Zhang *Advances in Difference Equations* (2020) 2020:85
https://doi.org/10.1186/s13662-020-2536-z

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

The non-uniqueness of solution for initial value problem of impulsive differential equations involving higher order Katugampola fractional derivative

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Abstract

In this paper we consider the initial value problem for some impulsive differential equations with higher order Katugampola fractional derivative (fractional order $q \in (1,2]$). The systems of impulsive higher order fractional differential equations can involve one or two kinds of impulses, and by analyzing the error between the approximate solution and exact solution it is found that these impulsive systems are equivalent to some integral equations with one or two undetermined constants correspondingly, which uncover the non-uniqueness of solution to these impulsive systems. Some numerical examples are offered to explain the obtained results.

MSC: 34A08; 34A37

Keywords: Fractional differential equations; Impulsive fractional differential equations; Generalized fractional derivative; Non-uniqueness of solution

1 Introduction

Fractional calculus serves as an important tool to characterize hereditary properties in many fields of science and engineering (such as chaotic behavior, epidemiology, thermal science, hydrology, and biology [1–17]). Since fractional calculus was put forward in the seventeenth century, there have appeared several definitions of fractional derivatives: Riemann–Liouville, Caputo, Hadamard, Grunwald–Letnikov etc. [18, 19]. To unify these fractional derivatives, some generalized fractional operators (such as Erdélyi–Kober fractional operator [18], Hilfer fractional operator [20, 21], Katugampola fractional operator [22, 23], and Atangana–Baleanu fractional operator [24] etc.) were presented, and some properties of these generalized fractional operators and differential equations involving these generalized fractional derivatives were widely studied [25–33]. The potential application in quantum mechanics was considered for some properties of the Katugampola fractional derivative in [34], and the existence and uniqueness of solutions was studied for fractional Langevin equation with the nonlocal Katugampola fractional integral conditions in [35].

Furthermore, impulsive differential equations are used in description of some processes with impulsive effects [36], and the subject of impulsive fractional differential equations (IFrDE) has been getting an enormous amount of attention recently [37–45]. In addition, IFrDE was considered from the short memory case that addressed the nonlocality and the impulsive conditions in [46]. For the studies of IFrDE, most of them considered impulsive differential equations involving the Caputo type fractional derivative, and a few of them were concerned with impulsive non-Caputo type fractional differential equations. Therefore, we consider the equivalent integral equation for the initial value problem (IVP) of impulsive differential equations involving higher order non-Caputo type fractional derivative (in the sense of Katugampola):

$$\begin{cases} {}_{t_0}^{K}\mathcal{D}_t^{q,\rho}x(t) = f(t,x(t)), \\ \quad t \in (t_0,T], t \neq t_k (k = 1,2,\ldots,m) \text{ and } t \neq \bar{t}_l (l = 1,2,\ldots,n), \\ {}_{t_0}^{K}\mathcal{D}_{t_k+}^{q-1,\rho}x(t_k+) - {}_{t_0}^{K}\mathcal{D}_{t_k-}^{q-1,\rho}x(t_k-) = J_k(x(t_k-)), \quad k = 1,2,\ldots,m, \\ {}_{t_0}^{K}\mathcal{I}_{\bar{t}_l+}^{2-q,\rho}x(\bar{t}_l+) - {}_{t_0}^{K}\mathcal{I}_{\bar{t}_l-}^{2-q,\rho}x(\bar{t}_l-) = \bar{J}_l(x(\bar{t}_l-)), \quad l = 1,2,\ldots,n, \\ {}_{t_0}^{K}\mathcal{D}_t^{q-1,\rho}x(t)|_{t\to t_0+} = x_1, \qquad {}_{t_0}^{K}\mathcal{I}_t^{2-q,\rho}x(t)|_{t\to t_0+} = x_2, \end{cases} \tag{1.1}$$

where ${}_{t_0}^{K}\mathcal{D}_t^{q,\rho}$ (here $q \in (1,2]$ and $\rho > 0$) denotes the left-sided Katugampola fractional derivative of order q. ${}_{t_0}^{K}\mathcal{D}_{t_k+}^{q-1,\rho}x(t_k+) = \lim_{\varepsilon\to 0+} {}_{t_0}^{K}\mathcal{D}_{t_k+\varepsilon}^{q-1,\rho}x(t_k+\varepsilon)$ and ${}_{t_0}^{K}\mathcal{D}_{t_k-}^{q-1,\rho}x(t_k-) = \lim_{\varepsilon\to 0-} {}_{t_0}^{K}\mathcal{D}_{t_k+\varepsilon}^{q-1,\rho}x(t_k+\varepsilon)$ represent the right and left limits of ${}_{t_0}^{K}\mathcal{D}_t^{q-1,\rho}x(t)$ at $t = t_k$, respectively. ${}_{t_0}^{K}\mathcal{I}_{\bar{t}_l+}^{2-q,\rho}x(\bar{t}_l+)$ and ${}_{t_0}^{K}\mathcal{I}_{\bar{t}_l-}^{2-q,\rho}x(\bar{t}_l-)$ denote the right and left limits of ${}_{t_0}^{K}\mathcal{I}_t^{2-q,\rho}x(t)$ at $t = \bar{t}_l$, respectively. Two kinds of impulsive points satisfy $0 \le t_0 < t_1 < \cdots < t_m < t_{m+1} = T$ and $t_0 < \bar{t}_1 < \cdots < \bar{t}_n < \bar{t}_{n+1} = T$, respectively. Moreover, for these impulsive points, two assumptions are given as follows:

(H1) Let $\{t_0,t_1,t_2,\ldots,t_m,\bar{t}_1,\bar{t}_2,\ldots,\bar{t}_n,T\} = \{t_0,t_1',t_2',\ldots,t_M',T\}$ satisfy

$$0 \le t_0 < t_1' < t_2' < \cdots < t_M' < t_{M+1}' = T.$$

(H2) For each $[t_0,t_k']$ $(k = 1,2,\ldots,M)$, suppose $[t_0,t_{k_1}] \subseteq [t_0,t_k'] \subset [t_0,t_{k_1+1}]$ (here $k_1 \in \{1,2,\ldots,m\}$) and $[t_0,\bar{t}_{k_2}] \subseteq [t_0,t_k'] \subset [t_0,\bar{t}_{k_2+1}]$ (here $k_2 \in \{1,2,\ldots,n\}$), respectively.

In particular, letting $J_k(x(t_k-)) = 0$ (for all $k \in \{1,2,\ldots,m\}$) and $\bar{J}_l(x(\bar{t}_l-)) = 0$ (for all $l \in \{1,2,\ldots,n\}$) in (1.1) respectively, we obtain two simple impulsive systems:

$$\begin{cases} {}_{t_0}^{K}\mathcal{D}_t^{q,\rho}x(t) = f(t,x(t)), \quad t \in (t_0,T] \text{ and } t \neq t_k \ (k = 1,2,\ldots,m), \\ {}_{t_0}^{K}\mathcal{D}_{t_k+}^{q-1,\rho}x(t_k+) - {}_{t_0}^{K}\mathcal{D}_{t_k-}^{q-1,\rho}x(t_k-) = J_k(x(t_k-)), \quad k = 1,2,\ldots,m, \\ {}_{t_0}^{K}\mathcal{D}_t^{q-1,\rho}x(t)|_{t\to t_0+} = x_1, \qquad {}_{t_0}^{K}\mathcal{I}_t^{2-q,\rho}x(t)|_{t\to t_0+} = x_2 \end{cases} \tag{1.2}$$

and

$$\begin{cases} {}_{t_0}^{K}\mathcal{D}_t^{q,\rho}x(t) = f(t,x(t)), \quad t \in (t_0,T] \text{ and } t \neq \bar{t}_l \ (l = 1,2,\ldots,n), \\ {}_{t_0}^{K}\mathcal{I}_{\bar{t}_l+}^{2-q,\rho}x(\bar{t}_l+) - {}_{t_0}^{K}\mathcal{I}_{\bar{t}_l-}^{2-q,\rho}x(\bar{t}_l-) = \bar{J}_l(x(\bar{t}_l-)), \quad l = 1,2,\ldots,n, \\ {}_{t_0}^{K}\mathcal{D}_t^{q-1,\rho}x(t)|_{t\to t_0+} = x_1, \qquad {}_{t_0}^{K}\mathcal{I}_t^{2-q,\rho}x(t)|_{t\to t_0+} = x_2. \end{cases} \tag{1.3}$$

Moreover, letting $\{t_1, t_2, \dots, t_m\} = \{\bar{t}_1, \bar{t}_2, \dots, \bar{t}_n\}$ in (1.1), we get the impulsive system

$$\begin{cases} {}_{t_0}^{K}\mathcal{D}_{t}^{q,\rho} x(t) = f(t, x(t)), \quad t \in (t_0, T] \text{ and } t \neq t_k \ (k = 1, 2, \dots, m), \\ {}_{t_0}^{K}\mathcal{D}_{t_k+}^{q-1,\rho} x(t_k+) - {}_{t_0}^{K}\mathcal{D}_{t_k-}^{q-1,\rho} x(t_k-) = J_k(x(t_k-)), \quad k = 1, 2, \dots, m, \\ {}_{t_0}^{K}\mathcal{I}_{t_k+}^{2-q,\rho} x(t_k+) - {}_{t_0}^{K}\mathcal{I}_{t_k-}^{2-q,\rho} x(t_k-) = \bar{J}_k(x(t_k-)), \quad k = 1, 2, \dots, m, \\ {}_{t_0}^{K}\mathcal{D}_{t}^{q-1,\rho} x(t)|_{t \to t_0+} = x_1, \qquad {}_{t_0}^{K}\mathcal{I}_{t}^{2-q,\rho} x(t)|_{t \to t_0+} = x_2. \end{cases} \tag{1.4}$$

Next we introduce some basic definitions and conclusions regarding the Katugampola fractional derivative in Sect. 2 and give some properties of IFrDEs (1.1)–(1.3) in Sect. 3. Then, we seek the equivalent integral equations of IFrDEs (1.1)–(1.4) in Sect. 4. Finally, we use some numerical examples to expound the obtained results in Sect. 5.

2 Preliminaries

Let $[a, b]$ $(-\infty \leq a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$ and $C[a, b]$ be the set of continuous functions on$[a, b]$. Define the function space

$$X_c^p(a, b) = \bigl\{x : [a, b] \to \mathbb{C} : \|x\|_{X_c^p} < \infty\bigr\} \quad (c \in \mathbb{R}, 1 \leq p \leq \infty) \tag{2.1}$$

endowed with the norm $\|x\|_{X_c^p} = (\int_a^b |t^c x(t)|^p \frac{dt}{t})^{1/p}$ $(1 \leq p < \infty)$ and $\|x\|_{X_c^\infty} = \operatorname{ess\,sup}_{t \in [a,b]} [t^c |x(t)|]$.

Definition 2.1 ([22]) The left-sided Katugampola fractional integrals of order $\alpha \in \mathbb{C}$ $(\Re(\alpha) > 0)$ of function $x \in X_c^p(a, b)$ are defined by

$$\bigl({}_{a}^{K}\mathcal{I}_{t}^{\alpha,\rho} x\bigr)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} \frac{x(s)\,ds}{s^{1-\rho}} \quad (t > a \geq 0). \tag{2.2}$$

Definition 2.2 ([23]) The left-sided Katugampola fractional derivatives of order $\alpha \in \mathbb{C}$ $(\Re(\alpha) > 0)$ are defined by

$$\begin{aligned} \bigl({}_{a}^{K}\mathcal{D}_{t}^{\alpha,\rho} x\bigr)(t) &= \gamma^n \bigl({}_{a}^{K}\mathcal{I}_{t}^{n-\alpha,\rho} x\bigr)(t) \\ &= \frac{\gamma^n}{\Gamma(n-\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{n-\alpha-1} \frac{x(s)\,ds}{s^{1-\rho}} \\ &\left(\rho > 0, t > a \geq 0, \gamma = t^{1-\rho} \frac{d}{dt}\right). \end{aligned} \tag{2.3}$$

Remark 2.3 From the L'Hospital rule, we have $\lim_{\rho \to 0+} (\frac{t^\rho - \tau^\rho}{\rho})^{q-1} = (\ln \frac{t}{\tau})^{q-1}$. The Katugampola fractional operators with $\rho \to 0+$ and $\rho = 1$ are the *Hadamard fractional operator* and the *Riemann–Liouville fractional operator*, respectively.

For $n - 1 < \alpha \leq n$ $(n \in \mathbb{N})$, a weighted space of continuous functions is defined by

$$\begin{aligned} &C_{n-\alpha,\rho}[a, b] = \bigl\{x(t) : \bigl(t^\rho - a^\rho\bigr)^{n-\alpha} x(t) \in C[a, b], \|x\|_{C_{n-\alpha,\rho}} = \bigl\|\bigl(t^\rho - a^\rho\bigr)^{n-\alpha} x(t)\bigr\|_C\bigr\} \\ &\quad (\rho \neq 0) \end{aligned} \tag{2.4}$$

and

$$C_{n-\alpha,0}[a,b] = \bigl\{x(t) : (\ln t - \ln a)^{n-\alpha} x(t) \in C[a,b], \\ \|x\|_{C_{n-\alpha,0}} = \bigl\|(\ln t - \ln a)^{n-\alpha} x(t)\bigr\|_C\bigr\}. \tag{2.5}$$

Moreover, let

$$C^{\alpha}_{n-\alpha,\rho}[a,b] = \bigl\{x(t) \in C_{n-\alpha,\rho}[a,b] : {}^{K}_{a}\mathcal{D}^{\alpha,\rho}_{t} x(t) \in C_{n-\alpha,\rho}[a,b]\bigr\} \tag{2.6}$$

and

$$C^{2}_{2-\alpha,\rho}[a,T] = \left\{x(t) \in C[a,b] : \gamma^2 x(t) \in C_{2-\alpha,\rho}[a,b], \gamma = t^{1-\rho}\frac{d}{dt}\right\}. \tag{2.7}$$

Lemma 2.4 *Let $q \in (1,2]$ and $a, \rho > 0$, and let $f : [a,T] \times \mathbb{R} \to \mathbb{R}$ be a function such that $f(\cdot, x(\cdot)) \in C_{2-q,\rho}[a,T]$ for any $x(\cdot) \in C_{2-q,\rho}[a,T]$.*

If $x(\cdot) \in C^{q}_{2-q,\rho}[a,T]$, then $x(t)$ is a solution of the fractional differential equation

$$\begin{cases} {}^{K}_{a}\mathcal{D}^{q,\rho}_{t} x(t) = f(t, x(t)), \quad t \in (a,T], \\ {}^{K}_{a}\mathcal{D}^{q-1,\rho}_{t} x(t)|_{t\to a+} = x_1, \quad {}^{K}_{a}\mathcal{I}^{2-q,\rho}_{t} x(t)|_{t\to a+} = x_2 \end{cases} \tag{2.8}$$

if, and only if, $x(t)$ satisfies the integral equation

$$\begin{aligned} x(t) = {} & \frac{x_2}{\Gamma(q-1)}\left[\frac{t^\rho - a^\rho}{\rho}\right]^{q-2} + \frac{x_1}{\Gamma(q)}\left[\frac{t^\rho - a^\rho}{\rho}\right]^{q-1} \\ & + \frac{1}{\Gamma(q)}\int_a^t \left[\frac{t^\rho - \tau^\rho}{\rho}\right]^{q-1} \frac{f\,d\tau}{\tau^{1-\rho}} \\ & \textit{for } t \in (a,T] \textit{ and } f = f\bigl(\tau, x(\tau)\bigr). \end{aligned} \tag{2.9}$$

Proof First, we prove the necessity. Let $x(t) \in C^{q}_{2-q,\rho}[a,T]$ be a solution of (2.8). By the hypotheses $x(t) \in C^{q}_{2-q,\rho}[a,T]$ and ${}^{K}_{a}\mathcal{D}^{q,\rho}_{t} x(t) = \gamma^2({}^{K}_{a}\mathcal{I}^{2-q,\rho}_{t} x)(t)$, we have ${}^{K}_{a}\mathcal{I}^{2-q,\rho}_{t} x(t) \in C[a,T]$ and ${}^{K}_{a}\mathcal{I}^{2-q,\rho}_{t} x \in C^{2}_{2-q,\rho}[a,T]$. Therefore, by (2.8), we get

$${}^{K}_{a}\mathcal{D}^{q,\rho}_{t} x(t) = \left(t^{1-\rho}\frac{d}{dt}\right)^2 \bigl({}^{K}_{a}\mathcal{I}^{2-q,\rho}_{t} x(t)\bigr) = f\bigl(t, x(t)\bigr).$$

Therefore

$${}^{K}_{a}\mathcal{I}^{2-q,\rho}_{t} x(t) = x_2 + \frac{t^\rho - a^\rho}{\rho} x_1 + \int_a^t \frac{t^\rho - \tau^\rho}{\rho} \frac{f\,d\tau}{\tau^{1-\rho}}. \tag{2.10}$$

Applying the operator ${}^{K}_{a}\mathcal{I}^{q,\rho}_{t}$ to two sides of (2.10), we have

$$\begin{aligned} {}^{K}_{a}\mathcal{I}^{2,\rho}_{t} x(t) = {} & \frac{x_2}{\Gamma(q+1)}\left[\frac{t^\rho - a^\rho}{\rho}\right]^{q} + \frac{x_1}{\Gamma(q+2)}\left[\frac{t^\rho - a^\rho}{\rho}\right]^{q+1} \\ & + \frac{1}{\Gamma(q+2)}\int_a^t \left[\frac{t^\rho - \tau^\rho}{\rho}\right]^{q+1} \frac{f\,d\tau}{\tau^{1-\rho}}. \end{aligned} \tag{2.11}$$

Using the operator γ^2 to two sides of (2.11), we obtain

$$\begin{aligned} x(t) &= \frac{x_2}{\Gamma(q-1)}\left[\frac{t^\rho - a^\rho}{\rho}\right]^{q-2} + \frac{x_1}{\Gamma(q)}\left[\frac{t^\rho - a^\rho}{\rho}\right]^{q-1} \\ &\quad + \frac{1}{\Gamma(q)}\int_a^t \left[\frac{t^\rho - \tau^\rho}{\rho}\right]^{q-1} \frac{f\,d\tau}{\tau^{1-\rho}} \\ &\quad \text{for } t \in (a,T]. \end{aligned}$$

Now we prove the sufficiency. Let $x(t) \in C^q_{2-q,\rho}[a,T]$ satisfy Eq. (2.9), which can be written as (2.9). Moreover, by the hypotheses of Lemma 2.4, for any $x(\cdot) \in C_{2-q,\rho}[a,T]$, we have $f(\cdot,x(\cdot)) \in C_{2-q,\rho}[a,T]$. Applying the operators ${}^K_a\mathcal{D}^{q,\rho}_t$, ${}^K_a\mathcal{D}^{q-1,\rho}_t$, and ${}^K_a\mathcal{I}^{2-q,\rho}_t$ to both sides of (2.9), respectively, we obtain

$$\begin{aligned} {}^K_a\mathcal{D}^{q,\rho}_t x(t) &= {}^K_a\mathcal{D}^{q,\rho}_t \left\{ \frac{x_2}{\Gamma(q-1)}\left[\frac{t^\rho - a^\rho}{\rho}\right]^{q-2} + \frac{x_1}{\Gamma(q)}\left[\frac{t^\rho - a^\rho}{\rho}\right]^{q-1} \right. \\ &\quad \left. + \frac{1}{\Gamma(q)}\int_a^t \left[\frac{t^\rho - \tau^\rho}{\rho}\right]^{q-1} \frac{f\,d\tau}{\tau^{1-\rho}} \right\} \\ &= f\bigl(t,x(t)\bigr) \quad \text{for } t \in (a,T], \\ {}^K_a\mathcal{D}^{q-1,\rho}_t x(t) &= x_1 + \int_a^t \frac{f\,d\tau}{\tau^{1-\rho}} \quad \text{for } t \in (a,T], \end{aligned}$$

and

$$ {}^K_a\mathcal{I}^{2-q,\rho}_t x(t) = x_2 + \frac{t^\rho - a^\rho}{\rho} x_1 + \int_a^t \frac{t^\rho - \tau^\rho}{\rho} \frac{f\,d\tau}{\tau^{1-\rho}} \quad \text{for } t \in (a,T].$$

By the hypothesis $f(\cdot,x(\cdot)) \in C_{2-q,\rho}[a,T]$, we have $(\tau^\rho - a^\rho)^{2-q} f(\tau,x(\tau)) \in C[a,T]$. Therefore $|(\tau^\rho - a^\rho)^{2-q} f| \le L$ (here L is a positive constant) and

$$\begin{aligned} \left|\int_a^t \frac{f\,d\tau}{\tau^{1-\rho}}\right| &\le \int_a^t \bigl|(\tau^\rho - a^\rho)^{q-2}\bigl[(\tau^\rho - a^\rho)^{2-q} f\bigr]\bigr| \frac{d\tau^\rho}{\rho} \\ &\le \frac{L(t^\rho - a^\rho)^{q-1}}{(q-1)\rho}, \end{aligned}$$

and

$$\begin{aligned} \left|\int_a^t \frac{t^\rho - \tau^\rho}{\rho} \frac{f\,d\tau}{\tau^{1-\rho}}\right| &\le \int_a^t \left| \frac{t^\rho - \tau^\rho}{\rho}\left[\frac{\tau^\rho - a^\rho}{\rho}\right]^{q-2} \bigl[(\tau^\rho - a^\rho)^{2-q} f\bigr]\right| \frac{d\tau^\rho}{\rho^{3-q}} \\ &\le \frac{LB(2,q-1)}{\rho^{2-q}}\left[\frac{t^\rho - a^\rho}{\rho}\right]^q. \end{aligned}$$

Thus ${}^K_a\mathcal{D}^{q-1,\rho}_t x(t)|_{t\to a+} = x_1$ and ${}^K_a\mathcal{I}^{2-q,\rho}_t x(t)|_{t\to a+} = x_2$. The proof is completed. □

3 Some properties of (1.1)–(1.3)

In this section, we give some properties of three impulsive systems (1.1)–(1.3):

(i)
$$\begin{aligned}
&\lim_{\substack{J_k(x(t_k-))\to 0 \text{ for all } k\in\{1,2,\ldots,m\}\\ \bar{J}_l(x(\bar{t}_l-))\to 0 \text{ for all } l\in\{1,2,\ldots,n\}}} \bigl\{\text{system (1.1)}\bigr\} \\
&= \lim_{J_k(x(t_k-))\to 0 \text{ for all } k\in\{1,2,\ldots,m\}} \bigl\{\text{system (1.2)}\bigr\} \\
&= \lim_{\bar{J}_l(x(\bar{t}_l-))\to 0 \text{ for all } l\in\{1,2,\ldots,n\}} \bigl\{\text{system (1.3)}\bigr\} \\
&= \begin{cases} {}^K_a\mathcal{D}^{q,\rho}_t x(t) = f(t,x(t)), & t\in(t_0,T], \\ {}^K_{t_0}\mathcal{D}^{q-1,\rho}_t x(t)|_{t\to t_0+} = x_1, & {}^K_{t_0}\mathcal{I}^{2-q,\rho}_t x(t)|_{t\to t_0+} = x_2. \end{cases} \\
&\Leftrightarrow \quad x(t) = \frac{x_2}{\Gamma(q-1)}\left[\frac{t^\rho-(t_0)^\rho}{\rho}\right]^{q-2} + \frac{x_1}{\Gamma(q)}\left[\frac{t^\rho-(t_0)^\rho}{\rho}\right]^{q-1} \\
&\qquad + \frac{1}{\Gamma(q)}\int_{t_0}^t \left[\frac{t^\rho-\tau^\rho}{\rho}\right]^{q-1}\frac{f\,d\tau}{\tau^{1-\rho}},
\end{aligned}$$

for $t\in(t_0,T]$ and $f=f\bigl(\tau,x(\tau)\bigr)$.

(ii)
$$\lim_{\bar{J}_l(x(\bar{t}_l-))\to 0 \text{ for all } l\in\{1,2,\ldots,n\}} \bigl\{\text{system (1.1)}\bigr\} = \bigl\{\text{system (1.2)}\bigr\}.$$

(iii)
$$\lim_{J_k(x(t_k-))\to 0 \text{ for all } k\in\{1,2,\ldots,m\}} \bigl\{\text{system (1.1)}\bigr\} = \bigl\{\text{system (1.3)}\bigr\}.$$

(iv)
$$\begin{aligned}
&\lim_{\substack{t_k\to t_p \text{ for all } k\in\{1,2,\ldots,m\} \text{ and } \forall p\in\{1,2,\ldots,m\},\\ \bar{t}_l\to \bar{t}_r \text{ for all } l\in\{1,2,\ldots,n\} \text{ and } \forall r\in\{1,2,\ldots,n\}}} \bigl\{\text{system (1.1)}\bigr\} \\
&= \begin{cases} {}^K_{t_0}\mathcal{D}^{q,\rho}_t x(t) = f(t,x(t)), & t\in(t_0,T], t\neq t_p \text{ and } t\neq \bar{t}_r, \\ {}^K_{t_0}\mathcal{D}^{q-1,\rho}_{t_p+} x(t_p+) - {}^K_{t_0}\mathcal{D}^{q-1,\rho}_{t_p-} x(t_p-) = \sum_{k=1}^m J_k(x(t_p-)), \\ {}^K_{t_0}\mathcal{I}^{2-q,\rho}_{\bar{t}_r+} x(\bar{t}_r+) - {}^K_{t_0}\mathcal{I}^{2-q,\rho}_{\bar{t}_r-} x(\bar{t}_r-) = \sum_{l=1}^n \bar{J}_l(x(\bar{t}_r-)), \\ {}^K_{t_0}\mathcal{D}^{q-1,\rho}_t x(t)|_{t\to t_0+} = x_1, & {}^K_{t_0}\mathcal{I}^{2-q,\rho}_t x(t)|_{t\to t_0+} = x_2. \end{cases}
\end{aligned}$$

(v)
$$\begin{aligned}
&\lim_{t_k\to t_p \text{ for all } k\in\{1,2,\ldots,m\} \text{ and } \forall p\in\{1,2,\ldots,m\}} \bigl\{\text{system (1.2)}\bigr\} \\
&= \begin{cases} {}^K_{t_0}\mathcal{D}^{q,\rho}_t x(t) = f(t,x(t)), & t\in(t_0,T] \text{ and } t\neq t_p, \\ {}^K_{t_0}\mathcal{D}^{q-1,\rho}_{t_p+} x(t_p+) - {}^K_{t_0}\mathcal{D}^{q-1,\rho}_{t_p-} x(t_p-) = \sum_{k=1}^m J_k(x(t_p-)), \\ {}^K_{t_0}\mathcal{D}^{q-1,\rho}_t x(t)|_{t\to t_0+} = x_1, & {}^K_{t_0}\mathcal{I}^{2-q,\rho}_t x(t)|_{t\to t_0+} = x_2. \end{cases}
\end{aligned}$$

(vi)
$$\begin{aligned}
&\lim_{\bar{t}_l\to \bar{t}_r \text{ for all } l\in\{1,2,\ldots,n\} \text{ and } \forall r\in\{1,2,\ldots,n\}} \bigl\{\text{system (1.3)}\bigr\} \\
&= \begin{cases} {}^K_{t_0}\mathcal{D}^{q,\rho}_t x(t) = f(t,x(t)), & t\in(t_0,T], t\neq \bar{t}_r, \\ {}^K_{t_0}\mathcal{I}^{2-q,\rho}_{\bar{t}_r+} x(\bar{t}_r+) - {}^K_{t_0}\mathcal{I}^{2-q,\rho}_{\bar{t}_r-} x(\bar{t}_r-) = \sum_{l=1}^n \bar{J}_l(x(\bar{t}_r-)), \\ {}^K_{t_0}\mathcal{D}^{q-1,\rho}_t x(t)|_{t\to t_0+} = x_1, & {}^K_{t_0}\mathcal{I}^{2-q,\rho}_t x(t)|_{t\to t_0+} = x_2. \end{cases}
\end{aligned}$$

4 The equivalent integral equations of (1.1)–(1.4)

For simplicity, let $f = f(\tau, x(\tau))$ and

$$
\begin{aligned}
y(\ell,t) ={}& \frac{x_2 + x_1 \frac{\ell^\rho - (t_0)^\rho}{\rho} + \int_{t_0}^{\ell} \frac{\ell^\rho - \tau^\rho}{\rho} \frac{f\,d\tau}{\tau^{1-\rho}}}{\Gamma(q-1)} \left[\frac{t^\rho - \ell^\rho}{\rho}\right]^{q-2} + \frac{x_1 + \int_{t_0}^{\ell} \frac{f\,d\tau}{\tau^{1-\rho}}}{\Gamma(q)} \left[\frac{t^\rho - \ell^\rho}{\rho}\right]^{q-1} \\
&+ \frac{1}{\Gamma(q)} \int_{\ell}^{t} \left[\frac{t^\rho - \tau^\rho}{\rho}\right]^{q-1} \frac{f\,d\tau}{\tau^{1-\rho}}, \\
&\text{here } \ell \in \{t_0, t_1, t_2, \ldots, t_m, \bar{t}_1, \bar{t}_2, \ldots, \bar{t}_n, T\}.
\end{aligned}
\tag{4.1}
$$

For $1 < q \leq 2$, define some function spaces:

$$
\begin{aligned}
&\hat{C}_{2-q,\rho}[t_0,T] = \left\{x : (t_0,T] \to \mathbb{R} : \left[t^\rho - \left(t_i'\right)^\rho\right]^{2-q} x(t) \in C\left[t_i', t_{i+1}'\right], i = 0,1,\ldots,M\right\} \\
&\quad (\rho \neq 0), \\
&\hat{C}_{2-q,\rho}[t_0,T] = \left\{x : (t_0,T] \to \mathbb{R} : \left[\ln t - \ln\left(t_i'\right)\right]^{2-q} x(t) \in C\left[t_i', t_{i+1}'\right], i = 0,1,\ldots,M\right\} \\
&\quad (\rho = 0), \\
&\hat{C}^q_{2-q,\rho}[t_0,T] = \left\{x(t) \in \hat{C}_{2-q,\rho}[t_0,T] : {}_{t_0}^{K}\mathcal{D}_t^{q,\rho} x(t) \in \hat{C}_{2-q,\rho}[t_0,T]\right\}, \\
&IC\left([t_0,T],\mathbb{R}\right) = \Big\{x(t) \in \hat{C}_{2-q,\rho}[t_0,T] : \\
&\qquad {}_{t_0}^{K}\mathcal{D}_{t_k-}^{q-1,\rho} x(t_k-) = \lim_{t\to t_k-} {}_{t_0}^{K}\mathcal{D}_t^{q-1,\rho} x(t) = {}_{t_0}^{K}\mathcal{D}_{t_k}^{q-1,\rho} x(t_k) < \infty \\
&\qquad \text{and } {}_{t_0}^{K}\mathcal{D}_{t_k+}^{q-1,\rho} x(t_k+) = \lim_{t\to t_k+} {}_{t_0}^{K}\mathcal{D}_t^{q-1,\rho} x(t) < \infty \text{ (here } k = 1,2,\ldots,m), \\
&\qquad \text{and } {}_{t_0}^{K}\mathcal{I}_{\bar{t}_l-}^{2-q,\rho} x(\bar{t}_l-) = \lim_{t\to \bar{t}_l-} {}_{t_0}^{K}\mathcal{I}_t^{2-q,\rho} x(t) = {}_{t_0}^{K}\mathcal{I}_{\bar{t}_l}^{2-q,\rho} x(\bar{t}_l) < \infty \text{ and} \\
&\qquad {}_{t_0}^{K}\mathcal{I}_{\bar{t}_l+}^{2-q,\rho} x(\bar{t}_l+) = \lim_{t\to \bar{t}_l+} {}_{t_0}^{K}\mathcal{I}_t^{2-q,\rho} x(t) < \infty \text{ (here } l = 1,2,\ldots,n)\Big\}; \\
&\tilde{C}_{2-q,\rho}[t_0,T] = \left\{x : (t_0,T] \to \mathbb{R} : \left[t^\rho - (t_i)^\rho\right]^{2-q} x(t) \in C[t_i, t_{i+1}], i = 0,1,\ldots,m\right\} \\
&\quad (\rho \neq 0), \\
&\tilde{C}_{2-q,\rho}[t_0,T] = \left\{x : (t_0,T] \to \mathbb{R} : \left[\ln t - \ln(t_i)\right]^{2-q} x(t) \in C[t_i, t_{i+1}], i = 0,1,\ldots,m\right\} \\
&\quad (\rho = 0), \\
&\tilde{C}^q_{2-q,\rho}[t_0,T] = \left\{x(t) \in \tilde{C}_{2-q,\rho}[t_0,T] : {}_{t_0}^{K}\mathcal{D}_t^{q,\rho} x(t) \in \tilde{C}_{2-q,\rho}[t_0,T]\right\}, \\
&IC_1\left([t_0,T],\mathbb{R}\right) = \Big\{x(t) \in \tilde{C}_{2-q,\rho}[t_0,T] : {}_{t_0}^{K}\mathcal{D}_{t_k+}^{q-1,\rho} x(t_k+) = \lim_{t\to t_k+} {}_{t_0}^{K}\mathcal{D}_t^{q-1,\rho} x(t) < \infty, \\
&\qquad {}_{t_0}^{K}\mathcal{D}_{t_k-}^{q-1,\rho} x(t_k-) = \lim_{t\to t_k-} {}_{t_0}^{K}\mathcal{D}_t^{q-1,\rho} x(t) = {}_{t_0}^{K}\mathcal{D}_{t_k}^{q-1,\rho} x(t_k) < \infty, \\
&\qquad \text{and } {}_{t_0}^{K}\mathcal{I}_{t_k+}^{2-q,\rho} x(t_k+) = {}_{t_0}^{K}\mathcal{I}_{t_k-}^{2-q,\rho} x(t_k-), \text{ here } k = 1,2,\ldots,m\Big\}; \\
&\bar{C}_{2-q,\rho}[t_0,T] = \left\{x : (t_0,T] \to \mathbb{R} : \left[t^\rho - (\bar{t}_j)^\rho\right]^{2-q} x(t) \in C[\bar{t}_j, \bar{t}_{j+1}], j = 0,1,\ldots,n\right\} \\
&\quad (\rho \neq 0), \\
&\bar{C}_{2-q,\rho}[t_0,T] = \left\{x : (t_0,T] \to \mathbb{R} : \left[\ln t - \ln(\bar{t}_j)\right]^{2-q} x(t) \in C[\bar{t}_j, \bar{t}_{j+1}], j = 0,1,\ldots,n\right\} \\
&\quad (\rho = 0),
\end{aligned}
$$

$$\bar{C}^q_{2-q,\rho}[t_0,T] = \left\{x(t) \in \bar{C}_{2-q,\rho}[t_0,T] : {}^K_{t_0}\mathcal{D}^{q,\rho}_t x(t) \in \bar{C}_{2-q,\rho}[t_0,T]\right\},$$

$$\begin{aligned} IC_2\big([t_0,T],\mathbb{R}\big) = \Big\{&x(t) \in \bar{C}_{2-q,\rho}[t_0,T] : {}^K_{t_0}\mathcal{I}^{2-q,\rho}_{\bar{t}_l+} x(\bar{t}_l+) = \lim_{t\to\bar{t}_l+} {}^K_{t_0}\mathcal{I}^{2-q,\rho}_t x(t) < \infty, \\ &{}^K_{t_0}\mathcal{I}^{2-q,\rho}_{\bar{t}_l-} x(\bar{t}_l-) = \lim_{t\to\bar{t}_l-} {}^K_{t_0}\mathcal{I}^{2-q,\rho}_t x(t) = {}^K_{t_0}\mathcal{I}^{2-q,\rho}_{\bar{t}_l} x(\bar{t}_l) < \infty \\ &\text{and } {}^K_{t_0}\mathcal{D}^{q-1,\rho}_{\bar{t}_l+} x(\bar{t}_l+) = {}^K_{t_0}\mathcal{D}^{q-1,\rho}_{\bar{t}_l-} x(\bar{t}_l-), \text{ here } l = 1,2,\ldots,n\Big\}. \end{aligned}$$

Next, we seek the equivalent integral equation of (1.2). Considering ${}^K_{t_0}\mathcal{D}^{q,\rho}_t x(t) = f(t,x(t))$ on each piecewise interval $(t_k,t_{k+1}]$ $(k = 1,2,\ldots,m)$ by Lemma 2.4, we find a piecewise function

$$\tilde{x}(t) = \begin{cases} y(t_0,t) \quad \text{for } t \in (t_0,t_1], \\ \frac{{}^K_{t_0}\mathcal{I}^{2-q,\rho}_{t_k+} x(t_k+)}{\Gamma(q-1)}[\frac{t^\rho-(t_k)^\rho}{\rho}]^{q-2} + \frac{{}^K_{t_0}\mathcal{D}^{q-1,\rho}_{t_k+} x(t_k+)}{\Gamma(q)}[\frac{t^\rho-(t_k)^\rho}{\rho}]^{q-1} \\ \quad + \frac{1}{\Gamma(q)}\int_{t_k}^t [\frac{t^\rho-\tau^\rho}{\rho}]^{q-1}\frac{f\,d\tau}{\tau^{1-\rho}} \quad \text{for } t \in (t_k,t_{k+1}], k = 1,2,\ldots,m, \end{cases} \tag{4.2}$$

with ${}^K_{t_0}\mathcal{D}^{q-1,\rho}_{t_k+} x(t_k+) = {}^K_{t_0}\mathcal{D}^{q-1,\rho}_{t_k-} x(t_k-) + J_k(x(t_k-))$ and ${}^K_{t_0}\mathcal{I}^{2-q,\rho}_{t_k+} x(t_k+) = {}^K_{t_0}\mathcal{I}^{2-q,\rho}_{t_k-} x(t_k-)$.

Because (4.2) does not satisfy property (i), $\tilde{x}(t)$ is only considered as *an approximate solution* of (1.2). And let

$$e_k(t) = x(t) - \tilde{x}(t), \quad \text{for } t \in (t_k,t_{k+1}]\ (k = 1,2,\ldots,m), \tag{4.3}$$

where $x(t)$ represents the exact solution of (1.2).

Lemma 4.1 *Let $q \in (1,2]$ and $t_0,\rho > 0$, and let $f : [t_0,T] \times \mathbb{R} \to \mathbb{R}$ be a function such that $f(\cdot,x(\cdot)) \in \tilde{C}_{2-q,\rho}[t_0,T]$ for any $x(\cdot) \in \tilde{C}_{2-q,\rho}[t_0,T]$.*

If $x(\cdot) \in IC_1([t_0,T],\mathbb{R})$, then $x(t)$ is a solution of (1.2) if, and only if, $x(t)$ satisfies

$$x(t) = \begin{cases} y(t_0,t) \quad \textit{for } t \in (t_0,t_1], \\ y(t_0,t) + \sum_{i=1}^k \frac{J_i(x(t_i-))}{\Gamma(q)}[\frac{t^\rho-(t_i)^\rho}{\rho}]^{q-1} + \xi\sum_{i=1}^k J_i(x(t_i-))[y(t_i,t) - y(t_0,t)] \\ \quad \textit{for } t \in (t_k,t_{k+1}], k = 1,2,\ldots,m, \end{cases} \tag{4.4}$$

where ξ is an arbitrary constant.

Proof First, we prove the necessity by applying mathematical induction. By Lemma 2.4, the solution of (1.2) as $t \in (t_0,t_1]$ satisfies

$$x(t) = y(t_0,t) \quad \text{for } t \in (t_0,t_1]. \tag{4.5}$$

Using two operators ${}^K_{t_0}\mathcal{D}^{q-1,\rho}_t$ and ${}^K_{t_0}\mathcal{I}^{2-q,\rho}_t$ to two sides of (4.5), respectively, we have

$$\begin{aligned} {}^K_{t_0}\mathcal{D}^{q-1,\rho}_{t_1+} x(t_1+) &= {}^K_{t_0}\mathcal{D}^{q-1,\rho}_{t_1-} x(t_1-) + J_1\big(x(t_1-)\big) \\ &= x_1 + \int_{t_0}^{t_1}\frac{f\,d\tau}{\tau^{1-\rho}} + J_1\big(x(t_1-)\big) \end{aligned} \tag{4.6}$$

and

$$
\begin{aligned}
{}^{K}_{t_0}\mathcal{I}^{2-q,\rho}_{t_1+}x(t_1+) &= {}^{K}_{t_0}\mathcal{I}^{2-q,\rho}_{t_1-}x(t_1-) \\
&= x_2 + x_1\frac{(t_1)^\rho - (t_0)^\rho}{\rho} + \int_{t_0}^{t_1}\frac{(t_1)^\rho - \tau^\rho}{\rho}\frac{f\,d\tau}{\tau^{1-\rho}}.
\end{aligned}
\tag{4.7}
$$

Substituting (4.6)–(4.7) into (4.2), the approximate solution of (1.2) as $t \in (t_1, t_2]$ is given as

$$
\tilde{x}(t) = y(t_1, t) + \frac{J_1(x(t_1-))}{\Gamma(q)}\left[\frac{t^\rho - (t_1)^\rho}{\rho}\right]^{q-1} \quad \text{for } t \in (t_1, t_2]. \tag{4.8}
$$

By (4.5) the exact solution of (1.2) as $t \in (t_1, t_2]$ satisfies

$$
\lim_{J_1(x(t_1-))\to 0} x(t) = y(t_0, t) \quad \text{for } t \in (t_1, t_2]. \tag{4.9}
$$

By (4.3) and (4.8)–(4.9), we get

$$
\lim_{J_1(x(t_1-))\to 0} e_1(t) = y(t_0, t) - y(t_1, t) \quad \text{for } t \in (t_1, t_2]. \tag{4.10}
$$

From (4.10), let $e_1(t) = \kappa(J_1(x(t_1^-)))\lim_{J_1(x(t_1^-))\to 0} e_1(t)$, where $\kappa(\cdot)$ is an undetermined function to satisfy $\kappa(0) = 1$, and

$$
e_1(t) = \kappa\big(J_1\big(x(t_1-)\big)\big)\lim_{J_1(x(t_1-))\to 0} e_1(t) = -\kappa\big(J_1\big(x(t_1-)\big)\big)\big[y(t_1, t) - y(t_0, t)\big]. \tag{4.11}
$$

Plugging (4.8) and (4.11) into (4.3), we obtain

$$
\begin{aligned}
x(t) &= y(t_0, t) + \frac{J_1(x(t_1-))}{\Gamma(q)}\left[\frac{t^\rho - (t_1)^\rho}{\rho}\right]^{q-1} + \big[1 - \kappa\big(J_1\big(x(t_1-)\big)\big)\big]\big[y(t_1, t) - y(t_0, t)\big] \\
&\quad \text{for } t \in (t_1, t_2].
\end{aligned}
\tag{4.12}
$$

Because ${}^{K}_{t_0}\mathcal{D}^{q,0+}_{t}$ (${}^{K}_{t_0}\mathcal{D}^{q,\rho}_{t}$ with $\rho \to 0+$) is the Hadamard fractional derivative, we get $1 - \kappa(J_1(x(t_1^-))) = \xi J_1(x(t_1^-))$ (here ξ is an arbitrary constant) by applying Lemma 3.3 in [44] to (1.2) and (4.12) with $\rho \to 0+$. Thus (4.12) is rewritten as

$$
\begin{aligned}
x(t) &= y(t_0, t) + \frac{J_1(x(t_1-))}{\Gamma(q)}\left[\frac{t^\rho - (t_1)^\rho}{\rho}\right]^{q-1} + \xi J_1\big(x(t_1-)\big)\big[y(t_1, t) - y(t_0, t)\big] \\
&\quad \text{for } t \in (t_1, t_2].
\end{aligned}
\tag{4.13}
$$

Therefore the solution of (1.2) satisfies (4.4) as $t \in (t_1, t_2]$.

For $t \in (t_k, t_{k+1}]$, suppose that the solution of (1.2) satisfies

$$
\begin{aligned}
x(t) &= y(t_0, t) + \sum_{i=1}^{k}\frac{J_i(x(t_i-))}{\Gamma(q)}\left[\frac{t^\rho - (t_i)^\rho}{\rho}\right]^{q-1} + \xi\sum_{i=1}^{k}J_i\big(x(t_i-)\big)\big[y(t_i, t) - y(t_0, t)\big] \\
&\quad \text{for } t \in (t_k, t_{k+1}]
\end{aligned}
\tag{4.14}
$$

to prove that the solution of (1.2) satisfies (4.4) as $t \in (t_{k+1}, t_{k+2}]$.

Using operators ${}_{t_0}^{K}\mathcal{D}_t^{q-1,\rho}$ and ${}_{t_0}^{K}\mathcal{I}_t^{2-q,\rho}$ to two sides of (4.14) respectively, we obtain

$$\begin{aligned}
{}_{t_0}^{K}\mathcal{D}_{t_{k+1}+}^{q-1,\rho}x(t_{k+1}+) &= {}_{t_0}^{K}\mathcal{D}_{t_{k+1}-}^{q-1,\rho}x(t_{k+1}-) + J_{k+1}\bigl(x(t_{k+1}-)\bigr) \\
&= x_1 + \int_{t_0}^{t_{k+1}} \frac{f\,d\tau}{\tau^{1-\rho}} + \sum_{i=1}^{k+1} J_i\bigl(x(t_i-)\bigr)
\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}
{}_{t_0}^{K}\mathcal{I}_{t_{k+1}+}^{2-q,\rho}x(t_{k+1}+) &= {}_{t_0}^{K}\mathcal{I}_{t_{k+1}-}^{2-q,\rho}x(t_{k+1}-) \\
&= x_2 + x_1\frac{(t_{k+1})^\rho - (t_0)^\rho}{\rho} + \int_{t_0}^{t_{k+1}} \frac{(t_{k+1})^\rho - \tau^\rho}{\rho}\frac{f\,d\tau}{\tau^{1-\rho}} \\
&\quad + \sum_{i=1}^{k} J_i\bigl(x(t_i-)\bigr)\frac{(t_{k+1})^\rho - (t_i)^\rho}{\rho}.
\end{aligned} \tag{4.16}$$

Plugging (4.15) and (4.16) into (4.2), the approximate solution of (1.2) as $t \in (t_{k+1}, t_{k+2}]$ is given by

$$\begin{aligned}
\tilde{x}(t) &= y(t_{k+1}, t) + \frac{\sum_{i=1}^{k} J_i(x(t_i-))\frac{(t_{k+1})^\rho - (t_i)^\rho}{\rho}}{\Gamma(q-1)}\left[\frac{t^\rho - (t_{k+1})^\rho}{\rho}\right]^{q-2} \\
&\quad + \frac{\sum_{i=1}^{k+1} J_i(x(t_i-))}{\Gamma(q)}\left[\frac{t^\rho - (t_{k+1})^\rho}{\rho}\right]^{q-1} \quad \text{for } t \in (t_{k+1}, t_{k+2}].
\end{aligned} \tag{4.17}$$

On the other hand, by (4.14) the exact solution of (1.2) as $t \in (t_{k+1}, t_{k+2}]$ satisfies

$$\lim_{J_i(x(t_i-))\to 0 \text{ for all } i\in\{1,2,\ldots,k+1\}} x(t) = y(t_0, t) \quad \text{for } t \in (t_{k+1}, t_{k+2}] \tag{4.18}$$

and

$$\begin{aligned}
\lim_{\substack{J_p(x(t_p-))\to 0 \text{ here} \\ p\in\{1,2,\ldots,k+1\}}} x(t) &= y(t_0, t) + \sum_{\substack{1\le i\le k+1 \\ \text{and } i\ne p}} \frac{J_i(x(t_i-))}{\Gamma(q)}\left[\frac{t^\rho - (t_i)^\rho}{\rho}\right]^{q-1} \\
&\quad + \xi \sum_{\substack{1\le i\le k+1 \\ \text{and } i\ne p}} J_i\bigl(x(t_i-)\bigr)\bigl[y(t_i, t) - y(t_0, t)\bigr] \quad \text{for } t \in (t_{k+1}, t_{k+2}].
\end{aligned} \tag{4.19}$$

By (4.3) and (4.17)–(4.19), we have

$$\lim_{\substack{J_i(x(t_i-))\to 0 \text{ for} \\ \text{all} i\in\{1,2,\ldots,k+1\}}} e_{k+1}(t) = \lim_{\substack{J_i(x(t_i-))\to 0 \text{ for} \\ \text{all } i\in\{1,2,\ldots,k+1\}}} \bigl\{x(t) - \tilde{x}(t)\bigr\} = -\bigl[y(t_{k+1}, t) - y(t_0, t)\bigr] \tag{4.20}$$

and

$$\begin{aligned}
\lim_{\substack{J_p(x(t_p-))\to 0 \text{ here} \\ p\in\{1,2,\ldots,k+1\}}} e_{k+1}(t) &= \lim_{\substack{J_p(x(t_p-))\to 0 \text{ here} \\ p\in\{1,2,\ldots,k+1\}}} \bigl\{x(t) - \tilde{x}(t)\bigr\} \\
&= -\bigl[y(t_{k+1}, t) - y(t_0, t)\bigr] + \xi \sum_{\substack{1\le i\le k+1 \\ \text{and } i\ne p}} J_i\bigl(x(t_i-)\bigr)\bigl[y(t_i, t) - y(t_0, t)\bigr]
\end{aligned}$$

$$+\sum_{\substack{1\le i\le k+1\\ \text{and } i\neq p}}\frac{J_i(x(t_i-))}{\Gamma(q)}\left\{\left[\frac{t^\rho-(t_i)^\rho}{\rho}\right]^{q-1}-\left[\frac{t^\rho-(t_{k+1})^\rho}{\rho}\right]^{q-1}\right\}$$
$$-\sum_{\substack{1\le i\le k+1\\ \text{and } i\neq p}}\frac{J_i(x(t_i-))\frac{(t_{k+1})^\rho-(t_i)^\rho}{\rho}}{\Gamma(q-1)}\left[\frac{t^\rho-(t_{k+1})^\rho}{\rho}\right]^{q-2}. \tag{4.21}$$

By (4.20) and (4.21), we obtain

$$\begin{aligned}e_{k+1}(t)={}&-\big[y(t_{k+1},t)-y(t_0,t)\big]+\xi\sum_{i=1}^{k+1}J_i\big(x(t_i-)\big)\big[y(t_i,t)-y(t_0,t)\big]\\&+\sum_{i=1}^{k+1}\frac{J_i(x(t_i-))}{\Gamma(q)}\left\{\left[\frac{t^\rho-(t_i)^\rho}{\rho}\right]^{q-1}-\left[\frac{t^\rho-(t_{k+1})^\rho}{\rho}\right]^{q-1}\right\}\\&-\sum_{i=1}^{k+1}\frac{J_i(x(t_i-))\frac{(t_{k+1})^\rho-(t_i)^\rho}{\rho}}{\Gamma(q-1)}\left[\frac{t^\rho-(t_{k+1})^\rho}{\rho}\right]^{q-2}.\end{aligned}\tag{4.22}$$

Thus, substituting (4.17) and (4.22) into (4.3), we get

$$x(t)=y(t_0,t)+\sum_{i=1}^{k+1}\frac{J_i(x(t_i-))}{\Gamma(q)}\left[\frac{t^\rho-(t_i)^\rho}{\rho}\right]^{q-1}+\xi\sum_{i=1}^{k+1}J_i\big(x(t_i-)\big)\big[y(t_i,t)-y(t_0,t)\big]$$
$$\text{for } t\in(t_{k+1},t_{k+2}].$$

Therefore the solution of (1.2) satisfies (4.4) as $t\in(t_{k+1},t_{k+2}]$. Hence the necessity is proved.

Now we prove the sufficiency. Applying the operators ${}_{t_0}^{K}\mathcal{D}_t^{q,\rho}$, ${}_{t_0}^{K}\mathcal{D}_t^{q-1,\rho}$, and ${}_{t_0}^{K}\mathcal{I}_t^{2-q,\rho}$ to two sides of (4.4) as $t\in(t_k,t_{k+1}]$, respectively, we have

$$\begin{aligned}&{}_{t_0}^{K}\mathcal{D}_t^{q,\rho}x(t)|_{t\in(t_k,t_{k+1}]}\\&\quad=\left\{f\big(t,x(t)\big)|_{t\ge t_0}+\xi\sum_{i=1}^{k}J_i\big(x(t_i-)\big)\big[f\big(t,x(t)\big)|_{t\ge t_i}-f\big(t,x(t)\big)|_{t\ge t_0}\big]\right\}_{t\in(t_k,t_{k+1}]}\\&\quad=f\big(t,x(t)\big)|_{t\in(t_k,t_{k+1}]},\\&{}_{t_0}^{K}\mathcal{D}_t^{q-1,\rho}x(t)|_{t\in(t_k,t_{k+1}]}\\&\quad=\left\{x_1+\int_{t_0}^{t}\frac{f\,d\tau}{\tau^{1-\rho}}+\sum_{i=1}^{k}J_i\big(x(t_i-)\big)\right.\\&\qquad\left.+\sum_{i=1}^{k}\xi J_i\big(x(t_i-)\big)\left[x_1+\int_{t_0}^{t_i}\frac{f\,d\tau}{\tau^{1-\rho}}+\int_{t_i}^{t}\frac{f\,d\tau}{\tau^{1-\rho}}-x_1-\int_{t_0}^{t}\frac{f\,d\tau}{\tau^{1-\rho}}\right]\right\}_{t\in(t_k,t_{k+1}]}\\&\quad=\left\{x_1+\int_{t_0}^{t}\frac{f\,d\tau}{\tau^{1-\rho}}+\sum_{i=1}^{k}J_i\big(x(t_i-)\big)\right\}_{t\in(t_k,t_{k+1}]},\end{aligned}$$

and

$$
\begin{aligned}
&{}_{t_0}^{K}\mathcal{I}_t^{2-q,\rho}x(t)|_{t\in(t_k,t_{k+1}]} \\
&\quad = \left\{x_2 + x_1\frac{t^\rho-(t_0)^\rho}{\rho} + \int_{t_0}^{t}\frac{t^\rho-\tau^\rho}{\rho}\frac{f\,d\tau}{\tau^{1-\rho}} + \sum_{i=1}^{k}J_i\bigl(x(t_i-)\bigr)\frac{t^\rho-(t_i)^\rho}{\rho}\right\}_{t\in(t_k,t_{k+1}]}.
\end{aligned}
$$

Thus ${}_{t_0}^{K}\mathcal{D}_t^{q-1,\rho}x(t)|_{t\to t_0+} = x_1$, ${}_{t_0}^{K}\mathcal{I}_t^{2-q,\rho}x(t)|_{t\to t_0+} = x_2$, ${}_{t_0}^{K}\mathcal{D}_{t_k+}^{q-1,\rho}x(t_k+) - {}_{t_0}^{K}\mathcal{D}_{t_k-}^{q-1,\rho}x(t_k-) = J_k(x(t_k-))$, and ${}_{t_0}^{K}\mathcal{I}_{t_k+}^{2-q,\rho}x(t_k+) = {}_{t_0}^{K}\mathcal{I}_{t_k-}^{2-q,\rho}x(t_k-)$, and (4.4) satisfies the condition of fractional derivative in (1.2).

Letting $J_k(x(t_k-)) = 0$ for all $k \in \{1,2,\dots,m\}$ in (4.3), we obtain

$$
\begin{aligned}
&\lim_{J_k(x(t_k-))\to 0 \text{ for all } k\in\{1,2,\dots,m\}}\bigl\{\text{Eq. (4.3)}\bigr\} \text{ is equivalent to} \\
&\quad \lim_{J_k(x(t_k-))\to 0 \text{ for all } k\in\{1,2,\dots,m\}}\bigl\{\text{system (1.2)}\bigr\}.
\end{aligned}
$$

Moreover, it is obvious that (4.4) satisfies condition (v). Therefore (4.4) satisfies all the conditions of (1.2). Hence, this proof is completed. □

Remark 4.2 Similar to (4.2), *an approximate solution* of (1.3) is presented by

$$
\tilde{\tilde{x}}(t) = \begin{cases} y(t_0,t) \quad \text{for } t\in(t_0,\bar{t}_1], \\ \frac{{}_{t_0}^{K}\mathcal{I}_{\bar{t}_l+}^{2-q,\rho}x(\bar{t}_l+)}{\Gamma(q-1)}[\frac{t^\rho-(\bar{t}_l)^\rho}{\rho}]^{q-2} + \frac{{}_{t_0}^{K}\mathcal{D}_{\bar{t}_l+}^{q-1,\rho}x(\bar{t}_l+)}{\Gamma(q)}[\frac{t^\rho-(\bar{t}_l)^\rho}{\rho}]^{q-1} \\ \quad + \frac{1}{\Gamma(q)}\int_{\bar{t}_l}^{t}[\frac{t^\rho-\tau^\rho}{\rho}]^{q-1}\frac{f\,d\tau}{\tau^{1-\rho}} \quad \text{for } t\in(\bar{t}_l,\bar{t}_{l+1}], l = 1,2,\dots,n, \end{cases} \tag{4.23}
$$

with ${}_{t_0}^{K}\mathcal{I}_{\bar{t}_l+}^{2-q,\rho}x(\bar{t}_l+) = {}_{t_0}^{K}\mathcal{I}_{\bar{t}_l-}^{2-q,\rho}x(\bar{t}_l-) + \bar{J}_l(x(\bar{t}_l-))$ and ${}_{t_0}^{K}\mathcal{D}_{\bar{t}_l+}^{q-1,\rho}x(\bar{t}_l+) = {}_{t_0}^{K}\mathcal{D}_{\bar{t}_l-}^{q-1,\rho}x(\bar{t}_l-)$.

Furthermore, using the thought of Lemma 4.1, we arrive at the following conclusion.

Lemma 4.3 *Let $q\in(1,2]$ and $t_0,\rho>0$, and let $f:[t_0,T]\times\mathbb{R}\to\mathbb{R}$ be a function such that $f(\cdot,x(\cdot))\in\bar{C}_{2-q,\rho}[t_0,T]$ for any $x(\cdot)\in\bar{C}_{2-q,\rho}[t_0,T]$.*

If $x(\cdot)\in IC_2([t_0,T],\mathbb{R})$, then $x(t)$ is a solution of (1.3) if, and only if, $x(t)$ satisfies the following integral equation:

$$
x(t) = \begin{cases} y(t_0,t) \quad \textit{for } t\in(t_0,\bar{t}_1], \\ y(t_0,t) + \sum_{j=1}^{l}\frac{\bar{J}_j(x(\bar{t}_j-))}{\Gamma(q-1)}[\frac{t^\rho-(\bar{t}_j)^\rho}{\rho}]^{q-2} + \eta\sum_{j=1}^{l}\bar{J}_j(x(\bar{t}_j-))[y(\bar{t}_j,t)-y(t_0,t)] \\ \quad \textit{for } t\in(\bar{t}_l,\bar{t}_{l+1}], l = 1,2,\dots,n, \end{cases} \tag{4.24}
$$

where η is an arbitrary constant.

The following theorem yields the equivalence between Cauchy problem (1.1) and the Volterra integral equation of the second kind:

$$x(t)=\begin{cases} y(t_0,t) \quad \text{for } t\in(t_0,t_1'], \\ y(t_0,t)+\sum_{i=1}^{k_1}\frac{J_i(x(t_i-))}{\Gamma(q)}[\frac{t^\rho-(t_i)^\rho}{\rho}]^{q-1}+\sum_{j=1}^{k_2}\frac{\bar{J}_j(x(\bar{t}_j-))}{\Gamma(q-1)}[\frac{t^\rho-(\bar{t}_j)^\rho}{\rho}]^{q-2} \\ \quad +\xi\sum_{i=1}^{k_1}J_i(x(t_i-))[y(t_i,t)-y(t_0,t)] \\ \quad +\eta\sum_{j=1}^{k_2}\bar{J}_j(x(\bar{t}_j-))[y(\bar{t}_j,t)-y(t_0,t)] \\ \quad \text{for } t\in(t_k',t_{k+1}'], \end{cases} \tag{4.25}$$

where ξ and η are two arbitrary constants.

Theorem 4.4 *Let $q\in(1,2]$ and $t_0,\rho>0$, and let $f:[a,T]\times\mathbb{R}\to\mathbb{R}$ be a function such that $f(\cdot,x(\cdot))\in\hat{C}_{2-q,\rho}[t_0,T]$ for any $x(\cdot)\in\hat{C}_{2-q,\rho}[a,T]$.*

If $x(\cdot)\in IC([t_0,T],\mathbb{R})$, then $x(t)$ is a solution of (1.1) if, and only if, $x(t)$ satisfies (4.25).

Proof First, we prove the necessity that the solution of (1.1) satisfies (4.25) by the mathematical induction. For $t\in(t_0,t_1']$, by Lemma 2.4, the solution of system (1.1) satisfies (4.25) and

$$x(t)=y(t_0,t)\quad \text{for } t\in\bigl(t_0,t_1'\bigr]. \tag{4.26}$$

For $t\in(t_1',t_2']$, there appear three cases $t_1'=t_1<\bar{t}_1$, $t_1'=\bar{t}_1<t_1$, and $t_1'=t_1=\bar{t}_1$. For $t_1'=t_1<\bar{t}_1$ and $t_1'=\bar{t}_1<t_1$, the solution of (1.1) satisfies (4.25) as $t\in(t_1',t_2']$ by Lemmas 4.1 and 4.3, respectively. Hence, we need only prove that the solution of (1.1) satisfies (4.25) as $t\in(t_1',t_2']$ with $t_1'=t_1=\bar{t}_1$. Applying ${}^K_{t_0}\mathcal{D}^{q-1,\rho}_t$ and ${}^K_{t_0}\mathcal{I}^{2-q,\rho}_t$ to two sides of (4.26), we have

$${}^K_{t_0}\mathcal{D}^{q-1,\rho}_{t_1'+}x\bigl(t_1'+\bigr)={}^K_{t_0}\mathcal{D}^{q-1,\rho}_{t_1'-}x\bigl(t_1'-\bigr)+J_1\bigl(x\bigl(t_1'-\bigr)\bigr)=x_1+\int_{t_0}^{t_1'}\frac{f\,d\tau}{\tau^{1-\rho}}+J_1\bigl(x\bigl(t_1'-\bigr)\bigr) \tag{4.27}$$

and

$$\begin{aligned}{}^K_{t_0}\mathcal{I}^{2-q,\rho}_{t_1'+}x\bigl(t_1'-\bigr)&={}^K_{t_0}\mathcal{I}^{2-q,\rho}_{t_1'-}x\bigl(t_1'-\bigr)+\bar{J}_1\bigl(x\bigl(t_1'-\bigr)\bigr)\\ &=x_2+x_1\frac{(t_1')^\rho-(t_0)^\rho}{\rho}+\int_{t_0}^{t_1'}\frac{(t_1')^\rho-\tau^\rho}{\rho}\frac{f\,d\tau}{\tau^{1-\rho}}+\bar{J}_1\bigl(x\bigl(t_1'-\bigr)\bigr).\end{aligned} \tag{4.28}$$

Therefore, the approximate solution of (1.1) is given as $t\in(t_1',t_2']$ by

$$\begin{aligned}\hat{x}(t)&=\frac{{}^K_{t_0}\mathcal{I}^{2-q,\rho}_{t_1'+}x(t_1'+)}{\Gamma(q-1)}\left[\frac{t^\rho-(t_1)^\rho}{\rho}\right]^{q-2}+\frac{{}^K_{t_0}\mathcal{D}^{q-1,\rho}_{t_1'+}x(t_1'+)}{\Gamma(q)}\left[\frac{t^\rho-(t_1')^\rho}{\rho}\right]^{q-1}\\ &\quad+\frac{1}{\Gamma(q)}\int_{t_1'}^{t}\left[\frac{t^\rho-\tau^\rho}{\rho}\right]^{q-1}\frac{f\,d\tau}{\tau^{1-\rho}}\quad \text{for } t\in\bigl(t_1',t_2'\bigr]\\ &=y\bigl(t_1',t\bigr)+\frac{\bar{J}_1(x(t_1'-))}{\Gamma(q-1)}\left[\frac{t^\rho-(t_1)^\rho}{\rho}\right]^{q-2}+\frac{J_1(x(t_1'-))}{\Gamma(q)}\left[\frac{t^\rho-(t_1')^\rho}{\rho}\right]^{q-1}\\ &\quad \text{for } t\in\bigl(t_1',t_2'\bigr],\end{aligned} \tag{4.29}$$

with the error $\hat{e}_1(t) = x(t) - \hat{x}(t)$ for $t \in (t'_1, t'_2]$, where $x(t)$ is the exact solution of (1.1). Moreover, by Lemmas 4.1 and 4.3, the exact solution $x(t)$ of (1.1) as $t \in (t'_1, t'_2]$ satisfies three conditions:

$$\lim_{J_1(x(t'_1-))\to 0, \bar{J}_1(x(t'_1-))\to 0} x(t) = y(t_0, t) \quad \text{for } t \in (t'_1, t'_2], \tag{4.30}$$

$$\lim_{\bar{J}_1(x(t'_1-))\to 0} x(t) = y(t_0, t) + \frac{J_1(x(t'_1-))}{\Gamma(q)}\left[\frac{t^\rho - (t'_1)^\rho}{\rho}\right]^{q-1} + \xi J_1\big(x(t'_1-)\big)\big[y(t'_1, t) - y(t_0, t)\big] \\ \text{for } t \in \big(t'_1, t'_2\big], \tag{4.31}$$

$$\lim_{J_1(x(t'_1-))\to 0} x(t) = y(t_0, t) + \frac{\bar{J}_1(x(t'_1-))}{\Gamma(q-1)}\left[\frac{t^\rho - (t'_1)^\rho}{\rho}\right]^{q-2} + \eta \bar{J}_1\big(x(t'_1-)\big)\big[y(t'_1, t) - y(t_0, t)\big] \\ \text{for } t \in \big(t'_1, t'_2\big]. \tag{4.32}$$

By (4.29)–(4.32), we get

$$\lim_{\substack{J_1(x(t'_1-))\to 0,\\ \bar{J}_1(x(t'_1-))\to 0}} \hat{e}_1(t) = y(t_0, t) - y\big(t'_1, t\big), \tag{4.33}$$

$$\lim_{\bar{J}_1(x(t'_1-))\to 0} \hat{e}_1(t) = \big[\xi J_1\big(x\big(t'_1-\big)\big) - 1\big]\big[y\big(t'_1, t\big) - y(t_0, t)\big], \tag{4.34}$$

$$\lim_{J_1(x(t'_1-))\to 0} \hat{e}_1(t) = \big[\eta \bar{J}_1\big(x\big(t'_1-\big)\big) - 1\big]\big[y\big(t'_1, t\big) - y(t_0, t)\big]. \tag{4.35}$$

By (4.33)–(4.35), we obtain

$$\hat{e}_1(t) = \big[\xi J_1\big(x\big(t'_1-\big)\big) + \eta \bar{J}_1\big(x\big(t'_1-\big)\big) - 1\big]\big[y\big(t'_1, t\big) - y(t_0, t)\big] \quad \text{for } t \in \big(t'_1, t'_2\big]. \tag{4.36}$$

By (4.29) and (4.36), we have

$$\begin{aligned} x(t) &= \hat{x}(t) + \hat{e}_1(t) \\ &= y(t_0, t) + \frac{J_1(x(t'_1-))}{\Gamma(q)}\left[\frac{t^\rho - (t'_1)^\rho}{\rho}\right]^{q-1} + \frac{\bar{J}_1(x(t'_1-))}{\Gamma(q-1)}\left[\frac{t^\rho - (t_1)^\rho}{\rho}\right]^{q-2} \\ &\quad + \big[\xi J_1\big(x\big(t'_1-\big)\big) + \eta \bar{J}_1\big(x\big(t'_1-\big)\big)\big]\big[y\big(t'_1, t\big) - y(t_0, t)\big] \quad \text{for } t \in \big(t'_1, t'_2\big]. \end{aligned} \tag{4.37}$$

Therefore the solution of (1.1) satisfies (4.25) as $t \in (t'_1, t'_2]$.

Next, for $t \in (t'_k, t'_{k+1}]$ ($k \in \{1, 2, \dots, M\}$), suppose that the solution of (1.1) satisfies

$$\begin{aligned} x(t) &= y(t_0, t) + \sum_{i=1}^{k_1} \frac{J_i(x(t_i-))}{\Gamma(q)}\left[\frac{t^\rho - (t_i)^\rho}{\rho}\right]^{q-1} + \sum_{j=1}^{k_2} \frac{\bar{J}_j(x(\bar{t}_j-))}{\Gamma(q-1)}\left[\frac{t^\rho - (\bar{t}_j)^\rho}{\rho}\right]^{q-2} \\ &\quad + \xi \sum_{i=1}^{k_1} J_i\big(x(t_i-)\big)\big[y(t_i, t) - y(t_0, t)\big] + \eta \sum_{j=1}^{k_2} \bar{J}_j\big(x(\bar{t}_j-)\big)\big[y(\bar{t}_j, t) - y(t_0, t)\big] \\ &\quad \text{for } t \in \big(t'_k, t'_{k+1}\big]. \end{aligned} \tag{4.38}$$

Using ${}_{t_0}^{K}\mathcal{D}_t^{q-1,\rho}$ and ${}_{t_0}^{K}\mathcal{I}_t^{2-q,\rho}$ to two sides of (4.38) respectively, we get

$$\begin{aligned}{}_{t_0}^{K}\mathcal{D}_{t'_{k+1}+}^{q-1,\rho}x\bigl(t'_{k+1}+\bigr) &= {}_{t_0}^{K}\mathcal{D}_{t'_{k+1}-}^{q-1,\rho}x\bigl(t'_{k+1}-\bigr) + \sum_{i=k_1+1}^{(k+1)_1} J_i\bigl(x(t_i-)\bigr) \\ &= x_1 + \int_{t_0}^{t'_{k+1}} \frac{f\,d\tau}{\tau^{1-\rho}} + \sum_{i=1}^{(k+1)_1} J_i\bigl(x(t_i-)\bigr)\end{aligned} \tag{4.39}$$

and

$$\begin{aligned}{}_{t_0}^{K}\mathcal{I}_{t'_{k+1}+}^{2-q,\rho}x\bigl(t'_{k+1}+\bigr) &= {}_{t_0}^{K}\mathcal{I}_{t'_{k+1}-}^{2-q,\rho}x\bigl(t'_{k+1}-\bigr) + \sum_{j=k_2+1}^{(k+1)_2} \bar{J}_j\bigl(x(\bar{t}_j-)\bigr) \\ &= x_2 + x_1\frac{(t'_{k+1})^\rho-(t_0)^\rho}{\rho} + \int_{t_0}^{t'_{k+1}} \frac{(t'_{k+1})^\rho-\tau^\rho}{\rho}\frac{f\,d\tau}{\tau^{1-\rho}} \\ &\quad + \sum_{j=1}^{(k+1)_2} \bar{J}_j\bigl(x(\bar{t}_j-)\bigr) + \sum_{i=1}^{k_1} J_i\bigl(x(t_i-)\bigr)\frac{(t'_{k+1})^\rho-(t_i)^\rho}{\rho}.\end{aligned} \tag{4.40}$$

Therefore, the approximate solution of (1.1) as $t\in(t'_{k+1},t'_{k+2}]$ is given by

$$\begin{aligned}\tilde{x}(t) &= \frac{{}_{t_0}^{K}\mathcal{I}_{t'_{k+1}+}^{2-q,\rho}x(t'_{k+1}+)}{\Gamma(q-1)}\left[\frac{t^\rho-(t'_{k+1})^\rho}{\rho}\right]^{q-2} + \frac{{}_{t_0}^{K}\mathcal{D}_{t'_{k+1}+}^{q-1,\rho}x(t'_{k+1}+)}{\Gamma(q)}\left[\frac{t^\rho-(t'_{k+1})^\rho}{\rho}\right]^{q-1} \\ &\quad + \frac{1}{\Gamma(q)}\int_{t'_{k+1}}^{t}\left[\frac{t^\rho-\tau^\rho}{\rho}\right]^{q-1}\frac{f\,d\tau}{\tau^{1-\rho}} \quad \text{for } t\in\bigl(t'_{k+1},t'_{k+2}\bigr] \\ &= y\bigl(t'_{k+1},t\bigr) + \frac{\sum_{j=1}^{(k+1)_2}\bar{J}_j(x(\bar{t}_j-)) + \sum_{i=1}^{k_1}J_i(x(t_i-))\frac{(t'_{k+1})^\rho-(t_i)^\rho}{\rho}}{\Gamma(q-1)}\left[\frac{t^\rho-(t'_{k+1})^\rho}{\rho}\right]^{q-2} \\ &\quad + \frac{\sum_{i=1}^{(k+1)_1}J_i(x(t_i-))}{\Gamma(q)}\left[\frac{t^\rho-(t'_{k+1})^\rho}{\rho}\right]^{q-1} \quad \text{for } t\in\bigl(t'_{k+1},t'_{k+2}\bigr],\end{aligned} \tag{4.41}$$

with $\hat{e}_{k+1}(t)=x(t)-\hat{x}(t)$ for $t\in(t'_{k+1},t'_{k+2}]$, where $x(t)$ is the exact solution of (1.1). By (4.38), the exact solution of (1.1) satisfies

$$\lim_{\substack{J_i(x(t_i-))\to0,\bar{J}_j(x(\bar{t}_j-))\to0\\ \text{for all } i \text{ and } j}} x(t) = y(t_0,t) \quad \text{for } t\in\bigl(t'_{k+1},t'_{k+2}\bigr], \tag{4.42}$$

$$\begin{aligned}&\lim_{\substack{J_i(x(t_i-))\to0 \text{ for all } i\in\{l_1+1,l_1+2,\dots,(l+1)_1\},\\ \bar{J}_j(x(\bar{t}_j-))\to0 \text{ for all } j\in\{l_2+1,l_2+2,\dots,(l+1)_2\}}} x(t) \\ &= y(t_0,t) + \sum_{\substack{1\le i\le(k+1)_1 \text{ and}\\ i\notin\{l_1+1,l_1+2,\dots,(l+1)_1\}}} \frac{J_i(x(t_i-))}{\Gamma(q)}\left[\frac{t^\rho-(t_i)^\rho}{\rho}\right]^{q-1} \\ &\quad + y(t_0,t) + \sum_{\substack{1\le j\le(k+1)_2 \text{ and}\\ j\notin\{l_2+1,l_2+2,\dots,(l+1)_2\}}} \frac{\bar{J}_j(x(\bar{t}_j-))}{\Gamma(q-1)}\left[\frac{t^\rho-(\bar{t}_j)^\rho}{\rho}\right]^{q-2} \\ &\quad + \xi \sum_{\substack{1\le i\le(k+1)_1 \text{ and}\\ i\notin\{l_1+1,l_1+2,\dots,(l+1)_1\}}} J_i\bigl(x(t_i-)\bigr)\bigl[y(t_i,t)-y(t_0,t)\bigr]\end{aligned}$$

$$+\eta \sum_{\substack{1\le j\le (k+1)_2 \text{ and} \\ j\notin\{l_2+1,l_2+2,\ldots,(l+1)_2\}}} \bar{J}_j\big(x(\bar{t}_j-)\big)\big[y(\bar{t}_j,t)-y(t_0,t)\big]$$

$$\text{for } t\in\big(t'_{k+1},t'_{k+2}\big], l=1,2,\ldots,k+1. \tag{4.43}$$

By (4.41)–(4.43), we obtain

$$\lim_{\substack{J_i(x(t_i-))\to 0,\bar{J}_j(x(\bar{t}_j-))\to 0 \\ \text{for all } i \text{ and } j}} \hat{e}_{k+1}(t) = -\big[y\big(t'_{k+1},t\big)-y(t_0,t)\big], \tag{4.44}$$

$$\begin{aligned}
&\lim_{\substack{J_i(x(t_i-))\to 0 \text{ for all } i\in\{l_1+1,l_1+2,\ldots,(l+1)_1\}, \\ \bar{J}_j(x(\bar{t}_j-))\to 0 \text{ for all } j\in\{l_2+1,l_2+2,\ldots,(l+1)_2\}}} \hat{e}_{k+1}(t) \\
&= -\big[y\big(t'_{k+1},t\big)-y(t_0,t)\big] + \xi \sum_{\substack{1\le i\le (k+1)_1 \text{ and} \\ i\notin\{l_1+1,\ldots,(l+1)_1\}}} J_i\big(x(t_i-)\big)\big[y(t_i,t)-y(t_0,t)\big] \\
&\quad + \eta \sum_{\substack{1\le j\le (k+1)_2 \text{ and} \\ j\notin\{l_2+1,\ldots,(l+1)_2\}}} \bar{J}_j\big(x(\bar{t}_j-)\big)\big[y(\bar{t}_j,t)-y(t_0,t)\big] \\
&\quad + \sum_{\substack{1\le i\le (k+1)_1 \text{ and} \\ i\notin\{l_1+1,l_1+2,\ldots,(l+1)_1\}}} \frac{J_i(x(t_i-))}{\Gamma(q)} \left\{\left[\frac{t^\rho-(t_i)^\rho}{\rho}\right]^{q-1} - \left[\frac{t^\rho-(t'_{k+1})^\rho}{\rho}\right]^{q-1}\right\} \\
&\quad + \sum_{\substack{1\le j\le (k+1)_2 \text{ and} \\ j\notin\{l_2+1,l_2+2,\ldots,(l+1)_2\}}} \frac{\bar{J}_j(x(\bar{t}_j-))}{\Gamma(q-1)} \left\{\left[\frac{t^\rho-(\bar{t}_j)^\rho}{\rho}\right]^{q-2} - \left[\frac{t^\rho-(t'_{k+1})^\rho}{\rho}\right]^{q-2}\right\} \\
&\quad - \sum_{\substack{1\le i\le k_1 \text{ and} \\ i\notin\{l_1+1,l_1+2,\ldots,(l+1)_1\}}} \frac{J_i(x(t_i-))}{\Gamma(q-1)} \frac{(t'_{k+1})^\rho-(t_i)^\rho}{\rho} \left[\frac{t^\rho-(t'_{k+1})^\rho}{\rho}\right]^{q-2}
\end{aligned}$$

$$\text{for } t\in\big(t'_{k+1},t'_{k+2}\big], l=1,2,\ldots,k+1. \tag{4.45}$$

By (4.44) and (4.45), we have

$$\begin{aligned}
\hat{e}_{k+1}(t) &= -\big[y\big(t'_{k+1},t\big)-y(t_0,t)\big] - \sum_{i=1}^{k_1} \frac{J_i(x(t_i-))}{\Gamma(q-1)} \frac{(t'_{k+1})^\rho-(t_i)^\rho}{\rho} \left[\frac{t^\rho-(t'_{k+1})^\rho}{\rho}\right]^{q-2} \\
&\quad + \xi \sum_{i=1}^{(k+1)_1} J_i\big(x(t_i-)\big)\big[y(t_i,t)-y(t_0,t)\big] \\
&\quad + \eta \sum_{j=1}^{(k+1)_2} \bar{J}_j\big(x(\bar{t}_j-)\big)\big[y(\bar{t}_j,t)-y(t_0,t)\big] \\
&\quad + \sum_{i=1}^{(k+1)_1} \frac{J_i(x(t_i-))}{\Gamma(q)} \left\{\left[\frac{t^\rho-(t_i)^\rho}{\rho}\right]^{q-1} - \left[\frac{t^\rho-(t'_{k+1})^\rho}{\rho}\right]^{q-1}\right\} \\
&\quad + \sum_{j=1}^{(k+1)_2} \frac{\bar{J}_j(x(\bar{t}_j-))}{\Gamma(q-1)} \left\{\left[\frac{t^\rho-(\bar{t}_j)^\rho}{\rho}\right]^{q-2} - \left[\frac{t^\rho-(t'_{k+1})^\rho}{\rho}\right]^{q-2}\right\}.
\end{aligned} \tag{4.46}$$

By (4.41) and (4.46), we get

$$\begin{aligned} x(t) &= \hat{x}(t) + \hat{e}_{k+1}(t) \\ &= y(t_0,t) + \sum_{i=1}^{(k+1)_1} \frac{J_i(x(t_i-))}{\Gamma(q)}\left[\frac{t^\rho-(t_i)^\rho}{\rho}\right]^{q-1} + \sum_{j=1}^{(k+1)_2} \frac{\bar{J}_j(x(\bar{t}_j-))}{\Gamma(q-1)}\left[\frac{t^\rho-(\bar{t}_j)^\rho}{\rho}\right]^{q-2} \\ &\quad + \xi \sum_{i=1}^{(k+1)_1} J_i\big(x(t_i-)\big)\big[y(t_i,t)-y(t_0,t)\big] + \eta \sum_{j=1}^{(k+1)_2} \bar{J}_j\big(x(\bar{t}_j-)\big)\big[y(\bar{t}_j,t)-y(t_0,t)\big] \\ &\quad \text{for } t \in \big(t'_{k+1}, t'_{k+2}\big]. \end{aligned} \tag{4.47}$$

Thus the solution of (1.1) satisfies (4.25) as $t \in (t'_{k+1}, t'_{k+2}]$, and the necessity is proved.

Now we verify the sufficiency that (4.25) satisfies all the conditions of system (1.1). It is easy to find that (4.25) satisfies conditions (i)–(iv) by Lemmas 4.1 and 4.3, and it is similar with the proof of Lemma 4.1 to verify that (4.25) satisfies the condition of generalized fractional derivative, impulsive conditions, and initial conditions in (1.1). The proof is completed. □

Corollary 4.5 *Let $q \in (1,2]$ and $t_0, \rho > 0$, and let $f : [t_0,T] \times \mathbb{R} \to \mathbb{R}$ be a function such that $f(\cdot,x(\cdot)) \in \hat{C}_{2-q,\rho}[t_0,T]$ for any $x(\cdot) \in \hat{C}_{2-q,\rho}[t_0,T]$.*

If $x(\cdot) \in IC([t_0,T],\mathbb{R})$, then $x(t)$ is a solution of (1.4) if, and only if, $x(t)$ satisfies the following integral equation:

$$x(t) = \begin{cases} y(t_0,t) \quad \textit{for } t \in (t_0,t_1], \\ y(t_0,t) + \sum_{i=1}^k \frac{J_i(x(t_i-))}{\Gamma(q)}[\frac{t^\rho-(t_i)^\rho}{\rho}]^{q-1} + \sum_{i=1}^k \frac{\bar{J}_i(x(t_i-))}{\Gamma(q-1)}[\frac{t^\rho-(t_i)^\rho}{\rho}]^{q-2} \\ \quad + \sum_{i=1}^k [\xi J_i(x(t_i-)) + \eta \bar{J}_i(x(t_i-))][y(t_i,t)-y(t_0,t)] \\ \quad \textit{for } t \in (t_k,t_{k+1}], k = 1,2,\ldots,m, \end{cases} \tag{4.48}$$

where ξ and η are two arbitrary constants.

5 Examples

In this section, we consider the following IVP of three IFrDEs:

$$\begin{cases} {}_1^K\mathcal{D}_t^{\frac{3}{2},\rho} x(t) = x(t), \quad t \in (1,5], t \neq 3 \\ {}_1^K\mathcal{D}_{3^+}^{\frac{1}{2},\rho} x(3^+) - {}_1^K\mathcal{D}_{3^-}^{\frac{1}{2},\rho} x(3^-) = 1, \\ {}_1^K\mathcal{D}_t^{\frac{1}{2},\rho} x(t)|_{t\to 1^+} = 1, \qquad {}_1^K\mathcal{I}_t^{\frac{1}{2},\rho} x(t)|_{t\to 1^+} = 0, \end{cases} \tag{5.1}$$

$$\begin{cases} {}_1^K\mathcal{D}_t^{\frac{3}{2},\rho} x(t) = x(t), \quad t \in (1,5], t \neq 3 \\ {}_1^K\mathcal{I}_{3^+}^{\frac{1}{2},\rho} x(3^+) - {}_1^K\mathcal{I}_{3^-}^{\frac{1}{2},\rho} x(3^-) = 1, \\ {}_1^K\mathcal{D}_t^{\frac{1}{2},\rho} x(t)|_{t\to 1^+} = 1, \qquad {}_1^K\mathcal{I}_t^{\frac{1}{2},\rho} x(t)|_{t\to 1^+} = 0, \end{cases} \tag{5.2}$$

$$\begin{cases} {}_1^K\mathcal{D}_t^{\frac{3}{2},\rho} x(t) = x(t), \quad t \in (1,5], t \neq 3 \\ {}_1^K\mathcal{D}_{3^+}^{\frac{1}{2},\rho} x(3^+) - {}_1^K\mathcal{D}_{3^-}^{\frac{1}{2},\rho} x(3^-) = 1, \\ {}_1^K\mathcal{I}_{3^+}^{\frac{1}{2},\rho} x(3^+) - {}_1^K\mathcal{I}_{3^-}^{\frac{1}{2},\rho} x(3^-) = 1, \\ {}_1^K\mathcal{D}_t^{\frac{1}{2},\rho} x(t)|_{t\to 1^+} = 1, \qquad {}_1^K\mathcal{I}_t^{\frac{1}{2},\rho} x(t)|_{t\to 1^+} = 0. \end{cases} \tag{5.3}$$

By Lemma 4.1, Lemma 4.3, and Corollary 4.5, the equivalent integral equations of three systems (5.1)–(5.3) as $t \in (1,3]$ are identical as follows:

$$x(t) = \frac{2}{\sqrt{\pi}}\left[\frac{t^\rho - 1}{\rho}\right]^{\frac{1}{2}} + \frac{2}{\sqrt{\pi}}\int_1^t \left[\frac{t^\rho - \tau^\rho}{\rho}\right]^{\frac{1}{2}} \frac{x(\tau)\,d\tau}{\tau^{1-\rho}} \quad \text{for } t \in (1,3], \tag{5.4}$$

and the equivalent integral equations of three systems (5.1)–(5.3) as $t \in (3,5]$ are respectively given by

$$\begin{aligned} x(t) ={}& \frac{2}{\sqrt{\pi}}\left[\frac{t^\rho - 1}{\rho}\right]^{\frac{1}{2}} + \frac{2}{\sqrt{\pi}}\int_1^t \left[\frac{t^\rho - \tau^\rho}{\rho}\right]^{\frac{1}{2}} \frac{x(\tau)\,d\tau}{\tau^{1-\rho}} + \frac{2}{\sqrt{\pi}}\left[\frac{t^\rho - 3^\rho}{\rho}\right]^{\frac{1}{2}} \\ &+ \xi\left\{\frac{\frac{3^\rho-1}{\rho} + \int_1^3 \frac{3^\rho-\tau^\rho}{\rho}\frac{x(\tau)\,d\tau}{\tau^{1-\rho}}}{\sqrt{\pi}}\left[\frac{t^\rho - 3^\rho}{\rho}\right]^{-\frac{1}{2}} + \frac{1+\int_1^3 \frac{x(\tau)\,d\tau}{\tau^{1-\rho}}}{\sqrt{\pi}/2}\left[\frac{t^\rho - 3^\rho}{\rho}\right]^{\frac{1}{2}}\right. \\ &+ \frac{2}{\sqrt{\pi}}\int_3^t \left[\frac{t^\rho - \tau^\rho}{\rho}\right]^{\frac{1}{2}} \frac{x(\tau)\,d\tau}{\tau^{1-\rho}} - \frac{2}{\sqrt{\pi}}\left[\frac{t^\rho - 1}{\rho}\right]^{\frac{1}{2}} \\ &\left.- \frac{2}{\sqrt{\pi}}\int_1^t \left[\frac{t^\rho - \tau^\rho}{\rho}\right]^{\frac{1}{2}} \frac{x(\tau)\,d\tau}{\tau^{1-\rho}}\right\} \\ &\text{for } t \in (3,5], \end{aligned} \tag{5.5}$$

$$\begin{aligned} x(t) ={}& \frac{2}{\sqrt{\pi}}\left[\frac{t^\rho - 1}{\rho}\right]^{\frac{1}{2}} + \frac{2}{\sqrt{\pi}}\int_1^t \left[\frac{t^\rho - \tau^\rho}{\rho}\right]^{\frac{1}{2}} \frac{x(\tau)\,d\tau}{\tau^{1-\rho}} + \frac{1}{\sqrt{\pi}}\left[\frac{t^\rho - 3^\rho}{\rho}\right]^{-\frac{1}{2}} \\ &+ \eta\left\{\frac{\frac{3^\rho-1}{\rho} + \int_1^3 \frac{3^\rho-\tau^\rho}{\rho}\frac{x(\tau)\,d\tau}{\tau^{1-\rho}}}{\sqrt{\pi}}\left[\frac{t^\rho - 3^\rho}{\rho}\right]^{-\frac{1}{2}} + \frac{1+\int_1^3 \frac{x(\tau)\,d\tau}{\tau^{1-\rho}}}{\sqrt{\pi}/2}\left[\frac{t^\rho - 3^\rho}{\rho}\right]^{\frac{1}{2}}\right. \\ &+ \frac{2}{\sqrt{\pi}}\int_3^t \left[\frac{t^\rho - \tau^\rho}{\rho}\right]^{\frac{1}{2}} \frac{x(\tau)\,d\tau}{\tau^{1-\rho}} - \frac{2}{\sqrt{\pi}}\left[\frac{t^\rho - 1}{\rho}\right]^{\frac{1}{2}} \\ &\left.- \frac{2}{\sqrt{\pi}}\int_1^t \left[\frac{t^\rho - \tau^\rho}{\rho}\right]^{\frac{1}{2}} \frac{x(\tau)\,d\tau}{\tau^{1-\rho}}\right\} \\ &\text{for } t \in (3,5], \end{aligned} \tag{5.6}$$

$$\begin{aligned} x(t) ={}& \frac{2}{\sqrt{\pi}}\left[\frac{t^\rho - 1}{\rho}\right]^{\frac{1}{2}} + \frac{2}{\sqrt{\pi}}\int_1^t \left[\frac{t^\rho - \tau^\rho}{\rho}\right]^{\frac{1}{2}} \frac{x(\tau)\,d\tau}{\tau^{1-\rho}} \\ &+ \frac{2}{\sqrt{\pi}}\left[\frac{t^\rho - 3^\rho}{\rho}\right]^{\frac{1}{2}} + \frac{1}{\sqrt{\pi}}\left[\frac{t^\rho - 3^\rho}{\rho}\right]^{-\frac{1}{2}} \\ &+ [\xi + \eta]\left\{\frac{\frac{3^\rho-1}{\rho} + \int_1^3 \frac{3^\rho-\tau^\rho}{\rho}\frac{x(\tau)\,d\tau}{\tau^{1-\rho}}}{\sqrt{\pi}}\left[\frac{t^\rho - 3^\rho}{\rho}\right]^{-\frac{1}{2}} + \frac{1+\int_1^3 \frac{x(\tau)\,d\tau}{\tau^{1-\rho}}}{\sqrt{\pi}/2}\left[\frac{t^\rho - 3^\rho}{\rho}\right]^{\frac{1}{2}}\right. \\ &+ \frac{2}{\sqrt{\pi}}\int_3^t \left[\frac{t^\rho - \tau^\rho}{\rho}\right]^{\frac{1}{2}} \frac{x(\tau)\,d\tau}{\tau^{1-\rho}} - \frac{2}{\sqrt{\pi}}\left[\frac{t^\rho - 1}{\rho}\right]^{\frac{1}{2}} \\ &\left.- \frac{2}{\sqrt{\pi}}\int_1^t \left[\frac{t^\rho - \tau^\rho}{\rho}\right]^{\frac{1}{2}} \frac{x(\tau)\,d\tau}{\tau^{1-\rho}}\right\} \\ &\text{for } t \in (3,5], \end{aligned} \tag{5.7}$$

where ξ and η in (5.5)–(5.7) are two arbitrary constants.

Next we realize numerical simulation of (5.4) and (5.5)–(5.7) by using the Euler method with variable step size to give some solution trajectories of three systems (5.1)–(5.3) with given ρ, respectively.

Figures 1–4 denote the solution trajectories of (5.1) with $\rho = 0.1, 0.5, 1, 2$, respectively. Moreover, in these figures three curves 'xi = $0, 1, -1$', which are drawn by numerical simulation of (5.4)–(5.5) with $\xi = 0, 1, -1$, respectively, represent three solutions of (5.1) with the corresponding ρ.

Figures 5–8 denote the solution trajectories of (5.2) with $\rho = 0.1, 0.5, 1, 2$, respectively. Moreover, in these figures three curves 'eta = $0, 1, -1$', which are drawn by numerical simulation of (5.4) and (5.6) with $\eta = 0, 1, -1$, respectively, represent three solutions of (5.2) with the corresponding ρ.

Figures 9–12 denote the solution trajectories of (5.3) with $\rho = 0.1, 0.5, 1, 2$, respectively. Moreover, in these figures five curves 'xi + eta = $2, 1, 0, -1, -2$', which are drawn by nu-

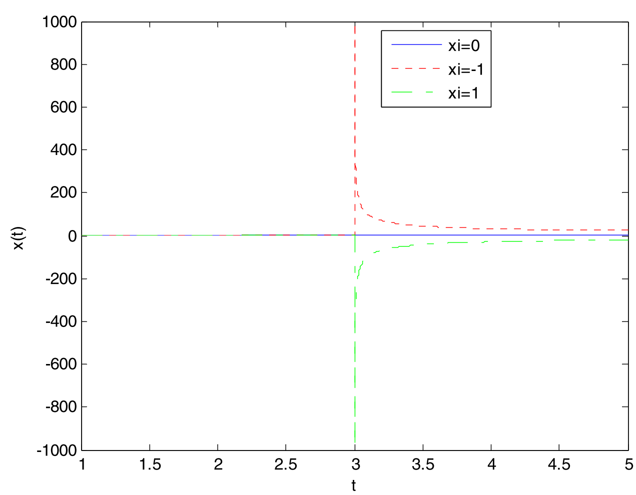

Figure 1 The solution trajectory of system (5.1) with $\rho = 0.1$

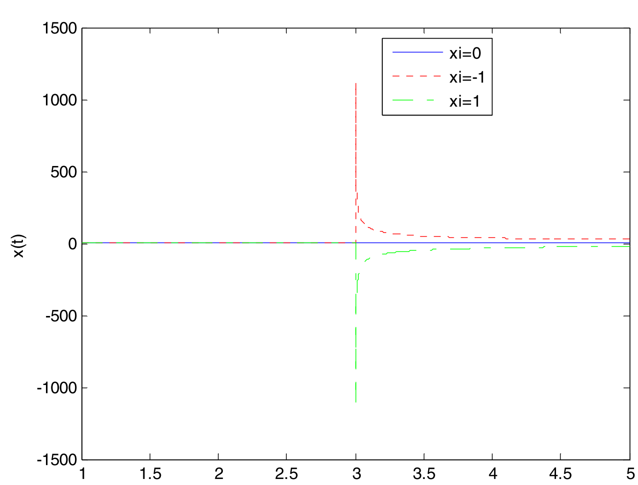

Figure 2 The solution trajectory of system (5.1) with $\rho = 0.5$

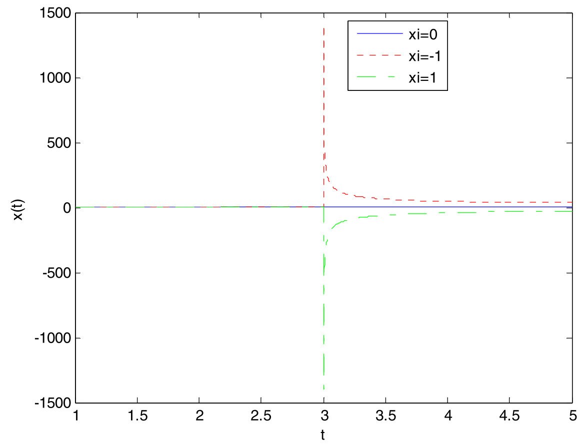

Figure 3 The solution trajectory of system (5.1) with $\rho = 1$

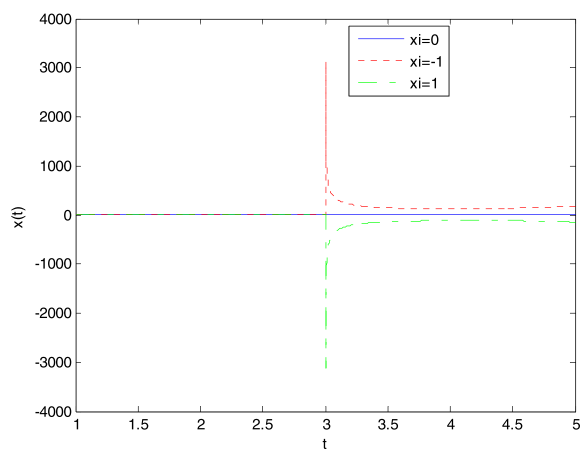

Figure 4 The solution trajectory of system (5.1) with $\rho = 2$

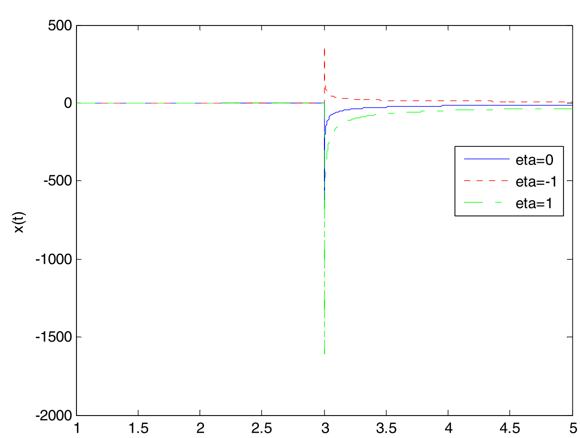

Figure 5 The solution trajectory of system (5.2) with $\rho = 0.1$

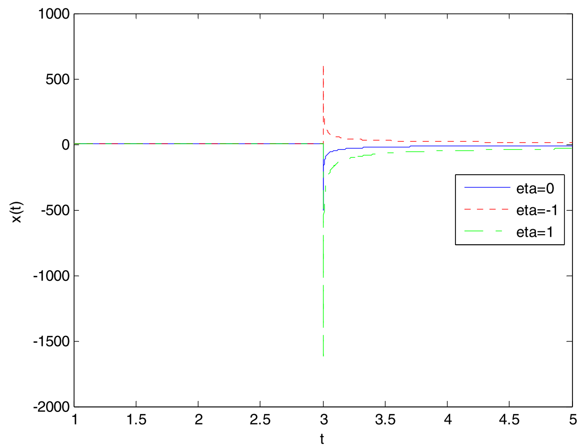

Figure 6 The solution trajectory of system (5.2) with $\rho = 0.5$

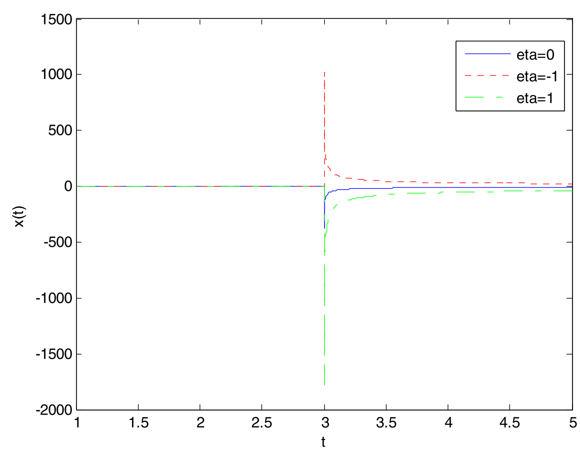

Figure 7 The solution trajectory of system (5.2) with $\rho = 1$

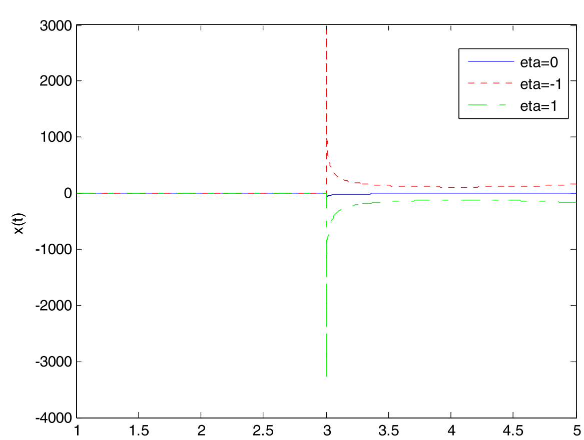

Figure 8 The solution trajectory of system (5.2) with $\rho = 2$

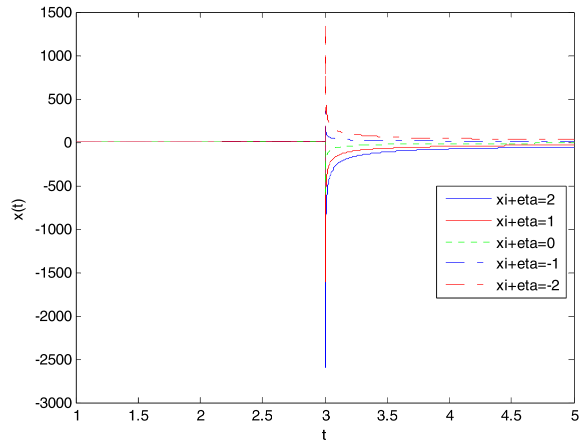

Figure 9 The solution trajectory of system (5.3) with $\rho = 0.1$

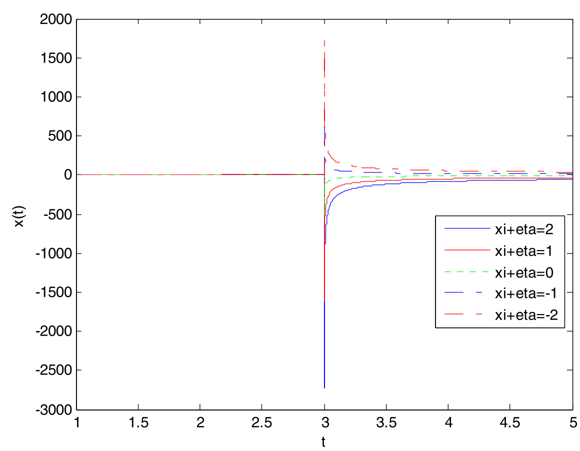

Figure 10 The solution trajectory of system (5.3) with $\rho = 0.5$

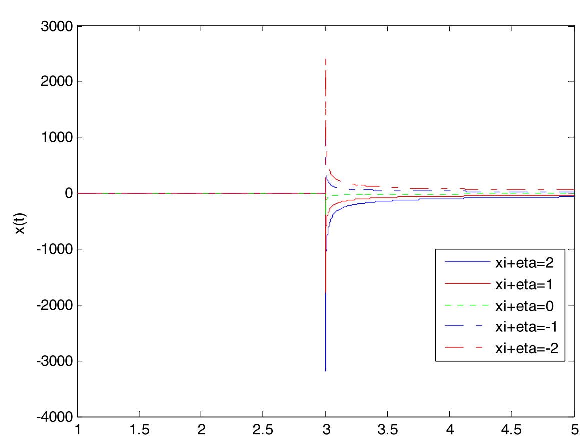

Figure 11 The solution trajectory of system (5.3) with $\rho = 1$

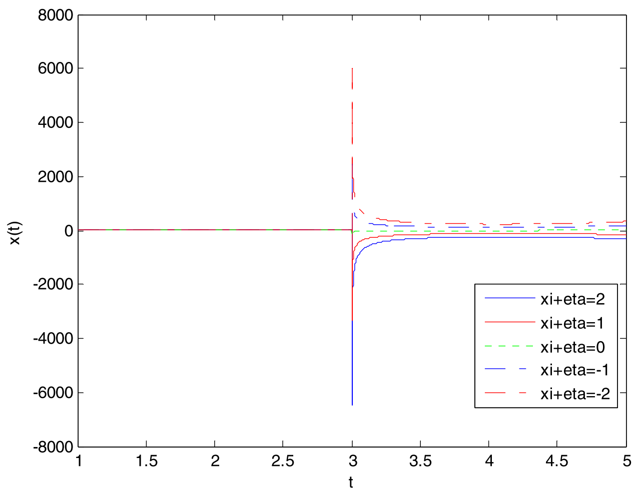

Figure 12 The solution trajectory of system (5.3) with $\rho = 2$

merical simulation of (5.4) and (5.7) with $\xi + \eta = 2, 1, 0, -1, -2$, respectively, represent five solutions of (5.3) with the corresponding ρ.

6 Conclusion

The systems of impulsive high order fractional differential equations can involve one or two kinds of impulses. As a result, their equivalent integral equations include one or two arbitrary constants which uncover the non-uniqueness of solution for the systems of impulsive high order fractional differential equations.

Acknowledgements
The author is deeply grateful to the anonymous referees for their kind comments, correcting errors, and improving written language, which have been very useful for improving the quality of this paper.

Funding
The work described in this paper is financially supported by the National Natural Science Foundation of China (Grant No. 21576033, 21636004).

Availability of data and materials
Not applicable.

Competing interests
The author declares that he has no competing interests.

Authors' contributions
The author wrote the first version of the manuscript and approved the final manuscript by himself.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 28 October 2019 Accepted: 3 February 2020 Published online: 22 February 2020

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