have*

$$\begin{aligned} |\mathcal{E}(f, d)| &\leq \frac{1}{2} \min\{\delta_1, \delta_2\} \sum_{k=1}^{m-1} (x_{k+1} - x_k)^2 \left( \left| g'\left(\frac{3x_k - x_{k+1}}{2}\right) \right|^q + \left| g'\left(\frac{3x_{k+1} - x_k}{2}\right) \right|^q \right)^{\frac{1}{q}} \\ &\leq \min\{\delta_1, \delta_2\} \sum_{k=1}^{m-1} (x_{k+1} - x_k)^2 \max\left( \left| g'\left(\frac{3x_k - x_{k+1}}{2}\right) \right|, \left| g'\left(\frac{3x_{k+1} - x_k}{2}\right) \right| \right). \end{aligned} \tag{55}$$

*Proof* Applying Corollary 2 with  $\vartheta = 1$  on the subinterval  $[\frac{3x_k - x_{k+1}}{2}, \frac{3x_{k+1} - x_k}{2}]$  ( $k = 0, 1, \dots, m - 1$ ) of the partition  $d$ , we obtain

$$\begin{aligned} &\left| \int_{x_k}^{x_{k+1}} g(x) dx - (x_{k+1} - x_k)g\left(\frac{x_k + x_{k+1}}{2}\right) \right| \\ &\leq \frac{1}{2} \min\{\delta_1, \delta_2\} (x_{k+1} - x_k)^2 \left( \left| g'\left(\frac{3x_k - x_{k+1}}{2}\right) \right|^q + \left| g'\left(\frac{3x_{k+1} - x_k}{2}\right) \right|^q \right)^{\frac{1}{q}}. \end{aligned}$$

Summing over  $k$  from 0 to  $m - 1$  and taking into account that  $|g'|$  is convex, we obtain, by the triangle inequality,

$$\begin{aligned} &\left| \int_u^v g(x) dx - T(g, d) \right| \\ &= \left| \sum_{k=0}^{m-1} \left[ \int_{x_k}^{x_{k+1}} g(x) dx - (x_{k+1} - x_k)g\left(\frac{x_k + x_{k+1}}{2}\right) \right] \right| \\ &\leq \sum_{k=0}^{m-1} \left| \int_{x_k}^{x_{k+1}} g(x) dx - (x_{k+1} - x_k)g\left(\frac{x_k + x_{k+1}}{2}\right) \right| \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{2} \min\{\delta_1, \delta_2\} \sum_{k=1}^{m-1} (x_{k+1} - x_k)^2 \left( \left| g' \left( \frac{3x_k - x_{k+1}}{2} \right) \right|^q + \left| g' \left( \frac{3x_{k+1} - x_k}{2} \right) \right|^q \right)^{\frac{1}{q}} \\ &\leq \min\{\delta_1, \delta_2\} \sum_{k=1}^{m-1} (x_{k+1} - x_k)^2 \max \left( \left| g' \left( \frac{3x_k - x_{k+1}}{2} \right) \right|, \left| g' \left( \frac{3x_{k+1} - x_k}{2} \right) \right| \right). \end{aligned}$$

This completes the proof of (55). □

### 5 Conclusion

In this paper, we generalized the modified Hermite–Hadamard inequality obtained by Mehrez and Agarwal in [12], it can be found in Lemma 3 and Theorems 1–6. Corollaries 4–7 confirm that our results modified the existing results of [12]. Furthermore, Theorems 7–8 modified the existing Theorems 5–6 of [5].

#### Acknowledgements

The authors would like to express their special thanks to the editor and the referees of this manuscript. The second author would like to thank Prince Sultan University for funding this work through research group Nonlinear Analysis Methods in Applied Mathematics (NAMAM) (group number RG-DES-2017-01-17).

#### Funding

Not applicable.

#### Availability of data and materials

Not applicable.

#### Competing interests

The authors declare that they have no competing interests.

#### Consent for publication

Not applicable.

#### Authors' contributions

All authors read and approved the final manuscript.

#### Author details

<sup>1</sup>Department of Mathematics, College of Education, University of Sulaimani, Sulaimani, Iraq. <sup>2</sup>Department of Mathematics and General Sciences, Prince Sultan University, Riyadh, Saudi Arabia. <sup>3</sup>Department of Medical Research, China Medical University, Taichung, Taiwan. <sup>4</sup>Department of Computer Sciences and Information Engineering, Asia University, Taichung, Taiwan.

### Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 23 August 2019 Accepted: 4 February 2020 Published online: 11 February 2020

#### References

1. Abdeljawad, T., Al-Mdallal, Q.M.: Discrete Mittag-Leffler kernel type fractional difference initial value problems and Gronwall's inequality. *J. Comput. Appl. Math.* **339**, 218–230 (2018). <https://doi.org/10.1016/j.cam.2017.10.021>
2. Avci, M., Kavurmaci, H., Odemir, M.E.: New inequalities of Hermite–Hadamard type via *s*-convex functions in the second sense with applications. *Appl. Math. Comput.* **217**, 5171–5176 (2011)
3. Ciatti, P., Cowling, M.G., Ricci, F.: Hardy and uncertainty inequalities on stratified Lie groups. *Adv. Math.* **277**, 365–387 (2015)
4. Dragomir, S.S.: Hermite–Hadamard's type inequalities for operator convex functions. *Appl. Math. Comput.* **218**, 766–772 (2011)
5. Dragomir, S.S., Bhatti, M.I., Iqbal, M., Muddassar, M.: Some new Hermite–Hadamard's type fractional integral inequalities. *J. Comput. Anal. Appl.* **18**(4), 655–661 (2015)
6. Dragomir, S.S., Pearce, C.E.M.: Selected topics on Hermite–Hadamard inequalities and applications, *RGMA Monographs*, Victoria University (2000)
7. Gavrea, B., Gavrea, I.: On some Ostrowski type inequalities. *Gen. Math.* **18**(1), 33–44 (2010)
8. Gunawan, H., Eridani: Fractional integrals and generalized Olsen inequalities. *Kyungpook Math. J.* **49**, 31–39 (2009)
9. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: *Theory and Applications of Fractional Differential Equations*. Elsevier, Amsterdam (2006)
10. Kirmaci, U.S.: Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula. *Appl. Math. Comput.* **147**, 137–146 (2004)

11. Lian, T., Tang, W., Zhou, R.: Fractional Hermite–Hadamard inequalities for  $(s, m)$ -convex or  $s$ -concave functions. *J. Inequal. Appl.* **2018**, 240 (2018)
12. Mehrez, K., Agarwal, P.: New Hermite–Hadamard type integral inequalities for convex functions and their applications. *J. Comput. Appl. Math.* **350**, 274–285 (2019)
13. Mohammed, P.O.: Inequalities of type Hermite–Hadamard for fractional integrals via differentiable convex functions. *Turk. J. Anal. Number Theory* **4**(5), 135–139 (2016)
14. Mohammed, P.O.: On new trapezoid type inequalities for  $h$ -convex functions via generalized fractional integral. *Turk. J. Anal. Number Theory* **6**(4), 125–128 (2018)
15. Mohammed, P.O., Sarikaya, M.Z.: Hermite–Hadamard type inequalities for  $F$ -convex function involving fractional integrals. *J. Inequal. Appl.* **2018**, 359 (2018)
16. Odemir, M.E., Avci, M., Set, E.: On some inequalities of Hermite–Hadamard type via  $m$ -convexity. *Appl. Math. Lett.* **23**, 1065–1070 (2010)
17. Pearce, C.E.M., Pečarić, J.E.: Inequalities for differentiable mappings with application to special means and quadrature formula. *Appl. Math. Lett.* **13**, 51–55 (2000)
18. Qi, F., Mohammed, P.O., Yao, J.-C., Yao, Y.-H.: Generalized fractional integral inequalities of Hermite–Hadamard type for  $(\alpha, m)$ -convex functions. *J. Inequal. Appl.* **2019**, 135 (2019)
19. Rahman, G., Abdeljawad, T., Jarad, F., Khan, A., Nisar, K.S.: Certain inequalities via generalized proportional Hadamard fractional integral operators. *Adv. Differ. Equ.* **2019**, 454 (2019)
20. Rashid, S., Abdeljawad, T., Jarad, F., Noor, M.A.: Some estimates for generalized Riemann–Liouville fractional integrals of exponentially convex functions and their applications. *Mathematics* **7**, 807 (2019). <https://doi.org/10.3390/math7090807>
21. Sarikaya, M.Z., Set, E., Yaldiz, H., Başak, N.: Hermite–Hadamard’s inequalities for fractional integrals and related fractional inequalities. *Math. Comput. Model.* **57**, 2403–2407 (2013)
22. Sawano, Y., Wadade, H.: On the Gagliardo–Nirenberg type inequality in the critical Sobolev–Morrey space. *J. Fourier Anal. Appl.* **19**(1), 20–47 (2013)
23. Zhu, C., Feckan, M., Wang, J.: Fractional integral inequalities for differential convex mappings and applications to special means and a midpoint formula. *J. Appl. Math. Stat. Inform.* **8**(2), 21–28 (2012)

Submit your manuscript to a SpringerOpen<sup>®</sup> journal and benefit from:

- Convenient online submission
- Rigorous peer review
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

---

Submit your next manuscript at ► [springeropen.com](https://www.springeropen.com)

---