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Dynamical analysis of a fractional-order eco-epidemiological model with disease in prey population

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Abstract

A fractional-order eco-epidemiological model with disease in the prey population is formulated and analyzed. Mathematical analysis and numerical simulations are performed to clarify the characteristics of the proposed fractional-order model. The existence, uniqueness, non-negativity and boundedness of the solutions are proved. The local and global asymptotic stability of all equilibrium points are investigated. Finally, numerical simulations are conducted to illustrate the analytical results. The occurrence of Hopf bifurcations and transcritical bifurcations for the fractional-order eco-epidemiological model are demonstrated. It is observed that the fractional order has a stabilization effect and it may help to control the coexistence between susceptible prey, infected prey and predator populations.

Keywords: Eco-epidemiological model; Predator–prey; Fractional-order system; Stability; Bifurcations; Numerical simulation

1 Introduction

The dynamics of the relationship between predators and their prey are topics of considerable interest in ecology and mathematical biology. The Lotka–Volterra model is the first system that modeled the interactions between prey and its predator [1]. Studies of the dynamics of prey–predator models include [2–5]. Kermack and MacKendrick [6] proposed the classical SIR model which has drawn much attention among the scholars. Li et al. [7] studied the global stability of an SI epidemic model with feedback controls in a patchy environment. The epidemiological models consider the spread of infectious diseases related to one species is one of the major issues in mathematical biology. The modeling of infectious diseases studies including [8, 9].

It is realistic to consider the impact of interacting species when epidemiological models are studied [10]. Eco-epidemiological models consider the relationships between prey and predator in which infectious diseases play a fundamental role in the dynamics of the system [11]. The eco-epidemiological model studies the spread of diseases among the prey and predator populations and one of the main aims is to control of infectious diseases. Hilker and Schmitz [12] observed that introducing disease into the Rosenzweig–

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Table 1 Table of the biological description of variables and parameters

Variables & parameters	Description
S	Susceptible prey population density.
I	Infected prey population density.
P	Predator population density.
\hat{r}	Recruitment rate into prey population.
$\hat{\beta}$	Transmission coefficient of disease in prey.
$\hat{\gamma}$	Recovery rate of infected prey.
$\hat{\alpha}$	Attack rate of predator.
\hat{a}	Half saturation constant.
\hat{c}	Conversion efficiency of predator.
\hat{d}_1	Natural death rate of susceptible prey.
\hat{d}_2	Natural death rate of infected prey.
\hat{d}_3	Natural death rate of predator.

MacArthur model can stabilize prey–predator oscillations. Some studies have been carried out on eco-epidemiological models with disease either in prey [13–16] or in predator [17–20] or in both populations [21–23]. The infectious diseases among prey and predator populations are disorders caused by bacteria or virus [24].

In [11] the authors considered a three-species eco-epidemiological model with Holling type II functional response of a predator. It is assumed that population can be divided into susceptible prey, infected prey and predator; disease spreads only among the preys; infected prey can recover and predator eats only infected prey.

Following [10, 11] a new version of an eco-epidemiological model is formulated by incorporating a predator’s attack rate and half saturation constant as follows:

$$\begin{aligned}
 \frac{dS}{dt} &= \hat{r} - \hat{\beta}SI - \hat{d}_1S + \hat{\gamma}I, \\
 \frac{dI}{dt} &= \hat{\beta}SI - \frac{\hat{\alpha}IP}{1 + \hat{a}I} - \hat{d}_2I - \hat{\gamma}I, \\
 \frac{dP}{dt} &= \frac{\hat{c}\hat{\alpha}IP}{1 + \hat{a}I} - \hat{d}_3P.
 \end{aligned}
 \tag{1}$$

All the parameters are assumed to be positive. The variables and parameters of system (1) are presented in Table 1.

Fractional-order differential equations can be considered as generalization of ordinary differential equations to an arbitrary (non-integer) order and have been successfully applied in engineering, chemistry and mathematical biology [25, 26]. The fractional-order derivative is a non-local operator in the sense that the system at present states depends on the recent past states [27].

The fractional-order differential equations exhibit richer dynamical behavior and this is because it incorporates the memory effect in the model [28]. Zhao and Luo [29] proposed a definition of general fractional derivatives to describe the dynamics with memory effects. Bolton et al. [30] conclude that the fractional-order Gompertz growth model is more realistic as regards a experimental dataset than the integer-order Gompertz model. Therefore, the use of fractional-order differential equations may improve the modeling of biological phenomena. Hence, the dynamics of the relations between predators and their prey can be more accurately described by fractional-order systems [31, 32]. Detailed background of the fractional-order differential equations can be found in [33–41]. Some

previous studies indicate that the fractional-order system cannot have a periodic solution [42, 43]. But the fractional derivative converges to periodic signals if it is defined on the whole real line [44, 45]. This paper will be focused on the final state of the trajectory [44]. Hence, the limit cycle attracts all nearby positive-valued trajectories (asymptotically stable limit cycle) [44].

In [33, 34] a kind of fractional order eco-epidemiological model with disease in the prey population was proposed and some issues related to theoretical and numerical analyses were investigated. However, the governing systems proposed in [33, 34] are different from our fractional-order model (3). In this paper, we consider the following fractional-order eco-epidemiological model incorporating a predator’s attack rate and half saturation constant:

$$\begin{aligned}
 {}^cD^q S(t) &= \hat{r}^q - \hat{\beta}^q SI - \hat{d}_1^q S + \hat{\gamma}^q I, \\
 {}^cD^q I(t) &= \hat{\beta}^q SI - \frac{\hat{\alpha}^q IP}{1 + \hat{a}I} - \hat{d}_2^q I - \hat{\gamma}^q I, \\
 {}^cD^q P(t) &= \frac{\hat{c}\hat{\alpha}^q IP}{1 + \hat{a}I} - \hat{d}_3^q P,
 \end{aligned} \tag{2}$$

with initial conditions

$$S(0) = S_0 \geq 0, \quad I(0) = I_0 \geq 0, \quad P(0) = P_0 \geq 0,$$

where $0 < q < 1$ and ${}^cD^q$ is the standard Caputo differentiation. The parameters in the fractional-order eco-epidemiological model (2) are assumed to be positive and are described in Table 1. The Caputo fractional derivative of order q is defined as [26]

$${}^cD^q f(t) = \frac{1}{\Gamma(n - q)} \int_0^t (t - s)^{n - q - 1} f^{(n)}(s) ds, \quad n - 1 < q < n, n \in \mathbb{N}.$$

However, it is important to note that the modified parameters, such as \hat{r}^q depending on the fractional order (q) and that the units of each differential equation terms are different [46, 47]. Also, the fractional derivative equations are formulated not with respect to the physical time but with respect to an intrinsic time variable that depends on q [48]. For simplification, this system can be redefined with the new parameters as follows [49]:

$$\begin{aligned}
 \hat{r}^q &= r, & \hat{\beta}^q &= \beta, & \hat{d}_1^q &= d_1, & \hat{d}_2^q &= d_2, & \hat{d}_3^q &= d_3, & \hat{c} &= c, \\
 \hat{\gamma}^q &= \gamma, & \hat{\alpha}^q &= \alpha, & \hat{a} &= a.
 \end{aligned}$$

Then, the system (2) becomes as follows:

$$\begin{aligned}
 {}^cD^q S(t) &= r - \beta SI - d_1 S + \gamma I, \\
 {}^cD^q I(t) &= \beta SI - \frac{\alpha IP}{1 + aI} - d_2 I - \gamma I, \\
 {}^cD^q P(t) &= \frac{c\alpha IP}{1 + aI} - d_3 P.
 \end{aligned} \tag{3}$$

So far as we are aware, no scholar has investigated the dynamics of the fractional-order eco-epidemiological model (3). Therefore, in this paper, a fractional order prey–predator

model with disease in the prey population is investigated. The mathematical analysis and numerical simulations are performed to clarify the characteristics of our fractional-order model (3). The aim of this study is to proposed and analyzed a fractional-order eco-epidemiological model incorporating predator’s attack rate (α) and half saturation constant (a) with infection in prey population. Also, we observe the effects of the prey’s disease (β), predator’s attack rate (α), half saturation constant (a), infected prey’s death rate (d_2) and fractional order (q) in the dynamics of the fractional-order eco-epidemiological model (3). The contributions of this paper are to prove existence, uniqueness, non-negativity, and boundedness of the solutions for model (3). The paper also investigates the local and global asymptotic stability of all equilibrium points of the fractional order model (3) by using Matignon’s condition and constructing suitable Lyapunov functions, respectively. Numerical simulations conducted indicate a rich dynamical behavior of the fractional order model (3) at the equilibrium points, which is in agreement with the theoretical analysis.

The paper is organized as follows. In the next section, the analysis of the fractional-order eco-epidemiological model (3) is presented. In Sect. 3, the numerical simulations of the fractional-order eco-epidemiological model (3) are provided to verify the theoretical results. Finally, Sect. 4 concludes the study with a brief discussion.

2 Mathematical analysis

The mathematical analysis of the fractional-order eco-epidemiological model (3) is presented in this section.

2.1 Existence and uniqueness

The existence and uniqueness of the solutions of the fractional-order system (3) are studied in the region $\Omega \times (0, T]$ where

$$\Omega = \{(S, I, P) \in \mathbb{R}^3 : \max(|S|, |I|, |P|) \leq \psi\}.$$

Theorem 1 *For each $X_0 = (S_0, I_0, P_0) \in \Omega$, there exists a unique solution $X(t) \in \Omega$ of the fractional-order system (3) with initial condition X_0 , which is defined for all $t \geq 0$.*

Proof The approach used by [50] is utilized. For $X, \bar{X} \in \Omega$, one can consider a mapping $M(X) = (M_1(X), M_2(X), M_3(X))$ where

$$\begin{aligned} M_1(X) &= r - \beta SI - d_1 S + \gamma I, \\ M_2(X) &= \beta SI - \frac{\alpha IP}{1 + aI} - d_2 I - \gamma I, \\ M_3(X) &= \frac{c\alpha IP}{1 + aI} - d_3 P. \end{aligned} \tag{4}$$

It follows from (4) that

$$\begin{aligned} &\|M(X) - M(\bar{X})\| \\ &= |M_1(X) - M_1(\bar{X})| + |M_2(X) - M_2(\bar{X})| + |M_3(X) - M_3(\bar{X})| \\ &= |r - \beta SI - d_1 S + \gamma I - r + \beta \bar{S}\bar{I} + d_1 \bar{S} - \gamma \bar{I}| \end{aligned}$$

$$\begin{aligned}
 & + \left| \beta SI - \frac{\alpha IP}{1+aI} - d_2 I - \gamma I - \beta \bar{S}\bar{I} + \frac{\alpha \bar{I}\bar{P}}{1+a\bar{I}} + d_2 \bar{I} + \gamma \bar{I} \right| \\
 & + \left| \frac{c\alpha IP}{1+aI} - d_3 P - \frac{c\alpha \bar{I}\bar{P}}{1+a\bar{I}} + d_3 \bar{P} \right| \\
 = & \left| -\beta(SI - \bar{S}\bar{I}) - d_1(S - \bar{S}) + \gamma(I - \bar{I}) \right| \\
 & + \left| -\frac{\alpha(IP + a\bar{I}\bar{P} - \bar{I}\bar{P} - a\bar{I}\bar{P})}{(1+aI)(1+a\bar{I})} + \beta(SI - \bar{S}\bar{I}) - d_2(I - \bar{I}) - \gamma(I - \bar{I}) \right| \\
 & + \left| \frac{c\alpha(IP + a\bar{I}\bar{P} - \bar{I}\bar{P} - a\bar{I}\bar{P})}{(1+aI)(1+a\bar{I})} - d_3(P - \bar{P}) \right| \\
 \leq & \beta\psi|S - \bar{S}| + \beta\psi|I - \bar{I}| + d_1|S - \bar{S}| + \gamma|I - \bar{I}| \\
 & + \frac{\alpha(1+c)|P - \bar{P}|}{a} + \beta\psi|S - \bar{S}| + \beta\psi|I - \bar{I}| + d_2|I - \bar{I}| + \gamma|I - \bar{I}| \\
 & + d_3|P - \bar{P}| + \alpha(1+c)\psi|I - \bar{I}| + \alpha(1+c)\psi|P - \bar{P}| \\
 \leq & (2\beta\psi + d_1)|S - \bar{S}| \\
 & + (2\beta\psi + 2\gamma + d_2 + \alpha(1+c)\psi)|I - \bar{I}| \\
 & + \left(\frac{\alpha(1+c)}{a} + d_3 + \alpha(1+c)\psi \right) |P - \bar{P}| \\
 \leq & G\|X - \bar{X}\|,
 \end{aligned}$$

where

$$G = \max \left\{ 2\beta\psi + d_1, 2\beta\psi + 2\gamma + d_2 + \alpha(1+c)\psi, \frac{\alpha(1+c)}{a} + d_3 + \alpha(1+c)\psi \right\}.$$

Thus, $M(X)$ satisfies the Lipschitz condition which proves the existence and uniqueness of the solutions of the fractional-order eco-epidemiological model (3). □

2.2 Non-negativity and boundedness

Now, the non-negativity of the solutions of the fractional-order system (3) is studied. From (3), one has

$$\begin{aligned}
 {}^c D^q S(t)|_{S=0} & = r + \gamma I \geq 0, \\
 {}^c D^q I(t)|_{I=0} & = 0, \\
 {}^c D^q P(t)|_{P=0} & = 0.
 \end{aligned}$$

According to Lemmas 5 and 6 in [51], one can deduce that the solutions of the fractional-order system (3) are non-negative.

Next, the boundedness of the solutions of the fractional-order system (3) is given.

Theorem 2 *The solutions of the fractional-order system (3) which start in \mathbb{R}_+^3 are uniformly bounded.*

Proof The approach of [50] is adopted. Let us consider the function $H(t) = S(t) + I(t) + \frac{1}{c}P(t)$. Then

$$\begin{aligned} {}^cD^q H(t) &= {}^cD^q S(t) + {}^cD^q I(t) + \frac{1}{c} {}^cD^q P(t) \\ &= r - d_1 S - d_2 I - \frac{d_3}{c} P. \end{aligned}$$

Hence, for each $\lambda > 0$,

$$\begin{aligned} {}^cD^q H(t) + \lambda H(t) &= r - d_1 S - d_2 I - \frac{d_3}{c} P + \lambda S + \lambda I + \frac{\lambda}{c} P \\ &= r + (\lambda - d_1) S + (\lambda - d_2) I + \frac{1}{c} (\lambda - d_3) P. \end{aligned}$$

One can choose $\lambda < \min\{d_1, d_2, d_3\}$. Thus

$${}^cD^q H(t) + \lambda H(t) \leq r.$$

According to Lemma 9 in [52], it follows that

$$0 \leq H(t) \leq H(0)E_q(-\lambda(t)^q) + r(t)^q E_{q,q+1}(-\lambda(t)^q),$$

where E_q is the well-known Mittag-Leffler function. By Lemma 5 and Corollary 6 in [52], one gets the following expression:

$$0 \leq H(t) \leq \frac{r}{\lambda}, \quad \text{as } t \rightarrow \infty.$$

Therefore, the solutions of fractional-order eco-epidemiological model (3) starting in \mathbb{R}_+^3 are uniformly bounded in the region W , where

$$W = \left\{ (S, I, P) \in \mathbb{R}_+^3 : H(t) \leq \frac{r}{\lambda} + \epsilon, \epsilon > 0 \right\}. \tag{5}$$

2.3 Equilibrium points and stability

In this section, we investigate the stability of equilibrium points of the fractional-order system (3). The fractional-order system (3) has the following equilibrium points:

1. The axial equilibrium point $E_1(\frac{r}{d_1}, 0, 0)$, which always exists.
2. The predator-extinction equilibrium point $E_2(\frac{\gamma+d_2}{\beta}, \frac{d_1(\gamma+d_2)(\mathfrak{R}_0-1)}{\beta d_2}, 0)$, which exists if $\mathfrak{R}_0 > 1$, where $\mathfrak{R}_0 = \frac{r\beta}{d_1(\gamma+d_2)}$ is the basic reproduction number obtained by using the next generation method [53].
3. The interior equilibrium point $E_3(S_3, I_3, P_3)$ where

$$\begin{aligned} S_3 &= \frac{r(c\alpha - ad_3) + \gamma d_3}{d_1(c\alpha - ad_3) + \beta d_3}, & I_3 &= \frac{d_3}{c\alpha - ad_3} \quad \text{and} \\ P_3 &= \frac{cd_1(\gamma + d_2)(\mathfrak{R}_0 - 1 - \frac{\beta d_2 d_3}{d_1(\gamma+d_2)(c\alpha - ad_3)})}{d_1(c\alpha - ad_3) + \beta d_3}. \end{aligned}$$

The interior equilibrium point E_3 exists if $\alpha > \frac{ad_3}{c}$ and $\mathfrak{R}_0 > 1 + \frac{\beta d_2 d_3}{d_1(\gamma+d_2)(c\alpha - ad_3)}$ which is equivalent to $\alpha > \frac{(\beta(ar+d_2) - ad_1(\gamma+d_2))d_3}{c(r\beta - d_1(\gamma+d_2))}$.

Thus, it is observed that the existence of interior equilibrium point E_3 depends on the threshold value of the predator’s attack rate (α) and the basic reproduction number (\mathfrak{R}_0), respectively. So, the attack rate of predator and basic reproduction number play some crucial roles in determining the dynamics of the fractional-order system (3). Next, we will discuss the stability of the equilibrium points of system (3). The Jacobian matrix of system (3) is given by

$$J(S, I, P) = \begin{pmatrix} -(I\beta + d_1) & \gamma - \beta S & 0 \\ \beta I & \frac{(1+aI)^2(\beta S - \gamma) - \alpha P}{(1+aI)^2} - d_2 & -\frac{\alpha I}{1+aI} \\ 0 & \frac{c\alpha P}{(1+aI)^2} & \frac{c\alpha I}{1+aI} - d_3 \end{pmatrix}. \tag{6}$$

Utilizing the Jacobian matrix (6) and the Matignon condition [32, 54], the local stability of the equilibrium points of the fractional-order system (3) is investigated.

Theorem 3 *The axial equilibrium point E_1 of the fractional-order system (3) is locally asymptotically stable if $\mathfrak{R}_0 < 1$ and unstable saddle point if $\mathfrak{R}_0 > 1$.*

Proof The Jacobian matrix (6) around the axial equilibrium point E_1 is as follows:

$$J(E_1) = \begin{pmatrix} -d_1 & \gamma - \frac{r\beta}{d_1} & 0 \\ 0 & \frac{r\beta}{d_1} - (\gamma + d_2) & 0 \\ 0 & 0 & -d_3 \end{pmatrix}.$$

The eigenvalues of the Jacobian matrix around the axial equilibrium point E_1 are $\mu_1 = -d_1$, $\mu_2 = -d_3$ and $\mu_3 = \frac{r\beta - d_1(\gamma + d_2)}{d_1}$. By using Matignon’s condition [32, 54], it can be observed that $|\arg(\mu_{1,2})| = \pi > \frac{q\pi}{2}$. If $\mathfrak{R}_0 < 1$ then $|\arg(\mu_3)| = \pi > \frac{q\pi}{2}$ for all $0 < q < 1$. Thus, the axial equilibrium point E_1 is locally asymptotically stable if $\mathfrak{R}_0 < 1$, which is equivalent to $\gamma > \frac{r\beta - d_1 d_2}{d_1}$. It is observed that the axial equilibrium point E_1 is locally asymptotically stable when the predator-extinction equilibrium point E_2 and interior equilibrium point E_3 do not exist. □

Now, the stability of the predator-extinction equilibrium point $E_2(S_2, I_2, 0)$ is discussed.

Theorem 4 *If $\mathfrak{R}_0 < 1 + \frac{\beta d_2 d_3}{d_1(\gamma + d_2)(c\alpha - ad_3)}$, then the predator-extinction equilibrium point E_2 of the fractional-order model (3) is locally asymptotically stable.*

Proof The Jacobian matrix (6) evaluated at E_2 is given by

$$J(E_2) = \begin{pmatrix} -(d_1 + \beta I_2) & -d_2 & 0 \\ \beta I_2 & 0 & -\frac{\alpha I_2}{1+aI_2} \\ 0 & 0 & \frac{c\alpha I_2}{1+aI_2} - d_3 \end{pmatrix}.$$

The eigenvalues of the Jacobian matrix around E_2 are the roots of the following equation:

$$\left(\frac{c\alpha I_2}{1 + aI_2} - d_3 - \mu\right) (\mu^2 + \mu(\beta I_2 + d_1) + \beta d_2 I_2) = 0. \tag{7}$$

The characteristic equation (7) has the following eigenvalues:

$$\mu_1 = \frac{c\alpha I_2}{1 + aI_2} - d_3 \quad \text{and} \quad \mu_{2,3} = \frac{1}{2}(-(\beta I_2 + d_1) \pm \sqrt{(\beta I_2 + d_1)^2 - 4\beta d_2 I_2}).$$

The eigenvalues $\mu_{2,3}$ have negative real parts, then $|\arg(\mu_{2,3})| = \pi > \frac{q\pi}{2}$. If $\Re_0 < 1 + \frac{\beta d_2 d_3}{d_1(\gamma + d_2)(c\alpha - ad_3)}$ then $|\arg(\mu_1)| = \pi > \frac{q\pi}{2}$ for all $0 < q < 1$. Thus, in accordance with Matignon’s condition [32, 54], the predator-extinction equilibrium point E_2 is locally asymptotically stable if $\Re_0 < 1 + \frac{\beta d_2 d_3}{d_1(\gamma + d_2)(c\alpha - ad_3)}$, which is equivalent to the attack rate of predator population (α) being less than $\frac{(\beta(ar+d_2)-ad_1(\gamma+d_2))d_3}{c(r\beta-d_1(\gamma+d_2))}$. □

Finally, the stability of interior equilibrium point $E_3(S_3, I_3, P_3)$ is investigated. The Jacobian matrix of system (3) around E_3 is given by

$$J(E_3) = \begin{pmatrix} -(\beta I_3 + d_1) & \gamma - \beta S_3 & 0 \\ \beta I_3 & \frac{\alpha P_3}{(1+aI_3)}(1 - \frac{1}{1+aI_3}) & -\frac{\alpha I_3}{1+aI_3} \\ 0 & \frac{c\alpha P_3}{(1+aI_3)^2} & 0 \end{pmatrix}.$$

The eigenvalues of $J(E_3)$ are the roots of the following equation:

$$F(\mu) = \mu^3 + K_1\mu^2 + K_2\mu + K_3 = 0, \tag{8}$$

where

$$K_1 = d_1 + I_3 \left(\beta - \frac{\alpha\alpha P_3}{(1 + aI_3)^2} \right),$$

$$K_2 = \frac{I_3(1 - \beta\gamma(+aI_3)^3 - \alpha(a(1 + aI_3)(d_1 + \beta I_3) - c\alpha)P_3 + \beta^2(1 + aI_3)^3 S_3}{(1 + aI_3)^3},$$

$$K_3 = \frac{c\alpha^2(\beta I_3 + d_1)I_3 P_3}{(1 + aI_3)^3}.$$

According to [55, 56], the local stability of the interior equilibrium point E_3 is determined by the following proposition.

Proposition 5 *The discriminant $D(V)$ of the cubic polynomial (8) is as follows:*

$$D(V) = 18K_1K_2K_3 + (K_1K_2)^2 - 4K_3K_1^3 - 4K_2^3 - 27K_3^2.$$

Then the conditions of local stability of the interior equilibrium point E_3 are given as follows:

- (i) *If $D(V) > 0, K_1 > 0, K_3 > 0$ and $K_1K_2 > K_3$, then E_3 is locally asymptotically stable for $0 < q < 1$.*
- (ii) *If $D(V) < 0, K_1 > 0, K_2 > 0, K_1K_2 = K_3$ and $0 < q < 1$, then E_3 is locally asymptotically stable.*

2.4 Global stability

The global asymptotic stability of the axial equilibrium point E_1 , predator-extinction equilibrium point E_2 and the interior equilibrium point E_3 of the fractional-order system (3) is now investigated.

Theorem 6 *If $\mathfrak{R}_0 < 1$, then the axial equilibrium point E_1 of the fractional-order system (3) is globally asymptotically stable.*

Proof We consider the following positive definite Lyapunov function:

$$V(S, I, P) = \left(S - \frac{r}{d_1} - \frac{r}{d_1} \ln \frac{d_1 S}{r} \right) + I + \frac{1}{c} P.$$

By calculating the q -order derivative of $V(S, I, P)$ along the solution of system (3) and utilizing Lemma 3.1 in [57], one has

$$\begin{aligned} {}^c D^q V(S, I, P) &\leq \left(\frac{d_1 S - r}{d_1 S} \right) {}^c D^q S(t) + {}^c D^q I(t) + \frac{1}{c} {}^c D^q P(t) \\ &\leq \frac{d_1 S - r}{d_1 S} (r - \beta SI - d_1 S + \gamma I) \\ &\quad + \beta SI - d_2 I - \gamma I - \frac{d_3}{c} P \\ &\leq \frac{d_1 S - r}{d_1 S} (-(Sd_1 + r) - (\beta S - \gamma)I) \\ &\quad + \beta SI - d_2 I - \gamma I - \frac{d_3}{c} P \\ &\leq \frac{-(d_1 S - r)^2}{d_1 S} - \frac{(Sd_1 - r)(\beta S - \gamma)I}{d_1 S} \\ &\quad + \beta SI - d_2 I - \gamma I - \frac{d_3}{c} P \\ &\leq \frac{-(d_1 S - r)^2}{d_1 S} + \frac{r\beta I}{d_1} - \frac{\gamma r I}{d_1 S} - d_2 I - \frac{d_3}{c} P \\ &\leq \frac{r\beta I}{d_1} - \frac{\gamma r I}{d_1 S} - d_2 I \\ &\leq \left(\frac{\beta r}{d_1} - \gamma - d_2 \right) I. \end{aligned}$$

Thus, ${}^c D^q V(S, I, P) \leq 0$ when $\frac{\beta r}{d_1} < \gamma + d_2$ which is equivalent to $\mathfrak{R}_0 < 1$. By Lemma 4.6 in [58], it is proof that the axial equilibrium point E_1 is globally asymptotically stable. \square

Theorem 7 *If $\mathfrak{R}_0 < 1 + \frac{\beta d_2 d_3}{d_1(\gamma + d_2)(c\alpha - ad_3)}$, then the predator-extinction equilibrium point E_2 is globally asymptotically stable.*

Proof To study the globally asymptotically stability of E_2 the following positive definite Lyapunov function is considered:

$$V(S, I, P) = L \left(S - S_2 - S_2 \ln \frac{S}{S_2} \right) + \left(I - I_2 - I_2 \ln \frac{I}{I_2} \right) + \frac{1}{c} P.$$

The q -order derivative of $V(S, I, P)$ is calculating along the solution of the fractional-order eco-epidemiological model (3) and by applying Lemma 3.1 in [57], we get

$${}^c D^q V(S, I, P) \leq L \left(\frac{S - S_2}{S} \right) {}^c D^q S(t) + \left(\frac{I - I_2}{I} \right) {}^c D^q I(t) + \frac{1}{c} {}^c D^q P(t)$$

$$\begin{aligned}
 &\leq L\left(\frac{S-S_2}{S}\right)(r-\beta SI-d_1S+\gamma I) \\
 &\quad + (I-I_2)\left(\beta S-\frac{\alpha P}{1+aI}-d_2-\gamma\right)+P\left(\frac{\alpha I}{1+aI}-\frac{d_3}{c}\right) \\
 &\leq L\left(\frac{S-S_2}{S}\right)(\beta S_2 I_2+d_1 S_2-\gamma I_2-\beta SI-d_1 S+\gamma I) \\
 &\quad + (I-I_2)\left(\beta S-\frac{\alpha P}{1+aI}-\beta S_2\right)+\frac{\alpha IP}{1+aI}-\frac{d_3}{c}P \\
 &\leq L\left(\frac{S-S_2}{S}\right)(-\beta I_2(S-S_2)-\beta S(I-I_2)-d_1(S-S_2)+\gamma(I-I_2)) \\
 &\quad + (I-I_2)\left(\beta(S-S_2)-\frac{\alpha P}{1+aI}\right)+\frac{\alpha IP}{1+aI}-\frac{d_3}{c}P,
 \end{aligned}$$

since $\frac{\gamma L}{S}(S-S_2)(I-I_2) = \frac{-\gamma L}{SS_2}(I-I_2)(S-S_2)^2 + \frac{\gamma L}{S_2}(S-S_2)(I-I_2)$,

$$\begin{aligned}
 {}^c D^q V(S, I, P) &\leq \left(\frac{\gamma L I_2}{SS_2}(S-S_2)^2 - \frac{(\beta I_2+d_1)L}{S}(S-S_2)^2\right) \\
 &\quad - \frac{\gamma L I}{SS_2}(S-S_2)^2 + \left(\beta - \beta L + \frac{\gamma L}{S_2}\right)(S-S_2)(I-I_2) \\
 &\quad + \left(\frac{\alpha I_2}{1+aI} - \frac{d_3}{c}\right)P \\
 &\leq -\frac{rL}{SS_2}(S-S_2)^2 + \frac{1}{S_2}(\beta S_2 - \beta L S_2 + \gamma L)(S-S_2)(I-I_2) \\
 &\quad + \left(\alpha I_2 - \frac{d_3}{c}\right)P.
 \end{aligned}$$

Suppose $L = \frac{\beta S_2}{\beta S_2 - \gamma} = 1 + \frac{\gamma}{d_2} > 0$. Hence, ${}^c D^q V(S, I, P) \leq 0$, when $\alpha I_2 - \frac{d_3}{c} < 0$, which equivalent to

$$\mathfrak{R}_0 < 1 + \frac{\beta d_2 d_3}{d_1(\gamma + d_2)(c\alpha - a d_3)}.$$

Hence the theorem is proved. □

Theorem 8 *The interior equilibrium point E_3 of the fractional-order system (3) is globally asymptotically stable if $\alpha a P_3 \theta S_3 (\theta^2 - 2\xi I_3 + I_3^2) < \gamma \xi M (1 + a I_3) (1 + a \xi) (\xi^2 - 2\theta S_3 + S_3^2)$.*

Proof The following positive definite Lyapunov function is considered to investigate the global asymptotic stability of the interior equilibrium point E_3 :

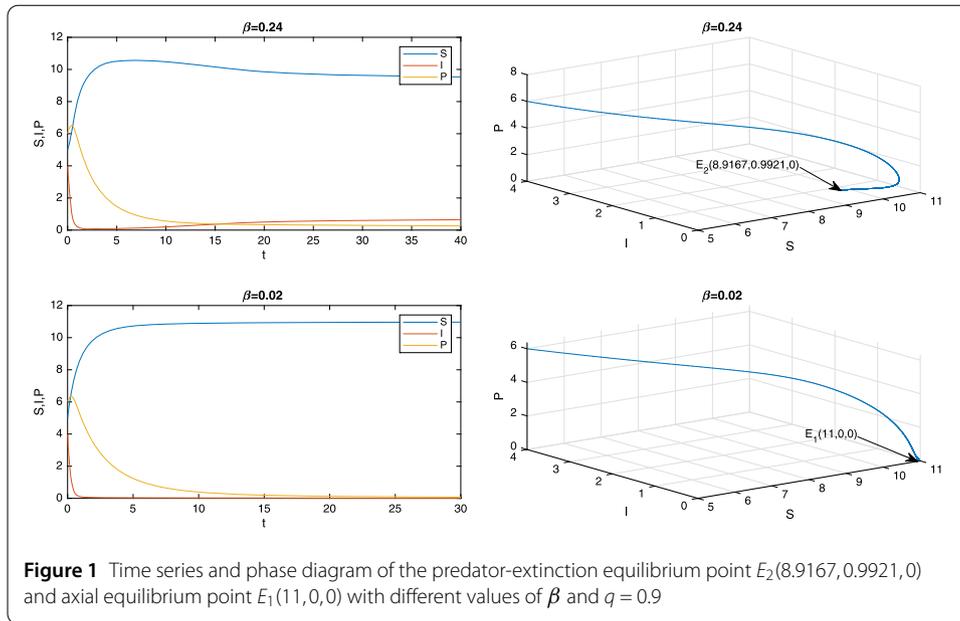
$$V(S, I, P) = M\left(S - S_3 - S_3 \ln \frac{S}{S_3}\right) + \left(I - I_3 - I_3 \ln \frac{I}{I_3}\right) + \left(\frac{1+aI_3}{c}\right)\left(P - P_3 - P_3 \ln \frac{P}{P_3}\right).$$

We calculate the time derivative of $V(S, I, P)$ along the solution of model (3) and by applying Lemma 3.1 in [57], one has

$${}^c D^q V(S, I, P)$$

$$\begin{aligned}
 &\leq M \left(\frac{S - S_3}{S} \right) {}^c D^q S(t) + \left(\frac{I - I_3}{I} \right) {}^c D^q I(t) + \left(\frac{1 + aI_3}{c} \right) \left(\frac{P - P_3}{P} \right) {}^c D^q P(t) \\
 &\leq M \left(\frac{S - S_3}{S} \right) (r - \beta SI - d_1 S + \gamma I) \\
 &\quad + (I - I_3) \left(\beta S - \frac{\alpha P}{1 + aI} - (d_2 + \gamma) \right) + (1 + aI_3)(P - P_3) \left(\frac{\alpha I}{1 + aI} - \frac{d_3}{c} \right) \\
 &\leq M \left(\frac{S - S_3}{S} \right) (-d_1(S - S_3) + \gamma(I - I_3) - \beta I_3(S - S_3) - \beta S(I - I_3)) \\
 &\quad + (I - I_3) \left(\beta S - \frac{\alpha P}{1 + aI} - \beta S_3 + \frac{\alpha P_3}{1 + aI_3} \right) \\
 &\quad + (1 + aI_3)(P - P_3) \alpha \left(\frac{I}{1 + aI} - \frac{I_3}{1 + aI_3} \right) \\
 &\leq -\frac{(d_1 + \beta I_3)M}{S} (S - S_3)^2 + \frac{\gamma M}{S} (S - S_3)(I - I_3) \\
 &\quad - \beta M(S - S_3)(I - I_3) + (I - I_3) \left(\beta S - \frac{\alpha P}{1 + aI} - \beta S_3 + \frac{\alpha P_3}{1 + aI_3} \right) \\
 &\quad + (1 + aI_3)(P - P_3) \alpha \left(\frac{I}{1 + aI} - \frac{I_3}{1 + aI_3} \right) \\
 &\leq -\frac{(d_1 + \beta I_3)M}{S} (S - S_3)^2 + \frac{\gamma M}{S_3} (S - S_3)(I - I_3) \\
 &\quad - \frac{\gamma M}{SS_3} (S - S_3)^2 (I - I_3) - \beta M(S - S_3)(I - I_3) \\
 &\quad + (I - I_3) \left(\beta S - \frac{\alpha P}{1 + aI} - \beta S_3 + \frac{\alpha P_3}{1 + aI_3} \right) \\
 &\quad + (1 + aI_3)(P - P_3) \alpha \left(\frac{I}{1 + aI} - \frac{I_3}{1 + aI_3} \right) \\
 &\leq \frac{M(S - S_3)^2}{SS_3} (\gamma I_3 - \beta I_3 S_3 - d_1 S_3) \\
 &\quad + \frac{1}{S_3} (S - S_3)(I - I_3) (\gamma M - \beta M S_3 + \beta S_3) \\
 &\quad + \frac{\alpha P_3(I - I_3)}{1 + aI_3} - \frac{\alpha P_3(I - I_3)}{1 + aI} - \frac{\gamma MI}{SS_3} (S - S_3)^2 \\
 &\leq -\frac{rM(S - S_3)^2}{SS_3} + \frac{1}{S_3} (S - S_3)(I - I_3) (\gamma M - \beta M S_3 + \beta S_3) \\
 &\quad + \frac{\alpha P_3(I^2 - 2II_3 + I_3^2)}{(1 + aI_3)(1 + aI)} - \frac{\gamma MI(S^2 - 2SS_3 + S_3^2)}{SS_3} \\
 &\leq -\frac{rM(S - S_3)^2}{SS_3} + \frac{1}{S_3} (S - S_3)(I - I_3) (\gamma M - \beta M S_3 + \beta S_3) \\
 &\quad + \frac{\alpha P_3(\theta^2 - 2\xi I_3 + I_3^2)}{(1 + aI_3)(1 + a\xi)} - \frac{\gamma M \xi (\xi^2 - 2\theta S_3 + S_3^2)}{\theta S_3}.
 \end{aligned}$$

Suppose $M = \frac{\beta S_3}{\beta S_3 - \gamma} = \frac{\beta S_3(1 + aI_3)}{\alpha P_3 + d_2(1 + aI_3)} > 0$ and $\xi < S, I < \theta$. Thus, ${}^c D^q V(S, I, P) \leq 0$, when $\alpha P_3 \theta S_3 (\theta^2 - 2\xi I_3 + I_3^2) < \gamma \xi M (1 + aI_3)(1 + a\xi) (\xi^2 - 2\theta S_3 + S_3^2)$. By Lemma 4.6 in [58], it is proof that the interior equilibrium point E_3 is globally asymptotically stable. \square



3 Numerical simulations

In this section, numerical simulations of the fractional-order eco-epidemiological model (3) are conducted to illustrate the theoretical results obtained before. The generalized Adams–Bashforth–Moulton type predictor-corrector scheme is applied in order to find an approximate solution for our fractional-order system [59, 60]. The role of prey’s disease (β), predator’s attack rate (α), half saturation constant (a), infected prey’s death rate (d_2) and fractional order (q) are discussed to validate our fractional-order model (3). We consider the following set of parameters:

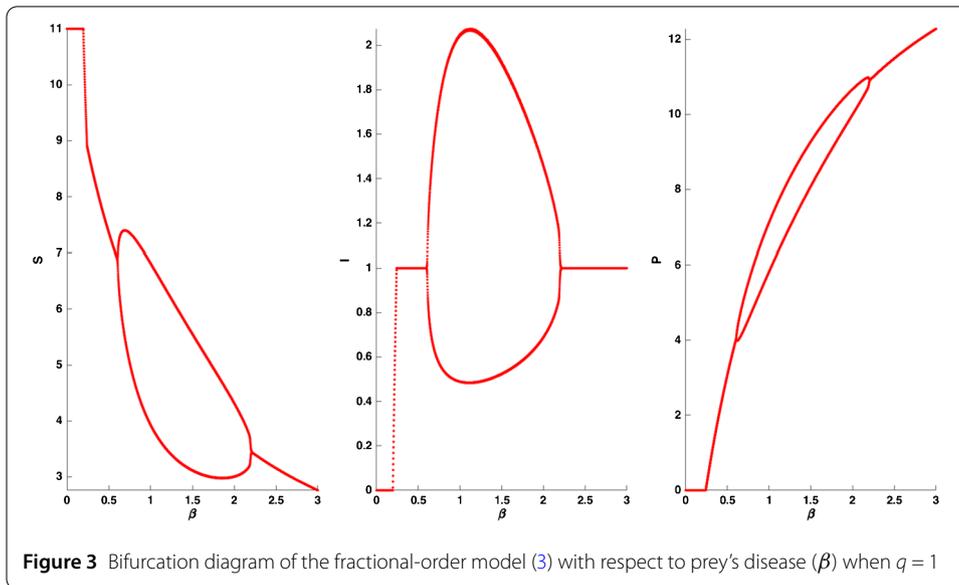
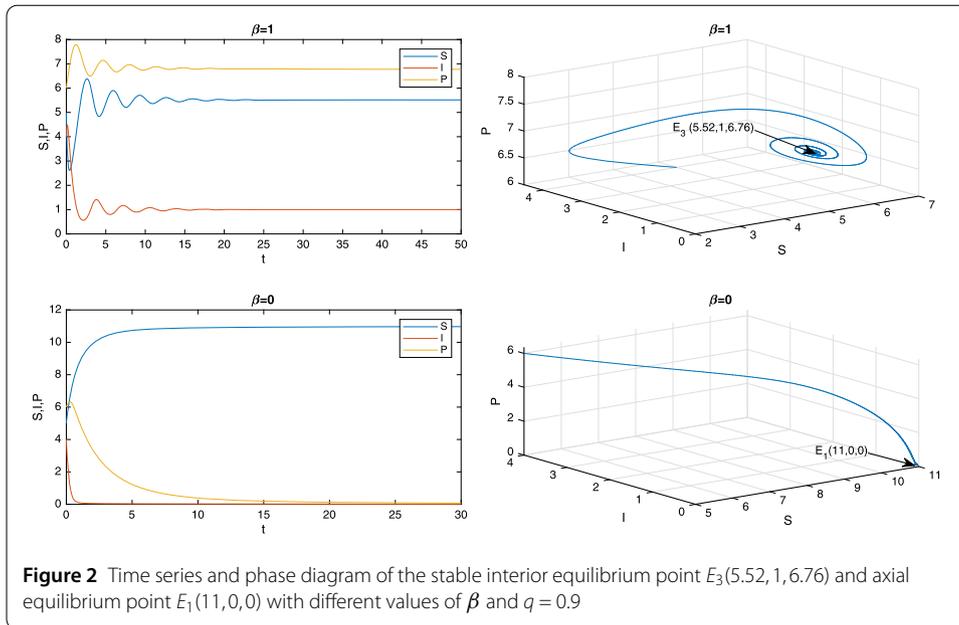
$$\begin{aligned}
 r = 11, \quad \beta = 1, \quad d_1 = 1, \quad d_2 = 2.1, \quad d_3 = 0.5, \quad c = 1, \\
 \gamma = 0.04, \quad \alpha = 1 \quad \text{and} \quad a = 1,
 \end{aligned}
 \tag{9}$$

which were also used for the integer-order model [11]. For the parameter set (9), the condition $\mathfrak{R}_0 = 5.14019 > 1$ holds and we get the predator-extinction equilibrium point $E_2(S_2, I_2, 0) = (2.14, 4.21905, 0)$, also the conditions $\alpha = 1 > \frac{ad_3}{c} = 0.5$ and $\mathfrak{R}_0 = 5.14019 > 1 + \frac{\beta d_2 d_3}{d_1(\gamma + d_2)(c\alpha - ad_3)} = 1.98131$ hold and one gets the interior equilibrium point $E_3(S_3, I_3, P_3) = (5.52, 1, 6.76)$. The axial equilibrium point is $E_1(S_1, 0, 0) = (11, 0, 0)$.

From Theorem 3, when $\beta = 0.02$ and $q = 0.9$, the condition of $\mathfrak{R}_0 = 0.102804 < 1$ holds and the axial equilibrium point $E_1(11, 0, 0)$ of the fractional-order system (3) is locally asymptotically stable as indicated in Fig. 1(b).

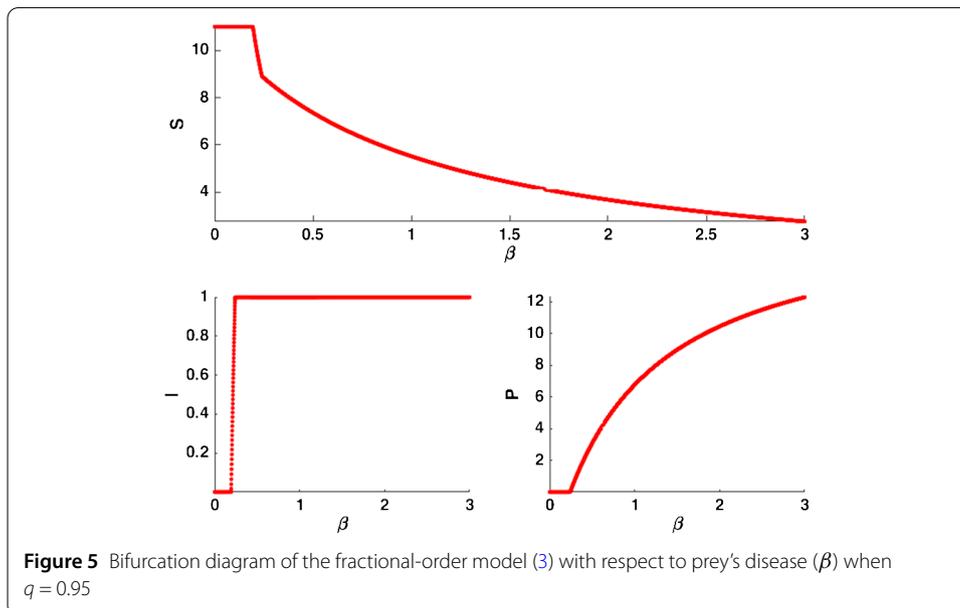
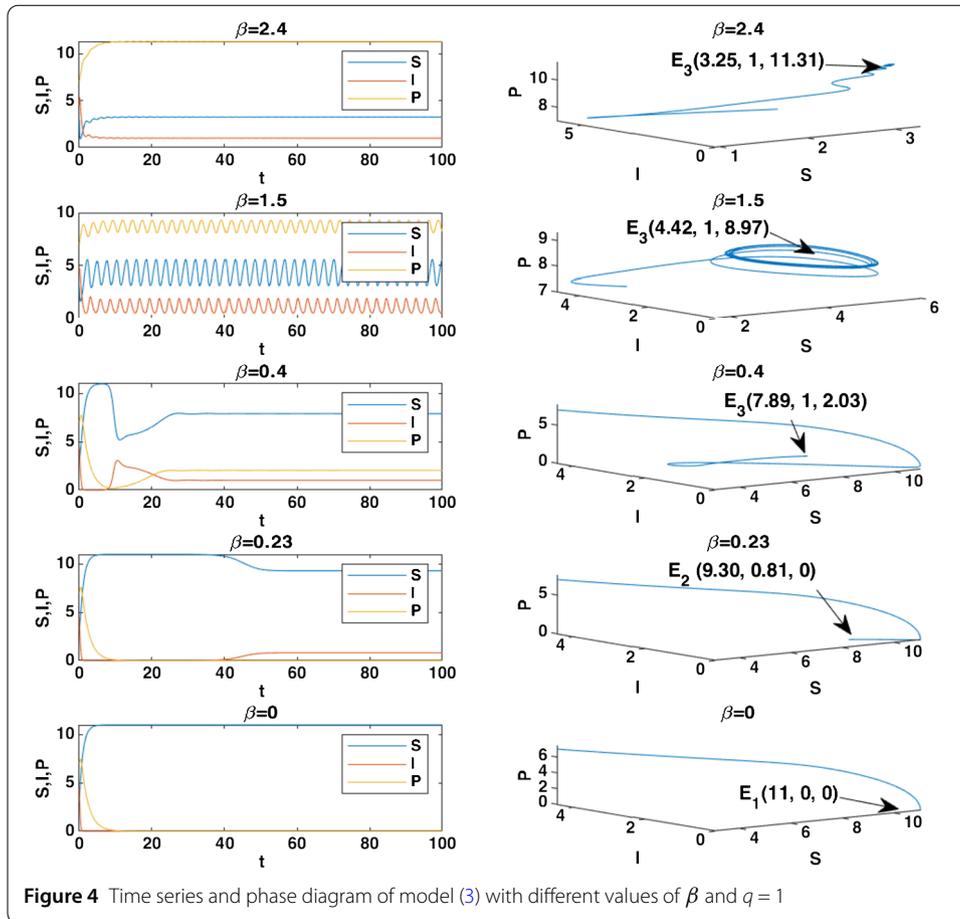
Following Theorem 4, when $\beta = 0.24$ and $q = 0.9$, the condition of $\mathfrak{R}_0 = 1.23364 < 1 + \frac{\beta d_2 d_3}{d_1(\gamma + d_2)(c\alpha - ad_3)} = 1.23551$ holds and the predator-extinction equilibrium point $E_2(8.91667, 0.992063, 0)$ of the fractional-order system (3) is locally asymptotically stable as shown in Fig. 1(a).

One can observe that, the dynamics of the population is stable around the interior equilibrium point $E_3(5.52, 1, 6.76)$ in the presence of a prey’s disease ($\beta = 1$) as shown in Fig. 2(a) coinciding with the bifurcation diagram 5. Furthermore, the infected prey and predator population go to extinction in the absence of prey’s disease ($\beta = 0$) as indicated

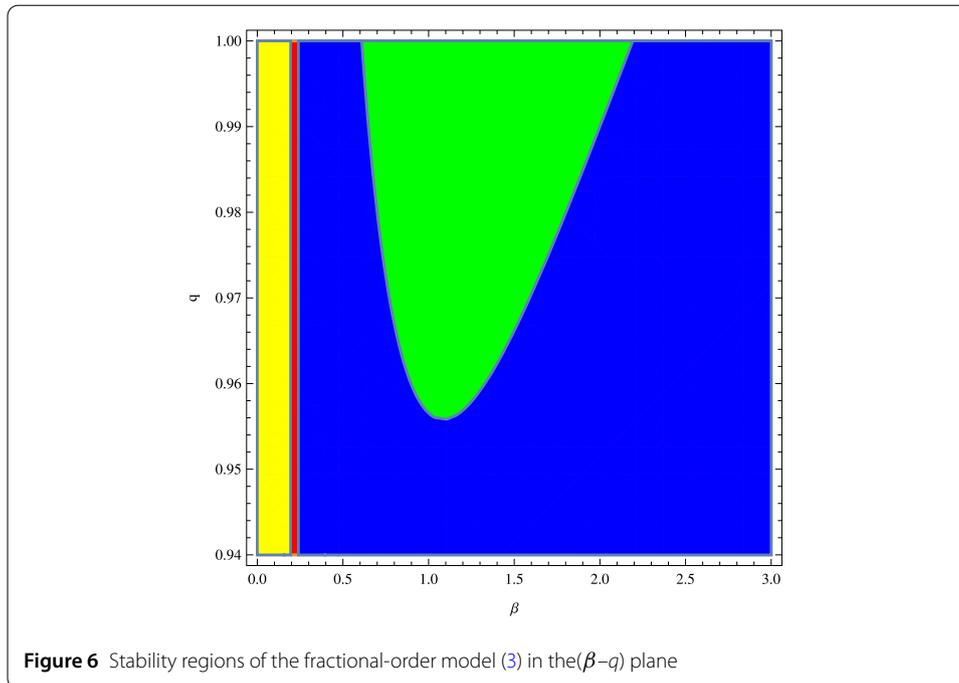


in Fig. 2(b) concurring with Fig. 5. Therefore, the prey's disease (β) has a crucial role in population dynamics and may make the dynamics stable between populations.

For better visualization of the impact of β on the dynamics of the integer-order eco-epidemiological model (when $q = 1$), we draw the bifurcation diagram with respect to β as shown in Fig. 3 coinciding with Fig. 4 and 6. The transcritical bifurcations and supercritical Hopf bifurcations values localized at $\beta_{tr1} = 0.194545$, $\beta_{tr2} = 0.240449$, $\beta_1^* = 0.608815$ and $\beta_2^* = 2.19262$, respectively, are shown in Fig. 3. It is observed that the system (3) shows stable extinction of the infected prey and predator populations for $\beta < \beta_{tr1}$, stable extinction of the predator population for $\beta_{tr1} < \beta < \beta_{tr2}$, stable coexistence of populations for $\beta_{tr2} < \beta < \beta_1^*$, cyclic dynamics occur for $\beta_1^* < \beta < \beta_2^*$ and stable coexistence of populations for $\beta > \beta_2^*$ as shown in Fig. 3 coinciding with Figs. 4 and 6.



To depict the impact of prey's disease (β) on the dynamics of the fractional-order eco-epidemiological model (3) (when $q = 0.9$) the bifurcation diagram is depicted in Fig. 3 with respect to β . The transcritical bifurcation value localized at $\beta_{tr} = 0.240449$ is shown



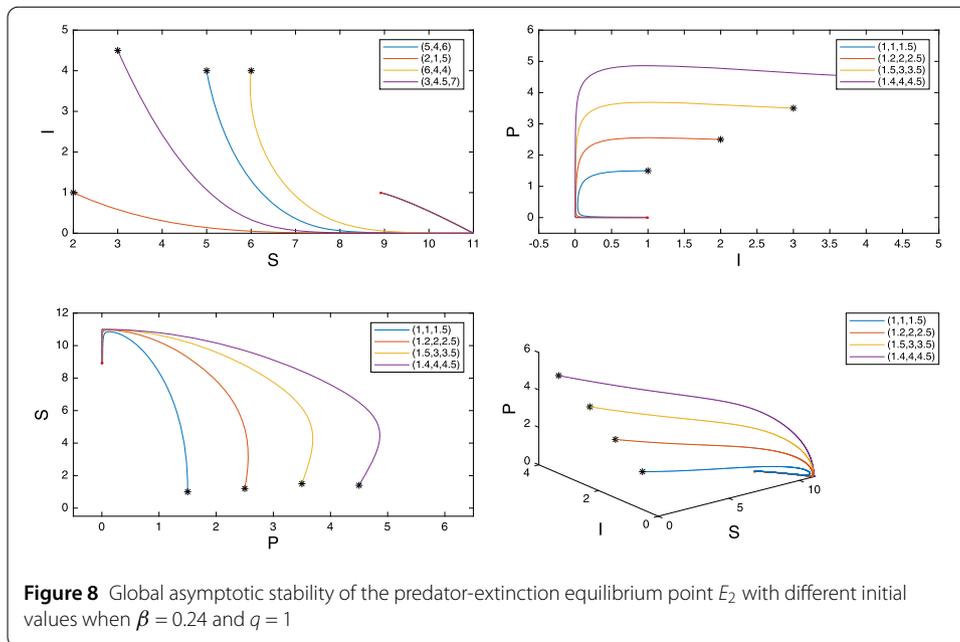
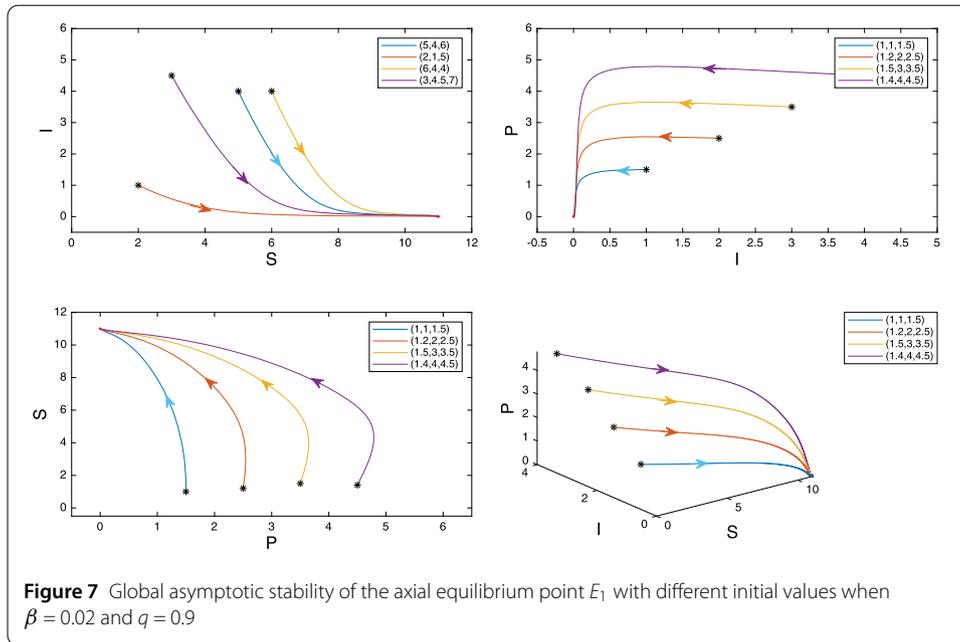
in Fig. 5. From Fig. 3 it is observed that the integer-order system (when $q = 1$) shows a limit cycle for $\beta_1^* < \beta < \beta_2^*$, which leads to the coexistence of populations in fractional-order case (when $q = 0.95$) as shown in Fig. 5 coinciding with Fig. 6. Therefore, the fractional order (q) and prey's disease (β) have stabilization effects and may help to control the disease.

Figure 6 shows the stability regions of the fractional-order system (3) with respect to the prey's disease (β) and fractional order (q). The regions are divided into four distinguished parts, where the yellow region shows the stable of axial equilibrium point E_1 , the red region shows stability of the predator-extinction equilibrium point E_2 , the blue region indicates the stability of the interior equilibrium point E_3 and the green region shows a limit cycle oscillation coinciding with theoretical results obtained before concurring with Figs. 3, 4 and 5. If we fix the parameter $\beta = 1.5$, then for $q < 0.966137$ the fractional-order system (3) is stable and the system changes its stability to show a limit cycle oscillation for $q > 0.966137$. It is to be noted that the three-species coexistence equilibrium point E_3 of the system (3) is stable in the fractional-order case when $q = 0.95$ becomes unstable in the integer-order case when $q = 1$. Therefore, the fractional order (q) has stabilization effects and can be used to control the persistence of species in the ecosystem.

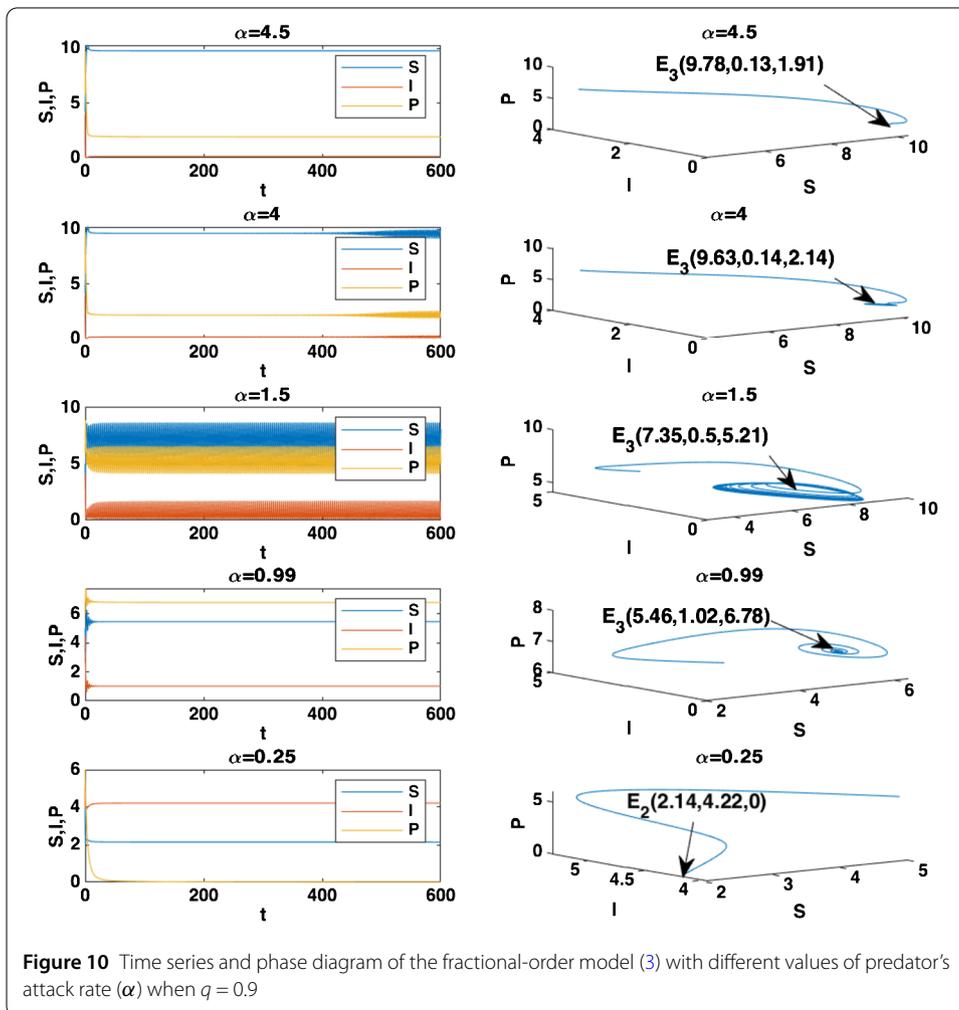
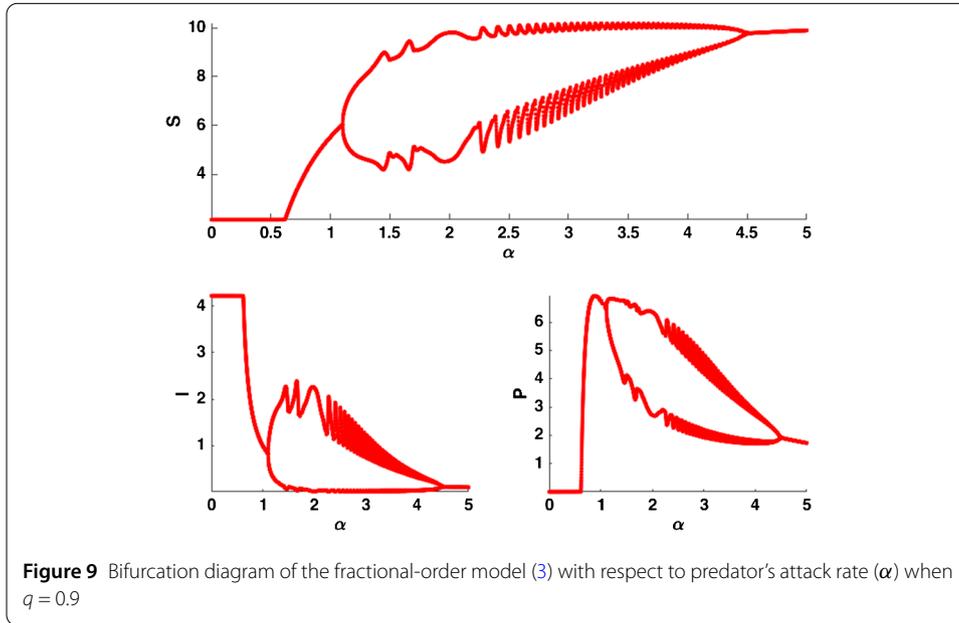
According to Theorem 6 one can observe that the condition of $\mathfrak{R}_0 = 0.1048 < 1$ holds when $\beta = 0.02$, and the axial equilibrium point $E_1(11, 0, 0)$ of the fractional-order system (3) is globally asymptotically stable as indicated in Fig. 7.

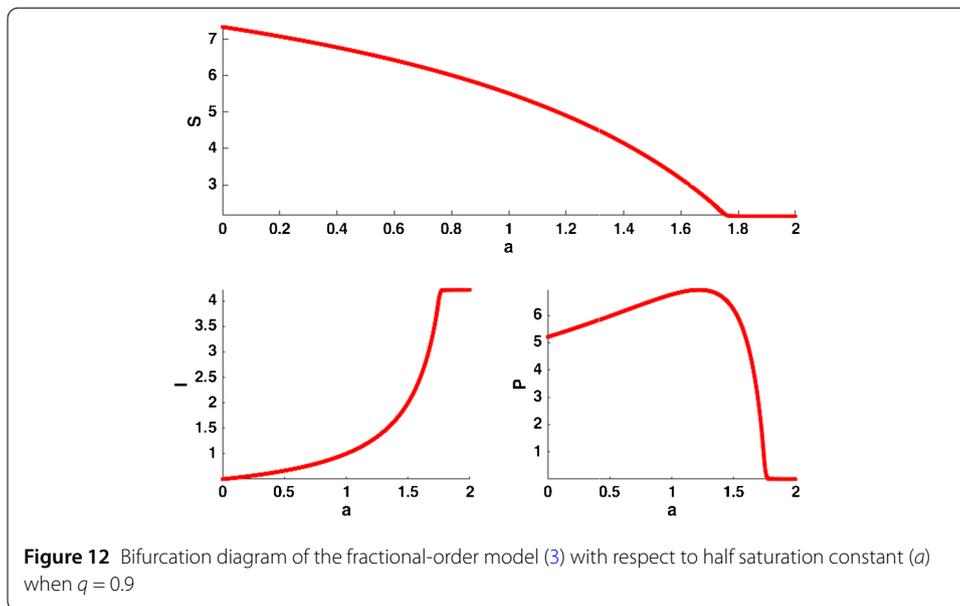
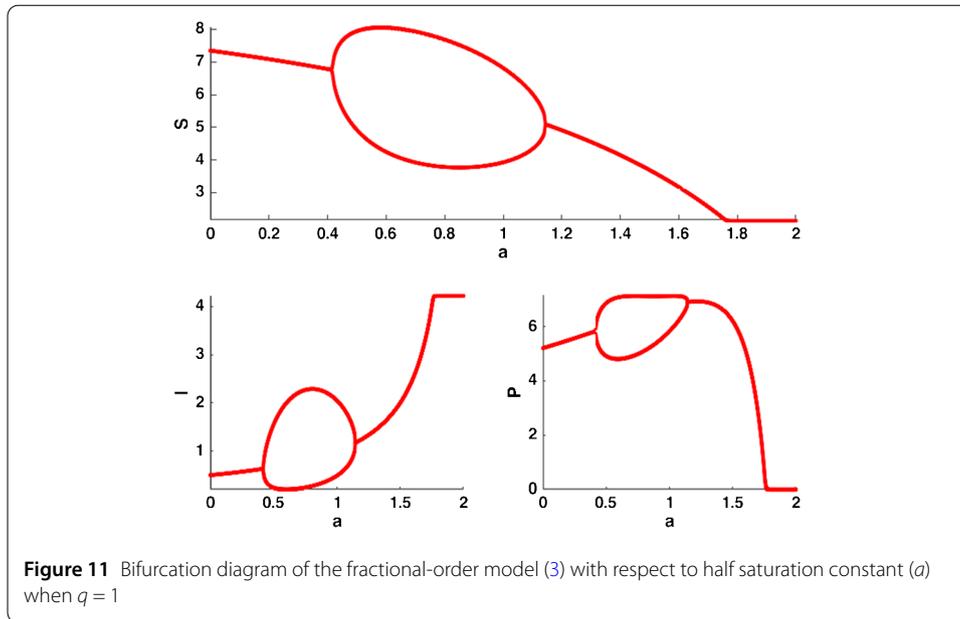
Figure 8 indicates that all trajectories with different positive initial conditions converge to the equilibria with predator-extinction outcome, E_2 when $\beta = 0.24$, which indicates that E_2 is globally asymptotically stable. In this case $\mathfrak{R}_0 = 1.23364 < 1 + \frac{\beta d_2 d_3}{d_1(\gamma + d_2)(c\alpha - ad_3)} = 1.23551$, concurring with the results of Theorem 7.

Figure 9 shows the bifurcation diagram around the interior equilibrium point E_3 with respect to the predator's attack rate (α) when $q = 0.9$. It can be observed that the fractional-order eco-epidemiological model (3) goes through the transcritical bifurcation value at $\alpha_{tr} = 0.61851$ and two supercritical Hopf bifurcation values at $\alpha_1^* = 1.10378$ and $\alpha_2^* =$



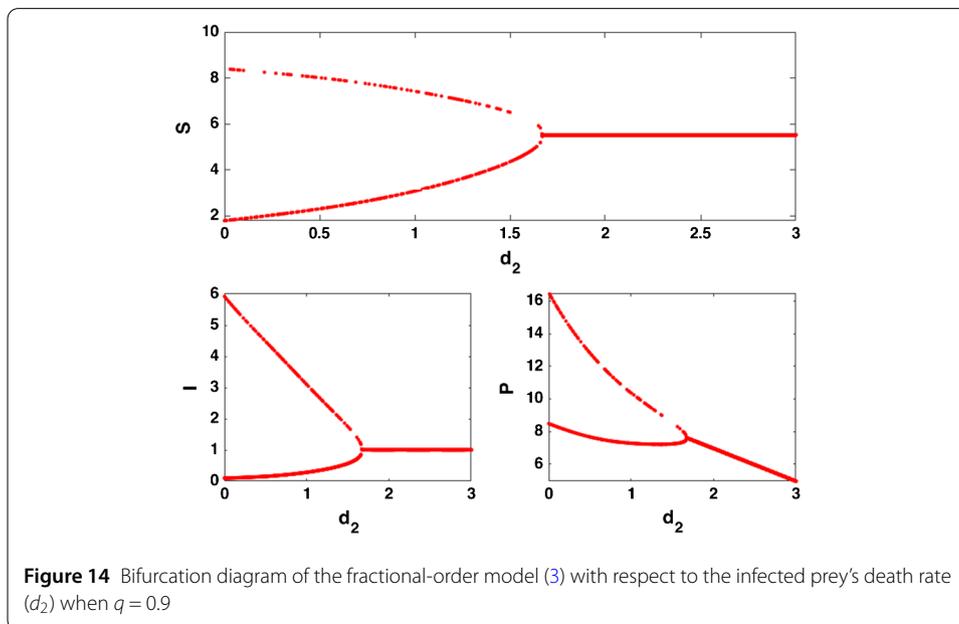
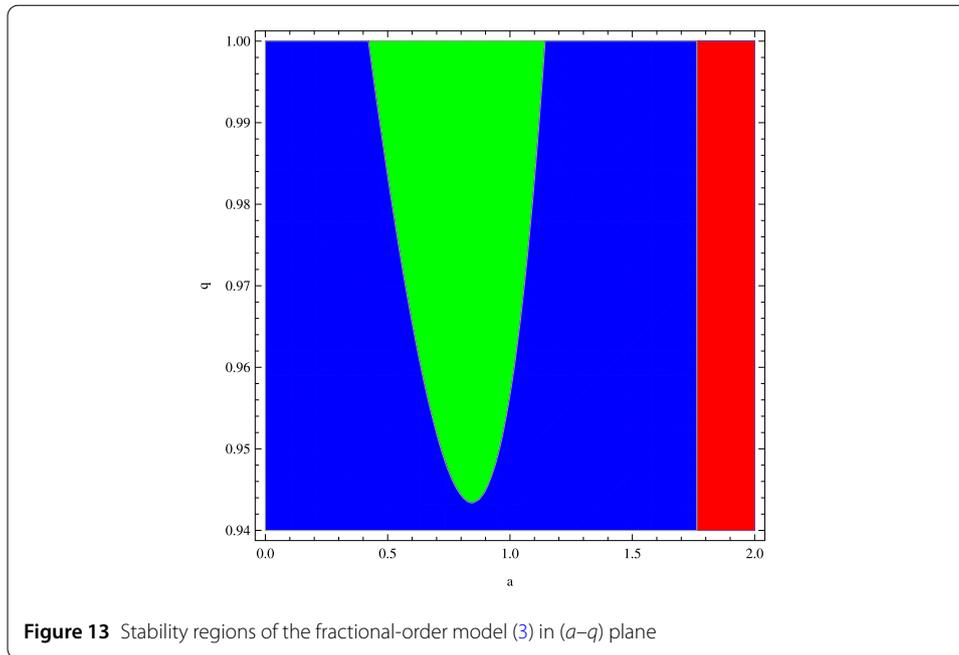
4.3697, respectively, as shown in Fig. 9. One can observe that below the transcritical bifurcation value, the predator population goes extinct and the populations of susceptible prey and infected prey survive as depicted in Fig. 9 corresponding to Fig. 10 when $\alpha = 0.25$. For the predator's attack rate (α) between the transcritical bifurcation value ($\alpha_{tr} = 0.61851$) and supercritical Hopf bifurcation value ($\alpha_1^* = 1.10378$) the system depicts the stable coexistence of susceptible prey, infected prey and predator populations as shown in Fig. 9 concurring with Fig. 10 when $\alpha = 0.99$. For the predator's attack rate (α) between supercritical Hopf bifurcation values ($\alpha_1^* = 1.10378$ and $\alpha_2^* = 4.3697$) the system shows a limit cycle oscillation as exhibited in Fig. 9 concurring with Fig. 10 when $\alpha = 1.5$ and $\alpha = 4$.





Above the supercritical Hopf bifurcation value ($\alpha_2^* = 4.3697$) the system exhibits the stable coexistence of populations as shown in Fig. 9 coinciding with Fig. 10 when $\alpha = 4.5$. Therefore, one can control the coexistence of populations by setting the predator’s attack rate between $\alpha_{tr} = 0.61851$ and $\alpha_1^* = 1.10378$ or above the supercritical Hopf bifurcation value ($\alpha_2^* = 4.3697$). Note that the predator’s attack rate (α) has a crucial effect on the dynamics of the fractional-order model (3) as shown in Fig. 9 coinciding with Fig. 10.

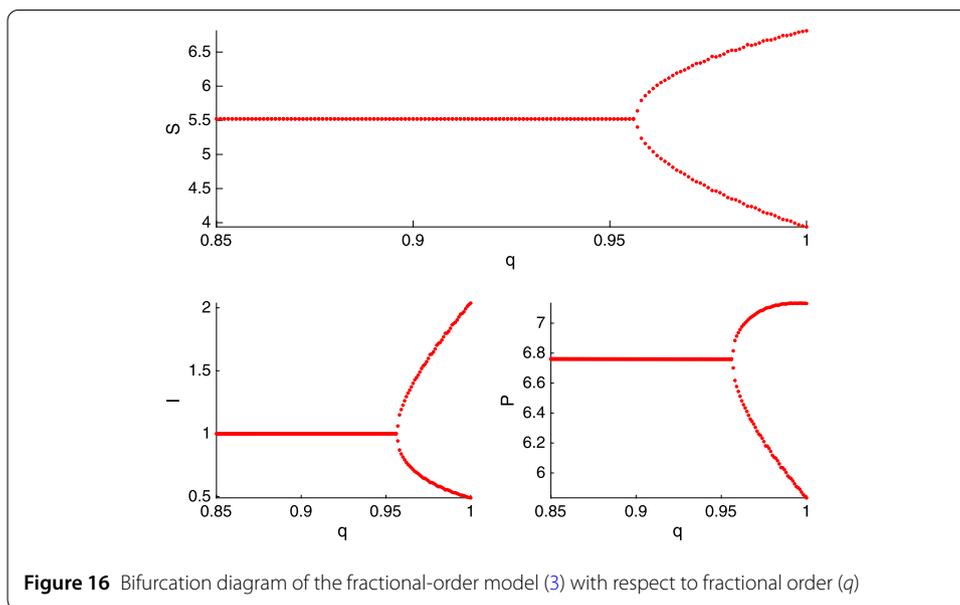
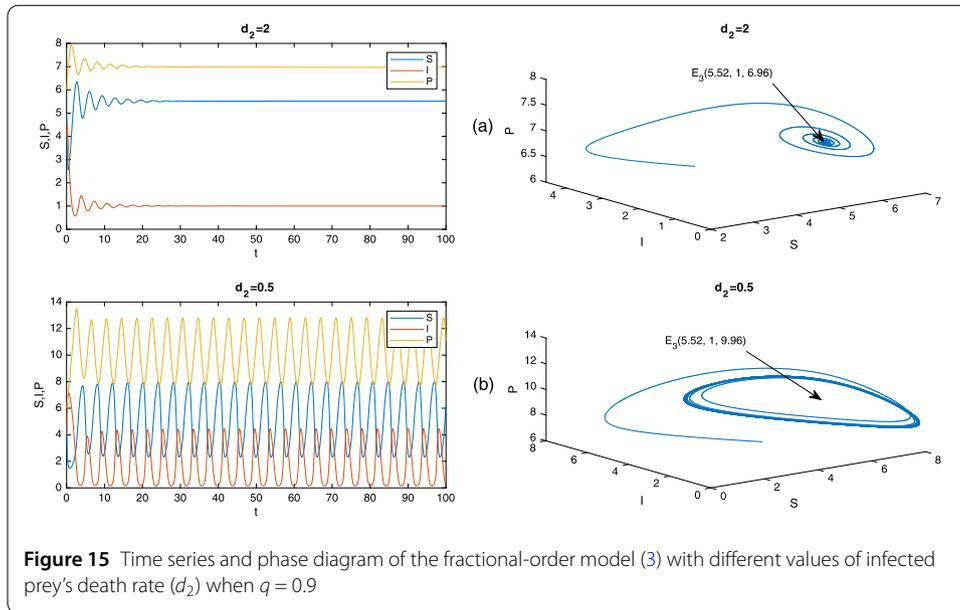
Figures 11 and 12 show the bifurcation analysis of the fractional-order system (3) with respect to half saturation constant (a) for integer-order case (when $q = 1$) and fractional-order case (when $q = 0.9$), respectively. Figure 11 shows that two Hopf bifurcation points and one transcritical bifurcation point emerge in the integer-order case (when $q = 1$). On



the other hand only one transcritical bifurcation point is observed in fractional-order case (when $q = 0.9$) and this is clearly shown in Fig. 12 coinciding with Fig. 13.

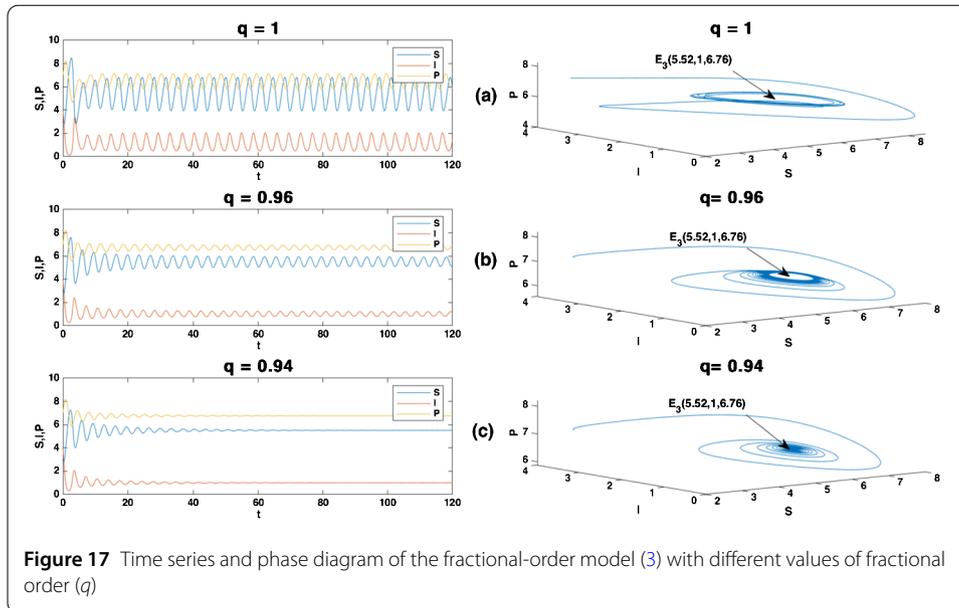
The stability regions of the fractional-order system (3) with respect to the half saturation constant (a) and fractional order (q) are presented in Fig. 13. The blue region shows the stability of the interior equilibrium point E_3 , the green region shows a limit cycle oscillation, the red region shows stability of predator-extinction equilibrium point E_2 .

In order to show the impact of infected prey's death rate (d_2) on the dynamics of the fractional-order eco-epidemiological model (3) the bifurcation diagram of the interior equilibrium point E_3 is drawn when ($q = 0.9$) as shown in Fig. 14. The supercritical Hopf



bifurcation value centralizes at $d_2^* = 1.67897$ as indicated in Fig. 14. When $d_2 > d_2^*$, the interior equilibrium point E_3 is locally asymptotically stable as depicted in Fig. 14 concurring with Fig. 15(a) when $d_2 = 2$. For $d_2 < d_2^*$, the system shows an unstable coexistence of susceptible prey, infected prey and predator populations as exhibited in Fig. 14 coinciding with Fig. 15(b) when $d_2 = 0.5$. Therefore, one can observe that the infected prey's death rate (d_2) has a critical role in the dynamics of the fractional-order model (3).

For better understand the effects of fractional order (q) on the dynamics of system (3) one can draw the bifurcation diagram considering fractional order (q) as a bifurcation parameter. Figure 16 exhibits the bifurcation diagrams for susceptible prey, infected prey and predator populations with respect to q . The supercritical Hopf bifurcation value centralizes at $q^* = 0.95653$ as indicated in Fig. 16. When $q < q^*$, the interior equilibrium



point $E_3(5.52, 1, 6.76)$ is locally asymptotically stable as shown in Fig. 16 concurring with Fig. 17(c) when $q = 0.94$. For $q > q^*$, the system undergoes limit cycle oscillations as exhibited in Fig. 16 coinciding with Fig. 17(b) when $q = 0.96$. It is important to notice that when $q = 1$ the fractional-order eco-epidemiological model (3) reduces to the epidemiological model [11]. Also, when $q = 1$ and $\gamma = 0$ the fractional-order eco-epidemiological model (3) reduces to the classical epidemiological model [10].

One can observe that the integer-order model (when $q = 1$) shows periodic limit cycle dynamics as shown in Fig. 16 concurring with Fig. 17(a) when $q = 1$. Furthermore, in the fractional-order case (when $q = 0.94$) the system exhibits the stable co-existence of susceptible prey, infected prey and predator as indicated in Fig. 16 coinciding with Fig. 17(c) when $q = 0.94$. Hence, the fractional order (q) has stabilization effects and it may help to control the coexistence between of susceptible prey, infected prey and predator populations when $0 < q < q^*$.

4 Conclusion

In this paper, a fractional-order eco-epidemiological model incorporating the predator’s attack rate (α) and half saturation constant (a) with infection in the prey population is formulated and analyzed in which the populations have been divided into susceptible prey, infected prey and predator. A sufficient condition for the existence as well as uniqueness of the fractional-order eco-epidemiological model (3) has been obtained. It has been proved that the solutions of the fractional-order system of differential equations (3) are uniformly bounded and non-negative. The local and global stability of the equilibrium points of the fractional-order eco-epidemiological model (3) has been investigated. The threshold parameter (\mathfrak{R}_0) has been used to determine the existence and stability conditions of the equilibrium points. It has noted that the axial equilibrium point E_1 is locally asymptotically stable when the predator-extinction equilibrium point E_2 and the interior equilibrium point E_3 do not exist. Numerical simulations have been conducted to illustrate our analytical results. For better visualization of the effects of the prey’s disease (β), predator’s attack rate (α), half saturation constant (a), infected prey’s death rate (d_2) and fractional order (q) on

the dynamics of the fractional-order eco-epidemiological model (3) we have drawn the bifurcation diagrams with respect to β, α, a, d_2 and q , respectively. It has been shown that the prey's disease may stabilize the system. Also, it has to be noted the predator's attack rate, the half saturation constant and the infected prey's death rate have crucial effect on the dynamics of the fractional-order model (3). Furthermore, it is observed that the fractional order (q) is important for the stabilization of the system and it may help to control the coexistence between of susceptible prey, infected prey and predator populations when $0 < q < q^*$.

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Authors' contributions

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